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Analysis / *Analyse*

New Inequalities of Simpson's type for differentiable functions via generalized convex function

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Abstract. This article presents some new inequalities of Simpson's type for differentiable functions by using (α, m)-convexity. Some results for concavity are also obtained. These new estimates improve on the previously known ones. Some applications for special means of real numbers are also provided.

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1. Introduction

A well known definition in the mathematical literature is known as convex function:

Definition 1. *A function $f : \Phi \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I, the following inequality holds:*

$$f\left(tx+(1-t)\,y\right) \leq t f(x) + (1-t)\, f\left(y\right), \text{ for all } x, y \in I, t \in [0,1].$$

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Many authors have been introduced new inequalities for convex functions but due to its large application and significance the well known Simpson's inequality is stated as [1]:

Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and

$$\|f^{(4)}\|_{\infty} = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty,$$

then we have the following inequality:

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\,\|f^{(4)}\|_{\infty}(b-a)^4. \tag{1}$$

In [2], Dragomir et al. proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L_1[a,b]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{3}\,\|f'\|_1, \tag{2}$$

where $\|f'\|_1 = \int_a^b |f'(x)|dx$.

The bound of inequality (2) for L-Lipschitzian mapping was given in [10] by $\frac{5}{36}(b-a)$.

Theorem 3. *Suppose $f : [a,b] \to \mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\,\|f'\|_p$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$.

In [5], Kirmaci established the following Hermite–Hadamard type inequality for differentiable convex functions as:

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 (interior of I), where $a,b \in I$ with $a<b$. If the mapping $\left|f'\right|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

In [3], Dragomir and Fitzpatrick presented the following inequalities:

Theorem 5. *Suppose that $f : [0,\infty) \to [0,\infty)$ is a convex function in the second sense, where $s \in (0,1)$ and let $a,b \in [0,\infty)$, $a<b$, $m \in (0,1)$. If $f \in L_1[ma,b]$, then*

$$2^{s-1}f\left(\frac{ma+b}{2}\right) \le \frac{1}{mb-a}\int_{ma}^{b} f(x)dx \le \frac{f(ma)+f(b)}{s+1}.$$

In [8], Qaisar et al. presented inequalities for differentiable convex functions which are linked with Simpson's inequality and the main inequality in [8], pointed out, is as follows.

Theorem 6. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a<b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \le 1$ on (a,b) with $a<b$. If $|f'|$ is (α,m)-convex on $[a,b]$, for $(\alpha,m) \in [0,1]^2$ then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right|$$
$$\le (b-ma)\left[v_1\left|f'(b)\right| + mv_2\left|f'(a)\right|\right], \tag{3}$$

$$v_1 = \frac{6^{-\alpha} - 9\,(2)^{-\alpha} + (5)^{\alpha+2}\,(6)^{-\alpha} + 3\alpha - 12}{18\,(\alpha+1)\,(\alpha+2)} \quad and \quad v_2 = \left(\frac{5}{36} - v_1\right).$$

Proposition 7. *Under the assumptions of Theorem 6 with $\alpha = 1$,* we have the following inequality,

$$\left|\frac{1}{6}\left[f\,(ma) + 4f\left(\frac{ma+b}{2}\right) + f\,(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \le \frac{5\,(b-ma)}{72}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \quad (4)$$

Theorem 8. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is (α, m)-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \ge 1$, then the following inequality holds:*

$$\frac{1}{6}\left[f\,(ma) + 4f\left(\frac{ma+b}{2}\right) + f\,(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx$$
$$\le \frac{(b-ma)}{2}\left(\frac{5}{72}\right)^{1-1/q} \times \left\{\left(u_1\left|f'(b)\right|^q + mu_2\left|f'(a)\right|^q\right)^{1/q} + \left(u_3\left|f'(b)\right|^q + u_4\left|f'(a)\right|^q\right)^{1/q}\right\}.$$

where

$$u_1 = \frac{(3^{-\alpha})\left(2^{1-\alpha}\right) + 3\,(\alpha)\left(2^{1-\alpha}\right) + 3\,(2^{-\alpha})}{6^3\,(\alpha+1)\,(\alpha+2)}, \qquad u_2 = \left(\frac{5}{72} - u_1\right),$$
$$u_3 = \frac{\left(5^{\alpha+2}\right)(3^{-\alpha})\left(2^{1-\alpha}\right) - 3\,(\alpha)\left(2^{1-\alpha}\right) - 21\,(2^{-\alpha}) + 6\,(\alpha) - 24}{6^3\,(\alpha+1)\,(\alpha+2)}, \qquad u_4 = \left(\frac{5}{72} - u_3\right).$$

