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Abstract. We compute a lower bound of Billingsley–Hausdorff dimension, defined by Gibbs measure, of the level set related to Birkhoff average in the countable symbolic space $\mathbb{N}^{\mathbb{N}}$.

1. Introduction and main result

Let $\mathcal{X}$ be the countable symbolic space $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology. Consider the shift map $\sigma : \mathcal{X} \to \mathcal{X}$ defined by $\sigma(x_1 x_2 \ldots) = x_2 x_3 \ldots$. An element $x_1 \ldots x_n \in \mathbb{N}^n$ is called an n-length word. Let $\mathcal{A}^* = \bigcup_{n \geq 0} \mathbb{N}^n$ stand for the set of all finite words, where $\mathbb{N}^0$ denotes the set of empty word. If $x \in \mathcal{A}^*$ and $y \in \mathcal{A}^* \cup \mathcal{X}$, then xy denote the concatenation of x and y. Let $x = x_1 x_2 \ldots \in \mathcal{X}$ and $m \geq n \geq 1$, we set $x|_n^m = x_n \ldots x_m$ denotes a subword of x. For $\omega = \omega_1 \ldots \omega_n \in \mathbb{N}^n$, the n-cylinder $[\omega]$ is defined by

$$[\omega] = \{x \in \mathcal{X};\ x|_1^n = \omega\}.$$

We will denote by $\mathcal{C}^n$ the set of all n-cylinders for $n \geq 0$. There is a one-to-one correspondence between $\mathbb{N}^n$ and $\mathcal{C}^n$. Let $\mathcal{C}^* = \bigcup_{n \geq 0} \mathcal{C}^n$ denote the set of all cylinders. For $k, N \geq 1$ we will write

$$\Sigma_N^k = \{1, \ldots, N\}^k \quad \text{and} \quad \mathcal{C}_N^k = \{[\omega];\ \omega \in \Sigma_N^k\}.$$

Let φ be a potential on $\mathcal{X}$ and ν be the Gibbs measure associated to φ (see Section 4.1 for the definition). without loss of generality, under the Remark 7, we suppose that $P(\varphi) = 0$ in the rest of this paper. Recall that ν induces a metric ρ_ν on $\mathcal{X}$: for any $x, y \in \mathcal{X}$, if $x = y$, we define $\rho_\nu(x, y) = 0$; otherwise

$$\rho_\nu(x, y) = \nu([x|_1^n]),$$

where $n = \min\{k \geq 0;\ x_{k+1} \neq y_{k+1}\}$. In addition the metric ρ_ν admits some kind of uniform property. In deed, if we define for $n \geq 1$, the integer

$$\delta_n = \sup\{\nu([u]);\ u \in \mathbb{N}^n\},$$

then from Proposition 9 in [11], we have $\lim_{n\to\infty} \delta_n = 0$. Let us recall the definition of Hausdorff dimension of set $E \subseteq \mathscr{X}$ with respect to ρ_v. Since v is non-atomic probability measure. For any $t \geq 0$ and $\delta > 0$,

$$\mathscr{H}^t_{v,\delta}(E) = \inf\left\{\sum_i v(E_i)^t : \{E_i\} \text{ countable cover of } E, \, v(E_i) \leq \delta\right\}$$

and

$$\mathscr{H}^t_v(E) = \lim_{\delta \to 0} \mathscr{H}^t_{v,\delta}(E).$$

The Billingsley dimension (for more details, see [3, 5]) $\dim_v(E)$ of E is

$$\dim_v(E) = \inf\left\{t \geq 0; \, \mathscr{H}^t_v(E) = 0\right\} = \sup\left\{t \geq 0; \, \mathscr{H}^t_v(E) = +\infty\right\}.$$

Let us come back to our problem. For $f \in C_b(\mathscr{X})$, the space of all bounded real-valued continuous functions on $\mathscr{X}$, we will be interested in the level sets determined by the Birkhoff averages of f defined as

$$E_f(\alpha) = \left\{x \in \mathscr{X}; \, \lim_{n\to+\infty} \frac{1}{n} S_n f(x) = \alpha\right\},$$

where $S_n f(x) = \sum_{k=0}^{n-1} f(\sigma^k x)$ is the n-th ergodic sum of f and $\alpha \in \mathbb{R}$.

