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On the structure of the h-fold sumsets

Jun-Yu Zhou [a] **and Quan-Hui Yang** [*, a]

[a] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mails: 1045908839@qq.com, yangquanhui01@163.com

Abstract. Let A be a set of nonnegative integers. Let $(hA)^{(t)}$ be the set of all integers in the sumset hA that have at least t representations as a sum of h elements of A. In this paper, we prove that, if $k \ge 2$, and $A = \{a_0, a_1, \ldots, a_k\}$ is a finite set of integers such that $0 = a_0 < a_1 < \cdots < a_k$ and $\gcd(a_1, a_2, \ldots, a_k) = 1$, then there exist integers c_t, d_t and sets $C_t \subseteq [0, c_t - 2]$, $D_t \subseteq [0, d_t - 2]$ such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_{k-1} - D_t)$$

for all $h \ge \sum_{i=2}^{k}(ta_i - 1) - 1$. This improves a recent result of Nathanson with the bound $h \ge (k-1)(ta_k - 1)a_k + 1$.

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1. Introduction

Let A and B be sets of integers. The sumsets and difference sets are defined by

$$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}$$

respectively. For any integer t, we define the sets

$$t + A = \{t\} + A, \quad t - A = \{t\} - A.$$

For $h \ge 2$, we denote by hA the h-fold sumset of A, which is the set of all integers n of the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h$ are elements of A and not necessarily distinct.

In [3, 4], Nathanson proved the following fundamental beautiful theorem on the structure of h-fold sumsets.

[*] Corresponding author.

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Nathanson's Theorem A. *Let $A = \{a_0, a_1, \ldots, a_k\}$ be a finite set of integers such that*

$$0 = a_0 < a_1 < \cdots < a_k \quad \textit{and} \quad \gcd(A) = 1.$$

Let $h_1 = (k-1)(a_k-1)a_k + 1$. There are nonnegative integers c_1 and d_1 and finite sets C_1 and D_1 with $C_1 \subseteq [0, c_1-2]$ and $D_1 \subseteq [0, d_1-2]$ such that

$$hA = C_1 \cup [c_1, ha_k - d_1] \cup (ha_k - D_1)$$

for all $h \geq h_1$.

Later, Wu, Chen and Chen [6] improved the lower bound of h_1 to $\sum_{i=2}^{k} a_i - k$. Recently, Granville and Shakan [1], and Granville and Walker [2] gave some further results on this topic.

Let A be a set of integers. For every positive integer h, the h-fold representation function $r_{A,h}(n)$ counts the number of representations of n as the sum of h elements of A. Thus,

$$r_{A,h}(n) = \sharp\left\{\left(a_{j_1}, \ldots, a_{j_h}\right) \in A^h : n = \sum_{i=1}^{h} a_{j_i} \text{ and } a_{j_1} \leq \cdots \leq a_{j_h}\right\}.$$

For every positive integer t, let $(hA)^{(t)}$ be the set of all integers n that have at least t representations as the sum of h elements of A, that is,

$$(hA)^{(t)} = \left\{n \in \mathbf{Z} : r_{A,h}(n) \geq t\right\}$$

Recently, Nathanson [5] found that the sumsets $(hA)^{(t)}$ have the same structure as the sumset hA and proved the following theorem.

Nathanson's Theorem B. *Let $k \geq 2$, and let $A = \{a_0, a_1, \ldots, a_k\}$ be a finite set of integers such that $0 = a_0 < a_1 < \cdots < a_k$ and $\gcd(A) = 1$. For every positive integer t, let $h_t = (k-1)(ta_k-1)a_k+1$. There are nonnegative integers c_t and d_t, and finite sets C_t and D_t with $C_t \subseteq [0, c_t-2]$ and $D_t \subseteq [0, d_t-2]$ such that*

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)$$

for all $h \geq h_t$.

In this paper, motivated by the idea of Wu, Chen and Chen [6], we improved the lower bound of h in Nathanson's Theorem B.

