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La conjecture de André–Oort effective pour les courbes non-compactes dans les variétés modulaires de Hilbert

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Abstract. In the proofs of most cases of the André–Oort conjecture, there are two different steps whose effectivity is unclear: the use of generalizations of Brauer–Siegel and the use of Pila–Wilkie. Only the case of curves in $\mathbb{C}^2$ is currently known effectively (by other methods).

We give an effective proof of André–Oort for non-compact curves in every Hilbert modular surface and every Hilbert modular variety of odd genus (under a minor generic simplicity condition). In particular we show that in these cases the first step may be replaced by the endomorphism estimates of Wüstholz and the second author together with the specialization method of André via G-functions, and the second step may be effectivized using the Q-functions of Novikov, Yakovenko and the first author.

Résumé. Dans les démonstrations de la plupart des cas de la conjecture de André–Oort, il y a deux étapes différentes dont l'effectivité n'est pas claire : l'utilisation de généralisations de Brauer–Siegel et l'utilisation de Pila–Wilkie. Seulement le cas des courbes dans $\mathbb{C}^2$ est couramment effectivement connu (par des autres méthodes).

Nous donnons une démonstration effective de la conjecture pour les courbes non-compactes dans chaque surface modulaire de Hilbert et chaque variété modulaire de Hilbert de genre impair (sous condition secondaire de simplicité générique). En particulier nous montrons que dans ces cas, la première étape peut être remplacée par les majorations d'endomorphismes de Wüstholz et le deuxième auteur combinées avec la méthode de spécialisation de André par les G-fonctions, et la deuxième étape peut être effectivisée en utilisant les Q-fonctions de Novikov, Yakovenko et le premier auteur.

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1. Introduction

In this note we discuss the known results on effective André-Oort, all of which are restricted to the context of modular curves, and announce a new effective result in the context of Hilbert modular varieties. We provide a sketch of the key ideas, and will provide full details in a paper under preparation.

1.1. *The André–Oort conjecture and effectivity*

In the André–Oort conjecture one considers a suitable ambient space $\mathscr{X}$ (in this note we will mention only $\mathscr{X} = \mathbb{C}^n$ ($n \geq 2$) or $\mathscr{X} = \mathscr{A}_g$ ($g \geq 2$) the moduli space of principally polarized abelian varieties of dimension g) equipped with irreducible algebraic subvarieties that are known as special subvarieties. Then one takes an algebraic subvariety V in $\mathscr{X}$, and one expects that if the special points of V are Zariski-dense in V, then V is itself special.

For $\mathscr{X} = \mathbb{C}^n$ the special points are the $(j_1, \ldots, j_n)$ whose coordinates are values of the elliptic modular function at quadratic points of the standard upper half-plane. A special variety is defined essentially by relations $F(x_r, x_s) = 0$ where the F are modular polynomials. The conjecture was proved for $n = 2$ by André himself [2] and a high point was reached with Pila's paper [21] for all n (and even more).

For $\mathscr{X} = \mathscr{A}_g$ the special points are the abelian varieties (not necessarily simple) with complex multiplication CM, in the sense that their endomorphism algebras contain a commutative algebra of dimension $2g$ over $\mathbb{Q}$. For special varieties the description is not so elementary, but when $g = 2, 3$ it can also be reduced to properties of the endomorphism algebras. The conjecture was proved for $g = 2$ by Pila and Tsimerman [22] and then for all g by Tsimerman [26].

In both situations $\mathscr{X} = \mathbb{C}^n$ or $\mathscr{X} = \mathscr{A}_g$ the proofs remain ineffective in the sense that when V is given, there is in general no method to find the set of its special points.

For $\mathscr{X} = \mathbb{C}^n$ some effective results are known in particular cases. The case of a general curve V has been treated independently by Kühne [14] and Bilu, Zannier and the second author [4], and this remains the only case where the André–Oort conjecture is known effectively. Since then some progress has been achieved for V of higher dimension: see Bilu and Kühne [3] for the linear case, and the first author [6] for an extension to arbitrary degree hypersurfaces under a genericity assumption on the leading homogeneous part. However no general result for say a surface V is known.

