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Further Equivalent Binomial Sums

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Abstract. Five binomial sums are extended by a free parameter m, that are shown, through the generating function method, to have the same value. These sums generalize the ones by Ruehr (1980), who discovered them in the study of two unexpected equalities between definite integrals.

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In 1980, Kimura [14] proposed a monthly problem about two curious identities of definite integrals. If f is continuous on $[-\frac{1}{2}, \frac{3}{2}]$, then for $\delta = 0,\ 1$, prove that

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} x^{\delta} f\left(3x^2 - 2x^3\right) \mathrm{dx} = 2\int_0^1 x^{\delta} f\left(3x^2 - 2x^3\right) \mathrm{dx}.$$

In his (trigonometric) proof, Ruehr [14] observed by linearity that to prove these identities, it is enough to verify them for monomials $f(x) = x^n$. This led him to discover the following interesting identities

$$A_n = C_n \quad \text{and} \quad B_n = D_n,$$

where for a natural number n, the four binomial sums are defined by

$$A_n = \sum_{j=0}^{n} 3^j \binom{3n-j}{2n},$$

$$B_n = \sum_{j=0}^{n} 2^j \binom{3n+1}{2n+j+1},$$

$$C_n = \sum_{j=0}^{2n} (-3)^j \binom{3n-j}{n},$$

$$D_n = \sum_{j=0}^{2n} (-4)^j \binom{3n+1}{n+j+1}.$$

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Allouche [1, 2] examined the related integrals and reviewed these identities in a more elegant manner. These identities were also reconfirmed by Meehan et al [16] who found, through the WZ-algorithm, that these four sequences satisfy also the common recurrence relation:

$$X_0 = 1 \quad \text{and} \quad X_{n+1} = \frac{27}{4} X_n - \frac{3\binom{3n+1}{n}}{4(n+1)}.$$

By introducing a variable, Alzer and Prodinger [3] recently considered the polynomial analogues, that were also examined by Kilic–Arikan [13] through bijections.

By applying the generating function approach to the binomial convolutions

$$\Omega_n = \sum_{k=0}^{n} \binom{3k}{k} \binom{3n-3k}{n-k}$$

the authors [4] not only confirmed the identities

$$\Omega_n = A_n = B_n = C_n = D_n; \tag{1}$$

but also found the two additional ones

$$\Omega_n = E_n = F_n, \tag{2}$$

where

$$E_n = \sum_{k=0}^{n} 3^k \binom{3n-k}{2n} \frac{2k(k+1)}{3n-k},$$
$$F_n = \sum_{k=0}^{n} 2^k \binom{3n+2}{n-k} \frac{(k+1)(3k+2)}{3n+2}.$$

Recall the following two binomial series due to Lambert [15] (see also [5–7, 11], [12, §5.4] and [17, §5.4]):

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+n\beta} \binom{\alpha+n\beta}{n} x^n = (1+y)^{\alpha}, \tag{3}$$

$$\sum_{n=0}^{\infty} \binom{\alpha+n\beta}{n} x^n = \frac{(1+y)^{\alpha+1}}{1+y-\beta y}; \tag{4}$$

where x and y are related by the equation $x = y/(1+y)^{\beta}$. According to the Lagrange inversion theorem (cf. Comtet [8, §3.8]), both (3) and (4) can be considered as "formal power series" equations in the variable x.

For two natural integers m and n, define the binomial sum

$$\Omega_m(n) = \sum_{k=0}^{n} \binom{mk}{k} \binom{mn-mk}{n-k}.$$

Then we can compute, in view of (4), its generating function

$$\begin{aligned} \Omega_m[x] := \sum_{n=0}^{\infty} \Omega_m(n) x^n &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{mk}{k} \binom{mn-mk}{n-k} \\ &= \sum_{k=0}^{\infty} \binom{mk}{k} x^k \sum_{n=k}^{\infty} \binom{mn-mk}{n-k} x^{n-k} \\ &= \frac{(1+y)^2}{(1+y-my)^2}, \quad \text{where} \quad \boxed{x = \frac{y}{(1+y)^m}}. \end{aligned}$$

By manipulating the above function, we shall prove that

$$\Omega_m(n) = A_m(n) = B_m(n) = E_m(n) = F_m(n), \tag{5}$$

where the four binomial sums are defined by

$$\begin{aligned}
A_m(n) &= \sum_{j=0}^{n} m^j \binom{nm-j}{n-j},\\
B_m(n) &= \sum_{j=0}^{n} (m-1)^j \binom{nm+1}{n-j},\\
E_m(n) &= \sum_{k=0}^{n} m^k \binom{mn-k}{mn-n} \frac{(m-1)k(k+1)}{mn-k},\\
F_m(n) &= \sum_{k=0}^{n} (m-1)^k \binom{mn+2}{n-k} \frac{(1+k)(2+mk)}{2+mn}.
\end{aligned}$$

When $m=3$, we recover the previously known results

$$\Omega_n = A_n = B_n = E_n = F_n.$$

As done by Ekhad and Zeilberger [10] for the above identities, it should also be possible to give, by the WZ-method, "automated" proofs for those displayed in (5).

