INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Xu Pan and Liangchen Wang

Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production

Volume 359, issue 2 (2021), p. 161-168

Published online: 17 March 2021

https://doi.org/10.5802/crmath.148

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2021, Vol. 359, 2, p. 161-168
https://doi.org/10.5802/crmath.148

Partial differential equations / *Équations aux dérivées partielles*

Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production

Xu Pan [a] **and Liangchen Wang** [*, a]

[a] School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China
E-mails: panx_math@163.com, wanglc@cqupt.edu.cn

Abstract. This paper deals with the chemotaxis system with nonlinear signal secretion

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - S(u)\nabla v), & x\in\Omega, \quad t>0,\\ v_t = \Delta v - v + g(u), & x\in\Omega, \quad t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$. The diffusion function $D(s)\in C^2([0,\infty))$ and the chemotactic sensitivity function $S(s)\in C^2([0,\infty))$ are given by $D(s)\geq C_d(1+s)^{-\alpha}$ and $0<S(s)\leq C_s s(1+s)^{\beta-1}$ for all $s\geq 0$ with $C_d, C_s>0$ and $\alpha,\beta\in\mathbb{R}$. The nonlinear signal secretion function $g(s)\in C^1([0,\infty))$ is supposed to satisfy $g(s)\leq C_g s^\gamma$ for all $s\geq 0$ with $C_g,\gamma>0$. Global boundedness of solution is established under the specific conditions:

$$0<\gamma\leq 1 \quad \text{and} \quad \alpha+\beta<\min\left\{1+\frac{1}{n}, 1+\frac{2}{n}-\gamma\right\}.$$

The purpose of this work is to remove the upper bound of the diffusion condition assumed in [9], and we also give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$, which is ignored in [9, Theorem 1.1].

Mathematical subject classification (2010). 35K35, 35A01, 35B44, 35B35, 92C17.

Funding. This work is supported by the Chongqing Research and Innovation Project of Graduate Students (No. CYS20271) and Chongqing Basic Science and Advanced Technology Research Program (No. cstc2017jcyjXB0037).

Manuscript received 1 September 2020, revised 9 October 2020 and 9 December 2020, accepted 10 December 2020.

* Corresponding author.

1. Introduction

In the present work, we consider the following system, which describes the fully parabolic chemotaxis system with nonlinear diffusion, sensitivity and signal secretion

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - S(u)\nabla v), & x\in\Omega, \quad t>0, \\ v_t = \Delta v - v + g(u), & x\in\Omega, \quad t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \ t>0, \\ (u,v)(x,0) = (u_0(x), v_0(x)), & x\in\Omega, \end{cases} \tag{1}$$

with homogeneous Neumann boundary conditions, where $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a bounded domain, and $\partial/\partial\nu$ is the derivative of the normal with respect to $\partial\Omega$. In system (1), $u = u(x,t)$ and $v = v(x,t)$ represent the density of population and the concentration of chemicals, respectively. In this article, the diffusion function $D \in C^2([0,\infty))$ and the chemotactic sensitivity function $S\in C^2([0,\infty))$ with $S(0)=0$ are given by

$$D(s)\geq C_d(1+s)^{-\alpha} \quad \text{and} \quad 0\leq S(s)\leq C_s s(1+s)^{\beta-1} \quad \text{for all } s\geq 0 \tag{2}$$

with $C_d, C_s > 0$ and $\alpha,\beta\in\mathbb{R}$. The signal secretion function $g\in C^1([0,\infty))$ is nonnegative and satisfies

$$g(s)\leq C_g s^{\gamma} \text{ for all } s\geq 0 \quad \text{with} \quad C_g,\gamma>0. \tag{3}$$

The well-known chemotaxis model for the chemotactic movement of one specie [4] proposed by Keller and Segel, which describes the aggregation phenomenon of the Dictyostelium discoideum, there are many results about this system [1, 3, 9, 12, 13, 15, 16]. For instance, in case $g(u) = u$, the asymptotics of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}}$ is critical to distinguish the blow-up and global boundedness: under the condition $\frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}-\epsilon}$ for all $u>1$ with $\epsilon>0$, Tao and Winkler [12] obtained the global boundedness of solution; while if $\frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}+\epsilon}$ for all $u>1$ [16], the solution of (1) blow-up either in infinite time or finite time. We note that in [9], global boundedness of solution is established under the conditions that $\alpha+\beta+\gamma<1+\frac{2}{n}$ and $d_0(1+u)^{\alpha}\leq D(u)\leq d_1(1+u)^{\alpha_1}$ with $d_0,d_1>0$ and $\alpha,\alpha_1\in\mathbb{R}$. The purpose of this work is to remove the upper bound of the diffusion condition and give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$ that is ignored in [9, Theorem 1.1]. The main result of this article is described below.