1.2. *Statement of the main result*

From now on V will be a curve; thus if V is not itself special then it contains at most finitely many special points. For $\mathscr{X} = \mathscr{A}_g$ there are no effective results up to now. But we can prove the following. For simplicity we assume that V is defined over $\overline{\mathbb{Q}}$; as special points are also defined over $\overline{\mathbb{Q}}$ there is no problem to extend to general fields provided these are "effectively given" as for example in Fröhlich and Shepherdson [12].

Theorem 1. *Let V be a curve in $\mathscr{A}_g$, defined over $\overline{\mathbb{Q}}$, that is not special. Suppose*

    (a) *$g = 2$ or $g > 2$ is odd,*
    (b) *V lies in a Hilbert modular variety,*

   (c) *V is not compact,*

   (d) *a generic point of V corresponds to a simple abelian variety.*

Then the finitely many special points on V can be found effectively.

We may identify V with a single abelian variety A, defined over the function field of a curve B, that is not isotrivial. Then (b) is equivalent to A having real multiplication RM, in the sense that its endomorphism algebra contains a totally real field of degree g over $\mathbb{Q}$. And (c) is equivalent to A having bad reduction somewhere. Also (d) is equivalent to A being simple. For all but finitely many points b of B, there is a specialized abelian variety A_b, and our claim is then that the finitely many b such that A_b has CM can be found effectively.

It seems plausible that condition (d) can be dropped. Aside from this technical issue, our main theorem applies to every non-compact curve in every Hilbert modular surface or Hilbert modular variety of odd genus, giving an analog for the effective result for curves in $\mathbb{C}^2$ by [4, 14] (where every curve is non-compact). In particular, in Hilbert modular surfaces every non-special curve is generically simple, and in these cases condition (d) is indeed superfluous[1]. To our knowledge no similar effective results are known for any compact families in $\mathscr{A}_g$, and treating such a problem would presumably require very different ideas, as all known effective proofs involve some form of asymptotic around a cusp.

For small genus we may express the fact that V is not special also in a more elementary way. Thus for $g = 2, 3$ we get the following consequences.

Corollary 2. *Let A be a simple abelian surface, defined over $\overline{\mathbb{Q}}(B)$, that is not isotrivial. Suppose that A has real multiplication but not quaternion multiplication, and bad reduction somewhere. Then the finitely many b in B such that A_b has complex multiplication can be found effectively.*

An example with $B = \mathbb{P}_1$ is the Jacobian of

$$y^2 = x^5 + (t-2)x^4 - (2t-3)x^3 + tx^2 - 2x + 1, \tag{1}$$

with real multiplication by $\mathbb{Q}(\sqrt{5})$. This comes from specializing a two-dimensional family due to Wilson [27].

Corollary 3. *Let A be a simple abelian threefold, defined over $\overline{\mathbb{Q}}(B)$, that is not isotrivial. Suppose that A has real multiplication and bad reduction somewhere. Then the finitely many b in B such that A_b has complex multiplication can be found effectively.*

An example again with $B = \mathbb{P}_1$ is the Jacobian of

$$y^2 = x^7 - tx^6 + 34x^5 + (8t+156)x^4 + 48x^3 - 16tx^2 - 32x + 64, \tag{2}$$

with real multiplication by $\mathbb{Q}(\cos(2\pi/7))$. This comes from specializing a two-dimensional family due to Mestre [18].

Remark 4. Of course such results are impossible for $g = 1$, because all elliptic curves have RM and among them are infinitely many non-isomorphic (and even non-isogenous) curves with CM. For example, there are infinitely many t such that

$$y^2 = x(x-1)(x-t) \tag{3}$$

has CM.

