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Comptes Rendus
Mathématique

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Volume 358, issue 11-12 (2020), p. 1187-1197

Published online: 25 January 2021

https://doi.org/10.5802/crmath.138

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

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Partial Differential Equations / *Équations aux dérivées partielles*

Asymptotic behavior of solutions of fully nonlinear equations over exterior domains

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Abstract. In this paper, we consider the asymptotic behavior at infinity of solutions of a class of fully nonlinear elliptic equations $F(D^2u) = f(x)$ over exterior domains, where the Hessian matrix (D^2u) tends to some symmetric positive definite matrix at infinity and $f(x) = O(|x|^{-t})$ at infinity with sharp condition $t > 2$. Moreover, we also obtain the same result if (D^2u) is only very close to some symmetric positive definite matrix at infinity.

2020 Mathematics Subject Classification. 35J60, 35B40.

Manuscript received 4th September 2020, revised 9th October 2020, accepted 25th October 2020.

1. Introduction

Rigidity theorem, such as Bernstein theorem, Liouville theorem and asymptotic behavior, is an important essay in both partial differential equations and geometric problems. For Laplace equation, lots of renowned mathematicians, such as Cauchy, Liouville and Bôcher, have done outstanding works on it. For Monge–Ampère equation, one of the most arresting fully nonlinear equations, Jögens [12] ($n = 2$), Calabi [6] ($n \leq 5$) and Pogorelov [20] ($n \geq 2$) established the well known Liouville theorem (also called Jögens–Calabi–Pogorelov Theorem), which states that any classical convex solution of

$$\det D^2u = 1 \text{ in } \mathbb{R}^n$$

is a quadratic polynomial. Later, Cheng and Yau [8] given an easier proof of Jögens–Calabi–Pogorelov Theorem by geometric method; Caffarelli [3] extended it to viscosity solutions; Caffarelli and Li [5] reported the asymptotic behavior at infinity of viscosity solution of $\det D^2u = 1$ outside a bounded domain of $\mathbb{R}^n$. Essentially, [5] provided a incisive observation to analyse the asymptotic behavior for fully nonlinear equations.

Recently, many researchers studied the Liouville theorem and asymptotic behavior for various types of fully nonlinear equations such as k-Hessian equations [2, 7], parabolic k-Hessian equations [19], Hessian quotient equations [13], special Lagrangian equations [14, 18], Lagrangian mean curvature equations [1], parabolic Monge–Ampère equations [21–23], some fully nonlinear degenerate equations [15–17], and the references therein. Especially, Li et al. [14] investigated the asymptotic behavior at infinity for general fully nonlinear elliptic equations

$$F\left(D^2u\right)=0 \text{ in } \mathbb{R}^n\backslash\overline{B}_1,$$

under the boundedness condition of Hessian matrix D^2u, where F is uniformly elliptic and either concave or convex (cf. [4] for definitions).

In this paper, we continue to consider the asymptotic behavior at infinity for more general fully nonlinear elliptic equations, say the right hand term is nontrivial. But, as a compensation for the loss of f, we need to assume that D^2u converges or is very close to some symmetric positive definite matrix at infinity.

The following is our main result.

Theorem 1. *Let u be a smooth solution of fully nonlinear equation*

$$F\left(D^2u\right)=f(x) \text{ in } \mathbb{R}^n\backslash\overline{B}_1, \tag{1}$$

where $n\geq 3$, $F\in C^m(\mathbb{R}^{2n})$ is concave and uniformly elliptic, and $f\in C^m(\mathbb{R}^n\backslash\overline{B}_1)$ satisfies

$$f(x)=O_m\left(|x|^{-t}\right) \text{ as } |x|\to\infty, \tag{2}$$

where $m\geq 2$ and $t>2$. Suppose that

$$D^2u\to A \text{ as } |x|\to\infty, \tag{3}$$

where A is some symmetric positive definite matrix with $F(A)=0$. Then there exists a unique quadratic polynomial

$$Q(x)=\frac{1}{2}x^TAx+b^Tx+c$$

such that

$$u-Q=\begin{cases} O_{m+1}\left(|x|^{2-t}\right), & \text{if } t<n,\\ o_{m+1}\left(|x|^{2-s}\right) \text{ for all } s\in(2,n), & \text{if } t\geq n,\\ O_{m+1}\left(|x|^{2-n}\right), & \text{if } t>n\end{cases} \quad \text{at infinity}, \tag{4}$$

where $b\in\mathbb{R}^n$ is some vector, $c\in\mathbb{R}$ is some constant and $\varphi(x)=O_k(|x|^\beta)$(or $o_k(|x|^\beta)$) means that $|D^\ell\varphi|=O(|x|^{\beta-\ell})$ (or $o(|x|^{\beta-\ell})$) for all $\ell=0,1,\cdots,k$.