Theorem 1. *Let $\Omega\subset R^n$ $(n\geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0,v_0)\in C^0(\overline{\Omega})\times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold. If $0<\gamma\leq 1$ and*

$$\alpha+\beta<\min\left\{1+\frac{1}{n}, 1+\frac{2}{n}-\gamma\right\},$$

then system (1) *possesses a unique global bounded classical solution (u,v) in the sense that there exists some constant $C>0$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} < C \quad \textit{for all } t>0.$$

Remark 2. Compared with the previous study in [9, Theorem 1.1], we give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$ that is ignored in it, and we also remove the restriction on the upper bound of the diffusion function $D(s)$.

2. Boundedness

Let us state the local existence result, which has been established in [1, 3, 8, 10, 17, 18].

Lemma 3. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold, then there exists $t \in (0, T_{\max})$ such that system* (1) *has a unique non-negative solution and satisfies*

$$u, v \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

In order to obtain the global boundedness of solution to system (1), we first establish a series of prior estimates; then we treat the dissipative terms on the right hand side of the inequality by using the Gagliardo–Nirenberg inequality; last, we get our final results by controlling the parameter range in the inequality. The ideas come from [9, 12–14].

Lemma 4. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold, then the first term of the solution to system* (1) *satisfies*

$$\|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{4}$$

Furthermore, assume that $0 < \gamma \leq 1$, if $s \in \left[1, \frac{n}{(n\gamma-1)_+}\right)$, then there exits $C > 0$ such that

$$\|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\max}). \tag{5}$$

Proof. Integrating the first equation of (1) over Ω, (4) can be easily obtained. From the Neumann semigroup estimates method in [5, Lemma 1], (5) can be obtained. □

Before we give the result of main part, we first select the appropriate parameters.

Lemma 5. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain, the nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$, assume that* (2) *and* (3) *hold. In case $0 < \gamma \leq 1$, if*

$$\alpha + \beta < \min\left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\right\},$$

then there exists $s \in \left[1, \frac{n}{(n\gamma-1)_+}\right)$ such that

$$\gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta. \tag{6}$$

Moreover, let $1 < a < \min\left\{\frac{n}{n-2}, \frac{s}{(s-2)_+}\right\}$ and $b > \max\left\{\frac{n}{2}, \frac{1}{2\gamma}\right\}$, we choose some $p_\star > 1 + \frac{n\alpha}{2}$ and $q_\star > 1 + \frac{s}{2}$ such that for all $p > p_\star$ and $q > q_\star$, then we have

$$\frac{n-2}{n} \cdot \frac{p + \alpha + 2\beta - 2}{p - \alpha} < \frac{1}{a} < p + \alpha + 2\beta - 2, \tag{7}$$

$$1 - \frac{2}{s} < \frac{1}{a} < 1 - \frac{n-2}{nq}, \tag{8}$$

$$\frac{n-2}{n} \cdot \frac{2\gamma}{p - \alpha} < \frac{1}{b} < \frac{2}{n} + \frac{1}{q}\left(1 - \frac{2}{n}\right) \quad \text{and} \quad \frac{2b(q-1)}{b-1} > s. \tag{9}$$

Proof. The proof is similar to [9] (also see [12]), so we omitted it here. □

In the following lemma, we obtain the uniform boundedness of $\|u\|_{L^p(\Omega)}$ by establishing a priori estimates and taking appropriate parameters.

Lemma 6. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) - (3) *and Lemma 5 hold. If*

$$0 < \gamma \leq 1 \quad and \quad \alpha + \beta < \min\left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\right\},$$

then there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq C \tag{10}$$

for all $t \in (0, T_{\max})$ with all $p \in [1,\infty) > p_\star$ and $q \in \left(\frac{3}{2},\infty\right) > q_\star$.