In fact our arguments are not restricted to real multiplication. For example, the Jacobian of the curve

$$y^2 = x(x-1)(x-t)(x-t^2)(x-t^4) \tag{4}$$

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[1] We thank Jonathan Pila for pointing out this special case to us.

has endomorphism ring $\mathbb{Z}$ (see [16, p. 296]) and we can effectively find all complex t for which it is simple with complex multiplication. Similarly for

$$y^2 = x(x-1)(x-t)(x-t^2)(x-t^4)(x-t^5)(x-t^8). \tag{5}$$

1.3. *G-functions and comparison to André's results*

Already André [1] had observed the applicability of G-functions to problems connected to special points in Shimura varieties. In particular, the theorem in [1, p. 201] implies a result similar to our main theorem, with the caveat that one counts CM points s only with bounded residual degree $[\mathbb{Q}(s) : \mathbb{Q}]$. This was one of the motivations leading up to André's formulation of his general conjecture (see in particular [1, p. 215–216]). Unlike the subsequent proof of the general conjecture, André's argument is effective. Our main contribution is to eliminate the condition of bounded residual degree in our situation while preserving the effectivity of the argument.

The recent papers of Daw and Orr [10,11] also use G-functions to study instances of the Zilber–Pink conjecture. Though they do not consider effectivity, it seems plausible that effective versions of their results may also be possible to derive using arguments similar to those employed in the present note.

Remark 5.   In fact [1] assumes that $g > 2$ is odd, and we are pleased to thank Yves André for helping us to see that his work can be extended to $g = 2$.

We may note that G-functions were used for a different purpose by Kühne in [15].

1.4. *Overview of the proof*

We use the strategy introduced by Pila–Zannier in [24], contrasting lower bounds on Galois orbits against upper bounds for rational points in definable sets. As mentioned, the proofs in [22] and [26] are ineffective, and this comes from two different sources. First, a Brauer–Siegel bound due to Tsimerman [25], and second the Pila–Wilkie bound [23] for rational points on definable sets.

We avoid Brauer–Siegel using a different version of the discriminant which can be controlled by refinements of "linear forms in abelian logarithms" due to Wüstholz and the second author based on transcendence theory. These introduce an extra height dependence which can be dealt with using "global relations for G-functions" due to André, also based on transcendence theory. It is for this that we need conditions (a) and (c) of our Theorem. And the combination of (b) and (c) leads to André's condition of multiplicative reduction. For more details see Section 2.

Several effective versions of the Pila–Wilkie theorem have been established in the literature. However, none of these are general enough to allow the type of counting needed in our context. More specifically, such results have been either restricted to the Pfaffian class of functions (see Jones and Thomas [13] for example) which is not known to contain period maps; or restricted to counting in compact subsets [5]. In our context it is necessary to count in the entire (non-compact) fundamental domain. We overcome the compactness issue by employing the theory of Q-functions due to Novikov, Yakovenko and the first author. This theory produces effective estimates for the number of zeros, and other analytic complexity measures, which crucially hold "all the way to the cusp". For more details see Section 4.

1.5. *Contents of this note*

In Section 2 we describe our method for achieving Galois orbit lower bounds effectively by combining G-function methods and endomorphism estimates. In Section 3 we describe how

combining these bounds with Pila–Wilkie type bounds gives the main theorem. In Section 4 we sketch how we obtain such bounds effectively using Q-functions.

For simplicity of the presentation we restrict attention to the specific case of (1) (essentially the same description applies to (2) above). Thus we assume that t is such that the Jacobian A_t has complex multiplication CM. It is easy to see that t is an algebraic number. Let d be its degree and h its absolute logarithmic height.

2. Effectively avoiding Brauer–Siegel.

As A_t has CM, the main result of Tsimerman [25, p. 1091] now implies that

$$|\mathrm{Disc}(Z(\mathrm{End}(A_t)))| \le cd^\kappa \tag{6}$$

for a suitable discriminant of the centre of the endomorphism ring over $\mathbb{C}$, where c and κ are absolute. The constant c is ineffective. Indeed this is so even in the analogue for (3) in dimension $g = 1$, where the left-hand side corresponds to a discriminant of an imaginary quadratic field and the right-hand side corresponds to a class number. In fact (6) holds for all g with suitable c, κ, as shown by Tsimerman [26, p. 385].

