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A Rudin–de Leeuw type theorem for functions with spectral gaps

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Abstract. Our starting point is a theorem of de Leeuw and Rudin that describes the extreme points of the unit ball in the Hardy space H^1. We extend this result to subspaces of H^1 formed by functions with smaller spectra. More precisely, given a finite set $\mathcal{K}$ of positive integers, we prove a Rudin–de Leeuw type theorem for the unit ball of $H^1_{\mathcal{K}}$, the space of functions $f \in H^1$ whose Fourier coefficients $\widehat{f}(k)$ vanish for all $k \in \mathcal{K}$.

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1. Introduction and main result

Let $\mathbb{T}$ stand for the unit circle $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$, endowed with normalized Lebesgue measure, and let $L^1 = L^1(\mathbb{T})$ be the space of all complex-valued integrable functions f on $\mathbb{T}$, with norm

$$\|f\|_1 := \frac{1}{2\pi}\int_{\mathbb{T}} |f(\zeta)|\,|d\zeta|. \tag{1}$$

The *Fourier coefficients* of a function $f \in L^1$ are the numbers

$$\widehat{f}(k) := \frac{1}{2\pi}\int_{\mathbb{T}} \overline{\zeta}^k f(\zeta)\,|d\zeta|, \quad k \in \mathbb{Z},$$

and the set

$$\operatorname{spec} f := \left\{k \in \mathbb{Z} : \widehat{f}(k) \neq 0\right\}$$

is called the *spectrum* of f.

Further, the *Hardy space* H^1 is defined by

$$H^1 := \left\{f \in L^1 : \operatorname{spec} f \subset \mathbb{Z}_+\right\}$$

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and normed as above; here $\mathbb{Z}_+ := \{0,1,2,\dots\}$. The harmonic extension (given by the Poisson integral) of a function $f \in H^1$ to the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is actually holomorphic there (see, e.g., [10, Chapter II]), so we may view elements of H^1 as holomorphic functions on $\mathbb{D}$. Recall also that a non-null function $F \in H^1$ is said to be *outer* if

$$\log|F(0)| = \frac{1}{2\pi}\int_{\mathbb{T}} \log|F(\zeta)|\,|d\zeta|,$$

whereas a function I of class $H^\infty := H^1 \cap L^\infty(\mathbb{T})$ is termed *inner* if $|I| = 1$ a.e. on $\mathbb{T}$. It is well known that a generic function $f \in H^1$, $f \not\equiv 0$, admits an (essentially unique) factorization of the form

$$f = IF, \tag{2}$$

where I is inner and F is outer. We refer to any of [10,11] or [12] for basic facts about Hardy spaces, including the canonical factorization theorem just mentioned.

This note is motivated by a beautiful theorem of de Leeuw and Rudin, which describes the extreme points of the unit ball in H^1. Before stating it, we need to introduce yet another piece of notation. Namely, given a Banach space $X = (X, \|\cdot\|)$, we write

$$\operatorname{ball}(X) := \{x \in X : \|x\| \le 1\}.$$

Finally, we recall that an element x of $\operatorname{ball}(X)$ is said to be an *extreme point* thereof if it is not an interior point of any line segment contained in $\operatorname{ball}(X)$. Of course, any such point x satisfies $\|x\| = 1$.

The Rudin–de Leeuw result that interests us here reads as follows.

Theorem A. *A function $f \in H^1$ with $\|f\|_1 = 1$ is an extreme point of* $\operatorname{ball}(H^1)$ *if and only if it is outer.*

The original proof can be found in [13]; see also [10, Chapter IV] and [11, Chapter 9] for alternative presentations.

We are concerned with certain finite-dimensional perturbations of Theorem A. Specifically, the question is what happens if H^1 gets replaced by a smaller subspace, whose elements are required to have some additional spectral holes (but not too many of them). To be more precise, we fix a finite number (say, M) of positive integers

$$k_1 < k_2 < \dots < k_M$$

and restrict our attention to the functions $f \in H^1$ that satisfy

$$\widehat{f}(k_1) = \dots = \widehat{f}(k_M) = 0.$$

The subspace comprised of such functions is thus

$$H^1_{\mathcal{K}} := \left\{f \in H^1 : \operatorname{spec} f \subset \mathbb{Z}_+ \setminus \mathcal{K}\right\},$$

where

$$\mathcal{K} := \{k_1, \dots, k_M\}. \tag{3}$$

Our purpose here is to characterize the extreme points of $\operatorname{ball}(H^1_{\mathcal{K}})$, the unit ball of $H^1_{\mathcal{K}}$ endowed with the L^1 norm (1).