Remark 2. Theorem 1 still holds if either F is convex, or $\{M \mid F(M)=0\}$ is convex.

Remark 3. For any fixed $e\in\partial B_1$, u_{ee} is a subsolution of the linearized equation of Equation (1) with right term f_{ee}. Indeed, for all $e,s\in\partial B_1$,

$$\frac{\partial F}{\partial x_e}=\frac{\partial F}{\partial u_{ij}}\frac{\partial u_{ij}}{\partial x_e}:=F_{ij}(u_e)_{ij}=f_e(x);$$

$$\begin{aligned}\frac{\partial^2 F}{\partial x_e\partial x_s}&=\frac{\partial\left(F_{ij}(u_e)_{ij}\right)}{\partial x_s}+F_{ij}(u_{es})_{ij}\\ &=\frac{\partial^2 F}{\partial u_{ij}\partial u_{kl}}\frac{\partial u_{ij}}{\partial x_e}\frac{\partial u_{kl}}{\partial x_s}+F_{ij}(u_{es})_{ij}\\ &:=F_{ij,kl}(u_e)_{ij}(u_s)_{kl}+F_{ij}(u_{es})_{ij}\\ &=f_{es}(x).\end{aligned}$$

Thus,

$$F_{ij}\left(D^2u\right)(u_e)_{ij}=f_e(x);\quad F_{ij}\left(D^2u\right)(u_{es})_{ij}=f_{es}(x)-F_{ij,kl}(u_e)_{ij}(u_s)_{kl}.$$

In particular, if $s=e$,

$$F_{ij}\left(D^2u\right)(u_{ee})_{ij}=f_{ee}(x)-F_{ij,kl}(u_e)_{ij}(u_e)_{kl}\geq f_{ee}(x),$$

where the concavity of F is applied. Thus, u_{ee} is a subsolution of

$$F_{ij}\left(D^2 u\right) w_{ij}(x) := a_{ij}(x) w_{ij}(x) := Lw = f_{ee}(x).$$

which is the linearized equation of Equation (1).

Next Theorem 4 states that the convergence condition (3) can be replaced by a weaker version.

Theorem 4. *There exists small $\sigma(n, A, F)$ depending only on n, A and F such that if*

$$\left|D^2 u - A\right| \le \sigma(n, A, F) \quad \text{at infinity,} \tag{5}$$

but not (3)*, then* (4) *in Theorem 1 still holds with a unique quadratic polynomial*

$$Q(x) = \frac{1}{2} x^T \widetilde{A} x + b^T x + c,$$

where $\widetilde{A}$ is some symmetric positive definite matrix with $F(\widetilde{A}) = 0$.

Remark 5. It is easy to see that $\widetilde{A}$ may not be A.

Remark 6. After proper rescaling, by (3) or (5), we can always assume that

$$\left\|D^2 u\right\|_{L^\infty\left(\mathbb{R}^n \setminus \overline{B}_1\right)} \le K < +\infty, \tag{6}$$

where $K > 0$ is some constant depending only on A. Therefore, in the remaining of this paper, we will take advantage of this fact directly and repeatedly.

For general fully nonlinear elliptic equations, including Equation (1), the usual method of obtaining the asymptotic behavior of its solution is to find proper function such that the difference between the function and the solution solves some kind of linear elliptic equations (we call this step is nonlinear approach), and then to investigate the asymptotic behavior of the kind of elliptic equations (we call this step is linear approach). Based on the nonlinear approach and the linear one, the desired result will be deduced.

This paper is organized as follows. In Section 1, we introduce our main results. In Section 2, we first study the first step (nonlinear approach) under weaker condition (5); and then to investigate the second step (linear approach), we give the asymptotic behavior for a class of non-divergence linear elliptic equations. In Section 3, combining above two approaches, the proof of Theorem 1 completes, which together with Lemma 7 follows Theorem 4.

2. Preliminaries

In this section, two parts are considered, the first one is to obtain (3) combining with (5) and the Equation (1); the second one is to study the asymptotic behavior for some kind of elliptic equations over exterior domains.

2.1. $D^2 u$ *tends to some symmetry positive definite matrix at infinity with* (5)

The following Lemma 7 shows that the Hessian matrix of u in Theorem 4 tends to some symmetric positive definite matrix at infinity by making use of the method given by Li et al. in [14, Lemma 2.1]. Then, the following auxiliary lemma together with Theorem 1 implies Theorem 4, immediately.