Proof. Multiplying both sides the first equation of (1) by $p(u+1)^{p-1}$ and integrating, then using Young's inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (1+u)^p &\leq -C_d p(p-1)\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 + C_s p(p-1)\int_\Omega (1+u)^{p+\beta-2}|\nabla u||\nabla v| \\ &\leq -\frac{C_d p(p-1)}{2}\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 + \frac{C_s^2 p(p-1)}{2C_d}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2.\end{aligned} \tag{11}$$

The first term on the right-hand side of the inequality (11) can be expressed as

$$\frac{C_d p(p-1)}{2}\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 = \frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega \left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2,$$

this together with (11) which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (1+u)^p + \frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega \left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2 \leq \frac{C_s^2 p(p-1)}{2C_d}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2 \tag{12}$$

for all $t \in (0, T_{\max})$. For a prior estimate of v, one can see [9,12,13], for completeness, a brief proof is given here. Applying the second equation of (1), the point-wise identity $\Delta|\nabla v|^2 = 2\left|D^2 v\right|^2 + 2\nabla v \cdot \nabla\Delta v$ and the fact $|\Delta v|^2 \leq n\left|D^2 v\right|^2$, we derive

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + 2\int_\Omega |\nabla v|^{2q} \\ &\quad\leq \int_\Omega |\nabla v|^{2(q-1)}\Delta|\nabla v|^2 + 2\int_\Omega |\nabla v|^{2(q-1)}\nabla v\cdot\nabla g(u) \\ &\quad= -\left(q-1\right)\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \int_{\partial\Omega}|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S \\ &\qquad -2(q-1)\int_\Omega |\nabla v|^{2(q-2)}\nabla|\nabla v|^2\cdot\nabla v\cdot g(u) - 2\int_\Omega |\nabla v|^{2(q-1)}\Delta v\cdot g(u)\end{aligned} \tag{13}$$

for all $t \in (0, T_{\max})$. Using the property of boundary integral without the convexity of domain [6, Lemma 4.2] and the trace inequality [2, Proposition 4.22, 4.24] we have

$$\int_{\partial\Omega}|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S \leq 2\kappa_\Omega\int_{\partial\Omega}|\nabla v|^{2q}dS \leq \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} \tag{14}$$

with some $\kappa_\Omega, C_1 > 0$. Combining (13) with (14) and using Young's inequality yield

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + 2\int_\Omega |\nabla v|^{2q} \\ &\quad\leq -\frac{q-1}{2}\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} \\ &\qquad + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + \left(2(q-1)+\frac{n}{2}\right)\int_\Omega |\nabla v|^{2(q-1)}g^2(u) \\ &\quad= -\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 \\ &\qquad + \left(2(q-1)+\frac{n}{2}\right)\int_\Omega |\nabla v|^{2(q-1)}g^2(u),\end{aligned}$$

thus, this together with (3) which implies

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q}+\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \le C_g^2\left(2(q-1)+\frac{n}{2}\right)\int_\Omega u^{2\gamma}|\nabla v|^{2(q-1)}+(C_1-2)\int_\Omega |\nabla v|^{2q} \quad (15)$$

for all $t\in(0,T_{\max})$. Combining (12) and (15) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left((1+u)^p+\frac{1}{q}|\nabla v|^{2q}\right)+\frac{q-1}{q^2}\int_\Omega|\nabla|\nabla v|^q|^2+\frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega|\nabla(1+u)^{\frac{p-\alpha}{2}}|^2\\ \le C_2\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2+C_2\int_\Omega(1+u)^{2\gamma}|\nabla v|^{2(q-1)}+C_2\int_\Omega|\nabla v|^{2q} \quad (16)\end{aligned}$$

for all $t\in(0,T_{\max})$ with $C_2:=\max\left\{\frac{C_s^2 p(p-1)}{2C_d}, C_1-2, C_g^2\left(2(q-1)+\frac{n}{2}\right)\right\}>0$. According to Lemma 5, $a,b>1$, let $a^{'}:=\frac{a}{a-1}>1$ and $b^{'}:=\frac{b}{b-1}>1$, applying Hölder's inequality to the first two terms on the right-hand side of the inequality (16), we infer

$$\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2\le\left(\int_\Omega(1+u)^{(p+\alpha+2\beta-2)a}\right)^{\frac{1}{a}}\left(\int_\Omega|\nabla v|^{2a^{'}}\right)^{\frac{1}{a^{'}}} \quad (17)$$

and

$$\int_\Omega(1+u)^{2\gamma}|\nabla v|^{2(q-1)}\le\left(\int_\Omega(1+u)^{2\gamma b}\right)^{\frac{1}{b}}\left(\int_\Omega|\nabla v|^{2(q-1)b^{'}}\right)^{\frac{1}{b^{'}}}. \quad (18)$$