Because $H^1_{\mathcal{K}}$ has finite codimension in H^1, one would not expect the situation to be very different from that in Theorem A. So, a priori, one feels that the extreme points f of $\operatorname{ball}(H^1_{\mathcal{K}})$ should probably be the unit-norm functions which are fairly close to being outer, in some sense or other. Our characterization, stated below in terms of the function's canonical factorization (2), justifies this guess and gives a precise meaning to the notion of a "nearly outer" function that arises.

First of all, it turns out that if $f \in \mathrm{ball}(H^1_{\mathcal{K}})$ is extreme, then its inner factor, I, must be a *finite Blaschke product* whose degree (i.e., the number of its zeros) does not exceed $M(=\#\mathcal{K})$. This means that I is writable, possibly after multiplication by a unimodular constant, as

$$I(z) = \prod_{j=1}^{m} \frac{z-a_j}{1-\overline{a}_j z}, \tag{4}$$

where $0 \le m \le M$ and $a_1, \dots, a_m$ are points in $\mathbb{D}$. (When $m = 0$, it is of course understood that $I(z) = 1$.) Secondly—and perhaps less predictably—there is an interplay between the two factors, I and F, in (2) which we now describe.

Assuming that I is given by (4) and $F \in H^1$ is outer, we consider the function

$$F_0(z) := F(z) \prod_{j=1}^{m} \left(1-\overline{a}_j z\right)^{-2} \tag{5}$$

and its coefficients

$$C_k := \widehat{F}_0(k), \quad k \in \mathbb{Z}.$$

Since $a_1, \dots, a_m \in \mathbb{D}$, it follows that $F_0 \in H^1$ and so $C_k = 0$ for all $k < 0$. Also, we define

$$A(k) := \mathrm{Re}\, C_k, \quad B(k) := \mathrm{Im}\, C_k \quad (k \in \mathbb{Z})$$

and introduce, for $j = 1, \dots, M$ and $l = 0, \dots, m$, the numbers

$$A^+_{j,l} := A\left(k_j + l - m\right) + A\left(k_j - l - m\right), \quad B^+_{j,l} := B\left(k_j + l - m\right) + B\left(k_j - l - m\right)$$

and

$$A^-_{j,l} := A\left(k_j + l - m\right) - A\left(k_j - l - m\right), \quad B^-_{j,l} := B\left(k_j + l - m\right) - B\left(k_j - l - m\right).$$

(The integers k_j are, of course, the same as in (3).) Next, we build the $M \times (m+1)$ matrices

$$\mathcal{A}^+ := \left\{A^+_{j,l}\right\}, \quad \mathcal{B}^+ := \left\{B^+_{j,l}\right\} \tag{6}$$

and the $M \times m$ matrices

$$\mathcal{A}^- := \left\{A^-_{j,l}\right\}, \quad \mathcal{B}^- := \left\{B^-_{j,l}\right\}. \tag{7}$$

Here, the row index j always runs from 1 to M. As to the column index l, it runs from 0 to m for each of the two matrices in (6), and from 1 to m for each of those in (7).

Finally, we need the block matrix

$$\mathfrak{M} = \mathfrak{M}_{\mathcal{K}}\left(F, \{a_j\}_{j=1}^m\right) := \begin{pmatrix} \mathcal{A}^+ & \mathcal{B}^- \\ \mathcal{B}^+ & -\mathcal{A}^- \end{pmatrix}, \tag{8}$$

which has $2M$ rows and $2m+1$ columns.

Now we are in a position to state our main result, which extends Theorem A from H^1 to $H^1_{\mathcal{K}}$. To keep on the safe side, we specify that the number $M := \#\mathcal{K}$ is also allowed to be 0; in this special case, we have $\mathcal{K} = \emptyset$, so that $H^1_{\mathcal{K}}$ reduces to H^1 and we are back to the classical situation. Our main concern is, however, the case where M is a positive integer.

Theorem 1. *Suppose that $f \in H^1_{\mathcal{K}}$ and $\|f\|_1 = 1$. Assume further that $f = IF$, where I is inner and F is outer. Then f is an extreme point of* $\mathrm{ball}(H^1_{\mathcal{K}})$ *if and only if the following two conditions hold:*

(a) *I is a finite Blaschke product whose degree, say m, does not exceed M.*
(b) *The matrix $\mathfrak{M} = \mathfrak{M}_{\mathcal{K}}(F, \{a_j\}_{j=1}^m)$, built as above from F and the zeros $\{a_j\}_{j=1}^m$ of I, has rank $2m$.*

