INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

René Dáger, Víctor Navarro and Mihaela Negreanu

Uniform boundedness of solutions for a predator-prey system with diffusion and chemotaxis

Volume 358, issue 1 (2020), p. 103-108

Published online: 18 March 2020

https://doi.org/10.5802/crmath.17

CENTRE MERSENNE

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus Mathématique
2020, 358, nº 1, p. 103-108
https://doi.org/10.5802/crmath.17

Partial Differential Equations / *Équations aux dérivées partielles*

Uniform boundedness of solutions for a predator-prey system with diffusion and chemotaxis

Limite uniforme des solutions pour un système prédateur-proie avec diffusion et chimiotaxie

René Dáger [a], Víctor Navarro [b] and Mihaela Negreanu [b]

[a] Departamento de Matemática Aplicada, Universidad Politécnica de Madrid, 28040 Madrid, Spain

[b] Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain.

E-mails: rene.dager@upm.es, vinavarr@ucm.es, negreanu@mat.ucm.es.

Abstract. In this Note we study a nonlinear system of reaction-diffusion differential equations consisting of an ordinary differential equation coupled to a fully parabolic chemotaxis system. This system constitutes a mathematical model for the evolution of a prey-predator biological population with chemotaxis and dormant predators. Under suitable assumptions we prove the global in time existence and boundedness of classical solutions of this system in any space dimension.

Résumé. Dans cette Note, nous étudions un système non linéaire d'équations différentielles partielles de type réaction-diffusion décrivant l'évolution d'un système biologique proie-prédateur avec chimiotaxie et prédateurs dormants. Nous considérons une équation ordinaire couplée à un système parabolique de chimiotaxie. Sous certaines hypothèses appropriées, nous obtenons l'existence globale en temps de solutions classiques du système considéré dans n'importe quelle dimension spatiale.

Manuscript received 4th October 2019, revised 25th November 2019, accepted 16th December 2019.

Version française abrégée

En considérant les interactions prédateur-proie, on peut arriver à la chimiotaxie, qui est un phénomàne durant lequel les organismes vivants orientent leur mouvement en rapport à un certain gradient de concentration chimique. Certains modèles mathématiques pour décrire la chimiotaxie ont été proposés au cours des dernières années, succédant aux premiers travaux de Keller et Segel dans la décennie des années 1970. Durant ce travail, nous considérons un

système d'équations ordinaires paraboliques avec chimiotaxie. Ce système considère des agents prédateurs actifs u, des proies v et des prédateurs dormants w. En outre, la chimiotaxie est incluse dans l'équation des prédateurs actifs, les faisant se déplacer vers les régions où le gradient de concentration des prédateurs dormants est plus élevé (les prédateurs actifs présentent une tendance protectrice envers ses œufs). Ce modèle peut être considéré comme un cas général des modèles proposé dans [4] et [9].

Ce travail est organisé de la forme suivante:

- Dans la section 2, en appliquant les résultats théoriques développés par Amann dans [2] et [3], nous donnons des résultats préliminaires afin d'obtenir une solution faible maximale (u, v, w) de (1).
- Dans la section 3, le système avec prédateurs dormants et chimiotaxies est étudié afin d'obtenir les limites de la solution. Comme argument crucial dans l'estimation d'une borne L_∞ pour u, nous employons une procédure itérative de type Moser–Alikakos.

En utilisant ces résultats et d'autres propriétés de régularité, nous étudions, dans le Théorème 1, l'existence globale et les limites de u, v et w.

1. Introduction

We consider the predator-prey model given by the system

$$\begin{cases} \partial_t u = d\Delta u - \nabla\cdot(u\chi(w)\nabla w) + \ell\xi(v) f(u,v,x) + a(v)w - m_u(u,x), & \text{in } \Omega\times(0,\infty),\\ \partial_t v = \Delta v + \phi(v,x) - f(u,v,x), & \text{in } \Omega\times(0,\infty),\\ \partial_t w = \ell\eta(v) f(u,v,x) - a(v)w - m_w(w)w, & \text{in } \Omega\times(0,\infty),\\ \frac{\partial u(x,t)}{\partial n} = \frac{\partial v(x,t)}{\partial n} = \frac{\partial w(x,t)}{\partial n} = 0, & \text{in } \partial\Omega\times(0,\infty) \end{cases} \tag{1}$$

with non-negative initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $w(x,0) = w_0(x)$, for $x \in \Omega$.