In view of (4) and Gagliardo–Nirenberg inequality [7, 11] we have

$$\begin{aligned}\left(\int_\Omega(1+u)^{(p+\alpha+2\beta-2)a}\right)^{\frac{1}{a}}&=\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2a(p+\alpha+2\beta-2)}{p-\alpha}}(\Omega)}\\ &\le C_3\left\|\nabla(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)\theta}{p-\alpha}}_{L^2(\Omega)}\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)(1-\theta)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)}\\ &\quad+C_3\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)}\\ &\le C_4\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}+C_4\end{aligned} \quad (19)$$

with $C_3,C_4>0$, and $\theta=\frac{\frac{p-\alpha}{2}-\frac{p-\alpha}{2a(p+\alpha+2\beta-2)}}{\frac{1}{n}-\frac{1}{2}+\frac{p-\alpha}{2}}\in(0,1)$ is guaranteed by (7). Similarly, according to (5) and Gagliardo–Nirenberg inequality again we have

$$\begin{aligned}\left(\int_\Omega|\nabla v|^{2a^{'}}\right)^{\frac{1}{a^{'}}}=\left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{2a^{'}}{q}}(\Omega)}&\le C_5\left\|\nabla|\nabla v|^q\right\|^{\frac{2\delta}{q}}_{L^2(\Omega)}\left\||\nabla v|^q\right\|^{\frac{2(1-\delta)}{q}}_{L^{\frac{s}{q}}(\Omega)}+C_5\left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{s}{q}}(\Omega)}\\ &\le C_6\left(\int_\Omega|\nabla|\nabla v|^q|^2\right)^{\frac{\delta}{q}}+C_6\end{aligned} \quad (20)$$

with $C_5,C_6>0$, and $\delta=\frac{\frac{q}{s}+\frac{q}{2a}-\frac{q}{2}}{\frac{1}{n}-\frac{1}{2}+\frac{q}{s}}\in(0,1)$ is guaranteed by (8). Combining (19) and (20) with (17), there exists a positive constant $C_7>0$ such that

$$C_2\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2\le C_7\left(\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}+1\right)\left(\left(\int_\Omega|\nabla|\nabla v|^q|^2\right)^{\frac{\delta}{q}}+1\right). \quad (21)$$

Similarly, in view of Lemma 4, (9) and Gagliardo–Nirenberg inequality again we derive

$$\left(\int_\Omega(1+u)^{2\gamma b}\right)^{\frac{1}{b}}=\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{4\gamma}{p-\alpha}}_{L^{\frac{4\gamma b}{p-\alpha}}(\Omega)}\le C_8\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{2\gamma\bar\theta}{p-\alpha}}+C_8 \quad (22)$$

and

$$\left(\int_\Omega |\nabla v|^{2(q-1)b'}\right)^{\frac{1}{b'}} = \left\| |\nabla v|^q \right\|^{\frac{2(q-1)}{q}}_{L^{\frac{2(q-1)b'}{q}}(\Omega)} \le C_9 \left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\frac{(q-1)\bar{\delta}}{q}} + C_9 \tag{23}$$

with some $C_8, C_9 > 0$, $\bar{\theta} = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{4\gamma b}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} \in (0,1)$ and $\bar{\delta} = \frac{\frac{q}{s} + \frac{q}{2(q-1)b} - \frac{q}{2(q-1)}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}} \in (0,1)$. Then combining (22) and (23) with (18), there exists a positive constant $C_{10} > 0$ such that

$$C_2 \int_\Omega (1+u)^{2\gamma} |\nabla v|^{2(q-1)} \le C_{10} \left(\left(\int_\Omega \left|\nabla (1+u)^{\frac{p-\alpha}{2}}\right|^2 \right)^{\frac{2\gamma\bar{\theta}}{p-\alpha}} + 1 \right) \left(\left(\int_\Omega |\nabla|\nabla v|^q|^2 \right)^{\frac{(q-1)\bar{\delta}}{q}} + 1 \right). \tag{24}$$