To see a simple example, suppose that $\mathcal{K}$ consists of a single element, an integer $k(=k_1)$ with $k \ge 2$. Thus, $\mathcal{K} = \{k\}$ and the subspace in question is

$$H^1_{\{k\}} := \left\{ f \in H^1 : \widehat{f}(k) = 0 \right\}.$$

Now let $F \in H^1$ be an outer function with $\|F\|_1 = 1$ and $\widehat{F}(k-1) = 0$; then put $f(z) := zF(z)$. Clearly, $f \in H^1_{\{k\}}$ and $\|f\|_1 = 1$. Using Theorem 1 with $M = m = 1$, we verify (via a calculation, which we omit) that f is an extreme point of $\operatorname{ball}(H^1_{\{k\}})$ if and only if $|\widehat{F}(k-2)| \ne |\widehat{F}(k)|$.

As regards possible applications of Theorem 1, one may recall first that Theorem A was crucial in describing the isometries of H^1; see [14] and [11, Chapter 9]. It is therefore conceivable that Theorem 1 might serve a similar purpose in the $H^1_{\mathcal{K}}$ setting.

We conclude this section by mentioning several types of subspaces in H^1, other than $H^1_{\mathcal{K}}$, where the geometry of the unit ball has been studied. This was done for shift-coinvariant subspaces [3, 4] and, more generally, for kernels of Toeplitz operators in H^1 [5]. Also considered were spaces of polynomials of fixed degree, along with their Paley–Wiener type counterparts [6], and quite recently, spaces of *lacunary* polynomials with prescribed spectral gaps [7]. This last-mentioned paper is especially close in spirit to our current topic.

The rest of this note is devoted to proving Theorem 1. The bulk of the proof is deferred to Section 3 below, while Section 2 provides a couple of preliminary lemmas. The proofs are somewhat sketchy; full details and a more complete discussion can be found in [8]. There, we also supplement Theorem 1 with a result concerning the exposed points of $\operatorname{ball}(H^1_{\mathcal{K}})$.

2. Preliminaries

Two lemmas will be needed. When stating them, we write $L^\infty_{\mathbb{R}}$ for the set of real-valued functions in $L^\infty = L^\infty(\mathbb{T})$.

Lemma 2. *Let X be a subspace of H^1. Suppose that $f \in X$ is a function with $\|f\|_1 = 1$ whose canonical factorization is $f = IF$, with I inner and F outer. The following conditions are equivalent.*

(i) *f is not an extreme point of* $\operatorname{ball}(X)$.
(ii) *There exists a function $G \in H^\infty$, other than a constant multiple of I, for which $G/I \in L^\infty_{\mathbb{R}}$ and $FG \in X$.*

Proof. We begin by restating condition (i). In fact, for X as above, it is known (see [9, Chapter V, Section 9]) that a unit-norm function $f \in X$ is a non-extreme point of $\operatorname{ball}(X)$ if and only if there is a nonconstant function $h \in L^\infty_{\mathbb{R}}$ satisfying $fh \in X$.

Now, if such an h can be found, then $g := fh$ is in X and condition (ii) is fulfilled with $G := Ih(= g/F)$. To check that this G is in H^∞, one may note that $Ih \in L^\infty$ and $g/F \in N^+$, where N^+ is the Smirnov class (see [10, Chapter II]).

Conversely, if (ii) holds with a certain $G \in H^\infty$, then $h := G/I$ is a nonconstant function in $L^\infty_{\mathbb{R}}$ and $fh(= FG) \in X$. $\square$

Before proceeding with the next result, we pause to introduce a certain class of polynomials that will be needed below.

Given a nonnegative integer N and a polynomial p, we say that p is *N-symmetric* if $\overline{z}^N p(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$. Equivalently, a polynomial p is N-symmetric if (and only if)

$$\widehat{p}(N-k) = \overline{\widehat{p}(N+k)}$$

for all $k \in \mathbb{Z}$; this accounts for the terminology. It follows that the general form of such a polynomial is

$$p(z) = \sum_{k=0}^{N-1} \left(\alpha_{N-k} - i\beta_{N-k}\right) z^k + 2\alpha_0 z^N + \sum_{k=N+1}^{2N} \left(\alpha_{k-N} + i\beta_{k-N}\right) z^k, \tag{9}$$

where $\alpha_0, \dots, \alpha_N$ and $\beta_1, \dots, \beta_N$ are real parameters. Arranging these into a vector

$$\left(\alpha_0, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\right) \in \mathbb{R}^{2N+1}, \tag{10}$$

which we call the *coefficient vector* of p, we arrive at a natural isomorphism between the space of N-symmetric polynomials and $\mathbb{R}^{2N+1}$.