The central question in the multifractal analysis of Birkhoff averages is to study geometrically the sets $E_f(\alpha)$ by computing their Hausdorff dimensions. Indeed, for any ergodic probability measure μ (see [3] for the definition), it follows from the Birkhoff ergodic theorem that,

$$\text{for } \mu\text{-almost all } x, \quad \lim_{n\to+\infty} \frac{1}{n} S_n f(x) = \int_{\mathscr{X}} f \, d\mu.$$

However, the Birkhoff ergodic theorem provides no structural information about the exceptional set of measure zero. The classical multifractal analysis of Birkhoff averages was studied initially by Pesin and Weiss in finite symbolic space and for Hölder potentials in [25]. Then, Fan et al., [8, 9] extended this to continuous potentials. We must also mention the work of [1, 2, 6, 7, 13, 17–19, 26], under always a different assumptions. Later, Li and Ma in [22], compute the Billingsley dimension, defined by Gibbs measure, of the set $E_f(\alpha)$. All these works deal with compact space.

The difference is the (countable) infinity of the alphabet and then the space $\mathscr{X}$ is not compact [27–29]. In addition, in this case, some particular phenomenon does not hold, see [16] for an example showing the difference between finite and infinite alphabet. There has also been a great deal of work dealing in the case of infinite symbolic space see [10–12, 14–16, 21, 23, 24].

For $n \geq 1$ and $x \in \mathscr{X}$, we define the orbit measure by

$$\Delta_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\sigma^k x}$$

and we denote by V_x the set of all limit points in w^*-topology of $\{\Delta_{x,n}\}_{n\geq 1}$. In the case of finite symbols, V_x is a non-empty connected compact while, in our case, V_x maybe empty for some $x \in \mathscr{X}$. For this raison, let us consider the set

$$\widehat{E}_f(\alpha) = \left\{x \in \mathscr{X}; \, V_x \neq \varnothing \text{ and } \lim_{n\to+\infty} \frac{1}{n} S_n f(x) = \alpha\right\}.$$

In this paper, we estimate the sizes of these levels sets by the Billingsley dimension, defined by Gibbs measures.

Let $\mathscr{P}(\mathscr{X})$ denotes the set of Borel probability measures on $\mathscr{X}$ and $\mathscr{P}_\sigma(\mathscr{X})$ denotes the set of σ-invariant Borel probability measures on $\mathscr{X}$. For $\mu \in \mathscr{P}_\sigma(\mathscr{X})$, define the entropy dimension of v with respect to μ by

$$\beta(v, \mu) = \limsup_{k\to\infty} \limsup_{N\to\infty} \frac{H_{k,N}(\mu, \mu)}{H_{k,N}(v, \mu)},$$

where

$$H_{k,N}(\nu,\mu) = -\sum_{\omega \in \Sigma_N^k} \mu([\omega]) \log \nu([\omega])$$

and define the convergence exponent of ν by

$$\alpha_\nu = \inf\left\{t > 0;\ \sum_{n=0}^{\infty} \nu([n])^t < +\infty\right\} \in [0,1].$$

Our main result is the following

Theorem 1. *Let φ be a potential function of summable variations. Assume that φ admits a unique Gibbs measure ν. We have*

$$\dim_\nu \widehat{E}_f(\alpha) \le \sup\left\{\gamma(\nu,\mu);\ \int f\,\mathrm{d}\mu = \alpha\right\},$$

where

$$\gamma(\nu,\mu) = \max\{\alpha_\nu, \beta(\nu,\mu)\}.$$

This result should be compared the result in [22] in the case of finite symbols. In deed in this case, the convergent exponent is zero and is not involved [11, 14].

2. Proof of the main result

We consider the set of quasi generic points with respect to ν,

$$E(\nu,\gamma) = \left\{x \in \mathcal{X};\ \exists \mu \in V_x \text{ with } \gamma(\nu,\mu) \le \gamma\right\}, \quad (0 \le \gamma \le 1).$$

Proposition 2. *Let φ be a potential function of summable variations. Assume that φ admits a unique Gibbs measure ν with convergence exponent α_ν. Then, for any $0 \le \gamma \le 1$, we have*

$$\dim_\nu E(\nu,\gamma) \le \gamma.$$

Remark 3. It's clear that Proposition 2 generalizes Bowen's result [4] in the case of infinite symbolic space.

Let $\alpha \in \mathbb{R}$ and $x \in \widehat{E}_f(\alpha)$, we have $\lim_{n \to +\infty} \int f\,d(\Delta_{x,n}) = \alpha$. That means, since V_x is not empty, there exists $\mu \in V_x$ such that $\int f\,\mathrm{d}\mu = \alpha$. It follows that

$$\widehat{E}_f(\alpha) \subseteq \left\{x \in \mathcal{X};\ \exists \mu \in V_x,\ \text{and}\ \int f\,\mathrm{d}\mu = \alpha\right\}.$$

Thus we have $\widehat{E}_f(\alpha) \subseteq E(\nu,\gamma_\alpha)$, where

$$\gamma_\alpha = \sup\left\{\gamma(\nu,\mu),\ \int f\,\mathrm{d}\mu = \alpha\right\}.$$

We deduce the result from Proposition 2.