Theorem 1. *Let $k \geq 2$, and let $A = \{a_0, a_1, \ldots, a_k\}$ be a finite set of integers such that*

$$0 = a_0 < a_1 < \cdots < a_k \quad \textit{and} \quad \gcd(A) = 1$$

For every positive integer t, let

$$h_t = \sum_{i=2}^{k} (ta_i - 1) - 1$$

There are nonnegative integers c_t and d_t and finite sets C_t and D_t with

$$C_t \subseteq [0, c_t - 2] \quad \textit{and} \quad D_t \subseteq [0, d_t - 2]$$

such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t) \tag{1}$$

for all $h \geq h_t$.

Remark 2. Theorem 1 is optimal.

We shall prove Theorem 1 and Remark 2 in Section 3. In Section 2, we give some lemmas.

2. Some Lemmas

Lemma 3 (see [5, Lemma 1]). *Let A be a set of integers. For any positive integer h and t, we have*

$$(hA)^{(t)} + A \subseteq ((h+1)A)^{(t)}.$$

Lemma 4. *Let $k \geq 2$, and let $A = \{a_0, a_1, \ldots, a_k\}$ be a set of integers satisfying $0 = a_0 < a_1 < \cdots < a_k$ and $\gcd(A) = 1$. For every positive integer t, let $h_t = \sum_{i=2}^{k}(ta_i - 1) - 1$ and $c'_t = \sum_{i=1}^{k-1} a_i(ta_{i+1} - 1)$. If $c'_t - a_k < n < c'_t$, then there exist at least t distinct nonnegative k-tuples $(x_{1,s}, x_{2,s}, \ldots, x_{k,s})$ $(1 \leq s \leq t)$ satisfying*

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$

and $x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h_t$ for $s = 1, 2, \ldots, t$.

Proof. Since $\gcd(a_1, \ldots, a_k) = 1$, there exist integers $x_1, \ldots, x_k$ such that

$$n = x_1 a_1 + \cdots + x_k a_k.$$

For any positive integer s, $[(s-1)a_2, sa_2 - 1]$ is a complete residue system modulo a_2. Hence there exists an integer q such that $x_1 = a_2 q + x_{1,s}$ with $(s-1)a_2 \leq x_{1,s} \leq sa_2 - 1$. This gives

$$n = x_{1,s}a_1 + \left(a_1 q + x_2\right) a_2 + \cdots + x_k a_k.$$

Let $x'_2 = a_1 q + x_2$. Similarly, there exists an integer q' such that $x'_2 = a_3 q' + x_{2,s}$ with $(s-1)a_3 \leq x_{2,s} \leq sa_3 - 1$. Now we have

$$n = x_{1,s}a_1 + x_{2,s}x_2 + (a_2 q' + x_3)a_3 + \cdots + x_k a_k.$$

By continuing this process, we obtain

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$

with $(s-1)a_{i+1} \leq x_{i,s} \leq sa_{i+1} - 1$ for $i = 1, \ldots, k-1$ and $x_{k,s}$ is some integer. Hence, for any integer $s \in [1, t]$, we have

$$0 \leq x_{i,s} \leq ta_{i+1} - 1.$$

Since $n > c'_t - a_k$, it follows that

$$\begin{aligned} x_{k,s}a_k &= n - \left(x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k-1,s}a_{k-1}\right) \\ &\geq n - (ta_2 - 1)\, a_1 - \cdots - (ta_k - 1)\, a_{k-1} = n - c'_t > -a_k, \end{aligned}$$

and then $x_{k,s} > -1$. Noting that $x_{k,s}$ is an integer, we have $x_{k,s} \geq 0$. By the bound of $x_{i,s}$, the following nonnegative k-tuples

$$(x_{1,s}, x_{2,s}, \ldots, x_{k-1,s}, x_{k,s}) \quad (1 \leq s \leq t)$$

are distinct.

Next, we shall prove that $x_{1,s} + x_{2,s} + \cdots + x_{k,s} \leq h_t$ for $s = 1, 2, \ldots, t$.