Lemma 7. *Let u be as in Theorem 4. Then there exists a symmetric positive definite matrix $\widetilde{A}$ such that*

$$D^2 u(x) \to \widetilde{A} \quad \text{as } |x| \to \infty.$$

Proof. It only needs to show that for any fixed $e\in\partial B_1$, u_{ee} converges to some constant at infinity. The symmetry and the positive definiteness can be deduced by the smoothness of u and (5), respectively.

Denote

$$w(x)=u_{ee}(x),\quad \overline{w}=\overline{\lim_{|x|\to\infty}}\,w(x),\quad \underline{w}=\underline{\lim_{|x|\to\infty}}\,w(x).$$

It's enough to prove that $\overline{w}=\underline{w}$.

Now we argue this by contradiction. If it is wrong, we have $\overline{w}-\underline{w}=:5d>0$. Clearly, for any $0<\varepsilon<d$, there exists some large constant $R=R(\varepsilon)>1$ such that

$$\underline{w}-\varepsilon\le w(x)\le\overline{w}+\varepsilon$$

for all $x\in B_{R/2}^{\mathscr{C}}$, and also there exists a sequence of $\underline{x}_k$ in $B_{R/2}^{\mathscr{C}}$, such that

$$w\left(\underline{x}_k\right)\le\underline{w}+\varepsilon,\quad \left|\underline{x}_k\right|\to\infty$$

for all $k\in\mathbb{Z}^+$. Then there exists a point $\overline{x}$ on the sphere $\partial B_{|\underline{x}|}$ for at least one $\underline{x}\in\left\{\underline{x}_k\right\}$, such that

$$w\left(\overline{x}\right)\ge\overline{w}-\varepsilon.$$

Otherwise, $w<\overline{w}-\varepsilon$ on the spheres $\partial B_{|\underline{x}_k|}$ for all $k\in\mathbb{Z}^+$.

By Remark 2, we have that for all $k\ge k_0$ large,

$$Lw:=F_{ij}\left(D^2u\right)w_{ij}(x)\ge f_{ee}\ \text{ in } B_{|\underline{x}_k|}^{\mathscr{C}},$$

Without loss of generality, we may assume that $F_{ij}(A)=I_n$, the identity matrix. Then, it follows from (5) that

$$\begin{aligned}L|x|^{-1/2}&=F_{ij}\left(A+\left(D^2u-A\right)\right)\left(|x|^{-1/2}\right)_{ij}\\&\le\Delta|x|^{-1/2}+\sigma(n,A)|x|^{-5/2}\\&=\left(-\frac{1}{2}\left(n-\frac{5}{2}\right)+\sigma(n,A)\right)|x|^{-5/2}\\&\le-\frac{1}{4}\left(n-\frac{5}{2}\right)|x|^{-5/2}\\&\le-\left|f_{ee}\right|\quad\text{in } B_{|\underline{x}_k|}^{\mathscr{C}}\end{aligned}$$

for all $|\underline{x}_k|$ large enough, say $k\ge k_0$.

By the comparison principle, we have that for all $k\ge k_0$ large,

$$w(x)<\overline{w}-\varepsilon+|x|^{-1/2}\quad\text{in}\quad B_{|\underline{x}_{k+1}|}\backslash\overline{B}_{|\underline{x}_k|}.$$

Then for all $|\underline{x}_k|\ge(2/\varepsilon)^2$, say $k\ge k_1$, we have

$$w(x)<\overline{w}-\varepsilon+|x|^{-1/2}\le\overline{w}-\frac{1}{2}\varepsilon\quad\text{in}\quad B_{\left|\underline{x}_{k_1}\right|}^{\mathscr{C}},$$

which leads to a contradiction.

Applying (2) and the Evans–Krylov estimate to u in $B_{|\underline{x}|/2}(\underline{x})$ (cf. [10]), we have

$$\begin{aligned}\underset{B_{\gamma|\underline{x}|}(\underline{x})}{\operatorname{osc}}u_{ee}&\le C\left(\frac{2\gamma\left|\underline{x}\right|}{\left|\underline{x}\right|}\right)^{\alpha}\left\{\underset{B_{|\underline{x}|/2}(\underline{x})}{\operatorname{osc}}D^2u+\left|\underline{x}\right|\left|Df\right|_{0,B_{|\underline{x}|/2}(\underline{x})}+\left|\underline{x}\right|^2\left|D^2f\right|_{0,B_{|\underline{x}|/2}(\underline{x})}\right\}\\&\le 2C\gamma^{\alpha}(K+1)\\&\le d,\end{aligned}$$

