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Number Theory / *Théorie des nombres*

A complete monotonicity property of the multiple gamma function

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Abstract. We consider the following functions

$$f_n(x) = 1 - \ln x + \frac{\ln G_n(x+1)}{x} \text{ and } g_n(x) = \frac{\sqrt[n]{G_n(x+1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N},$$

where $G_n(z) = (\Gamma_n(z))^{(-1)^{n-1}}$ and Γ_n is the multiple gamma function of order n . In this work, our aim is to establish that $f_{2n}^{(2n)}(x)$ and $(\ln g_{2n}(x))^{(2n)}$ are strictly completely monotonic on the positive half line for any positive integer n . In particular, we show that $f_2(x)$ and $g_2(x)$ are strictly completely monotonic and strictly logarithmically completely monotonic respectively on $(0, 3]$. As application, we obtain new bounds for the Barnes G-function.

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1. Introduction

E. W. Barnes introduced multiple gamma functions Γ_n around 1899 in a series of papers [2, 3], which generalize the Euler's gamma function Γ . Γ_n appears in functional equations for the Selberg zeta functions associated to higher rank symmetric spaces. Γ_n are also useful to study the determinant of Laplacians on the n -dimensional unit sphere [5, 6]. For these reasons, the multiple gamma functions have recently attracted many researchers' concern. Γ_n plays vital role in mathematical physics, quantum physics, theoretical physics, number theory, approximation theory and in many branches of applied science and engineering. One of the most applicable research area of recent interest for the gamma functions is fractional calculus. It is important to note that Γ_n are of higher order transcendence (known as transcendently transcendent function) and cannot be obtained by solving algebraic equations and algebraic differential equations. O. Hölder proved this result for the Euler's gamma function in 1887. Γ_n is characterized by the generalized Bohr–Møllerup theorem, which is given below.

Theorem 1. [13] Γ_n ($n \in \mathbb{N}$) satisfies the following relations:

- (i) $\Gamma_n(z) = \frac{\Gamma_{n+1}(z)}{\Gamma_{n+1}(z+1)}$ for $z \in \mathbb{C}$,
- (ii) $1/\Gamma_n(x)$ is \mathbb{C}^∞ on \mathbb{R} ,
- (iii) $(-1)^n \frac{d^{n+1}}{dx^{n+1}} \log \Gamma_n(x) \geq 0$ for $x > 0$,
- (iv) $\Gamma_n(1) = 1$,
- (v) $\Gamma_1(z) = \Gamma(z)$ and $\Gamma_0(z) = 1/z$.

It is well-known that $(\Gamma_n(z))^{-1}$ is an entire function with zeros at $z = -k$, $k \in \mathbb{N} \cup \{0\}$ with multiplicities given by

$$\binom{n+k-1}{n-1} \quad n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}. \tag{1}$$

Using (1), the following explicit form of Γ_n , in terms of Weierstrass canonical product can be obtained [7]:

$$\Gamma_n(1+z) = \exp [Q_n(z)] \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right)^{-\binom{n+k-2}{n-1}} \exp \left[\binom{n+k-2}{n-1} \left(\sum_{j=1}^n \frac{(-1)^{j-1} z^j}{j k^j} \right) \right] \right), \tag{2}$$

where $Q_n(z)$ is a polynomial of degree n given by

$$\begin{aligned} Q_n(z) &:= (-1)^{n-1} \left[-zA_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left(f_{n-1}^{(k)}(1) \right) \right], \\ f_n(z) &:= -zA_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left(f_{n-1}^{(k)}(1) \right) + A_n(z), \\ A_n(z) &:= \sum_{k=1}^{\infty} (-1)^{n-1} \binom{n+k-2}{n-1} \left[-\log \left(1 + \frac{z}{k}\right) + \sum_{j=1}^n \frac{(-1)^{j-1} z^j}{j k^j} \right] \end{aligned}$$

and

$$p_n(z) = \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} z^k, \quad n \in \mathbb{N},$$

with B_k being the Bernoulli numbers.

By introducing a hierarchy of functions

$$G_n(z) = (\Gamma_n(z))^{(-1)^{n-1}}, \tag{3}$$

M. F. Vignéras [14] redefined multiple gamma functions which satisfy the conditions of generalized Bohr–Mollerup theorem [7, 12, 14].

Theorem 2. [7] For all $n \in \mathbb{N}$, there exists a unique meromorphic function $G_n(z)$ satisfying each of the following properties:

- (i) $G_n(z+1) = G_n(z)G_{n-1}(z)$ for $z \in \mathbb{C}$;
- (ii) $G_n(1) = 1$;
- (iii) For $x \geq 0$, $G_n(x+1)$ are infinitely differentiable and

$$\frac{d^{n+1}}{dx^{n+1}} \log G_n(x+1) \geq 0;$$

- (iv) $G_0(z) = z$.