For any integer $s \in [1, t]$, let $x_{1,s} + x_{2,s} + \cdots + x_{k,s} = u_s$. Since $n < c'_t$, it follows that

$$\begin{aligned} n &= x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k \\ &= x_{1,s}a_1 + \cdots + x_{k-1,s}a_{k-1} + \left(u_s - x_{1,s} - x_{2,s} - \cdots - x_{k-1,s}\right) a_k \\ &= u_s a_k - x_{1,s}\,(a_k - a_1) - \cdots - x_{k-1,s}\,(a_k - a_{k-1}) \\ &\geq u_s a_k - (ta_2 - 1)\,(a_k - a_1) - \cdots - (ta_k - 1)\,(a_k - a_{k-1}) \\ &= u_s a_k - a_k\,[(ta_2 - 1) + \cdots + (ta_k - 1)] + a_1\,(ta_2 - 1) + \cdots + a_{k-1}\,(ta_k - 1) \\ &= u_s a_k - (h_t + 1)\, a_k + c'_t \\ &> u_s a_k - (h_t + 1)\, a_k + n. \end{aligned}$$

Hence $u_s a_k - (h_t + 1)\, a_k < 0$, and then $u_s < h_t + 1$. Therefore, $u_s \leq h_t$.

This completes the proof of Lemma 4. □

Lemma 5. *Let c'_t and h_t be defined in Lemma 4. Then*

$$c'_t = \sum_{i=1}^{k-1} a_i\,(ta_{i+1}-1) \in ((h_t+1)\,A)^{(t)}.$$

Proof. For $i = 1,2,\dots,k-1$, let $p_i = ta_{i+1}-1$. Then

$$c'_t = (ta_2-1)\,a_1 + \cdots + (ta_k-1)\,a_{k-1} = p_1 a_1 + \cdots + p_{k-1}a_{k-1}.$$

Noting that

$$p_1+\cdots+p_{k-1} = \sum_{i=2}^{k}(ta_i-1) = h_t+1,$$

we have $c'_t \in (h_t+1)\,A$.

Moreover, for any integer $r \in [0, t-1]$, we have

$$\begin{aligned}
c'_t &= \sum_{i=1}^{k-1}(ta_{i+1}-1)\,a_i = \sum_{i=1}^{k-1}((t-r)a_{i+1}-1)\,a_i + r\sum_{i=1}^{k-1}a_i a_{i+1}\\
&= ((t-r)a_2-1)\,a_1 + \sum_{i=2}^{k-1}((t-r)a_{i+1}-1+ra_{i-1})a_i + ra_{k-1}a_k\\
&:= p_{1,r}a_1 + p_{2,r}a_2 + \cdots + p_{k-1,r}a_{k-1} + p_{k,r}a_k,
\end{aligned}$$

where $p_{1,r} = (t-r)a_2-1$, $p_{k,r} = ra_{k-1}$ and $p_{i,r} = (t-r)a_{i+1}-1+ra_{i-1}$ $(2 \le i \le k-1)$. Hence $p_{i,r} \ge 0$ for all $i \in [1,k]$ and

$$\begin{aligned}
\sum_{i=1}^{k} p_{i,r} &= (t-r)a_2-1+(t-r)a_3-1+ra_1+\cdots+(t-r)a_k-1+ra_{k-2}+ra_{k-1}\\
&= h_t+1-r\,(a_2+\cdots+a_k)+r\,(a_1+\cdots+a_{k-1})\\
&= h_t+1-r\,(a_k-a_1) \le h_t+1.
\end{aligned}$$

Thus, $r_{A,h_t+1}(c'_t) \ge t$, and so $c'_t \in ((h_t+1)\,A)^{(t)}$. □

Lemma 6. *Let n and a_1, a_2 be positive integers with* $\gcd(a_1,a_2)=1$. *For any positive integer t, if $n > ta_1a_2-a_1-a_2$, then the diophantine equation*

$$a_1x + a_2y = n \tag{2}$$

has at least t nonnegative integer solutions. The lower bound of n is also best possible.

Proof. Suppose that $n > ta_1a_2-a_1-a_2$. Let (x_0,y_0) be a solution of the equation (2). Then all the integer solutions of the equation (2) is

$$\begin{cases} x = x_0 + ka_2, \\ y = y_0 - ka_1, \end{cases} \quad k \in \mathbb{Z}. \tag{3}$$

In order to have $x \ge 0$ and $y \ge 0$, we only need $x > -1$ and $y > -1$, that is,

$$\frac{-1-x_0}{a_2} < k < \frac{y_0+1}{a_1}. \tag{4}$$

Since

$$\frac{y_0+1}{a_1} - \frac{-1-x_0}{a_2} = \frac{a_1+a_2+a_1x_0+a_2y_0}{a_1a_2} = \frac{a_1+a_2+n}{a_1a_2} > \frac{a_1+a_2+ta_1a_2-a_1-a_2}{a_1a_2} = t,$$

there exist at least t integers k such that (4) holds.