where $\alpha=\alpha(n,\lambda,\Lambda)$, $\gamma=\gamma(n,\lambda,\Lambda,K,d)=:\min\{1/10,(d/(2C(K+1)))^{1/\alpha}\}$. Thus,

$$w(x)\le\underline{w}+\varepsilon+d\le\overline{w}-3d\quad\text{for } x\in B_{\gamma|\underline{x}|}\left(\underline{x}\right),$$

which yields

$$\overline{w} - w(x) \geq 3d \quad \text{for } x \in B_{\gamma|\underline{x}|}(\underline{x}).$$

Let $v(x) = \overline{w} + \varepsilon - w(x)$. Then

$$Lv \leq -f \quad \text{in } |x| \geq 1.$$

Applying the weak Harnack inequality to v in $B_{(1+3\gamma)|\underline{x}|}\backslash\overline{B}_{(1-3\gamma)|\underline{x}|}$ (cf. [10]), we obtain that

$$\begin{aligned}\left(\frac{1}{\left|B_{\gamma|\underline{x}|}\right|}\int_{B_{(1+\gamma)|\underline{x}|}\backslash\overline{B}_{(1-\gamma)|\underline{x}|}} v^{\delta}\right)^{1/\delta} &\leq C\left\{\inf_{B_{(1+\gamma)|\underline{x}|}\backslash\overline{B}_{(1-\gamma)|\underline{x}|}} v + \left|\underline{x}\right| \|f\|_{L^n\left(B_{(1+3\gamma)|\underline{x}|}\backslash\overline{B}_{(1-3\gamma)|\underline{x}|}\right)}\right\} \\ &\leq C\left\{2\varepsilon + \left|\underline{x}\right|^2 \sup_{B_{(1+3\gamma)|\underline{x}|}\backslash\overline{B}_{(1-3\gamma)}} f\right\} \\ &\leq C\left(2\varepsilon + \left|\underline{x}\right|^{2-t}\right) \\ &\leq 3C\varepsilon\end{aligned}$$

for $|\underline{x}|$ large, depending only on t and ε, as $t > 2$. Then $3d \leq 3C\varepsilon$, where C is independent of ε. Letting $\varepsilon \to 0$, we get $d = 0$, a contradiction. □

2.2. *Asymptotic behavior at infinity of solutions of a class of linear equations in non-divergence form*

In this subsection, the key technology of linear approach will be introduced. Precisely, we consider the asymptotic behavior at infinity of solutions of

$$a_{ij}(x)v_{ij} = f \ \text{ in } \mathbb{R}^n\backslash\overline{B}_R \tag{7}$$

where a_{ij} is uniformly elliptic, $a_{ij}(x) \to a_{ij}^{\infty}$ and $f = O(|x|^{-t})$ for some $t > 2$ at infinity. For more details on asymptotic behavior with $f \equiv 1$, one can refer to [9, Theorem 3', Theorem 4], [14, Theorem 2.2, Corollary 2.1], [11] and the references therein.

Firstly, let $w(x) = |x|^p$ and then one can easily check that:

(1) for any fixed $p \in (2-n, 0)$,

$$\Delta w = p(n+p-2)|x|^{p-2} < 0;$$

(2) for any fixed $t \geq 2$ and any fixed $p \in (2 - \min(n, t), 0)$,

$$\begin{aligned} Lw = \Delta w + \left(a_{ij} - \delta_{ij}\right) D_{ij} w &\leq p\left(n+p-2\right)|x|^{p-2} + C\left|a_{ij} - \delta_{ij}\right| |x|^{p-2} \\ &\leq \frac{1}{2} p\left(n+p-2\right)|x|^{p-2} \\ &\leq -|f|, \end{aligned} \tag{8}$$

where $|x|$ is large enough;

(3) if $t \in (2, n)$, then for any fixed $p \in [2-t, 0)$,

$$\begin{aligned} L(Aw) = A\left(\Delta w + \left(a_{ij} - \delta_{ij}\right) D_{ij} w\right) &\leq A\left(p\left(n+p-2\right)|x|^{p-2} + C\left|a_{ij} - \delta_{ij}\right| |x|^{p-2}\right) \\ &\leq \frac{1}{2} Ap\left(n+p-2\right)|x|^{p-2} \\ &\leq -|f|, \end{aligned} \tag{9}$$

where A and $|x|$ are large enough;