Here Ω is an open bounded domain with smooth boundary $\partial\Omega$ of measure $|\Omega| = 1$.

Model (1) extends those proposed in [4] and [8]. It is a prey-predator reaction-diffusion system with chemotaxis effect to include predator dormancy. The functions appearing in (1) are smooth and have the following biological meaning.

- u, v and w are the densities of predators, preys and predators with dormant state (resting eggs), respectively.
- $d \ge 0$ is the diffusion coefficient in the predators term. It is assumed to be small relative to 1,
- $\chi := \chi(w)$ is the function responsible of chemotaxis,
- $\phi(v,x)$ represents the production of preys,
- $m_u(u,x)$ is the mortality rate of predators,
- $f(u,v,x)$ is related to the predators consumption of preys,
- ℓ represents the fraction of preys biomass density that can be turned into predator biomass density,
- $\xi(v)$ and $\eta(v)$ are bounded functions related to the distribution of reproduction energy of predators between active and dormant states and verify $\xi(v) + \eta(v) = 1$,
- $a(v)$ denotes the average dormancy period,
- $m_w(w)$ represents the mortality rate of dormant predators.

In (1) we assume that f, ξ, m_u and m_w are increasing functions in (u,v), v, u and w, respectively, while a is non-decreasing in v. Besides, m_w is supposed to be small relative to m_u.

Based on biological reasons (in accordance with [4], [5], [8] and [9]), we consider the hypotheses:

- (H_0) The functions $m_u, m_w : [0,\infty) \times \overline{\Omega} \to [0,\infty)$, $a : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \times \overline{\Omega} \to \mathbb{R}$, $f : [0,\infty) \times [0,\infty) \times \overline{\Omega} \to [0,\infty)$ and $\xi, \eta : [0,\infty) \to [0,1]$ are at least of class C^2 and verify $\phi(0,x) = 0$, $m_u(0,x) = 0$, $a(0) = 0$, $f(u,0,x) = f(0,v,x) = 0$ for all $u, v \geq 0$, $x \in \overline{\Omega}$.
- (H_1) There exists $\beta > 0$ such that $f(u,v,x) \leq \beta u$ for all $u, v \geq 0$, $x \in \overline{\Omega}$.
- (H_2) The chemotaxis response function $\chi : [0,\infty) \to [0,\infty)$ satisfies $\chi \in C^{1+\theta} \cap L^1[0,\infty)$ for $\theta > 0$.
- (H_3) There exists $\gamma > 0$ such that $m_u(u,x) \geq \gamma u$ for all $u \geq 0$, $x \in \overline{\Omega}$, $m_w(w) \geq \gamma$ for $w \geq 0$, and $\gamma > \ell\beta$ with β given previously.
- (H_4) There exists $\delta, \zeta > 0$ such that $\phi(v,x) \leq \delta v - \zeta v^2$ for all $v \geq 0$, $x \in \overline{\Omega}$.
- (H_5) The function $\ell\eta(v) f(u,v,x) - a(v)w$ is an increasing function in v for all u, v and w.
- (H_6) There exists $M > 0$ verifying $0 < M < \gamma - \ell\beta$ such that $m_w(w)w \leq M/\chi(w)$, for all w.

The following theorem states our main result on the global existence and boundedness for solutions of system (1).

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$. Let $d, \ell \geq 0$, χ, ϕ, m_u and f be smooth and non-negative functions satisfying* (H_0)–(H_6). *Then, for every $(u_0, v_0, w_0) \in (W^{1,p}(\Omega) \cap L^1(\Omega))^3$, with $p > n$ and $u_0 \geq 0$, $v_0 \geq 0$, $w_0 \geq 0$ in $\overline{\Omega}$, system* (1) *has a unique global classical solution $(u(x,t), v(x,t), w(x,t))$ satisfying*

$$(u, v, w) \in (C([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^3. \tag{2}$$

In addition, $(u(x,t), v(x,t), w(x,t))$ is uniformly bounded in $\Omega \times (0,\infty)$.

2. Local existence and preliminary results. Basic a priori bounds.

In this section we assume that the initial data $(u_0, v_0, w_0) \in (W^{1,p}(\Omega) \cap L^1(\Omega))^3$ with $p > n$ are non-negative and that hypotheses (H_0)–(H_6) hold.