Lemma 3. *Given $N \in \mathbb{Z}_+$ and points $a_1, \dots, a_N \in \mathbb{D}$, let*

$$B(z) := \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a}_j z}.$$

The general form of a function $\psi \in H^\infty$ satisfying $\psi/B \in L^\infty_{\mathbb{R}}$ is then $\psi = p\Phi$, where

$$\Phi(z) := \prod_{j=1}^{N} \left(1 - \overline{a}_j z\right)^{-2} \tag{11}$$

and p is an N-symmetric polynomial. (If $N = 0$, the products are taken to be 1.)

Proof. If $\psi = p\Phi$, with p an N-symmetric polynomial, then it is indeed true that the ratio ψ/B is real-valued on $\mathbb{T}$ (and hence lies in $L^\infty_{\mathbb{R}}$). To see why, use the identity

$$\psi/B = \left(\overline{z}^N p\right) \cdot \left(z^N \Phi/B\right) \tag{12}$$

and the inequality $z^N \Phi/B \ge 0$, both valid on $\mathbb{T}$.

Conversely, suppose $\psi \in H^\infty$ is such that $\psi/B \in L^\infty_{\mathbb{R}}$. Using this last property in the form $\psi/B = \overline{\psi}/\overline{B}$, we infer that ψ is orthogonal (in the Hardy space H^2) to the shift-invariant subspace θH^2, where $\theta := zB^2$. In other words, ψ belongs to the *star-invariant* (or *model*) subspace $H^2 \ominus \theta H^2$. Furthermore, because θ is a finite Blaschke product, it follows (see, e.g., [2] or [15]) that ψ is a rational function whose poles, counted with their multiplicities, are contained among those of θ and which satisfies $\lim_{z\to\infty} \psi(z)/\theta(z) = 0$. This means that ψ is expressible as $p\Phi$ for *some* polynomial p of degree at most $2N$. Once this is known, we finally verify that p is N-symmetric by invoking (12) and the inequality stated next to it. □

3. Proof of Theorem 1

We shall proceed in two steps. First, we prove the necessity of condition (a). Second, we show that condition (b) characterizes the extreme points among those unit-norm functions which obey (a).

Step 1. Assuming that I, the inner factor of f, does not reduce to a finite Blaschke product of degree at most M (so that (a) fails), we want to conclude that f is not an extreme point of $\operatorname{ball}(H^1_{\mathcal{K}})$. By Lemma 2, it suffices to construct a function $G \in H^\infty$, not a constant multiple of I, with the properties that

$$G/I \in L^\infty_{\mathbb{R}} \tag{13}$$

and

$$FG \in H^1_{\mathcal{K}}. \tag{14}$$

We know from Frostman's theorem (see [10, Chapter II]) that there exists a point $w \in \mathbb{D}$ for which

$$\varphi := \frac{I - w}{1 - \overline{w} I}$$

is a (finite or infinite) Blaschke product. Furthermore, our current assumption on I guarantees that φ has at least $M+1$ zeros. Consequently, φ admits a factorization

$$\varphi = \varphi_1 \varphi_2, \tag{15}$$

where φ_1, φ_2 are Blaschke products and φ_1 has precisely $M+1$ zeros. Setting $N := M+1$, we therefore have

$$\varphi_1(z) = \prod_{j=1}^{N} \frac{z-a_j}{1-\overline{a}_j z} \tag{16}$$

for some $a_1, \dots, a_N \in \mathbb{D}$. Next, we consider the function $g := 1 - \overline{w} I (\in H^\infty)$ and observe that $I/\varphi = g/\overline{g}$ a.e. on $\mathbb{T}$. When coupled with (15), this yields

$$I = \varphi_1 \varphi_2 g/\overline{g}. \tag{17}$$

Our plan is to construct a function $G \in H^\infty$ satisfying (13) and (14) in the form

$$G = g^2 p \Phi \varphi_2, \tag{18}$$

where p is an N-symmetric polynomial and Φ is given by (11), with the a_j's coming from (16). In fact, any such G belongs to H^∞ and makes (13) true. To verify the latter claim, combine (17) and (18) to find that

$$G/I = G\overline{I} = |g|^2 p \Phi \overline{\varphi}_1$$

a.e. on $\mathbb{T}$; then apply Lemma 3 with $B = \varphi_1$ to deduce that the function $p\Phi\overline{\varphi}_1$ (and hence also $|g|^2 p \Phi \overline{\varphi}_1$) is in $L^\infty_{\mathbb{R}}$.