(4) additionally, if $t > n$ and $a_{ij}(x) - a_{ij}^{\infty} = O(|x|^{-s})$ at infinity for some $s > 0$, then

$$\begin{aligned} L\left(|x|^{2-n} - |x|^{2-n-\ell}\right) &= -\Delta|x|^{2-n-\ell} + \left(a_{ij} - \delta_{ij}\right)\left(|x|^{2-n} - |x|^{2-n-\ell}\right) \\ &\le (2-n-\ell)\,\ell|x|^{-n-\ell} + C|x|^{-s}\left(|x|^{-n} + |x|^{-n-\ell}\right) \\ &\le \frac{1}{2}(2-n-\ell)\ell|x|^{-n-\ell} \\ &\le -|f| \end{aligned} \tag{10}$$

for any fixed $\ell \in (0, \min(s, t-n))$, where $|x|$ is large enough.

Now, based on above several observations on the supersolutions of Equation (7) under different assumptions, we can state our main Theorem 8 of this section as follows.

Theorem 8. *Let v be a positive solution of*

$$Lv := a_{ij}(x)v_{ij} = f \quad in\ \mathbb{R}^n \backslash \overline{B}_1, \tag{11}$$

where $n \ge 3$, $a_{ij}(x)$ is uniformly elliptic with

$$a_{ij}(x) \to a_{ij}^{\infty} \quad as\ |x| \to \infty,$$

and for some $t > 2$,

$$f = O\left(|x|^{-t}\right) \quad as\ |x| \to \infty.$$

Then, if $t \in (2, n)$, there exists a constant v_∞ such that

$$v(x) = v_\infty + O\left(|x|^{2-t}\right) \quad as\ |x| \to \infty; \tag{12}$$

if $t \ge n$, there exists a constant v_∞ such that

$$v(x) = v_\infty + o\left(|x|^{2-m}\right) \quad as\ |x| \to \infty \tag{13}$$

for all $m < n$; if $t > n$ and

$$a_{ij}(x) - a_{ij}^{\infty} = O\left(|x|^{-s}\right) \quad as\ |x| \to \infty \tag{14}$$

for some $s > 0$, there exists a constant v_∞ such that

$$v(x) = v_\infty + O\left(|x|^{2-n}\right) \quad as\ |x| \to \infty. \tag{15}$$

Proof. After proper rescaling, we can assume $a_{ij}^{\infty} = \delta_{ij}$. The proof will be divided into two steps.

Step 1. We show that $\lim_{|x|\to\infty} v(x)$ exists and is finite.

Let

$$\overline{v} = \overline{\lim_{|x|\to\infty}}\, v(x), \quad \underline{v} = \underline{\lim_{|x|\to\infty}}\, v(x).$$

Then $\overline{v} \ge \underline{v} \ge 0$.

(*Step 1.1*) We first prove that $\underline{v} < +\infty$.

Otherwise, we have

$$v(x) \to +\infty \quad \text{as} \quad |x| \to \infty.$$

It follows from (8) that

$$L\left(2|x|^{-\delta} + \varepsilon v\right) \le -2\left|f\right| + \varepsilon f \le -\left|f\right| \le 0 \quad \text{in } B_{R_\varepsilon} \backslash \overline{B}_R,$$

where R is large and we can take $\delta = \frac{1}{2}\min(t, n) - 1$.

For any $\varepsilon > 0$, there exists $R_\varepsilon > R$ such that $\varepsilon v(x) > 2$ if $|x| \ge R_\varepsilon$. Then

$$0 \le 2\left(|x|^{-\delta} - R^{-\delta}\right) + \varepsilon v \quad \text{on} \quad \partial B_{R_\varepsilon} \cup \partial B_R.$$

By the comparison principle, we have

$$0 \le 2\left(|x|^{-\delta} - R^{-\delta}\right) + \varepsilon v \quad \text{in} \quad B_{R_\varepsilon} \backslash \overline{B}_R.$$

In particular, at $x^* = (M, 0, \ldots, 0)$,

$$0 \le 2\left(M^{-\delta} - R^{-\delta}\right) + \varepsilon v\left(x^*\right) \le -R^{-\delta} + \varepsilon v\left(x^*\right),$$

where M is chosen such that $M^{-\delta} \le \frac{1}{2}R^{-\delta}$.

Letting $\varepsilon \to 0$, we get $0 \le -R^{-\delta}$, which is a contradiction.

(*Step 1.2*) Now we prove that $\overline{v} \le \underline{v}$.