The following lemma provides a local in time existence result of a classical solution to (1).

Lemma 2. *There exists a maximal existence time $T_{max} > 0$ such that system* (1) *has a unique non-negative classical solution $(u(x,t), v(x,t), w(x,t)) \in (C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})))^3$.*

The proof of this fact follows a well-known scheme developed by Amann in [2] and [3] to obtain a maximal weak solution (u, v, w) of (1).

Lemma 3. *Let (u, v, w) be the solution of* (1) *over $[0, T_{\max})$. The total masses of $u(x,t)$, $v(x,t)$, $w(x,t)$ satisfy*

$$\int_\Omega u(x,t)\mathrm{d}x \leq C_0, \int_\Omega v(x,t)\mathrm{d}x \leq C_1, \int_\Omega w(x,t)\mathrm{d}x \leq C_0, \ \forall\ t \in (0, T_{\max}), \tag{3}$$

with

$$C_0 = \max\left\{\|u_0 + w_0 + \ell v_0\|_{L^1(\Omega)}, \frac{\ell(\delta+\gamma)C_1}{\gamma}\right\}, \ C_1 = \max\left\{\|v_0\|_{L^1(\Omega)}, \frac{\delta}{\zeta}.\right\} \tag{4}$$

Sketch of the proof. Integrating directly over Ω the second equation (1), using the hypotheses and the Cauchy–Schwarz inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v = \int_\Omega \phi(v) - \int_\Omega f(u,v) \leq \int_\Omega (\delta v - \zeta v^2) \leq \delta \int_\Omega v - \zeta \left(\int_\Omega v\right)^2, \tag{5}$$

and hence

$$\int_\Omega v \leq C_1 = \max\left\{\|v_0\|_{L^1(\Omega)}, \frac{\delta}{\zeta}\right\}. \tag{6}$$

In order to obtain bounds for u and w in $L^1(\Omega)$ we consider the linear combination $u+w+\ell v$. Its derivate is

$$u_t + w_t + \ell v_t = d\Delta u + \ell\Delta v - \nabla\cdot(u\chi(w)\nabla w) - m_u(u) + \ell\phi(v) - m_w(w)w. \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+w+\ell v) = \int_\Omega (\ell\phi(v) - m_u(u) - m_w(w)w) \le \ell\delta\int_\Omega v - \gamma\int_\Omega u - \int_\Omega m_w(w)w. \tag{8}$$

Using the assumptions (H_0)–(H_6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+w+\ell v) = -\gamma\int_\Omega (u+w+\ell v) + \ell(\delta+\gamma)\int_\Omega v + \int_\Omega (\gamma - m_w(w))w, \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+w+\ell v) \le -\gamma\int_\Omega (u+w+\ell v) + \ell(\delta+\gamma)C_1, \tag{10}$$

thus,

$$\int_\Omega (u+w+\ell v) \le C_0 = \max\left\{\|u_0+w_0+\ell v_0\|_{L^1(\Omega)}, \frac{\ell(\delta+\gamma)C_1}{\gamma}\right\}. \tag{11}$$

□

Lemma 4. *The solution (u,v,w) given by Lemma 2 satisfies, $0\le u(x,t)$, $0\le w(x,t)$, $0\le v(x,t)\le A$ with $x\in\Omega$, $0\le t<T_{\max}$ for some positive constant A.*

Sketch of the proof. We have that $w_t \ge -m_w(w)w$. Then using the Maximum Principle we get $w(t)\ge 0$ for all $t\in(0,T_{max})$. To prove the positivity of the functions u and v we define

$$F(w) = \exp\left\{\int_0^w \chi(s)\mathrm{d}s\right\}, \tag{12}$$

so that $F'(w) = \chi(w)F(w)$ as $\chi\in C^{1+\theta}\cap L^1[0,\infty)$ (see [6]–[7] for more details). After the change of variable $u = F(w)\tilde{u}$, the first equation in system (1) is reduced to $\tilde{u}_t = \Delta\tilde{u} + \chi(w)\nabla\tilde{u}\cdot\nabla w - \tilde{u}\chi(w)h_3 + G(\tilde{u},v,w)$, where

$$G(\tilde{u},v,w) = F(w)^{-1}\left[\ell\xi(v)f(F(w)\tilde{u},v) + a(v)w - m_u(F(w)\tilde{u})\right]. \tag{13}$$

Treating the second equation of the system as a scalar linear equation in v we see, in view of (H_0), that $v=0$ is a lower solution. Therefore, applying the Maximum Principle for parabolic equations, it holds that $v(x,t)\ge 0$ for all $x\in\Omega$ and $t\in[0,T_{\max}]$.