We will modify (6) and factorize it through two effective inequalities. First we prove that

$$0 < \mathrm{Disc}_r(\mathrm{End}(A_t)) \le c_1(h+d)^{\kappa_1}, \tag{7}$$

for a slightly different "polarized discriminant" and without taking the centre. This result comes directly from certain "endomorphism estimates" of Wüstholz and the second author [17, p. 642–650] obtained through transcendence techniques (with linear forms in abelian logarithms). It holds much more generally for any g and with no assumptions on the endomorphism ring. Second we prove that

$$h \le c_2 d^{\kappa_2}, \tag{8}$$

which comes from a slight extension of a specialization result of André [1, p. 201], also obtained through transcendence techniques (with global relations for G-functions). The latter was originally only for odd dimension $g \ge 3$ (the effectivity is still unknown for $g = 1$), and with a different sort of assumption about the endomorphism ring; furthermore there is a new assumption involving multiplicative reduction. This is automatically satisfied in the case of real multiplication RM as soon as there is bad reduction somewhere. For our (1) the latter is also automatic because $B = \mathbb{P}_1$ (and in fact it holds at $t = \infty$, where there is multiplicative reduction for (4) and (5) too). It is the use of transcendence techniques that makes both (7) and (8) effective.

It is relatively easy to compare the discriminants in (6) and (7), provided A_t is isogenous to a power of a simple abelian variety, also an automatic consequence of RM (which explains the extra simplicity assumption for (4) and (5) above). Then (7) and (8) give the required effective version of (6), assuming multiplicative reduction.

This (6) is combined with an estimate for the Siegel matrix τ_t of A_t when normalized to lie in a fundamental domain of the Siegel upper half-space $\mathcal{H}$. As A_t has CM, it is well-known that the entries are algebraic numbers, and so we can speak of a (non-logarithmic absolute) height $H(\tau_t)$. For this Theorem 3.1 of [22, p. 206] implies that

$$H(\tau_t) \le c|\mathrm{Disc}(Z(\mathrm{End}(A_t)))|^\lambda \tag{9}$$

where c, λ are absolute (here in dimension $g = 2$) and effective.

3. Reduction to point-counting

This (9) is now set up for counting. The traditional way is to embed $\mathscr{H}$ into $\mathbb{R}^6$ by taking the real and imaginary parts of the three entries of the (symmetric) matrix. For our particular t and τ_t it is easy to see that we get a point P_t whose coordinates are algebraic numbers of degree at most 16 with heights at most $4H(\tau_t)^2$. If we allow t for the moment to range over $\mathbb{C}$ we get a real surface Z in $\mathbb{R}^6$, actually definable in the sense of [23]. Now by Theorem 1.6 of [20, p. 153] the number of our particular P_t with height at most H which lie on the transcendental part Z^{trans} is

$$N \le c(\epsilon)H^\epsilon \tag{10}$$

for any $\epsilon > 0$. Here c is up to now not effective. It is not hard to show using Ax–Lindemann–Weierstrass as in Theorem 1.2 of [22, p. 204] that $Z^{\mathrm{trans}} = Z$.

Now combining this with (6) and (9) we get $N \le c_1(\epsilon)d^{2\kappa\lambda\epsilon}$. A rather primitive zero estimate (but also non-effective) yields a similar estimate for the number of t. On the other hand we could have taken many conjugates of just our single t, giving $N \ge d$. And now choosing $\epsilon < 1/(2\kappa\lambda)$ shows that d is bounded above. Thus by (8) so is h, and we finish with Northcott. Thus we have to make (10) effective.

4. Effective Pila–Wilkie counting

The proof of (10) is based on the ideas of Bombieri and Pila [9] and Pila and Wilkie [23] and involves two steps. For simplicity we will explain the strategy for counting points only over $\mathbb{Q}$ - the same ideas extend to points of any fixed degree, such as 16 above. First, one parametrizes the surface Z using smooth charts with bounded derivatives, and shows that in each of the charts the set of rational points of height at most H is interpolated by $c(\epsilon)H^\epsilon$ algebraic hypersurfaces of degree $C(\epsilon)$. Second, one consider the intersection of Z with each of these algebraic hypersurfaces separately and proceeds by induction.

It has been shown in [5] that this strategy can be carried out effectively when the functions defining Z satisfy certain overdetermined systems of differential equations. In particular, period maps fall within this class, and this result can therefore be applied to our context. However, a crucial assumption in [5] is that one works in a compact domain where the corresponding system of differential equations is non-singular. In our context it is necessary to work with the non-compact set Z, and this introduces significant technical difficulties, in particular in regard to the parametrization step of the Pila–Wilkie proof.