Proposition 9. *Under the assumptions of Theorem 8 with $\alpha = m = q = 1$, if $|f'(x)|^q \le Q \ \ \forall\ x \in I$, we have the following inequality,*

$$\left|\frac{1}{6}\left[f\,(a) + 4f\left(\frac{a+b}{2}\right) + f\,(b)\right] - \frac{1}{b-a}\int_{a}^{b} f(x)\,dx\right| \le \frac{5\,(b-a)}{36}Q.$$

For some results that generalize, improve and extend the inequality and new Simpson's type inequalities, see ([9,11] and [4,6]).

The main aim of this paper is to establish new Simpson's type inequalities for (α, m)- convexity for the class of functions whose derivatives in absolute value at certain powers are $(\alpha, m)-$convex functions. Some results for concave functions are also obtained.

2. Main Results

In [7], Mihesan presented the class of (α, m)-convex functions as:

Definition 10. *A function $f : [0,d] \to \mathbb{R}$, $d > 0$ is said to be (α, m)-convex functions where $(\alpha, m) \in [0,1]^2$, if for every $x, y \in [0,\infty]$ and $\lambda \in [0,1]$, the following inequality holds:*

$$f\left(\lambda y + m\,(1-\lambda)\,x\right) \le \lambda^{\alpha} f\left(y\right) + m\left(1-\lambda^{\alpha}\right) f\,(x),$$

Denote by $K_m^{\alpha}(d)$ the class of all (α, m)-convex functions on $[0,d]$ for which $f(0) \le 0$.

Definition 11. *If we choose $(\alpha, m) = (1, m)$, we can obtain $m-$convex functions and for $(\alpha, m) = (1,1)$, we have ordinary convex functions on* $[0,d]$.

In order to prove our main results, we need the following integral equality.

The following essential definitions and Lemmas play a key role to establish our main results:

In this paper, for the simplicity of the notation, let

$$s_{ma}^{b}(f)(h,n) = \frac{1}{n}\left[f(ma) + (n-2)f(hb + m(1-h)a) + f(b)\right] - \frac{1}{mb-a}\int_{ma}^{b} f(x)dx,$$

for $h \in (0,1)$ with $1/n \le h \le (n-1)/n$ for any integer $n \ge 2$.Then we have following equality:

Lemma 12. *Let $f:[0,b^*]\to\mathbb{R}$, $b^*>0$, be differentiable mapping on I^0 such that $f'\in L_1([ma,b])$ where $a,b\in I$ with $a<b$ and $[a,b]\subset[0,b^*]$ and $m\in(0,1)$. If $h\in(0,1)$ with $1/n\le h\le(n-1)/n$ for any integer $n\ge 2$, then we have:*

$$s_{ma}^{b}(f)(h,n)=(b-ma)\int^{1}k(\lambda,h)f'(\lambda b+m(1-\lambda)a)\,d\lambda,$$

where

$$k(\lambda)=\begin{cases}\left(\lambda-\frac{1}{n}\right), & \lambda\in[0,h],\\ \lambda-\frac{n-1}{n}, & \lambda\in[h,1].\end{cases}$$

Proof. Using integrating by parts, we have

$$\begin{aligned}s_{ma}^{b}(f)(h,n)&=\int_0^h\left(\lambda-\frac{1}{n}\right)f'(\lambda b+m(1-\lambda)a)\,d\lambda+\int_h^1\left(\lambda-\frac{n-1}{n}\right)f'(\lambda b+m(1-\lambda)a)d\lambda\\&=\frac{1}{mb-a}\left[\frac{n-2}{2}f(hb+m(1-h)a)+\frac{1}{n}\left(f(b)+f(ma)\right)\right]-\frac{1}{(mb-a)^2}\int_{ma}^{b}f(x)dx.\end{aligned}$$

□

This proves as required.