§1. Firstly, we can reformulate the generating function $\Omega_m[x]$ as

$$\begin{aligned}
\Omega_m[x] &= \frac{(1+y)^2}{(1+y-my)^2} = \frac{1+y}{1+y-my} \times \frac{1}{1-my/(1+y)}\\
&= \frac{1+y}{1+y-my} \times \frac{1}{1-mx(1+y)^{m-1}} \qquad \boxed{y = x(1+y)^m}\\
&= \sum_{j=0}^{\infty} (mx)^j \frac{(1+y)^{(m-1)j+1}}{1+y-my}.
\end{aligned}$$

Letting $\alpha = (m-1)j$ and $\beta = m$ in (4), we have further

$$\begin{aligned}
\Omega_m[x] &= \sum_{j=0}^{\infty} (mx)^j \sum_{k=0}^{\infty} x^k \binom{(m-1)j+km}{k}\\
&= \sum_{n=0}^{\infty} x^n \sum_{j=0}^{n} m^j \binom{nm-j}{n-j} \qquad \boxed{n = k+j}
\end{aligned}$$

which implies that $\Omega_m(n) = A_m(n)$. □

§2. Secondly, the generating function $\Omega_m[x]$ can also be restated as

$$\begin{aligned}
\Omega_m[x] &= \frac{(1+y)^2}{(1+y-my)^2} = \frac{(1+y)^2}{1+y-my} \times \frac{1}{1+y-my}\\
&= \frac{(1+y)^2}{1+y-my} \times \frac{1}{1-x(1+y)^m(m-1)} \qquad \boxed{y = x(1+y)^m}\\
&= \sum_{j=0}^{\infty} (m-1)^j x^j \frac{(1+y)^{2+mj}}{1+y-my}.
\end{aligned}$$

By letting $\alpha = 1+mj$ and $\beta = m$ in (4), we can expand further

$$\begin{aligned}
\Omega_m[x] &= \sum_{j=0}^{\infty} (m-1)^j x^j \sum_{k=0}^{\infty} x^k \binom{mj+km+1}{k}\\
&= \sum_{n=0}^{\infty} x^n \sum_{j=0}^{n} (m-1)^j \binom{nm+1}{n-j}. \qquad \boxed{n = k+j}.
\end{aligned}$$

This shows that $\Omega_m(n) = B_m(n)$. □

§3. Thirdly, rewrite the generating function $\Omega_m[x]$ as

$$\begin{aligned}\Omega_m[x] &= \frac{(1+y)^2}{(1+y-my)^2} = \frac{1}{\left(1-\frac{my}{1+y}\right)^2}\\ &= \frac{1}{(1-mx(1+y)^{m-1})^2} && \boxed{y = x(1+y)^m}\\ &= \sum_{k=0}^{\infty}(k+1)(mx)^k(1+y)^{(m-1)k}.\end{aligned}$$

Letting $\alpha = (m-1)k, \beta = m$ in (3), we have further

$$\begin{aligned}\Omega_m[x] &= \sum_{k=0}^{\infty}(mx)^k \sum_{i=0}^{\infty} x^i \binom{(m-1)k+mi}{i}\frac{(m-1)k(k+1)}{(m-1)k+mi}\\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} m^k \binom{mn-k}{mn-n}\frac{(m-1)k(k+1)}{mn-k}. && \boxed{n = k+i}\end{aligned}$$

This gives rise to the equality $\Omega_m(n) = E_m(n)$. □

§4. Finally, the generating function $\Omega_m[x]$ can be expressed alternatively as

$$\begin{aligned}\Omega_m[x] &= \frac{(1+y)^2}{(1+y-my)^2} = \frac{(1+y)^2}{(1-(m-1)x(1+y)^m)^2}\\ &= \sum_{k=0}^{\infty}(k+1)(m-1)^k x^k(1+y)^{mk+2}. && \boxed{y = x(1+y)^m}\end{aligned}$$

By making use of (3) with $\alpha = 2+mk$ and $\beta = m$, we deduce that

$$\begin{aligned}\Omega_m[x] &= \sum_{k=0}^{\infty}(m-1)^k x^k \sum_{i=0}^{\infty} x^i \binom{2+mk+im}{i}\frac{(k+1)(mk+2)}{2+mk+im}\\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}(m-1)^k \binom{2+mn}{n-k}\frac{(k+1)(2+mk)}{2+mn} && \boxed{n = k+i}\end{aligned}$$

which leads us to the fourth identity $\Omega_m(n) = F_m(n)$. □

§5. Before concluding the paper, it is worthwhile making the following comments.

- As done by Duarte and Guedes de Oliveira [9], for an arbitrary real number λ, we have the identity
$$\Omega_m(n) = \sum_{k=0}^{n}\binom{mk+\lambda}{k}\binom{mn-mk-\lambda}{n-k}.$$
This can be justified easily by the functional relation
$$\Omega_m[x] = \frac{(1+y)^2}{(1+y-my)^2} = \frac{(1+y)^{1+\lambda}}{1+y-my} \times \frac{(1+y)^{1-\lambda}}{1+y-my}.$$
- Observe that the equalities in (5) hold for infinitely many integers m. According to the fundamental theorem of polynomial algebra, these identities are valid also when m is replaced by a variable and any complex number.
- For the remaining two sequences C_n and D_n, we failed to find their generalized expressions $C_m(n)$ and $D_m(n)$. The interested reader is encouraged to make further attempts.

□

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