Therefore, using (16) in conjunction with (21) and (24), we infer

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left((1+u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{2C_d p(p-1)}{(p-\alpha)^2} \int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 + \frac{q-1}{q^2} \int_\Omega |\nabla|\nabla v|^q|^2 \\
&\quad \le C_{11} \left(\left(\int_\Omega \left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2 \right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta} + 1 \right) \left(\left(\int_\Omega |\nabla|\nabla v|^q|^2 \right)^{\frac{\delta}{q}} + 1 \right) \\
&\qquad + C_{11} \left(\left(\int_\Omega \left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2 \right)^{\frac{2\gamma\bar{\theta}}{p-\alpha}} + 1 \right) \left(\left(\int_\Omega |\nabla|\nabla v|^q|^2 \right)^{\frac{(q-1)\bar{\delta}}{q}} + 1 \right) + C_2 \int_\Omega |\nabla v|^{2q}
\end{aligned} \tag{25}$$

for all $t \in (0, T_{\max})$ with some $C_{11} > 0$. Thus, according to [12, Lemma 3.1] and Young's inequality, we can obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left((1+u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{C_d p(p-1)}{(p-\alpha)^2} \int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 + \frac{q-1}{2q^2} \int_\Omega |\nabla|\nabla v|^q|^2 \\
&\qquad \le C_2 \int_\Omega |\nabla v|^{2q} + C_{12}
\end{aligned} \tag{26}$$

with $C_{12} > 0$ if the assumptions

$$\frac{p+\alpha+2\beta-2}{p-\alpha}\theta + \frac{\delta}{q} < 1 \quad \text{and} \quad \frac{2\gamma\bar{\theta}}{p-\alpha} + \frac{(q-1)\bar{\delta}}{q} < 1 \tag{27}$$

are satisfied. Therefore, in order for the assumptions in (27) to be satisfied, let

$$h(q) := \frac{p+\alpha+2\beta-2}{p-\alpha}\theta + \frac{\delta}{q} = \frac{\frac{p+\alpha+2\beta-2}{2} - \frac{1}{2a}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} + \frac{\frac{1}{s} + \frac{1}{2a} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}}$$

and

$$\bar{h}(q) := \frac{2\gamma\bar{\theta}}{p-\alpha} + \frac{(q-1)\bar{\delta}}{q} = \frac{\gamma - \frac{1}{2b}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} + \frac{\frac{q-1}{s} + \frac{1}{2b} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}},$$

according to the condition (6) of Lemma 5, we have

$$h(q(p)) < 1 \quad \text{and} \quad \bar{h}(q(p)) < 1$$

with $q(p) := \frac{p-\alpha}{2} s$. Since $q(p) \to +\infty$ as $p \to \infty$, for all $p \ge p_\star$, there exists $q \ge q_\star$ such that

$$h(q) < 1 \quad \text{and} \quad \bar{h}(q) < 1,$$

thus, the assumptions in (27) are satisfied. In order for the inequality (26) to satisfy the form of Gronwall's inequality, using Gagliardo–Nirenberg inequality and Lemma 4 imply

$$\int_\Omega (1+u)^p = \left\| (1+u)^{\frac{p-\alpha}{2}} \right\|^{\frac{2p}{p-\alpha}}_{L^{\frac{2p}{p-\alpha}}(\Omega)} \le C_{13} \left(\int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 \right)^{\frac{p\sigma}{p-\alpha}} + C_{13} \tag{28}$$

with some $C_{13} > 0$, and $\sigma = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{2p}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} \in (0,1)$ is satisfied because of the condition $p > 1 + \frac{n\alpha}{2}$ in Lemma 5. In the same way, we obtain

$$\begin{aligned}\left(\frac{1}{q} + C_2\right)\int_\Omega |\nabla v|^{2q} = \left(\frac{1}{q} + C_2\right)\left\||\nabla v|^q\right\|^2_{L^2(\Omega)} &\le C_{14}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\bar{\sigma}} + C_{14} \\ &\le \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_{15}\end{aligned} \tag{29}$$

with some $C_{14}, C_{15} > 0$, and $\bar{\sigma} = \frac{\frac{q}{s} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}} \in (0,1)$ is satisfied because of the condition $q > 1 + \frac{s}{2}$ in Lemma 5. Therefore, combining (28) and (29) with (26), which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left((1+u)^p + \frac{1}{q}|\nabla v|^{2q}\right) + C_{16}\left(\int_\Omega (1+u)^p\right)^{\frac{p-\alpha}{p\sigma}} + \frac{1}{q}\int_\Omega |\nabla v|^{2q} \le C_{17} \tag{30}$$

for all $t \in (0, T_{\max})$ with some $C_{16}, C_{17} > 0$, therefore, according to the ODI comparison principle with (30), which implies (10). □