4.1. *Q-functions*

To overcome the difficulty involving non-compact domains, we appeal to the theory of *Q-functions* developed by Novikov, Yakovenko and the first author [7, 8]. Briefly, consider a system of linear ordinary differential equations

$$\mathrm{d}X = \Omega \cdot X \tag{11}$$

where Ω is a matrix of $\overline{\mathbb{Q}}$-rational meromorphic one-forms on V. Suppose further that the singularities of the system are regular, and that the monodromy is quasi-unipotent. Then, if X denotes a fundamental solution for this system, each entry of X is called a Q-function. The *complexity* of these Q-functions is defined to be the maximum of the degrees and heights of the entries of Ω. By the direct-sum and tensor operations on connections, Q-functions form an algebra and the complexity of sums and products is suitably controlled in terms of the individual terms. In our application, we will consider the equation (11) associated to the Gauss-Manin

connection for the H^1-cohomology of the family of abelian varieties over V. Then τ_t is given by a ratio of the two upper blocks of X (thought of as a 2×2 block matrix), and in particular each entry of τ_t is a ratio of Q-functions (we call such ratios *RQ-functions*). For example, the classical case (3) corresponds to $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ \frac{1/4}{t(1-t)} & \frac{-1+2t}{t(1-t)} \end{smallmatrix} \right) \mathrm{d}t$ and $X = \left(\begin{smallmatrix} F & G \\ F' & G' \end{smallmatrix} \right)$ for the classical hypergeometric function $F = F(1/2, 1/2, 1; t)$ and its derivative (see below for G).

Fix an étale coordinate $z : V \to \mathbb{C}$, and denote by $\Sigma \subset \mathbb{C}$ its (finite) ramification locus. We may think of the Q-functions as (multivalued) functions in the z-plane. The main result of [7] shows that the number of zeros of a Q-function in any disc $D \subset \mathbb{C} \setminus \Sigma$ is explicitly bounded in terms of the complexity; and moreover, the same holds for a punctured disc $D_\circ \subset \mathbb{C} \setminus \Sigma$ if one makes a branch cut along some straight real line and counts zeros in a single branch. This easily extends to RQ-functions. It is this ability to count "all the way to the cusp" that we employ to avoid the compactness restrictions.

4.2. *The point-counting strategy*

Another advantage of working with the complex analytic theory of Q-functions in place of general real sets of o-minimal structures is that we can view Z as a one-dimensional *complex* curve rather than a real surface. Thus the Pila–Wilkie induction mentioned above terminates after a single step, and the main issue is to suitably parametrize Z, or at least the part of Z that contains points up to height H.

Our strategy is to cut up $\mathbb{C} \setminus \Sigma$ into finitely many (punctured) discs and annuli such that in each piece, we can indeed interpolate all points of height at most H by $O(H^\epsilon)$ algebraic curves of degree $C(\epsilon)$. Our basic technical tool is the following theorem.

Theorem 6. *Let f_1, f_2 be holomorphic on a disc in $\mathbb{C}$ and p-valent there. Then for z in the concentric disc of half the radius, the points $(f_1(z), f_2(z))$ of height at most H can be interpolated by $c(p, \epsilon) H^\epsilon$ curves of degree $C(\epsilon)$ for some explicit constants $c(p, \epsilon), C(\epsilon)$.*

Theorem 6 is similar to the main lemma of Bombieri–Pila [9, p. 343], which has appeared in numerous variations in the literature. However, the constants C, c normally depend on the maximum of the functions f_1, f_2 and their derivatives, and we replace this by the valency of these functions. Since RQ-functions are known to be p-valent with p effective, we can apply this to any disc $D \subset \mathbb{C} \setminus \Sigma$ such that the concentric disc of twice the radius does not meet Σ without having to control derivatives. In this way we cover the part of the curve that lies away from the singularities Σ.