Therefore, the equation (2) has at least t nonnegative integer solutions.

Now suppose that $l = ta_1a_2 - a_1 - a_2$. Then $(ta_2 - 1, -1)$ is a solution of (2). Take $x_0 = ta_2 - 1$ and $y_0 = -1$. Then (3) becomes

$$\begin{cases} x = ta_2 - 1 - ka_2, \\ y = -1 + ka_1, \end{cases} \quad k \in \mathbb{Z}.$$

Since $x \geq 0$ and $y \geq 0$, it follows that $1 \leq k \leq t-1$. Hence there exist at most $t-1$ nonnegative integer solutions.

This completes the proof of Lemma 6. □

3. Proofs

Proof of Theorem 1. Let $c_t' = \sum_{i=1}^{k-1} a_i(ta_{i+1} - 1)$. By Lemma 4, there exist smallest integers c_t and d_t satisfying

$$\left[c_t' - a_k + 1, c_t' - 1\right] \subseteq [c_t, h_t a_k - d_t] \subseteq (h_t A)^{(t)}.$$

It follows that $c_t - 1 \notin (h_t A)^{(t)}$ and $h_t a_k - d_t + 1 \notin (h_t A)^{(t)}$. Additionally

$$c_t \leq c_t' - a_k + 1, \tag{5}$$

$$c_t' - 1 \leq h_t a_k - d_t. \tag{6}$$

Define the finite sets C_t and D_t by

$$C_t = (h_t A)^{(t)} \cap [0, c_t - 2]$$

and

$$h_t a_k - D_t = (h_t A)^{(t)} \cap [h_t a_k - (d_t - 2), h_t a_k].$$

Then

$$(h_t A)^{(t)} = C_t \cup [c_t, h_t a_k - d_t] \cup (h_t a_k - D_t). \tag{7}$$

Therefore, (1) holds for $h = h_t$.

Now we prove (1) by induction on h. Suppose that (1) holds for some $h \geq h_t$. Define

$$B^{(t)} = C_t \cup [c_t, (h+1)a_k - d_t] \cup ((h+1)a_k - D_t).$$

Firstly we prove that $B^{(t)} \subseteq ((h+1)A)^{(t)}$.

Take an arbitrary integer $b \in B^{(t)}$.

Case 1: $b \in C_t \cup [c_t, h_t a_k - d_t]$. By (7), we have

$$b \in (h_t A)^{(t)} \subseteq ((h+1)A)^{(t)}.$$

Case 2: $b \in [c_t + a_k, (h+1)a_k - d_t] \cup ((h+1)a_k - D_t)$. It follows that

$$b - a_k \in [c_t, ha_k - d_t] \cup (ha_k - D_t) \subseteq (hA)^{(t)}.$$

Thus, By Lemma 3, $b \in (hA)^{(t)} + a_k \subseteq ((h+1)A)^{(t)}$.

Case 3: $h_t a_k - d_t + 1 \leq b \leq c_t + a_k - 1$. By (5) and (6), we have

$$c_t + a_k - 1 \leq c_t' \leq h_t a_k - d_t + 1. \tag{8}$$

Thus $b = c_t'$. By Lemma 5, we have

$$b = c_t' \in ((h_t + 1)A)^{(t)} \subseteq ((h+1)A)^{(t)}.$$

Therefore, $B^{(t)} \subseteq ((h+1)A)^{(t)}$.

Next we shall prove that $((h+1)A)^{(t)} \subseteq B^{(t)}$. Take an arbitrary integer $a \in ((h+1)A)^{(t)}$.

Case 1: $a = c'_t$. By (8) and $h \ge h_t$, we have
$$c_t \le c'_t \le h_t a_k - d_t + 1 \le (h+1)a_k - d_t.$$
Hence $a = c'_t \in B^{(t)}$.