Now, we can easily prove Theorem 1.

Proof of Theorem 1. In view of [12, Lemmas 3.3 and A.1], we obtain the desired results. □

Acknowledgments

The authors thank the anonymous referee for their positive and useful comments, which helped him improve further the exposition of the paper.

References

[1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, "Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues", *Math. Models Methods Appl. Sci.* **25** (2015), no. 9, p. 1663-1763.

[2] D. D. Haroske, H. Triebel, *Distributions, Sobolev Spaces, Elliptic Equations*, EMS Textbooks in Mathematics, European Mathematical Society, 2008.

[3] D. Horstmann, M. Winkler, "Boundedness vs. blow-up in a chemotaxis system", *J. Differ. Equations* **215** (2005), no. 1, p. 52-107.

[4] E. F. Keller, L. A. Segel, "Initiation of slime mold aggregation viewed as an instability", *J. Theor. Biol.* **26** (1970), no. 3, p. 399-415.

[5] R. Kowalczyk, Z. Szymańska, "On the global existence of solutions to an aggregation model", *J. Math. Anal. Appl.* **343** (2008), no. 1, p. 379-398.

[6] N. Mizoguchi, P. Souplet, "Nondegeneracy of blow-up points for the parabolic Keller-Segel system", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **31** (2014), no. 4, p. 851-875, corrigendum *ibid.* **36** (2019), no. 4, p. 1181.

[7] L. Nirenberg, "An extended interpolation inequality", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **20** (1966), p. 733-737.

[8] C. Stinner, J. I. Tello, M. Winkler, "Competitive exclusion in a two-species chemotaxis model", *J. Math. Biol.* **68** (2014), no. 7, p. 1607-1626.

[9] X. Tao, S. Zhou, M. Ding, "Boundedness of solutions to a quasilinear parabolic-parabolic chemotaxis model with nonlinear signal production", *J. Math. Anal. Appl.* **474** (2019), no. 1, p. 733-747.

[10] Y. Tao, "Boundedness in a chemotaxis model with oxygen consumption by bacteria", *J. Math. Anal. Appl.* **381** (2011), no. 2, p. 521-529.

[11] Y. Tao, Z.-A. Wang, "Competing effects of attraction vs. repulsion in chemotaxis", *Math. Models Methods Appl. Sci.* **23** (2013), no. 1, p. 1-36.

[12] Y. Tao, M. Winkler, "Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity", *J. Differ. Equations* **252** (2012), no. 1, p. 692-715.

[13] L. Wang, Y. Li, C. Mu, "Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source", *Discrete Contin. Dyn. Syst.* **34** (2014), no. 2, p. 789-802.

[14] L. Wang, C. Mu, X. Hu, P. Zheng, "Boundedness and asymptotic stability of solutions to a two-species chemotaxis system with consumption of chemoattractant", *J. Differ. Equations* **264** (2018), no. 5, p. 3369-3401.

[15] L. Wang, C. Mu, P. Zheng, "On a quasilinear parabolic-elliptic chemotaxis system with logistic source", *J. Differ. Equations* **256** (2014), no. 5, p. 1847-1872.

[16] M. Winkler, "Does a "volume-filling effect" always prevent chemotactic collapse?", *Math. Methods Appl. Sci.* **33** (2010), no. 1, p. 12-24.

[17] M. Winkler, Y. Tao, "Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals", *Discrete Contin. Dyn. Syst.* **20** (2015), no. 9, p. 3165-3183.

[18] Q. Zhang, X. Liu, X. Yang, "Global existence and asymptotic behavior of solutions to a two-species chemotaxis system with two chemicals", *J. Math. Phys.* **58** (2017), no. 11, article no. 111504 (9 pages).