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Group Theory / *Théorie des groupes*

Finite groups with Quaternion Sylow subgroup

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Abstract. In this paper we show that a finite group G with Quaternion Sylow 2-subgroup is 2-nilpotent if, either $3 \nmid |G|$ or G is solvable and the order of its Sylow 2-subgroup is strictly greater than 16.

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1. Introduction

A 2-group of order 2^n, with a cyclic maximal subgroup is isomorphic to one of the following group:

(i) Cyclic group $\mathbb{Z}_{2^n}$ and abelian group $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$.
(ii) Dihedral group D_{2^n}.
(iii) Semi-dihedral group SD_{2^n}.
(iv) Modular group $M(2^n)$.
(v) Quaternion group Q_{2^n}.

Let G be a group with a Sylow 2-subgroup S of type (i). Then G is 2-nilpotent, since $\mathrm{Aut}(S)$ is a 2-group and so $\mathcal{N}_G(S) = \mathscr{C}_G(S)$. In [5], D. Gorenstein and J. Walter gave the characterization of finite groups with dihedral Sylow 2-subgroups. They proved that, a finite group G with a dihedral Sylow 2-subgroup is 2-nilpotent if G contains a normal subgroup of index 4 (see [5, Lemma 2.1 (iii)]). In [8, 9], W. Wong obtained the structure of finite groups whose Sylow 2-subgroups are semi-dihedral or modular 2-group. In the latter case G has a normal 2-complement (see [8, Theorem 1]), so a modular 2-group can not be a Sylow 2-subgroup of a non-solvable group. In the semi-dihedral case, G is 2-nilpotent if $\mathrm{foc}(S) = S'$, where S denotes a Sylow 2-subgroup of G and $\mathrm{foc}(S)$, the focal subgroup of S [8, Theorem 2 (I)].

In [1], R. Brauer and M. Suzuki proved that any group with a Quaternion Sylow 2-subgroup is not simple.

Now this question seems to be natural: when a group G with a Quaternion Sylow 2-subgroup is 2-nilpotent?

In this paper we obtain sufficient conditions for a group G with a Quaternion (ordinary or generalized) Sylow 2-subgroup to be 2-nilpotent.

Our notations are standard and can be found in [6].

Main Theorem. *Let G be a finite group with a Quaternion (ordinary or generalized) Sylow 2-subgroup. Then G is 2-nilpotent if:*

(i) *either $3 \nmid |G|$;*
(ii) *or G is solvable with a Sylow 2-subgroup of order strictly greater than 16.*

Proof. Assume that $Q \cong Q_{2^n}$ is a Sylow 2-subgroup of G. If $Q \trianglelefteq G$ then $G = QN$, where N is a complement of Q. Now N acts trivially on Q, since $\mathrm{Aut}(Q_{2^n})$ is a 2-group unless for $n = 3$, in the latter case $\mathrm{Aut}(Q_8) \cong S_4$ and $3 \nmid |G|$, again N acts trivially on Q. Hence $N \trianglelefteq G$ and $G \cong Q \times N$. Therefore we can assume that $Q \ntrianglelefteq G$.

(i). By [1], G is not simple. Assume that G is a minimal counterexample and M is a maximal normal subgroup of G. If $|M|$ is odd, then G/M is a simple group with a Quaternion Sylow subgroup which is a contradiction, so $2 \mid |M|$. By assumption $M \neq Q$. Suppose that $M \leqslant Q$, as G/M is a non-abelian simple group and $3 \nmid |G/M|$, as the Suzuki groups are the only non-abelian simple groups which 3 does not divides its order [4], then G/M is a Suzuki group with Q/M as its Sylow 2-subgroup, which is a contradiction for Q/M is either cyclic or dihedral but a Sylow 2-subgroup of a Suzuki group is of exponent 4 and order greater than or equal 64, (see [7, Lemma 1.6(4)] and [2, Lemma 1 & Proposition 3]).