Theorem 13. *Let $f:[0,b^*]\to\mathbb{R}$ $b^*>0$,be differentiable mapping on I^0 such that $f'\in L_1([ma,b])$ where $a,b\in I$ with $a<b$ and $[a,b]\subset[0,b]$. If the mapping $|f'|$ is (α,m)-convex on $[a,b]$ for some $(\alpha,m)\in[0,1]^2$ then for $h\in(0,1)$ with $1/n\le h\le(n-1)/n$ for any integer $n\ge 2$, then we have the inequality:*

$$\left|s_{ma}^{b}(f)(h,n)\right|\le(b-ma)\left[v_1\left|f'(b)\right|+v_2m\left|f'(a)\right|\right],$$

where

$$v_1=\frac{2+2(n-1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{2+\alpha-n}{n(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(\alpha(2h-1)+2(h-1))}{(\alpha+1)(\alpha+2)},$$

and

$$v_2=\left(\frac{n^2+2n^2h^2+2(n-1)^2-2n(n-1)(h+1)+2-2nh}{2n^2}-v_1\right).$$

Proof. Taking modulus on both sides of Lemma 12 and using (α,m)-convexity we get

$$\begin{aligned}\left|s_{ma}^{b}(f)(h,n)\right|&=(b-ma)\int_0^1k(\lambda,h)f'(\lambda b+m(1-\lambda)a)\,d\lambda,\\&\le(b-ma)\int_0^h\left|\lambda-\frac{1}{n}\right|f'(\lambda b+m(1-\lambda)a)\,d\lambda\\&+(b-ma)\int_h^1\left|\lambda-\frac{n-1}{n}\right|f'(\lambda b+m(1-\lambda)a)\,d\lambda\\&\le(b-ma)\int_0^h\left|\lambda-\frac{1}{n}\right|\lambda^\alpha\left|f'(b)\right|+m\left(1-\lambda^\alpha\right)\left|f'(a)\right|d\lambda\\&+(b-ma)\int_h^1\left|\lambda-\frac{n-1}{n}\right|\lambda^\alpha\left|f'(b)\right|+m\left(1-\lambda^\alpha\right)\left|f'(a)\right|d\lambda.\end{aligned}$$

By simple calculations, we have

$$\begin{aligned}&\int_0^h\lambda^\alpha\left|\lambda-\frac{1}{n}\right|d\lambda+\int_h^1\lambda^\alpha\left|\lambda-\frac{n-1}{n}\right|d\lambda\\&\qquad=\frac{2+2(n-1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{2+\alpha-n}{n(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(\alpha(2h-1)+2(h-1))}{(\alpha+1)(\alpha+2)},\end{aligned}$$

and

$$\int_0^h (1-\lambda^\alpha)\left|\lambda-\frac{1}{n}\right|d\lambda+\int_h^1 \left(1-\lambda^\alpha\right)\left|\lambda-\frac{n-1}{n}\right|d\lambda$$
$$=\frac{n^2+2n^2h^2+2(n-1)^2-2n(n-1)(h+1)+2-2nh}{2n^2}$$
$$-\frac{2+2(n-1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{2+\alpha-n}{n(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(\alpha(2h-1)+2(h-1))}{(\alpha+1)(\alpha+2)}.$$

The proof of Theorem 13 is completed. □

Now, we conclude the following Corollaries 14 and 15.

Corollary 14. *Under the assumption of Theorem 13, we get*

$$\left|S^b_{ma}(f)\left(\frac{1}{2},6\right)\right|\le (b-ma)\left[v_3\left|f'\right|(b)+mv_4\left|f'\right|(a)\right],$$

where

$$v_3=\frac{6^{-\alpha}(1+(5)^{\alpha+2}-9(2)^{-\alpha}+3\alpha-12}{18(\alpha^2+3\alpha+2)}\quad \textit{and}\quad v_4=\left(\frac{5}{36}-v_3\right).$$

Corollary 15. *Putting $\alpha=1$ and $m=1$, in the above Corollary 14, we get,*

$$\left|S^b_{a}(f)\left(\frac{1}{2},6\right)\right|\le \frac{5(b-a)}{72}\left[\left|f'\right|(b)+\left|f'\right|(a)\right].$$

Remark 16. It is observed that Corollary 14 and Corollary 15 are better than the Theorems presented by Dragomir and Kiramic. Moreover Corollary 14 was proved by Qaisar et al. in [5]. Hence, our results in Theorem 13 are generalizations of the corresponding results of Qaisar et al. in [5].

By applying Hölder's inequality, we obtain the following Theorem 17.