3. Proof of Proposition 2

Let $\mu \in \mathscr{P}_\sigma(\mathcal{X})$, define the relative entropy of ν with respect to μ by

$$h(\nu,\mu) = \limsup_{k \to \infty} -\frac{1}{k} \sum_{\omega \in \mathbb{N}^k} \mu([\omega]) \log \nu([\omega])$$

and, when $\mu = \nu$, $h(\mu,\mu)$ will be denoted by h_μ. We have $h(\nu,\mu) = -\int_{\mathcal{X}} \varphi\,\mathrm{d}\mu$ (see [11, Proposition 8]). We define for $a \in \mathbb{Q}_+^*$,

$$E_a(\nu,\gamma) = \left\{x \in \mathcal{X};\ \exists \mu \in V_x \text{ with } h(\nu,\mu) \ge a \text{ and } \gamma(\nu,\mu) \le \gamma\right\}$$

and

$$E_0(\nu,\gamma) = \Big\{x \in \mathscr{X}; \ \exists \mu \in V_x \ \text{with} \ h(\nu,\mu) = 0 \ \text{and} \ \gamma(\nu,\mu) \le \gamma\Big\}.$$

It is clair that

$$E(\nu,\gamma) = \left(\bigcup_{a \in \mathbb{Q}_+^*} E_a(\nu,\gamma)\right) \cup E_0(\nu,\gamma).$$

Proposition 4. $\dim_\nu\big(E_0(\nu,1)\big) \le 1$ *and, for all* $\gamma \in [0,1[$*, we have* $\dim_\nu\big(E_0(\nu,\gamma)\big) = 0$.

Proof. Recall that $h(\nu,\mu) = 0$ imply that $\mu = \nu$ (Subsection 4.3), then

$$E_0(\nu,\gamma) \subseteq \Big\{x \in \mathscr{X}; \ \exists \mu \in V_x \ \text{with} \ \nu = \mu \ \text{and} \ \gamma(\nu,\mu) \le \gamma\Big\}$$
$$= \Big\{x \in \mathscr{X}; \ V_x = \{\nu\} \ \text{with} \ \gamma(\nu,\nu) = 1 \le \gamma\Big\}.$$

It's clear that, if $\gamma \ne 1$, we have $E_0(\nu,\gamma) = \emptyset$ and then $\dim_\nu\big(E_0(\nu,\gamma)\big) = 0$. Else, the set $E_0(\nu,1)$ is a subset of the set of generic points G_ν defined by

$$G_\nu = \Big\{x \in \mathscr{X}; \ V_x = \{\nu\}\Big\}.$$

We deduce the result since $\dim_\nu G_\nu = 1$ ([11]).   □

Suppose that we have shown, for all $a \in \mathbb{Q}_+^*$,

$$\dim_\nu E_a(\nu,\gamma) \le \gamma. \tag{1}$$

Then by the stability of ν-Hausdorff dimension, we have, for all $\gamma \in [0,1]$

$$\dim_\nu E(\nu,\gamma) = \sup\left\{\sup_{a \in \mathbb{Q}_+^*} \dim_\nu E_a(\nu,\gamma), \dim_\nu E_0(\nu,\gamma)\right\} \le \gamma.$$

We only need to prove (1). Let $\varepsilon > 0$ and $x \in E_a(\nu,\gamma)$, there exist $\mu \in V_x$ such that for any k, N large enough,

$$\frac{H_{k,N}(\mu,\mu) + 5k\varepsilon}{H_{k,N}(\nu,\mu)} \le \frac{h_\mu + 6\varepsilon}{h(\nu,\mu)} \le \gamma + \frac{6\varepsilon}{a}. \tag{2}$$

Thus, we have

$$E(\nu,\gamma) \subseteq \bigcup_{l \ge 1} \bigcap_{k \ge l} \bigcup_{l_1 \ge 1} \bigcap_{N \ge l_1} A(\varepsilon,k,N),$$

where

$$A(\varepsilon,k,N) = \Big\{x \in \mathscr{X}; \ \exists \mu \in V_x \ \text{with} \ h(\nu,\mu) \ge a \ \text{and} \ (2) \ \text{hold}\Big\}.$$