Further, in view of (H_0)–(H_4), the function G verifies

$$-G(\tilde{u},v,w) = -\ell\xi(v)\beta\tilde{u} - \frac{a(v)w}{F(w)} + \gamma\tilde{u} = (-\ell\xi(v)\beta+\gamma)\tilde{u} - \frac{a(v)w}{F(w)}. \tag{14}$$

So $-G(\tilde{u},v,w)\ge 0$. Therefore, since $u_0\ge 0$, from the Maximum Principle for parabolic equations, we get that $\tilde{u}(x,t)\ge 0$ and consequently, $u(x,t)\ge 0$ for all $x\in\Omega$ and $t\in[0,T_{\max}]$.

Now note that in account of assumptions (H_1) and (H_4), the function v satisties $v_t = \Delta v + \phi(v) - f(u,v) \le \Delta v + \delta v - \zeta v^2 \le \Delta v + \delta v$. Thus, from the comparison principle we obtain

$$\sup_{t\in[0,T_{\max})}\|v\|_{L^\infty} \le \max\left\{1, \|v_0\|_{L^\infty}, \sup_{t\in[0,T_{\max})}\|v\|_{L^1}\right\} = A. \tag{15}$$

□

Lemma 5. *If for each $T>0$ there exists a constant $M_0(T)$ depending only on T and $\|(u_0,v_0,w_0)\|_{1,p}$ such that*

$$\|(u(t),v(t),w(t))\|_{L^\infty} \le M_0(T), \quad 0<T<\min\{T,T_{\max}\}, \tag{16}$$

then $T_{\max}=+\infty$.

This fact is obtained by applying Theorem 15.5 in [3] to (1).

3. Proof of Theorem 1

In this section we complete the proof of Theorem 1. First, we prove the L^∞-boundedness of the functions u and w defined by (1) following a Moser–Alikakos iteration method [1]. Note that for v this property has been already proved in the previous section.

Let $p>1$ and (u,v,w) be the solution of (1). We assume that (H_0)–(H_6) are satisfied and we proceed en 3 steps:

Step 1. For the positive function $F(w)$ given by (12), there exists a constant $K>0$, such that

$$\int_\Omega u^p F(w)^{1-p} \le K. \tag{17}$$

Proof. For every $p\ge 1$, from the first equation in (1) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p F(w)^{1-p}&\\ &= p\int_\Omega u^{p-1}u_t F(w)^{1-p} + (1-p)\int_\Omega u^p F(w)^{-p}F'(w)w_t\\ &= p\int_\Omega u^{p-1}F(w)^{1-p}[d\Delta u+\ell\xi(v)f(u,v)+a(v)w-m_u(u)-\nabla\cdot(u\chi(w)\nabla w)]\\ &\quad +(1-p)\int_\Omega u^pF(w)^{1-p}\chi(w)h_3(u,v,w),\end{aligned} \tag{18}$$

where

$$h_3(u,v,w) := \ell\eta(v)f(u,v,x)-a(v)w-m_w(w)w.$$

Since $u\ge 0$, $F>0$ and $p>1$, from (18) it holds

$$p\int_\Omega u^{p-1}F(w)^{1-p}[d\Delta u-\nabla\cdot(u\chi(w)\nabla w)] = -p(p-1)\int_\Omega u^{p-2}F(w)^{1-p}|\nabla u-u\chi(w)\nabla w|^2 \le 0. \tag{19}$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p F(w)^{1-p}&\\ &\le p\int_\Omega u^{p-1}F(w)^{1-p}(\ell\xi(v)f(u,v)+a(v)w-m_u(u))+(1-p)\int_\Omega u^pF(w)^{1-p}\chi(w)h_3.\end{aligned} \tag{20}$$

Now, from (H_0)–(H_6) we can see

$$-(p-1)\int_\Omega u^pF(w)^{1-p}\chi(w)h_3(u,v,w) \le M(p-1)\int_\Omega u^pF(w)^{1-p}. \tag{21}$$

and

$$\ell\xi(v)f(u,v)+a(v)w-m_u(u) = \ell f(u,v)-m_u(u)-m_w(w)w-h_3(u,v,w) \le \ell\beta u-\gamma u. \tag{22}$$