Theorem 17. *Let $f: I\subset 0,\infty)\to\mathbb{R}$ be a differentiable function on I^0 (interior of I) $a,b\in I$ with $a<b$. If the mapping $|f'|^{\frac{p}{(p-1)}}$ is (α,m)-convex on $[a,b]$ and for some fixed $(\alpha,m)\in[0,1]^2$ and $p>1$, the following inequality holds:*

$$\left|S^b_{ma}(f)\,(h,n)\right|\le(b-ma)\left(\frac{h}{\alpha+1}\right)^{\frac{1}{q}}\left(\frac{1+(nh-1)^{p+1}}{n^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left|f'(hb+m(1-h)\alpha)\right|^q+\left|f'(a)\right|^q\right\}^{\frac{1}{q}}$$
$$+(b-ma)\left(\frac{1-h}{\alpha+1}\right)^{\frac{1}{q}}\left(\frac{1+(n-nh-1)}{n^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left|f'(hb+m(1-h)\alpha)\right|^q+\left|f'(a)\right|^q\right\}^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Hölder's inequality and by Lemma 12, we get

$$\le(b-ma)\left(\int_0^h\left|\left(\lambda-\frac{1}{n}\right)\right|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^h f'\,|(\lambda b+m(1-\lambda)\,a|^q\,d\lambda\right)^{\frac{1}{q}}$$
$$+(b-ma)\left(\int_h^1\left|\left(\lambda-\frac{n-1}{n}\right)\right|^p d\lambda\right)^{\frac{1}{p}}\left(\int_h^1 f'\,|(\lambda b+m(1-\lambda)\,a|^q\,d\lambda\right)^{\frac{1}{q}}.$$

Also the (α,m)-convexity of $|f'|^{\frac{p}{(p-1)}}$ implies that

$$\int_0^h\left|f'(\lambda b+m(1-\lambda))a\right|^q d\lambda\le\left(\frac{h}{\alpha+1}\right)^{\frac{1}{q}}\left\{\left|f'(hb+m(1-h)\alpha)\right|^q+\left|f'(a)\right|^q\right\},$$
$$\int_h^1\left|f'(\lambda b+m(1-\lambda))a\right|^q d\lambda\le\left(\frac{1-h}{\alpha+1}\right)^{\frac{1}{q}}\left\{\left|f'(hb+m(1-h)\alpha)\right|^q+\left|f'(a)\right|^q\right\}.$$

Therefore, by combining above inequalities, we get the required result.
The proof of Theorem 17 is completed. □

Corollary 18. *In Theorem 17 putting $n=6$ and $h=\frac{1}{2}$, $\alpha=1$, and $m=1$, then we have*

$$\left|S_a^b(f)\left(\frac{1}{2},6\right)\right| \le 4^{\frac{-1}{q}}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \times\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].$$

In the following Theorem 19, we obtain another form of Simpson's inequality for powers in term of the first derivative.

Theorem 19. *Let f be defined as in Theorem 17. If the mapping $|f'|^{\frac{p}{(p-1)}}$ is (α,m)-convex on $[a,b]$, for some fixed $(\alpha,m)\in[0,1]^2$ and $q>1$, we have the following inequality:*

$$\left|S_{ma}^b(f)(h,n)\right| \le (b-ma)\left(\frac{1+(nh-1)^2}{2n^2}\right)^{\frac{1}{p}}\left(u_1\left|f'(b)\right|^q+u_2\left|f'(a)\right|^q\right)^{\frac{1}{q}} + (b-ma)\left(\frac{1+(nh-n+1)^2}{2n^2}\right)^{\frac{1}{p}}\left(u_3\left|f'(b)\right|^q+u_4 m\left|f'(a)\right|^q\right)^{\frac{1}{q}},$$

where

$$u_1=\frac{2}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(nh(\alpha+1)-\alpha-2)}{n(\alpha+1)(\alpha+2)},\ u_2=\frac{1+(nh-1)^2}{2n^2}-u_1,$$
$$u_3=\frac{2(n-1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{(1+h^{\alpha+1})(\alpha+2-n)}{n(\alpha+1)(\alpha+2)}-\frac{h^{\alpha+1}(1-h)}{(\alpha+2)},\ u_4=\frac{1+(nh-n+1)^2}{2n^2}-u_3.$$