For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $\widetilde{v}(x) = v(x) - \underline{v} + \varepsilon > 0$ for all $x \in B_{R_\varepsilon}^{\mathscr{C}}$ since

$$\varliminf_{|x|\to\infty} \widetilde{v}(x) = \varepsilon.$$

And then there exist $\{x_k\}_{k=1}^{\infty}$ such that

$$2R_\varepsilon \le r_k = |x_k| \to +\infty, \quad r_k < r_{k+1} \quad \text{and} \quad \widetilde{v}(x_k) \le 2\varepsilon.$$

Applying the Krylov–Safonov's Harnack inequality to $\widetilde{v}$ in $B_{(1+1/4)r_k} \backslash \overline{B}_{(1-1/4)r_k}$, we obtain

$$\begin{aligned}
\widetilde{v}(x) &\le C\left(\widetilde{v}(x_k) + (1+1/4)r_k \|f\|_{L^n\left(B_{(1+1/4)r_k} \backslash \overline{B}_{(1-1/4)r_k}\right)}\right) \\
&\le C\left(2\varepsilon + r_k^2 \sup_{B_{(1+1/4)r_k} \backslash \overline{B}_{(1-1/4)r_k}} f\right) \\
&\le C\left(2\varepsilon + r_k^{2-t}\right) \\
&\le 3C\varepsilon
\end{aligned}$$

for all $x \in \partial B_{r_k}$ and all $k \in \mathbb{Z}^+$.

By the comparison principle, we have

$$\widetilde{v}(x) \le 3C\varepsilon + |x|^{-\delta} \quad \text{in} \quad B_{r_{k+1}} \backslash \overline{B}_{r_k}$$

for all $k \in \mathbb{Z}^+$, which yields

$$\widetilde{v}(x) \le 3C\varepsilon + |x|^{-\delta} \quad \text{in} \quad B_{r_1}^{\mathscr{C}}.$$

By letting $|x| \to \infty$ and taking limit superior, we get $\overline{v} - \underline{v} + \varepsilon \le 3C\varepsilon$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain $\overline{v} \le \underline{v}$.

Therefore,*Step 1.1* and *Step 1.2* follows that $v(x)$ tends to a finite constant at infinity.

Step 2. We give a simple outline of proof of (12), (13) and (15).

Without loss of generality, we assume $v_\infty = 0$ and $|v| \le 1$. Or, we consider $\frac{v - v_\infty}{\sup_{B_1^{\mathscr{C}}} |v|}$.

(*Step 2.1*) First we prove (12).

By (9), $A|x|^{2-t}$ is a supersolution of (11) in $B_R^{\mathscr{C}}$, where A and R are large constants. For any $\varepsilon > 0$ small, there exists $R_\varepsilon > R$ such that

$$v \le A|x|^{2-t} + \varepsilon \quad \text{on } \partial\left(B_{R_\varepsilon} \backslash \overline{B}_R\right).$$

Applying the comparison principle, we get

$$v \le A|x|^{2-t} + \varepsilon \quad \text{in } B_{R_\varepsilon} \backslash \overline{B}_R.$$

Letting $\varepsilon \to 0$, the assertion (12) is proved.

(*Step 2.2*) Next we prove (13).

For any fixed $m < n$, it only needs to show

$$v(x) = v_\infty + O\left(|x|^{2-m'}\right) \quad \text{as } |x| \to \infty,$$

where $m' = \frac{1}{2}(n-m) + m \in (m, n)$. By (8), we have that $|x|^{2-m'}$ a supersolution of (11) in $B_R^{\mathscr{C}}$, where R is some large constant. Similar to *Step 2.1*, by the comparison principle, one can obtain the desired result in this step.

(*Step 2.3*) Finally we show (15).

By (10), for any fixed $\ell \in (0, \min(s, t-n))$, $|x|^{2-n} - |x|^{2-n-\ell}$ is a supersolution of (11) in $B_R^{\mathscr{C}}$ with (14) and $t > n$, where R is a large constant. Then, by the comparison principle, one can obtain

$$v = O\left(|x|^{2-n} - |x|^{2-n-\ell}\right) \quad \text{as } |x| \to \infty,$$

which implies (15) immediately.