Case 2: $a \ne c'_t$ and $a \notin (hA)^{(t)}$. Since $a \in ((h+1)A)^{(t)}$, there exist t nonnegative integer k-tuples $(x_{1,s}, x_{2,s}, \dots x_{k,s})$ $(1 \le s \le t)$ satisfying
$$a = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k \quad \text{and} \quad x_{1,s} + x_{2,s} + \cdots + x_{k,s} \le h+1.$$
Furthermore, we can get
$$0 \le x_{i,s} \le t a_{i+1} - 1,\ 1 \le i \le k-1,\ 1 \le s \le t. \tag{9}$$
Otherwise, without loss of generality, assume that $x_{1,1} \ge t a_2$, then for $j = 1, 2, \dots, t$, we have
$$\begin{aligned} a &= x_{1,1}a_1 + x_{2,1}a_2 + \cdots + x_{k,1}a_k \\ &= \left(x_{1,1} - j a_2\right) a_1 + \left(x_{2,1} + j a_1\right) a_2 + \cdots + x_{k,1} a_k. \end{aligned}$$
Noting that for $j = 1, 2, \dots, t$,
$$(x_{1,1} - j a_2) + (x_{2,1} + j a_1) + x_{3,1} + \cdots + x_{k,1} = h + 1 - j(a_2 - a_1) < h+1,$$
we have $a \in (hA)^{(t)}$, a contradiction. Hence the inequality (9) holds.

By $a \notin (hA)^{(t)}$, there exists $s \in [1, t]$ such that $a = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$ and
$$x_{1,s} + x_{2,s} + \cdots + x_{k,s} = h+1.$$
By (9), we have
$$\begin{aligned} a &= x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k \\ &= x_{1,s}a_1 + \cdots + x_{k-1,s}a_{k-1} + \left(h + 1 - x_{1,s} - x_{2,s} - \cdots - x_{k-1,s}\right) a_k \\ &= (h+1)a_k - x_{1,s}(a_k - a_1) - \cdots - x_{k-1,s}(a_k - a_{k-1}) \\ &\ge (h+1)a_k - (t a_2 - 1)(a_k - a_1) - \cdots - (t a_k - 1)(a_k - a_{k-1}) \\ &= (h+1)a_k - a_k\left[(t a_2 - 1) + \cdots + (t a_k - 1)\right] + a_1(t a_2 - 1) + \cdots + a_{k-1}(t a_k - 1) \\ &= (h+1)a_k - (h_t + 1)a_k + c'_t \\ &\ge c'_t. \end{aligned}$$
Since $a \ne c'_t$, it follows that $a \ge c'_t + 1$. By (5), we have $a \ge c'_t + 1 \ge c_t + a_k$.

If $x_{k,s} = 0$ for some s with $1 \le s \le t$, by (9), then
$$a \le (t a_2 - 1)a_1 + \cdots + (t a_k - 1)a_{k-1} = c'_t,$$
a contradiction.

Hence $x_{k,s} \ge 1$ for all integers $s = 1, 2, \dots, t$.

Therefore, $a - a_k \in (hA)^{(t)}$ and $a - a_k \ge c_t$. By the induction hypothesis,
$$a \in a_k + [c_t, h a_k - d_t] \cup (h a_k - D_t) = [c_t + a_k, (h+1)a_k - d_t] \cup ((h+1)a_k - D_t) \subseteq B^{(t)}.$$

Case 3: $a \ne c'_t$ and $a \in (hA)^{(t)}$. By the induction hypothesis, we have
$$(hA)^{(t)} = C_t \cup [c_t, h a_k - d_t] \cup (h a_k - D_t).$$
Since $C_t \cup [c_t, (h+1)a_k - d_t] \subseteq B^{(t)}$, we can suppose that $a > (h+1)a_k - d_t$. By $a \in (hA)^{(t)}$, there exist at least t distinct nonnegative k-tuples $(x_{1,s}, x_{2,s}, \dots, x_{k,s})$ $(1 \le s \le t)$ such that
$$a = x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k,s}a_k$$
and
$$x_{1,s} + x_{2,s} + \cdots + x_{k,s} \le h.$$

