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A two-sided Faulhaber-like formula involving Bernoulli polynomials

Une formule bilatérale de type Faulhaber utilisant les polynômes de Bernoulli

J. Fernando Barbero G. [a, b], **Juan Margalef-Bentabol** [b, c, d]
and Eduardo J.S. Villaseñor [b, e]

[a] Instituto de Estructura de la Materia, CSIC. Serrano 123, 28006 Madrid, Spain

[b] Grupo de Teorías de Campos y Física Estadística. Instituto Gregorio Millán (UC3M). Unidad Asociada al Instituto de Estructura de la Materia, CSIC, Madrid, Spain

[c] Laboratory of Geometry and Dynamical Systems, Department of Mathematics, EPSEB, Universitat Politècnica de Catalunya, BGSMath, Barcelona, Spain

[d] Institute for Gravitation and the Cosmos & Physics Department, Penn State, University Park, PA 16802, USA

[e] Departamento de Matemáticas, Universidad Carlos III de Madrid. Avda. de la Universidad 30, 28911 Leganés, Spain.

Abstract. We give a new identity involving Bernoulli polynomials and combinatorial numbers. This provides, in particular, a Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.

Résumé. Nous donnons une nouvelle identité utilisant les polynômes de Bernoulli et les coefficient binomiaux. Ceci fournit, en particulier, une formule de type Faulhaber pour des sommes de la forme $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ où m et n sont des entiers positifs.

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1. Introduction

Bernoulli numbers B_k are given by the exponential generating function $z/(e^z - 1)$,

$$B_k = k![z^k]\frac{z}{e^z - 1},$$

where $[z^n]f(z)$ is the n-th coefficient of the Taylor expansion of f around $z = 0$.

In the course of studying the distribution of the eigenvalues of the so-called *area operator* in loop quantum gravity [1] we were led to believe that the following identity held

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{2m-k+1}}{2m-k+1}2^k=\frac{(-1)^m}{2}\left(1-2^{2m+1}\frac{\Gamma(1+m)^2}{\Gamma(2m+2)}\right), \tag{1}$$

for $m\in\mathbb{N}\cup\{0\}$. The purpose of this short note is to prove this formula by proving a generalization of it. Particular cases of this general formula involve what we called a two-sided Faulhaber-like formula. A Faulhaber formula (also called Bernoulli's formula as Jacob Bernoulli was the first to write it) is given by

$$\sum_{k=1}^{n}k^p=\frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}.$$

Notice that in

$$\sum_{k=1}^{n-1}k^p=1^p+2^p+\cdots+(n-2)^p+(n-1)^p$$

there is an increasing sequence of addends given by powers of the integers. A particular and interesting case of the aforementioned generalized formula will involve instead a "two-sided" version of it:

$$\sum_{k=1}^{n-1}k^p(n-k)^p=1^p(n-1)^p+2^p(n-2)^p+\cdots+(n-2)^p2^p+(n-1)^p1^p.$$

Likewise, the Bernoulli numbers are generalized by considering the Bernoulli polynomials:

$$B_k(x)=k![z^k]\frac{ze^{xz}}{e^z-1}.$$

2. Main theorem

The main result of the paper is the following

Theorem 1. *Given $N\in\mathbb{Z}$, $m\in\mathbb{N}$ and $w\in\mathbb{C}$, we have*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}&\frac{B_{m+k+1}\left(\frac{N-w}{2}\right)}{m+k+1}w^{m-k}\\ &=\frac{(-1)^{m+1}}{2^{2m+1}}\left[\frac{(2w)^{2m+1}}{2(2m+1)\binom{2m}{m}}-\operatorname{sign}(N-1)\sum_{k=1}^{|N-1|}\Big(w^2-(|N-1|-2k+1)^2\Big)^m\right].\end{aligned}\tag{2}$$

Before proceeding with the proof let us discuss some consequences of this formula

Remark 2. It is possible to get a number of Faulhaber-like formulas from (2). The simplest one can be obtained by taking both w and N to be equal to a natural number $n\geq 2$.