By the σ-stability and monotony of ν-Hausdorff dimension, we have

$$
\begin{aligned}
\dim_\nu E(\nu,\gamma) &\le \sup_{l \ge 1} \dim_\nu \left(\bigcap_{k \ge l} \bigcup_{l_1 \ge 1} \bigcap_{N \ge l_1} A(\varepsilon,k,N)\right) \\
&= \lim_{l \to +\infty} \dim_\nu \left(\bigcap_{k \ge l} \bigcup_{l_1 \ge 1} \bigcap_{N \ge l_1} A(\varepsilon,k,N)\right) \\
&\le \liminf_{k \to +\infty} \dim_\nu \left(\bigcup_{l_1 \ge 1} \bigcap_{N \ge l_1} A(\varepsilon,k,N)\right) \\
&= \liminf_{k \to +\infty} \sup_{l_1 \ge 1} \dim_\nu \left(\bigcap_{N \ge l_1} A(\varepsilon,k,N)\right) \\
&= \liminf_{k \to +\infty} \lim_{l_1 \to +\infty} \dim_\nu \left(\bigcap_{N \ge l_1} A(\varepsilon,k,N)\right) \\
&\le \liminf_{k \to +\infty} \liminf_{N \to +\infty} \dim_\nu \Big(A(\varepsilon,k,N)\Big). \tag{3}
\end{aligned}
$$

Suppose that, for all $t > \gamma, \varepsilon > 0$ and $N, k \geq 1$, we have

$$\mathcal{H}_\nu^t\big(A(\varepsilon, k, N)\big) < \infty, \tag{4}$$

then $\dim_\nu(A(\varepsilon, k, N)) \leq \gamma$ and we get the desire upper bound under (3).

Let us prove (4). Let $\mu \in V_x$, then, for any $s \geq 1$, there exists $n(s)$ such that, for all $n \geq n(s)$, we have

$$d^*(\Delta_{x,n}, \mu) \leq \frac{1}{s}.$$

Then, according to (2), we get

$$\frac{H_{k,N}(\Delta_{x,n}, \Delta_{x,n}) + 5k\varepsilon}{H_{k,N}(\nu, \Delta_{x,n})} \leq \frac{h_\mu + 7\epsilon}{h(\nu, \mu)} \leq \gamma + \frac{7\epsilon}{a}. \tag{5}$$

By the uniform continuity of $H_{k,N}(.,.)$, we have

$$A(\varepsilon, k, N) \subseteq \bigcap_{s \geq 1} \bigcup_{n \geq s} C_n(\varepsilon, k, N), \tag{6}$$

where

$$C_n(\varepsilon, k, N) = \Big\{x \in \mathscr{X}; \ \exists \mu \in \mathscr{M}_\sigma(\mathscr{X}) \ \text{with} \ h(\nu, \mu) \geq a \ \text{and} \ (5) \ \text{hold}\Big\}.$$

In order to estimate the dimension of $A(\varepsilon, k, N)$, let us consider the $(n + k - 1)$-prefixes of the points in $C_n(\varepsilon, k, N)$ defined as

$$\Lambda_n(\varepsilon, k, N) = \Big\{x|_1^{n+k-1} \in \mathbb{N}^{n+k-1}; \ x \in C_n(\varepsilon, k, N)\Big\}.$$

Let $\delta_n = \{\nu([u]); \ u \in \mathbb{N}^n\}$ and recall that $\lim_{n \to \infty} \delta_n = 0$. Let $s \geq 1$, then the cylinder set $\{[u]; \ u \in \bigcup_{n \geq n(s)} \Lambda_n(\varepsilon, k, N)\}$ forms a $\delta_{n(s)+k-1}$-covering of $A(\varepsilon, k, N)$.

By the definition of (ν, t)-Hausdorff measure, we have, for $t > \gamma$,

$$\mathcal{H}_{\delta_{n(s)+k-1}}^t\big(A(\varepsilon, k, N)\big) \leq \sum_{n \geq n(s)} \sum_{u \in \Lambda_n(\varepsilon, k, N)} \nu([u])^t.$$

3.1. Decomposition of $\Lambda_n(\varepsilon, k, N)$

Here we use a general decomposition introduced in [11]. For $u \in \Sigma_N^k$ and $\omega \in \Lambda(n, k)$, we define the number

$$\tau_u(\omega) = \sharp\Big\{1 \leq j \leq n - k + 1; \ \ \omega_j \dots \omega_{j+k-1} = u\Big\} \in \{0, 1, \dots, n\}.$$

Then, for $\omega \in \Lambda(n, k)$, we consider the appearance distribution with respect to Σ_N^k of ω denoted by $(\tau_u(\omega))_{u \in \Sigma_N^k}$. We set

$$\mathscr{D}_n(\varepsilon, k, N) = \Big\{(\tau_u(\omega))_{u \in \Sigma_N^k}; \ \omega \in \Lambda_n(\varepsilon, k, N)\Big\}$$

and, for a distribution $(\tau_u) \in \mathscr{D}$,

$$E\big((\tau_u)\big) = \Big\{\omega \in \Lambda_n(\varepsilon, k, N); \ \tau_u(\omega) = \tau_u, \ \forall u \in \Sigma_N^k\Big\}.$$

Then $\Lambda_n(\varepsilon, k, N)$ is partitioned into $E((\tau_u))$'s. Since, there are N^k possible words u in Σ_N^k, It follows that

$$\sharp\mathscr{D}_n(\varepsilon, k, N) \leq (2n)^{N^k}. \tag{7}$$

Now we will decompose the set $E((\tau_u))$ into disjoint union of some sets. For $\omega = \omega_1 \dots \omega_{n+k-1}$ in $E((\tau_u))$, we say $\omega_j \omega_{j+1} \dots \omega_{j+m-1}$ is a maximal (N, k)-run subword of ω if the following conditions are satisfied

(1) $m \geq k$,
(2) $\forall 0 \leq i \leq m-1, \omega_{j+i} \leq N$ and $\omega_{j-1} > N, \omega_{j+m} > N$.