As in the proof of Lemma 4, assume that $0 \le x_{i,s} \le ta_{i+1}-1$ for $i = 1,2,\dots,k-1$. If $x_{k,s} \le 0$, then by (6) we have

$$\begin{aligned} a &\le x_{1,s}a_1 + x_{2,s}a_2 + \cdots + x_{k-1,s}a_{k-1} \\ &\le a_1(ta_2-1) + \cdots + a_{k-1}(ta_k-1) \\ &= c_t' \le h_t a_k - d_t + 1 \\ &\le (h_t+1)a_k - d_t \le (h+1)a_k - d_t, \end{aligned}$$

which contradicts with $a > (h+1)a_k - d_t$. Therefore $x_{k,s} \ge 1$ and $a - a_k \in (hA)^{(t)}$. Since $a > (h+1)a_k - d_t$, it follows that $a - a_k \in ha_k - D_t$. Hence

$$a \in (h+1)a_k - D_t \subseteq B^{(t)},$$

and so $((h+1)A)^{(t)} \subseteq B^{(t)}$.

This completes the proof of Theorem 1. □

Proof of Remark 2. Let $n \ge 3$ be an integer and $A = \{0,\ n,\ n+1\}$. By Theorem 1, there exist integers c_t, d_t and sets $C_t \subseteq [0, c_t-2]$, $D_t \subseteq [0, d_t-2]$ such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)$$

for all $h \ge h_t = t(n+1)-2$.

For any integer $m \ge c_t$, choose an integer $h' \ge t(n+1)-2$ such that $h'a_k - d_t \ge m$, then we have $m \in (h'A)^{(t)}$.

Hence, there exist t nonnegative integer tuples (u_i, v_i) $(1 \le i \le t)$ such that $m = u_i n + v_i(n+1)$.

On the other hand, there does not exist t nonnegative integer tuples (u_i, v_i) $(1 \le i \le t)$ such that $c_t - 1 = u_i n + v_i(n+1)$. Otherwise, if exist, choose $h > \max_{1\le i\le t}\{u_i + v_i\}$, then we have $c_t - 1 \in (hA)^{(t)}$, a contradiction. Hence, by Lemma 6, it follows that $c_t - 1 = ta_1a_2 - a_1 - a_2 = tn(n+1) - n - (n+1)$, and then $c_t = tn(n+1) - 2n$.

Let $p \in ((h_t-1)A)^{(t)}$. Then there exist t nonnegative integer tuples (u_i, v_i) $(1 \le i \le t)$ such that $p = u_i n + v_i(n+1)$ and $u_1 > u_2 > \cdots > u_t$ are the maximal t numbers in all the representations. Hence

$$\begin{aligned} p = u_1 n + v_1(n+1) &= [u_1 - (n+1)]n + (v_1+n)(n+1) \\ &= [u_1 - 2(n+1)]n + (v_1+2n)(n+1) \\ &= \dots \\ &= [u_1 - (t-1)(n+1)]n + [v_1 + (t-1)n](n+1). \end{aligned}$$

It follows that $u_t = u_1 - (t-1)(n+1)$, $v_t = v_1 + (t-1)n$. Noting that

$$u_t + v_t < u_{t-1} + v_{t-1} < \cdots < u_1 + v_1 \le h_t - 1,$$

we have

$$\begin{aligned} u_t + v_t &= u_1 - (t-1)(n+1) + v_1 + (t-1)n \\ &= u_1 + v_1 - (t-1) \le h_t - 1 - (t-1) = tn - 2. \end{aligned}$$

Hence, for every $p \in ((h_t-1)A)^{(t)}$,

$$\begin{aligned} p &= u_t n + v_t(n+1) \le (u_t+v_t)(n+1) \le (tn-2)(n+1) \\ &= tn(n+1) - 2(n+1) < tn(n+1) - 2n = c_t. \end{aligned}$$

By (1), it follows that

$$((h_t-1)A)^{(t)} \subseteq [0, tn(n+1) - 2(n+1)].$$

Therefore, (1) cannot hold for $h = h_t - 1$, and so Theorem 1 is optimal. □

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