$$\begin{aligned}\sum_{k=1}^{n-1}k^m(n-k)^m&=\frac{n^{2m+1}}{(2m+1)\binom{2m}{m}}+2(-1)^m\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+1}}{m+k+1}n^{m-k}\\ &=\frac{n^{2m+1}}{(2m+1)\binom{2m}{m}}-2(-1)^m\sum_{k=0}^{m}\binom{m}{k}\zeta(-m-k)n^{m-k}.\end{aligned}\tag{3}$$

where we have used the well known relation between the zeta Riemann function and the Bernoulli numbers

$$\zeta(1-N)=-\frac{B_N}{N}.$$

Equation (3) appears often in the literature obtained through different methods (see for instance [2, p. 10]).

Remark 3. For $N=1$, Equation (2) gives the beautiful expression (equivalent to equation (1.17) of [4]) valid for any $w\in\mathbb{C}$,

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+1}\left(\frac{1-w}{2}\right)}{m+k+1}w^{m-k}=\frac{(-1)^{m+1}}{2}\frac{w^{2m+1}}{(2m+1)\binom{2m}{m}}. \tag{4}$$

Remark 4. Sums involving

$$\frac{B_{\beta m+k+1}}{\beta m+k+1}=-\zeta(-k-\beta m)$$

with integer $\beta\geq 2$ can also be studied although a more complicated approach is needed involving complex analysis and combinatorial identities. Nonetheless, the results are not as neat as (2) and each case has to be studied separately.

Remark 5. It is also possible to generalize (2) for fractional values of N but, again, no systematic approach has been found. One such expression is when $w=N=1/2$

$$(-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m+k+1}}{m+k+1}2^k=\frac{1}{2^{m+2}(2m+1)\binom{2m}{m}}+\frac{1}{2^{3m+2}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}E_{2k}$$

where the E_n are the Euler numbers [3, entry A122045].

Proof of Theorem 1. The result is a consequence, on one hand, of the following easy-to-prove formula for the Bernoulli polynomials

$$B_n(x+r)-B_n(x)=n\,\mathrm{sign}(r)\left(\sum_{k=1}^{|r|-1}(x+k\,\mathrm{sign}(r)-1)^{n-1}\right.$$
$$\left.+\frac{1+\mathrm{sign}(r)}{2}(x+r-1)^{n-1}+\frac{1-\mathrm{sign}(r)}{2}(x-1)^{n-1}\right), \tag{5}$$

valid for $r\in\mathbb{Z}$ and $x\in\mathbb{C}$, which is a direct consequence of

$$B_n(x+1)-B_n(x)=nx^{n-1},$$

and, on the other hand, of the remarkable identity obtained by Sun (equation (1.14) of [4])

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{B_{\ell+j+1}(y)}{\ell+j+1}+(-1)^\ell\sum_{j=0}^{\ell}\binom{\ell}{j}x^{\ell-j}\frac{B_{k+j+1}(z)}{k+j+1}=\frac{(-x)^{k+\ell+1}}{(k+\ell+1)\binom{k+\ell}{k}} \tag{6}$$

where $k,\ell\in\mathbb{N}$ and $x+y+z=1$.

Taking now $x=w$, $y=(N-w)/2$, $z=1-(N+w)/2$ and $k=\ell=m\in\mathbb{N}$ in (6) we obtain

$$(-1)^m\sum_{j=0}^{m}\binom{m}{j}w^{m-j}\frac{B_{m+k+1}\left(\frac{N-w}{2}\right)}{m+j+1}=\frac{(-w)^{2m+1}}{(2m+1)\binom{2m}{m}}+(-1)^{m+1}\sum_{j=0}^{m}\binom{m}{j}w^{m-j}\frac{B_{m+j+1}\left(1-\frac{N+w}{2}\right)}{m+j+1}.$$

Using now (5) to rewrite the last term in terms of $B_{m+j+1}\left(\frac{N-w}{2}\right)$, we finally obtain (2). □

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