The reciprocal of the double gamma function is the well-known Barnes G-function $G(z) = G_2(z) = \frac{1}{\Gamma_2(z)}$. Note that $G_1(z) = \Gamma_1(z) = \Gamma(z)$, $G_2(z) = 1/\Gamma_2(z) = G(z)$, $G_3(z) = \Gamma_3(z)$ and so on. The multiple psi function Ψ_n can be defined as the logarithmic derivative of Γ_n . i.e. $\Psi_n = \frac{\Gamma'_n}{\Gamma_n}$. The m^{th} order derivative of Ψ_n (where, $m, n \in \mathbb{N}$) is known as the poly multiple gamma function and denoted by $\Psi_n^{(m)}$. Similarly, one can define multiple psi function $\psi_n = \frac{G'_n}{G_n}$ involving G_n and poly multiple gamma function $\Phi_n^{(m)}$ (involving G_n) as the m^{th} order derivative of Φ_n , where, $m, n \in \mathbb{N}$. Using (3), it can be easily verify that Φ_n and Ψ_n satisfy the following relations:

$$\Phi_n(z) = (-1)^{n-1} \Psi_n(z) \quad \text{and} \quad \Phi_n^{(m)}(z) = (-1)^{n-1} \Psi_n^{(m)}(z), \quad n, m \in \mathbb{N}. \tag{4}$$

For further information on multiple gamma functions we refer to [1–3, 7–10] and references cited therein.

If a positive function $g(x)$ has derivatives of all orders on an interval I and satisfy the following nonnegativity condition

$$(-1)^k g^{(k)}(x) \geq 0 \quad \text{for all } x \in I \quad \text{and} \quad k \geq 0, \tag{5}$$

then the function $g(x)$ is called completely monotonic on I . If the inequality (5) is strict for any $x \in I$, then $g(x)$ is known as strictly completely monotonic on I . A function $g(x)$ is called logarithmically completely monotonic if

$$(-1)^k (\ln g(x))^{(k)} \geq 0 \quad \text{for all } x \in I \quad \text{and} \quad k \geq 1. \tag{6}$$

If the inequality (6) is strict, then $g(x)$ is called strictly logarithmically completely monotonic on I .

In [11], Qi and Chen proved that the function

$$f(x) = 1 - \ln x + \frac{\ln \Gamma(x+1)}{x}$$

is strictly completely monotonic on $(0, \infty)$. In addition, they [11] established that the function $g(x) = \sqrt[x]{\Gamma(x+1)}/x$ is strictly logarithmically completely monotonic on the positive half line. They [11] also proved the following result.

Theorem 3. [11] *A (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.*

The above results motivate us to discuss the complete monotonicity properties of the following functions:

$$f_n(x) = 1 - \ln x + \frac{\ln G_n(x+1)}{x} \quad \text{and} \quad g_n(x) = \frac{\sqrt[x]{G_n(x+1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N}.$$

2. Main Results

Let $m \geq n$ be any natural number. Then taking logarithm on both sides of (2) and differentiating $m + 1$ times, we have (see also [8])

$$\Psi_n^{(m)}(x) = (-1)^{m-1} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{m!}{(x+k)^{m+1}}, \quad x > 0, \tag{7}$$

where $(\alpha)_n$ is the Pochhammer symbol defined as $(\alpha)_0 = 1, (\alpha)_n = \prod_{k=1}^n (\alpha + k - 1), n \in \mathbb{N}$. Combining (4) and (7), we obtain

$$\Phi_n^{(m)}(x) = (-1)^{m+n} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{m!}{(x+k)^{m+1}}, \quad m \geq n \geq 1, \quad x > 0. \tag{8}$$

This leads to the following result.

Theorem 4. Let n and p be natural numbers. Then $\psi_n^{(n)}(x)$ and $(-1)^p \psi_n^{(n+p)}(x)$ are positive for any positive real number x .

Now, we proceed to prove the main result.

Theorem 5. Let $f_n(x)$ and $g_n(x)$ be defined as

$$f_n(x) = 1 - \ln x + \frac{1}{x} \ln G_n(x + 1), \tag{9}$$

$$g_n(x) = \frac{\sqrt[n]{G_n(x + 1)}}{x}, \quad x \in (0, \infty), \quad n \in \mathbb{N}. \tag{10}$$

Then $f_{2n}^{(2n)}(x)$ and $(\ln g_{2n})^{(2n)}(x)$ are strictly completely monotonic on $(0, \infty)$.

Proof. Let n and p be any positive integer. Then with the help of Leibnitz's rule,

$$(u(x)v(x))^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned} f_n^{(n+p)}(x) &= \sum_{k=0}^{n+p} \binom{n+p}{k} \left(\frac{1}{x}\right)^{(n+p-k)} (\ln G_n(x + 1))^{(k)} - \frac{(-1)^{n+p-1}(n+p-1)!}{x^{n+p}} \\ &= \left(\frac{1}{x}\right)^{(n+p)} \ln G_n(x + 1) + \sum_{k=1}^{n+p} \binom{n+p}{k} \left(\frac{1}{x}\right)^{(n+p-k)} (\Phi_n(x + 1))^{(k-1)} \\ &\quad + \frac{(-1)^n(n+p)!}{(n+p)x^{n+p}} \\ &= \frac{(-1)^{n+p}(n+p)!}{x^{n+p+1}} \ln G_n(x + 1) + \sum_{k=1}^{n+p} \frac{(n+p)!}{k!} \frac{(-1)^{n+p-k}}{x^{n+p-k+1}} (\Phi_n(x + 1))^{(k-1)} \\ &\quad + \frac{(-1)^{n+p}(n+p)!}{(n+p)x^{n+p}} \\ &\triangleq (-1)^{n+p} \frac{(n+p)!}{x^{n+p+1}} g_{n,p}(x), \end{aligned} \tag{11}$$