Consequently, (20) is reduced to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^pF(w)^{1-p} &\le p(\ell\beta-\gamma)\int_\Omega u^pF(w)^{1-p}+M(p-1)\int_\Omega u^pF(w)^{1-p}\\ &= (p(\ell\beta-\gamma+M)-M)\int_\Omega u^pF(w)^{1-p},\end{aligned} \tag{23}$$

and hence,

$$\left(\int_\Omega u^pF^{1-p}(w)\right)^{1/p} \le \left(\int_\Omega u_0^pF^{1-p}(w_0)\right)^{1/p} = K^{1/p}. \tag{24}$$

□

Step 2. For each $u_0 \in L^\infty(\Omega)$ the solution u of (1) satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le e^{\|\chi\|_1} \left\| \frac{u_0}{F(w_0)} \right\|_{L^\infty(\Omega)} := C_\infty \tag{25}$$

for any $t < 0$, where $\|\chi\|_1$ denotes the bounded $L^1[0,\infty)$ norm.

Proof. From (17), we have

$$\int_\Omega u^p F(w)^{1-p} \le \int_\Omega u_0^p F(w_0)^{1-p} = \int_\Omega \left(\frac{u_0}{F(w_0)}\right)^p F(w_0). \tag{26}$$

Besides, from (12) we see that $1 \le F(w) \le e^{\|\chi\|_{L^1(0,\infty)}}$ and then, as $p \to \infty$, we get (25). □

Step 3. For $w_0 \in L^\infty(\Omega)$, there exists a constant $W > 0$ such that $\|w\|_{L^\infty(\Omega)} < W$ for all $x \in \Omega$ and $t \in [0,\infty)$. Moreover, W is given by the explicit formula

$$W = \max\left\{\|w_0\|_\infty, \frac{\ell\beta C_\infty}{\gamma}\right\}, \tag{27}$$

with C_∞ defined in (25).

Proof. Since w verifies the ordinary differential equation in (1), under the hypotheses of the theorem, we have

$$w_t \le \ell\beta u - \gamma w \le \ell\beta C_\infty - \gamma w. \tag{28}$$

Thus, using the Comparison Lemma we get the bound (27). □

Proof of Theorem 1. The global existence of (u,v,w) over $\Omega \times (0,\infty)$ is a direct consequence of the local existence and the uniform boundedness of (u,v,w) in L^∞ established in the previous steps. □

Acknowledgments

This work was supported by Project MTM2017-83391-P DGICT Spain.

References

[1] N. D. Alikakos, “L^p bounds of solutions of reaction-diffusion equations”, *Commun. Partial Differ. Equations* **4** (1979), p. 827-868.

[2] H. Amann, “Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems”, *Differ. Integral Equ.* **3** (1990), no. 1, p. 13-75.

[3] ———, “Nonhomogeneous linear and quasilinear elliptic and parabollic boundary value problems”, in *Function spaces, differential operators and nonlinear analysis*, Teubner-Texte zur Mathematik, vol. 133, Teubner, 1993, p. 9-126.

[4] M. Kuwamura, “Turing instabilities in prey-predator systems with dormancy of predators”, *J. Math. Biol.* **71** (2015), no. 1, p. 125-149.

[5] M. Kuwamura, T. Nakazawa, “Dormancy of predators dependent on the rate of variation in prey density”, *SIAM J. Appl. Math.* **71** (2011), no. 1, p. 169-179.

[6] M. Negreanu, J. I. Tello, “On a two species chemotaxis model with slow chemical diffusion”, *SIAM J. Math. Anal.* **46** (2014), no. 6, p. 3761-3781.

[7] ———, “Global existence and asymptotic behavior of solutions to a Predator-Prey chemotaxis system with two chemicals”, *J. Math. Anal. Appl.* **474** (2019), no. 2, p. 1116-1131.

[8] S. Wu, J. Shi, B. Wu, “Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis”, *J. Differ. Equations* **260** (2016), no. 7, p. 5847-5874.

[9] S. Wu, J. Wang, J. Shi, “Dynamics and pattern formation of a diffusive predator-prey model with predator-taxis”, *Math. Models Methods Appl. Sci.* **28** (2018), p. 2275-2312.