On the other hand, a subword between maximal (N, k)-run subwords is called "bad subword". The set $E((\tau_u))$ is just a collection of words like

$$\omega = B_{r_1} W_{n_1} B_{r_2} \dots W_{n_p} B_{r_{p+1}} \tag{8}$$

where B_{r_i} denotes "bad subword" with length r_i and W_{n_i} denotes maximal (N, k)-run subword with length n_i. Write

$$K = \sum_{u \in \Sigma_N^k} \tau_u \quad \text{and} \quad q = \left\lfloor \frac{n - K}{k} \right\rfloor + 1.$$

Note that $p \le q$, in other word, every element in $E((\tau_u))$ has at most s maximal (N, k)-run subwords. Furthermore, by writing $K_p = \sum_{i=1}^p n_i$, we have

$$K_p = K + p(k - 1)$$

and

$$r_1 \ge 0, \ r_{p+1} \ge 0, \ r_i \ge 1 \quad \text{and} \quad \sum_{i=1}^{p+1} r_i = n + k - 1 - K_p. \tag{9}$$

For $0 \le p \le q$, we denote by E_p the set of words in $E((\tau_u))$ with p maximal (N, k)-run subwords. It is clear that $E((\tau_u))$ is partitioned into E_p's, i.e.

$$E\big((\tau_u)\big) = \bigcup_{p=1}^q E_p. \tag{10}$$

Next, we partition E_p by the length pattern of "bad subword" and maximal (N, k)-run subword. Recall (8), then for every word $\omega \in E_p$, we associate the length pattern of "bad subword" and maximal (N, k)-run subword $(r_1, n_1, r_2, \dots, n_p, r_{p+1})$. Denote by $\mathscr{L}_p$ the set of all such length pattern of ω in E_p. For a length pattern $(r, n) := (r_1, n_1, \dots, n_p, r_{p+1}) \in \mathscr{L}_p$, we set

$$B(r, n) = \Big\{ \omega \in E_p \ \text{with the length pattern } (r, n) \Big\}$$

so that we have, for $N, k \ge 1$,

$$\Lambda_n(\varepsilon, k, N) = \bigcup_{(\tau_u) \in \mathscr{D}_n(\varepsilon, k, N)} E\big((\tau_u)\big) = \bigcup_{(\tau_u) \in \mathscr{D}_n(\varepsilon, k, N)} \bigcup_{p=1}^q E_p$$

$$= \bigcup_{(\tau_u) \in \mathscr{D}_n(\varepsilon, k, N)} \bigcup_{p=1}^q \bigcup_{(r,n) \in \mathscr{L}_p} B(r, n). \tag{11}$$

Finally, we consider E_p' the set of finite words by deleting all "bad subwords" of ω in E_p.

3.2. Estimation of $\sharp \mathscr{L}_p$

We start with two technical lemmas, the first is a general fact in the element combinatorial theory and the second lemma is a consequence of Stirling formula.

Lemma 5. *For $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$, then*

$$\sharp \Big\{ (x_1, \dots, x_n) \in \mathbb{N}^n \ x_1 + \dots + x_n = m \Big\} = \frac{(n + m - 1)!}{(n - 1)! \, m!}.$$

Lemma 6. *Let $n, N \in \mathbb{N}^*$ and $m_1, \dots, m_N \in \mathbb{N}$, such that $n = \sum_{k=1}^N m_k$. For n large enough, we have*

$$\frac{1}{n} \log \frac{n!}{m_1! \dots m_N!} = \sum_{k=1}^N \phi\Big(\frac{m_k}{n}\Big) + O\Big(\frac{\log n}{n}\Big),$$

where $\phi(t) = -t \log t$.

Which will be useful in the estimation of cardinally of $\mathscr{L}_p$. Recall that

(1) $\sum_{i=1}^{p} n_i = K_p$, $n_i \geq k$ $(1 \leq i \leq p)$.

(2) $\sum_{i=1}^{p+1} r_i = n + k - 1 - K_p$, $r_1 \geq 0$, $r_{p+1} \geq 0$, $r_i \geq 1 (2 \leq i \leq p)$.