Proof. By Lemma 12 and using power mean inequality, we get

$$\left|S_{ma}^b(f)(h,n)\right| \le (b-ma)\left(\int_0^h\left|\left(\lambda-\frac{1}{n}\right)\right|d\lambda\right)^{\frac{1}{p}}\left(\int_0^h\left(\lambda-\frac{1}{n}\right)f'|(\lambda b+m(1-\lambda)a|^q\,d\lambda\right)^{\frac{1}{q}} + (b-ma)\left(\int_h^1\left|\left(\lambda-\frac{n-1}{n}\right)\right|d\lambda\right)^{\frac{1}{p}}\left(\int_h^1\left(\lambda-\frac{n-1}{n}\right)f'|(\lambda b+m(1-\lambda)a|^q\,d\lambda\right)^{\frac{1}{q}}.$$

The (α,m)-convexity of $|f'|^{\frac{p}{(p-1)}}$ gives that

$$\int_0^h\left|\left(\lambda-\frac{1}{n}\right)\right|\left|f'(\lambda b+m(1-\lambda)a\right|^q d\lambda \le u_1\left|f'(b)\right|^q+mu_2\left|f'(a)\right|^q,$$
$$\int_h^1\left|\left(\lambda-\frac{n-1}{n}\right)\right|\left|f'(\lambda b+m(1-\lambda)a\right|^q d\lambda \le u_3\left|f'(b)\right|^q+mu_4\left|f'(a)\right|^q.$$

Our required result is obtained by combining above inequalities.
The proof of Theorem 19 is completed. □

Corollary 20. *Let f be defined as in Theorem 19 and $\alpha=1$, $m=1$, the inequality holds for convex functions:*

$$\left|S_a^b(f)\left(\frac{1}{2},6\right)\right| \le (b-a)\left(\frac{5}{72}\right)^{\frac{1}{p}} \times\left[\left(\frac{29}{1296}\left|f'(b)\right|^q+\frac{61}{1296}\left|f'(a)\right|^q+\frac{61}{1296}\left|f'(b)\right|^q+\frac{29}{1296}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right].$$

The next result, gives an inequality of Simpson's type for concave functions.

Theorem 21. *Let $f : I \subset 0,\infty) \to \mathbb{R}$ be a differentiable function on I^0 (interior of I) $a, b \in I$ with if the mapping $|f'|^{\frac{p}{(p-1)}}$ is concave on $[a,b]$ and $p > 1$ the following inequality holds:*

$$\left|S_a^b(f)(h,n)\right| \leq (b-a)$$

$$\times\left\{\left[\frac{1+(nh-1)^2}{2n^2}\right] f'\left(\frac{2+n^2h^2(2nh-3)b+\left[3n(1+nh-1)^2-\left(2+n^2h^2(2nh-3)\right)\right]a}{3n\left[1+(nh-1)^2\right]}\right)+\left[\frac{1+(nh-n+1)^2}{2n^2}\right]\right.$$
$$\left.\times\left|f'\left(\frac{\left[n^2\left[h^2(2nh-3n+3)+(n-3)\right]+6n-2+3n\left(1+(nh-n+1)^2\right)\right]b-\left[n^2\left(h^2(2nh-3n+3)+(n-3)+6n-2\right)\right]a}{3n\left[1+(nh-n+1)^2\right]}\right)\right|\right\}.$$

Proof. We note that by concavity of $|f'|^{\frac{p}{(p-1)}}$ and the power mean inequality and we have

$$\left|f'(\lambda b+(1-\lambda)b\right|^q \geq \lambda\left|f'(a)\right|^q+(1-\lambda)\left|f'(b)\right|^q.$$

Hence

$$\left|f'(\lambda a+(1-\lambda)b\right| \geq \lambda\left|f'(a)\right|+(1-\lambda)\left|f'(b)\right|,$$

so, $|f'|$ is also concave.

Accordingly, by the Jensen's integral inequality, we have

$$\int_0^h\left|\left(\lambda-\frac{1}{n}\right)\right||f'|(\lambda b+(1-\lambda)a|\,d\lambda \leq \left(\int_0^h\left|\lambda-\frac{1}{n}\right|d\lambda\right)\left|f'\left(\frac{\int_0^h\left|\lambda-\frac{1}{n}\right||\lambda b+(1-\lambda)\,a|\,d\lambda}{\int_0^h\left|\lambda-\frac{1}{n}\right|d\lambda}\right)\right|$$
$$\left[\frac{1+(nh-1)^2}{2n^2}\right] f'\left(\frac{2+n^2h^2(2nh-3)b+\left[3n(1+nh-1)^2-\left(2+n^2h^2(2nh-3)\right)\right]a}{3n\left[1+(nh-1)^2\right]}\right),$$