A different strategy is required for studying punctured discs $D_\circ$ around the singular points. It is well-known that sections of regular-singular connections with quasi-unipotent monodromy admit converging expansions of the form

$$f(z) = \sum_{\lambda \in \Lambda} z^\lambda P_\lambda(\log z) \tag{12}$$

where $\Lambda \subset \mathbb{R}$ is a union of finitely many $\mathbb{N}$-cosets, and each P_λ is a polynomial of degree bounded by some fixed integer. The expansion holds in a branch defined on $D_\circ$ with a branch cut along a straight real line.

For example with (3) these can be taken as essentially as F and $G = iF \log z + \widetilde{F}$ where $F = F(1/2, 1/2, 1; z)$ is as above and $\widetilde{F}$ is a related power series in z.

With the theory of Q-functions one has good control over the rate of asymptotic convergence in (12). Thus we can further subdivide $D_\circ$ into discs and annuli such that Theorem 6 applies to all the discs; and in each remaining annulus one of the terms $z^\lambda P_\lambda(\log z)$ strongly dominates (the sum of) all remaining terms.

The same applies for RQ-functions, with the dominant term now given in the form $z^\lambda R_\lambda(\log z)$ for a rational function R_λ. Suppose now that f_1, f_2 are two RQ-functions on an annulus $E \subset D_\circ$, with dominant terms $z^{\lambda_j} R_{\lambda_j}(\log z)$ for $j = 1, 2$. If $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$ then the corresponding f_1 or f_2 can take values of height H only in a sub-annulus $E' \subset E$ of logarithmic width $O(\log H)$. In this case it is a simple matter to cover E' by $O(\log H)$ discs and reduce to Theorem 6.

The remaining case is $\lambda_1 = \lambda_2 = 0$. It is a small miracle that in this particular case, the leading terms of f_1, f_2 are algebraically dependent (both being rational functions of $\log z$). Since the remaining terms in the asymptotic expansion are much smaller than these leading terms, it turns out that $(f_1, f_2)(E)$ is extremely close to an algebraic curve, and in this case one is able to use a different, more naive argument to prove the interpolation result. Briefly, we look for an upper-bound for an interpolation determinant, as in the Bombieri–Pila method. But here the interpolation determinant vanishes identically on the leading terms (as they are algebraically dependent), and the remaining terms are so small that a trivial majoration suffices for our purposes.

4.3. *Under the rug: complications around branch cuts*

Our approach is to parametrize Z by means of discs and punctured discs in $D, D_\circ \subset \mathbb{C} \setminus \Sigma$. Recall that in the punctured discs we make branch cuts along straight real lines. This means that we count rational values in the set $\tau(D_\circ)$ (of Siegel matrices) for such a branch. In general the image $\tau(D_\circ)$ will not be contained in a single fundamental domain: the straight lines in the z-coordinate do not exactly correspond to the semialgebraic walls of the fundamental domain in $\mathcal{H}$. If a point $\tau(z)$ does not belong to the standard fundamental domain then it is not possible, a priori, to estimate its height in terms of the corresponding discriminant.

Remark 7. It is true that each $\tau(D_\circ)$ will meet finitely many fundamental domains, for instance by the definability of the universal covering map of the Siegel modular variety in an o-minimal structure [19]. However, computing this number effectively for a general curve V does not seem straightforward.

We overcome this problem in a different manner. We show instead that a suitable norm $\|\tau(z)\|$ is effectively $O(|\log z|)$; for this we consider τ as a map between the hyperbolic Riemann surface $D_\circ$ and the hyperbolic Siegel space $\mathcal{H}$ and use the Schwarz–Pick lemma. If z corresponds to a CM point then $|\log z|$ is bounded by a polynomial in the corresponding discriminant. Thus $\|\tau(z)\|$ is similarly bounded, and this allows us to suitably estimate its height by a polynomial in the discriminant even if $\tau(z)$ does not belong to the standard fundamental domain.

References

[1] Y. André, *G-functions and geometry*, Aspects of Mathematics, vol. 13, Vieweg & Sohn, 1989.

[2] ———, "Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire", *J. Reine Angew. Math.* **505** (1998), p. 203-208.

[3] Y. Bilu, L. Kühne, "Linear equations in singular moduli", *Int. Math. Res. Not.* **2020** (2020), no. 21, p. 7617-7643.