Then, by Lemma 5, we get the following estimate

$$\sharp \mathscr{L}_p \leq \frac{(K-1)!}{(K-p)!(p-1)!} \frac{(n-K-(p-1)k+p)!}{p!(n-K-(p-1)k)!}.$$

For any $\delta' > 0$, one can choose N large enough such that

$$1 - \frac{K}{n} \leq 2\delta'. \tag{12}$$

Noting that k is a fixed integer relative to n, by Lemma 6, for $\delta > 0$, for n large enough, we have

$$\sharp \mathscr{L}_p \leq \frac{(K-1)!}{(K-p)!(p-1)!} \frac{n!}{p!\big(n-K-(p-1)k\big)!(K+(p-1)k-p)!} \leq e^{\frac{n\delta}{2}}. \tag{13}$$

3.3. *Estimation of $\sharp E'_p$*

In order to estimate the cardinally of the set E'_p we will introduce the set

$$\widetilde{E}_p = \left\{ \widetilde{B}_{r_1} W_{n_1} \widetilde{B}_{r_2} \ldots W_{n_p} \widetilde{B}_{r_{p+1}}; \, \omega = B_{r_1} W_{n_1} B_{r_2} \ldots W_{n_p} B_{r_{p+1}} \in E_p \right\},$$

where $\widetilde{B}_{r_i}$ is a finite word composed of digit $N+1$ with length r_i. Thus $\sharp \widetilde{E}_p = \sharp E'_p$ and each subword $u \in \Sigma_N^k$ appears τ_u times in ω of $\widetilde{E}_p$. Take

$$h = \frac{1}{k} \left(\sum_{u \in \Sigma_N^k} -\frac{\tau_u}{n} \log \frac{\tau_u}{n} - \frac{n-K}{n} \log \frac{n-K}{n} \right) \quad \text{and} \quad \delta = \varepsilon \tag{14}$$

in Lemma 7.6 in [20]. Then, for n large enough we have

$$\sharp E'_p = \sharp \widetilde{E}_p \leq \exp\big(n(h+\varepsilon)\big). \tag{15}$$

Let us return to the proof of Proposition 2. It follow, from the decomposition (11), that

$$\sum_{u \in \Lambda_n(\varepsilon,k,N)} \nu([u])^t \leq \sharp \mathscr{D}_n(\varepsilon,k,N) \max_{(\tau_u) \in \mathscr{D}_n(\varepsilon,k,N)} q \max_{1 \leq p \leq q} \sharp \mathscr{L}_p \max_{(r,n) \in \mathscr{L}_p} \sum_{\omega \in B(r,n)} \nu([\omega])^t$$

$$\leqq (2n)^{N^k} \max_{(\tau_u) \in \mathscr{D}_n(\varepsilon,k,N)} \max_{1 \leq p \leq q} \sharp \mathscr{L}_p \max_{(r,n) \in \mathscr{L}_p} \sum_{\omega \in B(r,n)} \nu([\omega])^t. \tag{16}$$

Thus by the quasi Bernoulli property (Lemma 8), we have

$$\sum_{\omega \in B(r,n)} \nu([\omega])^t \leq C^{2t(p+1)} \sum_{\omega \in B(r,n)} \prod_{i=1}^{p+1} \nu([B_{r_i}(\omega)])^t \prod_{i=1}^{p} \nu([W_{n_i}(\omega)])^t$$

$$\leq C^{2t(p+1)} \sum_{\omega \in B(r,n)} \prod_{i=1}^{p+1} \nu([B_{r_i}(\omega)])^t \sum_{\omega \in B(r,n)} \prod_{i=1}^{p} \nu([W_{n_i}(\omega)])^t$$

$$\leq C^{t(4p+5)} \left[\sum_{\omega \in B(r,n)} \nu\big([B_{r_1}(\omega) \ldots B_{r_{p+1}}(\omega)]\big)^t \right] \sum_{\omega \in E'_p} \nu([\omega])^t. \tag{17}$$

Remark that $\sum_{i=1}^{p+1} r_i \leq n - K$ and recall that $t > \gamma \geq \alpha_\nu$. This yields, by Lemma 10

$$\sum_{\omega \in B(r,n)} \nu([B_{r_1}(\omega) \ldots B_{r_{p+1}}(\omega)])^t \leq \sum_{\omega \in \mathbb{N}^{n-K}} \nu([\omega])^t \leq C_0 M^{n-K}.$$

Then,

$$\sum_{\omega \in B(r,n)} \nu([\omega])^t \leq C_0 C^{t(4p+5)} M^{n-K} \sum_{\omega \in E'_p} \nu([\omega])^t. \tag{18}$$