where

$$g'_{n,p}(x) = \frac{(-1)^{n+p}}{(n+p)!} x^{n+p} \Phi_n^{(n+p)}(x + 1) + \frac{1}{n+p}. \tag{12}$$

Using Theorem 4, we have $g'_{n,p}(x) > 0$, which implies that $g_{2n,p}(x)$ is strictly increasing on $(0, \infty)$. Hence, $g_{2n,p}(x) > g_{2n,p}(0) = 0$. Consequently, $(-1)^p f_{2n}^{(2n+p)}(x) > 0$ for $p \geq 0$. This proves that $f_{2n}^{(2n)}$ is strictly completely monotonic.

To prove the remaining part of the Theorem 5, let us consider the following function:

$$h_n(x) = \sqrt[n]{G_n(x + 1)}/x.$$

Then

$$\ln h_n(x) = f_n(x) - 1 \implies (\ln h_{2n}(x))^{(2n)} = f_{2n}^{(2n)}(x),$$

which proves the Theorem 5. □

Remark 6. We note the following points.

- (i) Using the series expansion [7] of $\ln G(x + 1)$, one can verify that $f_2(x) > 0$ for any $x \in (0, \infty)$ and

$$f'_2(x) = \begin{cases} < 0, & \text{if } x \in (0, 3] \\ > 0, & \text{if } x \in [3.1, \infty). \end{cases} \tag{13}$$

Figure 1 verifies our claim. Therefore, $f_2(x)$ is strictly completely monotonic on $(0, 3]$ and consequently using Theorem 3, we obtain that $g_2(x)$ is strictly logarithmically completely monotonic on $(0, 3]$.

(ii) From Theorem 5, we have $f_2^{(2)}(x) > 0$. Again, we have $f_2(x) > 0$ for all $x > 0$. Hence for any positive real x , the Barnes G-function satisfies the following inequality:

$$\left(\frac{x}{e}\right)^x < G(x+1) < \exp\left[\frac{x}{2}(2\Phi_2(x+1) - 1) - \frac{x^2}{2}\Phi_2^{(1)}(x+1)\right]. \tag{14}$$

(iii) In [4], Batir established various bounds for $G(x+1)$. One of such bounds is given below:

$$(\Gamma(x))^{x/2} x^x (2\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)} < G(x+1) < \left(\frac{\Gamma(x)}{\Gamma(x/2)}\right)^x (8\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)}. \tag{15}$$

Comparing the lower bounds in (14) and (15) using Mathematica software, we can verify that lower bound in (14) is sharper than the lower bound in (15) in the interval [2.4, 4.7]. Figure 2 verifies our claim.

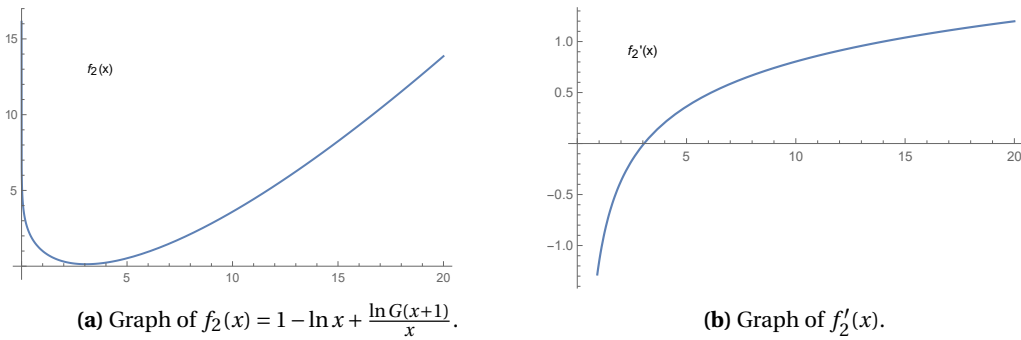


Figure 1. Graph of $f_2(x)$ and $f_2'(x)$.

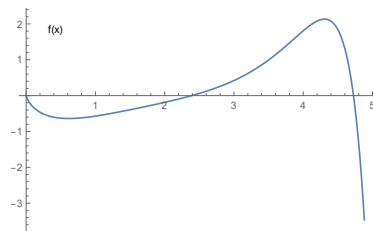


Figure 2. Graph of $f(x) = \left(\frac{x}{e}\right)^x - (\Gamma(x))^{x/2} x^x (2\pi)^{x/2} e^{\left(-\frac{x}{2} - \frac{x^2}{2}\right)}$.

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