This finishes the proof of the Theorem 8. □

Theorem 8 requires that v is positive, that is, v must be bounded from below (or from above). The following theorem shows that Theorem 8 still holds if the boundedness of v from one side is weakened by

$$|Dv(x)| = O\left(|x|^{-1}\right) \quad \text{as } |x| \to \infty.$$

Theorem 9. *Let v be a smooth solution of*

$$a^{ij}(x) v_{ij} = f \quad \textit{in} \quad \mathbb{R}^n \setminus \overline{B}_1,$$

where $n \geq 3$, $a^{ij}(x)$ is uniformly elliptic with

$$a^{ij}(x) \to a_\infty^{ij} \quad \textit{as} \quad |x| \to \infty,$$

and f satisfies

$$f = O\left(|x|^{-t}\right) \quad \textit{as} \quad |x| \to \infty$$

for some $t > 2$. Suppose

$$|Dv(x)| = O\left(|x|^{-1}\right) \quad \textit{as} \quad |x| \to \infty.$$

Then there exists a constant v_∞ such that (12) *holds if $t < n$,* (13) *if $t \geq n$, and* (15) *if $t > n$ with* (14).

Proof. The proof of this theorem is standard (cf. [5, Theorem 4] and [14, Corollary 2.1]) and therefore we omit it here. Notice that, to show the boundedness from one side of v, one should apply the comparison principle between v and $C + |x|^{-\delta}$ in $\mathbb{R}^n \setminus \overline{B}_R$ for large $R > 1$, where we can set $\delta = \min(t-2, n-2)$. □

3. Proof of Theorem 1 and Theorem 4

The only aim of this section is to show Theorem 1, since which together with Lemma 7 implies Theorem 4. The idea of showing Theorem 1 is standard (cf. [5] and [14, Theorem 2.1]). Specifically, Lemma 7 will be used in the nonlinear approach and Theorem 9 in the linear one repeatedly.

Proof.

Step 1. Nonlinear approach

Let

$$v(x) = u(x) - \frac{1}{2} x^T A x,$$

where A was given by Lemma 7. Then for all $e \in \partial B_1$, v, v_e and v_{ee} satisfy

$$\overline{a}_{ij} v_{ij} = f, \quad \widehat{a}_{ij} (v_e)_{ij} = f_e, \quad \widehat{a}_{ij} (v_{ee})_{ij} \geq f_{ee} \quad \text{in} \quad \mathbb{R}^n \setminus \overline{B}_1; \tag{16}$$

where

$$\overline{a}_{ij}(x) = \int_0^1 F_{M_{ij}}\left(t D^2 v(x) + A\right) dt \quad \text{and} \quad \widehat{a}_{ij}(x) = F_{M_{ij}}\left(D^2 v(x) + A\right)$$

are uniformly elliptic. Applying Lemma 7, we have

$$\overline{a}_{ij}(x) \to F_{M_{ij}}(A) \quad \text{and} \quad \widehat{a}_{ij}(x) \to F_{M_{ij}}(A).$$

After proper rotation, we may assume that $F_{M_{ij}}(A) = \delta_{ij}$.

Then for any fixed $\delta \in (0, \min(t,n)-2)$,

$$\varphi(x) = |x|^{-\delta}$$

is a supersolution of

$$\widehat{a}_{ij}(x) w_{ij} = f_{ee} \quad \text{in } \overline{B}_{R_0}^{\mathscr{C}},$$

where $R_0 > 1$ large. Since for any $e \in \partial B_1$

$$\widehat{a}_{ij}(v_{ee})_{ij} \ge f_{ee} \quad \text{and} \quad v_{ee}(x) \to 0 \quad \text{as } |x| \to \infty,$$

we can use φ as a barrier function to obtain that

$$v_{ee}(x) \le C\varphi(x) \le C|x|^{-\delta}$$

for some $C > 0$. This together with $\overline{a}_{ij}(x) v_{ij} = f$ and $\overline{a}_{ij}(x)$ is uniformly elliptic follows that

$$\left|D^2 v(x)\right| \le C|x|^{-\delta},$$

which yields that

$$\left|\overline{a}_{ij}(x) - \overline{a}_{ij}^{\infty}\right| + \left|\widehat{a}_{ij}(x) - \widehat{a}_{ij}^{\infty}\right| \le C|x|^{-\delta}, \quad x \in B_1^{\mathscr{C}}.$$

In particular, if $n \ge 4$, we have

$$\left|D^2 v(x)\right| \le C|x|^{-2};$$

and if $n = 3$, we have $t > 3$ and then we can replace above φ by $\varphi = |x|^{2-n} - |x|^{2-n+s}$. Then

$$\left|D^2 v(x)\right| \le |x|^{2-n} - |x|^{2-n+s} \le |x|^{2-n} = |x|^{-1}.$$

Therefore, for all $n \ge 3$, we have

$$\left|\overline{a}_{ij}(x) - \overline{a}_{ij}^{\infty}\right| + \left|\widehat{a}_{ij}(x) - \widehat{a}_{ij}^{\infty}\right| \le C|x|^{-1}, \quad x \in B_1^{\mathscr{C}}.$$