According to the definition of q and the fact that $p \leq q$, for any $\delta > 0$, when N is taken big enough (recall (12)), we have

$$C_0 C^{t(4p+5)} M^{n-K} \leq C_0 C^{9t} \, e^{\frac{n\delta}{2}}. \tag{19}$$

Given $\omega \in E_p'$, denote by (τ_u') the appearance distribution with respect to Σ_N^k of ω. Then, we have

$$|\omega| = K_p \quad \text{and} \quad \tau_u \leq \tau_u' \leq \tau_u + (p-1)(k-1) \leq \tau_u + n - K. \tag{20}$$

By the Gibbsian property (see (24)) and (25), we have

$$k \log v([w_1 \ldots w_{K_p}]) \leq k \log C + k \sum_{i=0}^{K_p+1} \varphi(\sigma^i x)$$

$$= k \log C + \sum_{i=0}^{k-2} (k-i-1)\varphi(\sigma^i x) + \sum_{i=K_p-j+1}^{K_p-1} (K_p - i)\varphi(\sigma^i x) + \sum_{i=0}^{K_p-k} \sum_{j=0}^{k-1} \varphi(\sigma^{i+j} x)$$

$$\leq (K_p + k^2 - k + 1) \log C + \sum_{u \in \Sigma_N^k} \tau_u' \log v([u]).$$

Together with (15), this yields

$$\sum_{\omega \in E_p'} v([\omega])^t \leq \sharp E_p' \max_{\omega \in E_p'} v([\omega])^t$$

$$\leq \exp \left\{ n(h+\varepsilon) + \frac{t}{k} \left\{ \sum_{u \in \Sigma_N^k} \tau_u' \log v([u]) + (K_p + k^2) \log C \right\} \right\}.$$

Given $\omega \in E((\tau_u))$, there exists $x \in [\omega]$ such that $x \in C_n(\varepsilon, k, N)$ and

$$\forall \, u \in \Sigma_N^k, \ \Delta_{x,n}([u]) = \frac{\tau_u}{n}.$$

From (20) and by the definition of $C_n(\varepsilon, k, N)$, there exists $\mu \in \mathscr{P}_\sigma(\mathscr{X})$ such that

$$h(\mu, v) \geq a \quad \text{and} \quad \frac{-\frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u}{n} \log \frac{\tau_u}{n} + 5\varepsilon}{-\frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u'}{n} \log v([u])} \leq \gamma + \frac{7\varepsilon}{a}.$$

Recall that $t > \gamma$. Take $\varepsilon > 0$ small enough such that $t > \gamma + \frac{7\varepsilon}{a}$. Then, we have

$$\frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u}{n} \log \frac{\tau_u}{n} + 5\varepsilon \geq \frac{t}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u'}{n} \log v([u]). \tag{21}$$

Choose n large enough such that

$$\frac{1}{kn} (K_p + k^2) \log C \leq \varepsilon. \tag{22}$$

It follows from (21), (22) and (14) that

$$\sum_{\omega \in E_p'} v([\omega])^t \leq \exp \left\{ n \left(h + \varepsilon + \frac{t}{k} \left\{ \sum_{u \in \Sigma_N^k} \frac{\tau_u'}{n} \log v([u]) + \frac{1}{n} (K_p + k^2) \log C \right\} \right) \right\}$$

$$\leq \exp \left\{ n \left(h + \frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u}{n} \log \frac{\tau_u}{n} - 3\varepsilon \right) \right\}$$

$$\leq \exp \left\{ n \left(-\frac{1}{k} \frac{n-K}{n} \log \frac{n-K}{n} - 3\varepsilon \right) \right\}.$$

By (12), we can take δ' small enough such that

$$-\frac{1}{k} \frac{n-K}{n} \log \frac{n-K}{n} \leq \varepsilon.$$

Thus, we have

$$\sum_{\omega \in E'_p} \nu([\omega])^t \le \exp(-2n\varepsilon). \tag{23}$$

Take $\delta = \varepsilon$ in (13) and (19). In combination with (16), (17) and (23), this yields

$$\sum_{u \in \Lambda_n(\varepsilon, k, N)} \nu([u])^t \le C_0 C^{9t} q \, (2n)^{N^k} \exp(-n\varepsilon).$$

Which implies that for any $t > \gamma \ge \gamma(\nu, \mu)$,

$$\mathcal{H}^t_{\nu, \delta_{n(s)+k-1}} \big(A(\varepsilon, k, N) \big) \le C_0 C^{9t} q \sum_{n \ge n(s)} (2n)^{N^k} \exp(-n\varepsilon)$$

and, finally, we get (4).