We also need to ensure (14), as well as the condition

$$G/I \neq \text{const}, \tag{19}$$

by choosing p appropriately. To this end, we set $\mathcal{F}_0 := F g^2 \Phi \varphi_2 (\in H^1)$ and note that $FG = \mathcal{F}_0 p$, so that (14) boils down to requiring that the numbers

$$\gamma_j(p) := \widehat{(\mathcal{F}_0 p)}(k_j), \quad j = 1, \dots, M,$$

be null. Now let T be the linear map, defined on the space of all N-symmetric polynomials, that takes p to the vector

$$\left(\operatorname{Re} \gamma_1(p), \operatorname{Im} \gamma_1(p), \dots, \operatorname{Re} \gamma_M(p), \operatorname{Im} \gamma_M(p)\right).$$

Identifying an N-symmetric polynomial p with its coefficient vector (see Section 2 above), we may view T as a linear operator from $\mathbb{R}^{2N+1} (= \mathbb{R}^{2M+3})$ to $\mathbb{R}^{2M}$. Its rank being obviously bounded by $2M$, we deduce from the rank-nullity theorem (see, e.g., [1, p. 63]) that the kernel of T, to be denoted by $\mathfrak{N}_T$, satisfies $\dim \mathfrak{N}_T \geq 3$. In particular, we can find two linearly independent N-symmetric polynomials, say p_1 and p_2, in $\mathfrak{N}_T$. When plugged into (18) in place of p, each of these makes (14) true, while at least one of them ensures (19) as well. This completes the construction and proves the necessity of condition (a) in the theorem.

Step 2. From now on, we assume that (a) holds, so that I is given by (4) with $0 \leq m \leq M$ and $a_1, \dots, a_m \in \mathbb{D}$. In view of Lemma 2, our function $f (= IF)$ will be an extreme point of $\operatorname{ball}(H^1_{\mathcal{K}})$ if and only if every $G \in H^\infty$ satisfying (13) and (14) is necessarily a constant multiple of I. Our purpose is therefore to prove that the latter condition is equivalent to (b).

The functions $G \in H^\infty$ we should consider are those of the form

$$G = p \Phi_0, \tag{20}$$

where p is an m-symmetric polynomial and

$$\Phi_0(z) := \prod_{j=1}^{m} \left(1 - \overline{a}_j z\right)^{-2}.$$

Indeed, Lemma 3 (coupled with our current assumption on I) tells us that these are precisely the G's that enjoy property (13). Now, if $F_0 \in H^1$ is the function defined by (5), or equivalently by $F_0 := F\Phi_0$, then we have $FG = F_0 p$ and condition (14) takes the form

$$\widehat{(F_0 p)}(k_j) = 0, \quad j = 1, \dots, M. \tag{21}$$

Rewriting these Fourier coefficients as convolutions and splitting each of the resulting equations into a real and imaginary part, we arrive at $2M$ real equations that recast (21) in terms of the coefficient vector

$$\left(\alpha_0, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m\right) \tag{22}$$

of p. (It is understood that p is related to (22) as (9) is to (10), but with m in place of N.) Once these routine calculations are performed, we eventually rephrase (21)—and hence (14)—by saying that the vector (22) belongs to the subspace $\mathcal{N} := \ker \mathfrak{M}$, the kernel of the linear operator $\mathfrak{M} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2M}$ given by (8).

This makes it easy to decide whether conditions (13) and (14) imply, for a function $G \in H^\infty$, that $G/I = \text{const}$. Namely, this implication is valid if and only if $\dim \mathcal{N} = 1$. To see why, one should observe first that $\mathcal{N}$ always contains a nonzero vector. Specifically, the m-symmetric polynomial $p_0 := I/\Phi_0$ solves (21), so its coefficient vector is in $\mathcal{N}$. Now, if $\dim \mathcal{N} = 1$, then any other m-symmetric polynomial p solving (21) is given by $p = cp_0$ with some $c \in \mathbb{R}$; accordingly, the functions G produced by (20) are all of the form $G = cI$. Conversely, if $\dim \mathcal{N} > 1$, then we can find an m-symmetric polynomial p with $p/p_0 \neq \text{const}$ that makes (21) true, so the associated function $G (= p\Phi_0)$ satisfies (19).

To summarize, among the unit-norm functions $f = IF \in H^1_{\mathcal{K}}$ that obey (a), the extreme points of $\text{ball}(H^1_{\mathcal{K}})$ are characterized by the property that $\mathcal{N}$, the kernel in $\mathbb{R}^{2m+1}$ of the linear map (8), has dimension 1. Finally, another application of the rank-nullity theorem allows us to restate the latter condition as $\operatorname{rank} \mathfrak{M} = 2m$, and we arrive at (b). The proof of Theorem 1 is complete. □

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