Therefore either $Q \leqslant M$, then M has a normal 2-complement, for $|M| < |G|$, or $M \cap Q$ is a proper subgroup of Q which is cyclic or $Q_{2^{n-1}}$, again M has a normal 2-complement. Thus in either case M has a normal 2-complement M_1 which is normal in G. Now G/M_1 has a Quaternion Sylow 2-subgroup. By the choice of G, $G \neq QM_1$ so G/M_1 has a normal 2-complement N/M_1. Hence $G = NQ$ which is a contradiction. Therefore G is 2-nilpotent.

(ii). Proof by induction on $|G|$. Assume that G is solvable and $n \geq 5$. Obviously $Q \neq G'$. If $Q \leqslant G'$, as $Q \ntrianglelefteq G'$, then by induction G' is 2-nilpotent with a normal 2-complement M which is normal in G. Now G/M has a normal Sylow 2-subgroup so is 2-nilpotent with a normal 2-complement N/M. Obviously N is normal a 2-complement of G. So assume that $G' \cap Q$ is a proper subgroup of Q, then either $G' \cap Q$ is cyclic which again implies that G' is 2-nilpotent and we are done, or $G' \cap Q \cong Q_{2^{n-1}}$ and in this case G' is 2-nilpotent unless $n = 5$, $3 \mid |G'|$ and $G' \cap Q$ is non-cyclic and a non-normal subgroup of G'. In the latter case, $Q_1 = G' \cap Q \cong Q_{16}$ and $G^{(\ell)} \neq Q_1$, for all $\ell \geq 2$. By solvability for some ℓ, $G^{(\ell)} \cap Q_1$ is a proper subgroup of Q_1. We can assume that ℓ is the smallest number such that $G^{(\ell)} \cap Q_1 \neq Q_1$, hence $G^{(\ell)} \cap Q_1$ is cyclic, since $G^{(\ell)} \cap Q_1 \trianglelefteq Q$. If $G^{(\ell)} \leqslant Q_1$, then $G^{(\ell-1)}/G^{(\ell)}$ is abelian with a normal 2-complement $M/G^{(\ell)}$, thus $MG^{(\ell)} \trianglelefteq G^{(\ell-1)}$ and so $M \trianglelefteq G^{(\ell-1)}$, for $G^{(\ell-1)} = MQ_1$. This implies that $M \trianglelefteq G$ and so G/M is 2-nilpotent by induction. Therefore G is 2-nilpotent. Otherwise $G^{(\ell)}$ is 2-nilpotent with a normal 2-complement M which is normal in G, again G/M is 2-nilpotent by induction and we are done. □

Corollary 1. *Let G be a finite group with Quaternion (ordinary or generalized) Sylow 2-subgroup. If a Sylow 3-subgroup of G is normal, then G is 2-nilpotent.*

Proof. Assume that P is the Sylow 3-subgroup of G. As $3 \nmid |G/P|$, G/P is 2-nilpotent by the main Theorem. So G is 2-nilpotent. □

Corollary 2. *Let G be a finite group with Quaternion (ordinary or generalized) Sylow 2-subgroup such that $3 \nmid |G|$, then G is solvable.*

Remark 3. In the above lemma for $n = 3$ and 4 where $3 \mid |G|$, there exist many examples for which G is not 2-nilpotent. The smallest order of such groups is a group of order 48. Assume that $G :=$ SmallGroup(48, 28) the small group of GAP library [3], in this group $G' \cong \mathrm{SL}(2,3)$ and a Sylow 2-subgroup of G is non-normal, obviously $G/G' \cong \mathbb{Z}_2$. For another example we consider $G \cong \mathbb{Z}_5 \times \mathrm{SL}(2,3)$. In this case $G' \cong Q_8$ and $G \cong (\mathbb{Z}_7 \times \mathbb{Z}_7) \rtimes \mathrm{SL}(2,3)$, where Q_8 acts irreducibly on $\mathbb{Z}_7 \times \mathbb{Z}_7$, in this case $G' \cong (\mathbb{Z}_7 \times \mathbb{Z}_7) \rtimes Q_8$. For non-solvable case let $G :=$ SmallGroup(672, 1045), in this group $G' \cong \mathrm{SL}(2,7)$ and a Sylow 2-subgroup of G is a Quaternion group of order 32.

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