and

$$\int_h^1\left|\left(\lambda-\frac{n-1}{n}\right)\right| f'(\lambda b+(1-\lambda a)\,d\lambda$$
$$\leq\left(\int_h^1\left|\lambda-\frac{n-1}{n}\right|d\lambda\right)\left|f'\left(\frac{\int_0^h\left|\lambda-\frac{n-1}{n}\right||\lambda b+(1-\lambda)\,a|\,d\lambda}{\int_0^h\left|\lambda-\frac{n-1}{n}\right|d\lambda}\right)\right|$$
$$=\frac{1+(nh-n+1)^2}{2n^2} f'\left(\frac{n^2\left[h^2(2nh-3+3n)+(n-3)\right]+6n-2}{3n\left[1+(nh-n+1)^2\right]}b\right.$$
$$\left.+\frac{3n[(1+nh-1)^2]-n^2\left[h^2(2nh-3+3n)+(n-3)\right]+6n-2}{3n\left[1+(nh-n+1)^2\right]}a\right).$$

A combination of the above inequalities gives the desired result, that is

$$\left|S_a^b(f)(h,n)\right| \leq (b-a)\left[\frac{1+(nh-1)^2}{2n^2}\right]$$
$$\times\left|f'\left(\frac{2+n^2h^2(2nh-3)\,b+\left[3n(1+(nh-1)^2-\left(2+n^2h^2(2nh-3)\right)\right]a}{3n\left[1+(nh-1)^2\right]}\right)\right|+\left[\frac{1+(nh-n+1)^2}{2n^2}\right]$$
$$\times\left|f'\left[n^2\left[h^2(2nh-3n+3)+(n-3)\right]+6n-2+3n\left(1+(nh-n+1)^2\right)\right]b.\right.$$
$$\left.-\left[n^2\left(h^2(2nh-3n+3)+(n-3)\right)+6n-2\right]a\right\}\big/\left\{3n\left[1+(nh-n+1)^2\right]\right\}\Big|.$$

This completes the proof of Theorem 21. □

Another result is considered as follows:

Theorem 22. *Let $f:[0,b]\to\mathbb{R}$, be a differentiable function on I^o (interior of I) $a,b\in I$ with $a<b$. If the mapping $|f'|^q$ is concave on $[a,b]$, and for some fixed $q>1$ the following inequality holds:*

$$\left|S_{ma}^{b}(f)(h,n)\right|\le(b-ma)\left(\frac{q-1}{2q-1}\right)\left(h^{\frac{q-1}{2q-1}}+1\right)\\ \times\left[\left|f'\left(h^2b+m\left(2h-h^2\right)a\right)\right|+\left|f'\left(\left(1-h^2\right)b+mh^2a\right)\right|\right].$$

Proof. From Lemma 12, we have

$$\left|S_{ma}^{b}(f)(h,n)\right|\le(b-ma)\int_0^h\left|\lambda-\frac{1}{n}\right|\left|f'(\lambda b+m(1-\lambda)a)\right|d\lambda\\ +(b-ma)\int_h^1\left|\lambda-\frac{n-1}{n}\right|\left|f'(\lambda b+m(1-\lambda)a)\right|d\lambda.$$

By applying Hölder's inequality, for $q>1$ and $p=\frac{q}{q-1}$, we get

$$(b-ma)\int_0^h\left|\lambda-\frac{1}{n}\right|\left|f'(\lambda b+m(1-\lambda)a)\right|d\lambda\\ \le(b-ma)\left(\int_0^h\left|\lambda-\frac{1}{n}\right|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\left(\int_0^h|(\lambda b+m(1-\lambda)a)|^q\,d\lambda\right)^{1/q}\\ +(b-ma)\left(\int_h^1\left|\lambda-\frac{n-1}{n}\right|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\left(\int_h^1|(\lambda b+m(1-\lambda)a)|^q\,d\lambda\right)^{1/q}$$

where we use the fact

$$\int_0^h\left|\lambda-\frac{1}{n}\right|^{\frac{q}{q-1}}d\lambda=\int_h^1\left|\lambda-\frac{n-1}{n}\right|^{\frac{q}{q-1}}d\lambda=\frac{1}{\frac{2q-1}{n^{q-1}}}\left(\frac{q-1}{2q-1}\right)\left(h^{\frac{q-1}{2q-1}}+1\right).$$