Step 2. Linear approach Step 1 follows that

$$|Dv_e| \le C|x|^{-1}.$$

By (16) and Theorem 9, we have that there exists some constant b_e such that

$$v_e(x) = b_e + \begin{cases} O\left(|x|^{2-(t+1)}\right), & \text{if } t+1 < n, \\ o\left(|x|^{2-s}\right) \text{ for all } s \in (2,n), & \text{if } t+1 \ge n, \\ O\left(|x|^{2-n}\right), & \text{if } t+1 > n \text{ with (14)} \end{cases} \quad \text{as } |x| \to \infty.$$

Let $b = \left(b_{e_1}, \dots, b_{e_n}\right)^T$ with $e_1, \dots, e_n$ being the coordinate unit vector in $\mathbb{R}^n$ and

$$\overline{v}(x) = v(x) - b^T x = u(x) - \left(\frac{1}{2} x^T A x + b^T x\right).$$

Then

$$\left|D\overline{v}(x)\right| = \begin{cases} O\left(|x|^{1-t}\right), & \text{if } t+1 < n, \\ o\left(|x|^{2-s}\right) \text{ for all } s \in (2,n), & \text{if } t+1 \ge n, \\ O\left(|x|^{2-n}\right), & \text{if } t+1 > n \text{ with (14)} \end{cases} \quad \text{as } |x| \to \infty.$$

In particular, since $t > 2$, $n \ge 3$ and the arbitrariness of s, we have that

$$\left|D\overline{v}(x)\right| \le C|x|^{-1}.$$

By Theorem 9 and (16) again, there exists a constant c such that

$$\overline{v}(x) = c + \begin{cases} O\left(|x|^{2-t}\right), & \text{if } t < n, \\ o\left(|x|^{2-s}\right) \text{ for all } 2 < s < n, & \text{if } t \ge n, \\ O\left(|x|^{2-n}\right), & \text{if } t > n \text{ with (14)} \end{cases} \quad \text{as } |x| \to \infty.$$

Set $Q(x)=\frac{1}{2}x^TAx+b^Tx+c$, and then

$$|u(x)-Q(x)|=\left|\overline{v}(x)-c\right| \quad =\begin{cases} O\left(|x|^{2-t}\right), & \text{if } t<n,\\ o\left(|x|^{2-s}\right) \text{ for all } 2<s<n, & \text{if } t\geq n,\\ O\left(|x|^{2-n}\right), & \text{if } t>n \text{ with (14)}\end{cases} \quad \text{as } |x|\to\infty. \tag{17}$$

Step 3. Complete the proof of (4)

For any fixed x with $|x|$ sufficiently large, let

$$E(y)=\left(\frac{2}{|x|}\right)^2(u-Q)\left(x+\frac{|x|}{2}y\right)$$

Then

$$\underline{a}_{ij}(y)E_{ij}(y)=F\left(A+D^2E(y)\right)-F(A)=f\left(x+\frac{|x|}{2}y\right),\quad y\in B_1$$

where

$$\underline{a}_{ij}(y)=\int_0^1 F_{M_{ij}}\left(A+tD^2E(y)\right)dt.$$

By the Evans–Krylov estimate and the Schauder estimate, we have that for all $k\in[2,m+1]$,

$$\left\|D^kE(0)\right\|_{C^\alpha(B_{1/2})}\leq C_k\left\{\|E\|_{L^\infty(B_1)}+\left\|D^{k-2}f\right\|_{C^\alpha(B_1)}\right\}.$$

It follows from (17) and (2) that

$$|u-Q|=\begin{cases} O_{m+1}\left(|x|^{2-t}\right), & \text{if } t<n,\\ o_{m+1}\left(|x|^{2-s}\right) \text{ for all } 2<s<n, & \text{if } t\geq n,\\ O_{m+1}\left(|x|^{2-n}\right), & \text{if } t>n \text{ with (14)}\end{cases} \quad \text{as } |x|\to\infty.$$

Step 4. Uniqueness of Q

By the comparison principle, one can obtain the uniqueness of the quadratic polynomial $Q(x)$ immediately, since the difference between any two asymptotic quadratic polynomial solves a uniformly elliptic equation. □

Acknowledgments

We would like to thank Z.S. Li, K. Zhang, S.S. Ma and Y. Li for their helpful discussions on the supersolutions of these equations we concerned. We would also be grateful to the anonymous referee for the valuable comments and suggestions.

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