4. Appendix

4.1. *Gibbs measure*

Let us recall some facts of Gibbs measures, to induce metrics on $\mathscr{X}$. For $n \in \mathbb{N}^*, a \in \mathbb{N}$ and potential function $\varphi : \mathscr{X} \to \mathbb{R}$ we define the function

$$P_n(\varphi) = \frac{1}{n} \log \sum_{\sigma^n x = x} \exp(S_n \varphi(x)) \mathbf{1}_{[a]}(x)$$

and the *n*-order variation of φ by

$$var_n \varphi = \sup_{x|_1^n = y|_1^n} \Big\{ |\varphi(x) - \varphi(y)|; \; x, y \in \mathscr{X} \Big\}.$$

We say that a potential φ has summable variations if $\sum_{n=2}^{\infty} var_n \varphi < +\infty$. In this case, φ is uniformly continuous on $\mathscr{X}$ and the Gurevich pressure of φ,

$$P(\varphi) = \lim_{n \to \infty} P_n(\varphi),$$

is well defined and is independent of a (see [27]).

An invariant probability measure ν is called a Gibbs measure associated to a potential function φ if it satisfies the Gibbsian property :

$$\exists \; C > 1, \, P \in \mathbb{R} \quad \text{such that} \quad C^{-1} \le \frac{\nu[x_1 x_2 \ldots x_n])}{\exp(S_n \varphi(x) - nP)} \le C \tag{24}$$

holds for any $n \ge 1$ and any $x \in \mathscr{X}$. It is known [11] that a potential function φ with summable variations admits a unique Gibbs measure ν if and only if $var_1 \varphi < +\infty$ and $P(\varphi) < +\infty$.

Remark 7. Assume that φ admits a unique Gibbs measure denoted by ν_φ. Then the constant P in (24) is equal to the Gurevich pressure $P(\varphi)$. If we consider the potential $\varphi^* = \varphi - P(\varphi)$, we get $P(\varphi^*) = 0$ and $\nu_{\varphi^*} = \nu_\varphi$.

A trivial fact is that the Gibbsian property (24) implies that

$$\forall \; x \in \mathscr{X}, \; \; \varphi(x) \le \log C. \tag{25}$$

As a consequence, for any probability measure μ, we have $\int_{\mathscr{X}} \varphi \mathrm{d}\mu$ is defined as a number in $[-\infty, +\infty)$ Also, the Gibbsian property implies the quasi Bernoulli property.

Lemma 8. *Let ν be a Gibbs measure associated to potential φ. For any k words $\omega_1, \ldots, \omega_k$, we have*

$$C^{-(k+1)} \nu([\omega_1 \ldots \omega_k]) \le \nu([\omega_1]) \ldots \nu([\omega_k]) \le C^{(k+1)} \nu([\omega_1 \ldots \omega_k]).$$

4.2. *Metrization of the w^*-topology*

Let $C_b(\mathcal{X})$ denote the set of bounded continuous functions on $\mathcal{X}$. We endow $\mathscr{P}(\mathcal{X})$ with the w^*-topology induced by $C_b(\mathcal{X})$. For μ and $v \in \mathscr{P}(\mathcal{X})$, we define

$$d^*(\mu, v) = \sum_{[\omega] \in \mathscr{C}^*} a_{[\omega]} |\mu([\omega]) - v([\omega])|,$$

where $a_{[\omega]}$ is a positive number such that $\sum_{[\omega] \in \mathscr{C}^*} a_{[\omega]} = 1$. In addition (see [11, Proposition 3]), for all sequence $(\mu_n) \in \mathscr{P}(\mathcal{X})$, we have

$$\mu_n \text{ converges in } w^*\text{-topology to } \mu \text{ if and only if } \lim_{n \to \infty} d^*(\mu_n, \mu) = 0$$

4.3. *Some useful inequalities*

Recall the definitions of $h(v, \mu)$ and $\beta(v, \mu)$. By the concavity of the logarithm function, it is easy to show $h(v, \mu) \geq h_\mu$. It follows that when $v \neq \mu$, we have $h(v, \mu) > h_\mu$, which implies $h(v, \mu) > 0$ (see [11] for more details). In addition, also from [11], we have the following result.

Lemma 9. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{X})$ and φ be a potential function of summable variations. Assume that φ admits a unique Gibbs measure v with convergence exponent α_v.*
If $v \neq \mu$ and $h(v, \mu) < +\infty$, then

$$\beta(v, \mu) = \frac{h_\mu}{h(v, \mu)},$$

if $h(v, \mu) = +\infty$, we have

$$\beta(v, \mu) \leq \alpha_v.$$

Finally, we recall a basic property of the convergence exponent a α_v.

Lemma 10. *Let a α_v be the convergence exponent of Gibbs measure v associated to a potential function φ. Then for any $\varepsilon > 0$ there exist constants C_0 and M such that*

$$\sum_{\omega \in \mathbb{N}^k} v([\omega])^{\alpha_v + \varepsilon} \leq C_0 M^k, \ \forall \ k \geq 1.$$

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