Since $|f'|^q$ is concave on $[a,b]$ using integral $Jensen's$ inequality, we have

$$\int_0^h\left|f'(\lambda b+m(1-\lambda)a)\right|^q d\lambda\\ =\int_0^h\lambda^0\left|f'(\lambda b+m(1-\lambda)a)\right|^q d\lambda\le\left(\int_0^h\lambda^0 d\lambda\right)\left|f'\left(\frac{\left(\int_0^h\lambda^0 d\lambda\right)|(\lambda b+m(1-\lambda)a)|\,d\lambda}{\int_0^h\lambda^0 d\lambda}\right)\right|^q\\ =\frac{1}{2}\left|f'\left(2\int_0^h(\lambda b+m(1-\lambda)ad\lambda)\right)\right|^q\\ =\frac{1}{2}\left|f'\left(h^2b+m\left(2h-h^2\right)a\right)\right|.$$

Analogously

$$\int_h^1\left|f'(\lambda b+m(1-\lambda)a)\right|^q d\lambda\le\frac{1}{2}\left|f'\left(\left(1-h^2\right)b+mh^2a\right)\right|^q.$$

By simple computation

$$\left|S_{ma}^{b}(f)(h,n)\right|\le(b-ma)\left(\frac{q-1}{2q-1}\right)\left(h^{\frac{q-1}{2q-1}}+1\right)\\ \times\left[\left|f'\left(h^2b+m\left(2h-h^2\right)a\right)\right|+\left|f'\left(\left(1-h^2\right)b+mh^2a\right)\right|\right].$$

This completes the proof of Theorem 22. $\square$

Theorem 23. *Let $f : [0,b] \to \mathbb{R}$, be a differentiable function on I^o (interior of I) $a,b \in I$ with $a < b$. If the mapping $|f'|^q$ is concave on $[a,b]$ and for some fixed $q > 1$, $h \in (0,1)$, the following inequality holds:*

$$\left|S_{ma}^{b}\left(f\right)(h,n)\right| \le (b-ma)\,2^{\frac{(\alpha-1)}{q}} \frac{1}{\frac{2q-1}{q-1}} \left(\frac{q-1}{2q-1}\right)\left(h^{\frac{q-1}{2q-1}}+1\right) \times \left[\left|f'\left(hb+m(1+h)\,a\right)\right| + \left|f'\left((1+h)\,b+mha\right)\right|\right].$$

Proof. It is similar as in the proof of Theorem 22, by utilizing Theorem 6 instead of the Jensen's integral inequality for concave functions. For the concavity of $|f'|^q$, we have

$$\int_0^h \left|f'(\lambda b+m(1-\lambda)\,a)\right|^q d\lambda \le 2^{\alpha-1}\left|f'(hb+m(1+h)\,a)\right|^q,$$
$$\int_h^1 \left|f'(\lambda b+m(1-\lambda)\,a)\right|^q d\lambda \le 2^{\alpha-1}\left|f'(1+h)\,b+mha\right|^q.$$

Hence

$$\left|S_{ma}^{b}\left(f\right)(h,n)\right| \le (b-ma)\,2^{\frac{(\alpha-1)}{q}} \frac{1}{\frac{2q-1}{q-1}} \left(\frac{q-1}{2q-1}\right)\left(h^{\frac{q-1}{2q-1}}+1\right) \times \left[\left|f'\left(hb+m(1+h)\,a\right)\right| + \left|f'(1+h)\,b+mha\right|\right].$$

This completes the proof of Theorem 23. □

3. Application to Some Special Means

Let us recall the following means for arbitrary real numbers a and b.

(1) The Arithmetic mean

$$A = A(a,b) = \frac{a+b}{2},\ a,b \ge 0.$$

(2) Generalized-logarithmic mean

$$L_n(a,b) = \begin{cases} a, & \text{if } a=b, \\ \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{1/n} & \text{if } a \neq b. \end{cases} \quad n \in \mathbb{R}\setminus\{-1,0\};\ a,b>0$$

(3) The Logarithmic mean

$$L = L(a,b) = \begin{cases} a, & \text{if } a=b, \\ \left[\frac{b-a}{\ln b-\ln a}\right]^{1/n} & \text{if } a \neq b. \end{cases} \quad a,b>0$$

Now utilizing outcomes of Section 2, some new inequalities are derived for the above means.

It is well known that L_p is monotonic non- decreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$L \le A.$$

In the following some new inequalities are derived for the above means.