[4] Y. Bilu, D. Masser, U. Zannier, "An effective 'Theorem of André' for *CM*-points on a plane curve", *Math. Proc. Camb. Philos. Soc.* **154** (2013), no. 1, p. 145-152.

[5] G. Binyamini, "Density of algebraic points on Noetherian varieties", *Geom. Funct. Anal.* **29** (2019), no. 1, p. 72-118.

[6] ———, "Some effective estimates for André–Oort in $Y(1)^n$", *J. Reine Angew. Math.* **767** (2020), p. 17-35, with appendix by E. Kowalski.

[7] G. Binyamini, D. Novikov, S. Yakovenko, "On the number of zeros of Abelian integrals. A constructive solution of the infinitesimal Hilbert sixteenth problem", *Invent. Math.* **181** (2010), no. 2, p. 227-289.

[8] ———, "Quasialgebraic functions", in *Algebraic methods in dynamical systems*, Banach Center Publications, vol. 94, Polish Academy of Sciences, 2011, p. 61-81.

[9] E. Bombieri, J. Pila, "The number of integral points on arcs and ovals", *Duke Math. J.* **59** (1989), no. 2, p. 337-357.

[10] C. Daw, M. Orr, "Quantitative reduction theory and unlikely intersections", https://arxiv.org/abs/1911.05618, 2019.

[11] ———, "Unlikely intersections with $E \times CM$ curves in $\mathscr{A}_2$", https://arxiv.org/abs/1902.10483, 2019.

[12] A. Fröhlich, J. C. Shepherdson, "Effective procedures in field theory", *Philos. Trans. Roy. Soc. London* **248** (1956), p. 407-432.

[13] G. Jones, M. E. M. Thomas, "Effective Pila–Wilkie bounds for unrestricted Pfaffian surfaces", https://arxiv.org/abs/1804.08232, to appear in *Math. Ann.*, 2018.

[14] L. Kühne, "An effective result of André–Oort type", *Ann. Math.* **176** (2012), no. 1, p. 651-671.

[15] ———, "Logarithms of algebraic numbers", *J. Théor. Nombres Bordeaux* **27** (2015), no. 2, p. 499-535.

[16] D. Masser, "Specializations of some hyperelliptic Jacobians", in *Number theory in progress*, Walter de Gruyter, 1999, p. 293-307.

[17] D. Masser, G. Wüstholz, "Endomorphism estimates for abelian varieties", *Math. Z.* **215** (1994), no. 4, p. 641-653.

[18] J.-F. Mestre, "Families of hyperelliptic curves with real multiplication", in *Arithmetic algebraic geometry*, Progress in Mathematics, vol. 89, Birkhäuser, 1991, p. 193-208.

[19] Y. Peterzil, S. Starchenko, "Definability of restricted theta functions and families of abelian varieties", *Duke Math. J.* **162** (2013), no. 4, p. 731-765.

[20] J. Pila, "On the algebraic points of a definable set", *Sel. Math., New Ser.* **15** (2009), no. 1, p. 151-170.

[21] ———, "O-minimality and the André–Oort conjecture for $\mathbb{C}^n$", *Ann. Math.* **173** (2011), no. 3, p. 1779-1840.

[22] J. Pila, J. Tsimerman, "The André–Oort conjecture for the moduli space of abelian surfaces", *Compos. Math.* **149** (2013), no. 2, p. 204-216.

[23] J. Pila, A. Wilkie, "The rational points of a definable set", *Duke Math. J.* **133** (2006), no. 3, p. 591-616.

[24] J. Pila, U. Zannier, "Rational points in periodic analytic sets and the Manin–Mumford conjecture", *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.* **19** (2008), no. 2, p. 149-162.

[25] J. Tsimerman, "Brauer–Siegel for arithmetic tori and lower bounds for Galois orbits of special points", *J. Am. Math. Soc.* **25** (2012), no. 4, p. 1091-1117.

[26] ———, "The André–Oort conjecture for $\mathscr{A}_g$", *Ann. Math.* **187** (2015), no. 2, p. 379-390.

[27] J. Wilson, "Explicit moduli for curves of genus 2 with real multiplication by $\mathbb{Q}(\sqrt{5})$", *Acta Arith.* **93** (2000), no. 2, p. 121-138.