Consider

$$f:[a,b]\to\mathbb{R},\quad (0<a<b),\ f(x)=x^{\alpha},\ \alpha\in(0,1],$$

then

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_{\alpha}^{\alpha}(a,b).$$
$$\frac{f(a)+f(b)}{2}=A\left(a^{\alpha},b^{\alpha}\right).$$
$$f\left(\frac{a+b}{2}\right)=A^{\alpha}(a,b).$$

Under the assumption of Corollary 15, we have

$$\left|\frac{2}{n}A\left(a^{\alpha},b^{\alpha}\right)+\frac{(n-2)}{n}A^{\alpha}(a,b)-L_{\alpha}^{\alpha}(a,b)\right|\le\alpha(b-a)\left[v_1b^{(\alpha-1)}+v_2a^{(\alpha-1)}\right],$$

where

$$v_1=\frac{2+2(n-1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{2+\alpha-n}{n(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(\alpha(2h-1)+2(h-1))}{(\alpha+1)(\alpha+2)}$$
$$v_2=\left(\frac{n^2+2n^2h^2+2(n-1)^2-2n(n-1)(h+1)+2-2nh}{2n^2}-v_1\right).$$

If $n=6$, $h=1/2$, $\alpha=1$, then we have

$$|A(a,b)-L(a,b)|\le\frac{5}{36}(b-a).$$

Under the assumption of Corollary 18, we have

$$\left|\frac{2}{n}A\left(a^{\alpha},b^{\alpha}\right)+\frac{(n-2)}{n}A^{\alpha}(a,b)-L_{\alpha}^{\alpha}(a,b)\right|$$
$$\le 4^{\frac{-1}{q}}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left\{A^{(\alpha-1)q}(A(a,b))+a^{(\alpha-1)q}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{A^{(\alpha-1)q}(A(a,b))+b^{(\alpha-1)q}\right\}^{\frac{1}{q}}\right].$$

Under the assumption of Corollary 20, we have

$$\left|\frac{2}{n}A\left(a^{\alpha},b^{\alpha}\right)+\frac{(n-2)}{n}A^{\alpha}(a,b)-L_{\alpha}^{\alpha}(a,b)\right|$$
$$\le(b-a)\left(\frac{5}{72}\right)^{\frac{1}{p}}\left[\left\{u_1b^{(\alpha-1)q}+u_2a^{(\alpha-1)q}\right\}^{\frac{1}{q}}+\left\{u_3b^{(\alpha-1)q}+u_4a^{(\alpha-1)q}\right\}^{\frac{1}{q}}\right].$$

where

$$u_1=\frac{2}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{h^{\alpha+1}(\alpha(2h-1)+2(h-1))}{(\alpha+1)(\alpha+2)},\qquad u_2=\frac{1+(nh-1)^2}{2n^2}-u_1,$$
$$u_3=\frac{2(n+1)^{\alpha+2}}{n^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{\left(1+h^{\alpha+1}\right)(\alpha+2-n)}{n(\alpha+1)(\alpha+2)}-\frac{h^{\alpha+1}(1-h)}{(\alpha+2)},\quad u_4=\frac{1+(nh-n-1)^2}{2n^2}-u_3.$$

$$\left|\frac{2}{n}A\left(a^{\alpha},b^{\alpha}\right)+\frac{(n-2)}{n}A^{\alpha}(a,b)-L_{\alpha}^{\alpha}(a,b)\right| \leq \left[\left(u_1\left|f'(b)\right|^q+u_2\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(u_3\left|f'(b)\right|^q+u_4\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right],$$

where

$$u_1=\frac{2}{6^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{(3(1+\alpha)-\alpha-2)}{6(\alpha+1)(\alpha+2)2^{\alpha+1}}, \qquad u_2=\frac{5}{72}-u_1,$$

$$u_3=\frac{2.5^{\alpha+2}}{6^{\alpha+2}(\alpha+1)(\alpha+2)}+\frac{\left(1+2^{\alpha+1}\right)(\alpha-4)}{6(\alpha+1)(\alpha+2)2^{\alpha+1}}-\frac{1}{2^{\alpha+2}(\alpha+2)}, \qquad u_4=\frac{5}{72}-u_3.$$

4. Conclusion

In this paper, we have presented some new results related to Simpson's type inequalities for powers in term of the first derivative using the (α, m)-convex functions. Few inequalities for concave functions are also obtained. Our existing results generalize some previously obtained results. Many other interesting inequalities could be derived from our newly obtained results by considering for different values of parameters. Some applications for special means of real numbers are also provided.

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