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LENGTH DERIVATIVE OF THE GENERATING FUNCTION OF WALKS CONFINED IN THE QUARTER PLANE

THOMAS DREYFUS AND CHARLOTTE HARDOUIN

Abstract. In the present paper, we use difference Galois theory to study the nature of the generating function counting walks with small steps in the quarter plane. These series are trivariate formal power series $Q(x,y,t)$ that count the number of walks confined in the first quadrant of the plane with a fixed set of admissible steps, called the model of the walk. While the variables x and y are associated to the ending point of the path, the variable t encodes its length. In this paper, we prove that in the unweighted case, $Q(x,y,t)$ satisfies an algebraic differential relation with respect to t if and only if it satisfies an algebraic differential relation with respect x (resp. y). Combined with [2, 3, 4, 9, 11], we are able to characterize the t-differential transcendence of the 79 models of walks listed by Bousquet-Mélou and Mishna.

Contents

Introduction

Classifying lattice walks in restricted domains is an important problem in enumerative combinatorics. Recently much progress has been made in the study of walks with small steps in the quarter plane. A small step model in the quarter plane $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is composed by a set of admissible cardinal directions $\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$. Given $\mathcal{D}$, we consider the walks that start at $(0,0)$, with directions in $\mathcal{D}$, and that stay in the quarter plane, see for instance Figure 1.

For a given model, one defines $q_{i,j,k}$ to be the number of walks confined to the first quadrant of the plane that begin at $(0,0)$ and end at (i,j) in k admissible steps. The algebraic nature of the associated complete generating function $Q(x,y,t) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j t^k$ captures many important combinatorial properties

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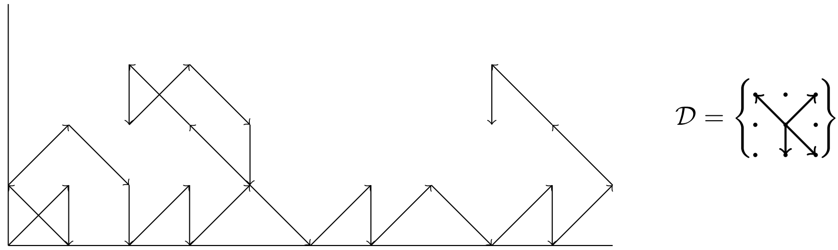


FIGURE 1. An example of a lattice walk

of the model: symmetries, asymptotic information, and recursive relations of the coefficients.

Among the $2^8 - 1 = 255$ models in the first quadrant of the plane, Bousquet-Mélou and Mishna proved in [4] that, after accounting for symmetries and eliminating the trivial cases, walks in the half plane, and one dimensional cases, only 79 models remained. It is worth mentioning that the generating function is algebraic in all the trivial cases, the half plane cases, and the one dimensional cases.

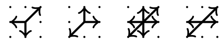
For any choice of a variable $\star$ among x, y, t, one classifies the algebraic nature of the generating series $Q(x, y, t)$ with respect to $\star$ as follows:

- *Algebraic cases:* The series $Q(x, y, t)$ satisfies a nontrivial polynomial relation with coefficients in $\mathbb{Q}(x, y, t)$.
- *Transcendental $\star$-holonomic cases:* The series $Q(x, y, t)$ is transcendental and holonomic with respect to $\star$, i.e. there exists $n \in \mathbb{Z}_{\geqslant 0}$, such that there exist $a_0, \dots, a_n \in \mathbb{Q}(x, y, t)$, not all zero, such that
$$0 = \sum_{\ell=0}^{n} a_\ell \frac{d}{d\star}^{\ell} Q(x, y, t).$$
- *Nonholonomic $\frac{d}{d\star}$-differentially algebraic cases:* The series $Q(x, y, t)$ is nonholonomic and $\frac{d}{d\star}$-differentially algebraic, i.e. there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that there exists a nonzero multivariate polynomial $P_\star \in \mathbb{Q}(x, y, t)[X_0, \dots, X_n]$ with
$$0 = P_\star(Q(x, y, t), \dots, \frac{d}{d\star}^{n} Q(x, y, t)).$$
We stress out the fact that in the above definition, it is equivalent to require that $P_\star \in \mathbb{Q}[X_0, \dots, X_n]$, see Remark C.7.
- *$\frac{d}{d\star}$-differentially transcendental cases:* The series is not $\frac{d}{d\star}$-differentially algebraic.

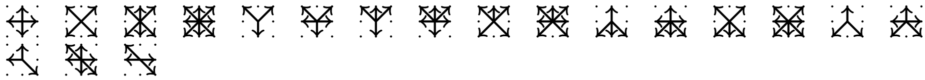
The authors of [2, 3, 4, 9, 11] proved that the algebraic nature of the generating series was identical for the variables x and y. The classification of the models of walks regarding the algebraic nature of their series with respect to the variables x and y is the culmination of ten years of research and the works of many researchers (see Figure 2 below).

Statement of the main result. In this paper, we address the question of the classification with respect to the variable t and we prove that this classification coincides with the classification with respect to x and y. There is a priori no relation between the $\frac{d}{dx}$ and $\frac{d}{dt}$ differential algebraic properties of a function in

Algebraic cases [4]



Transcendental $\frac{d}{dx}$-holonomic cases [3, 4, 15]



Nonholonomic $\frac{d}{dx}$-differentially algebraic cases ([2, 23, 26])

$\frac{d}{dx}$-differentially transcendental cases [9, 11]

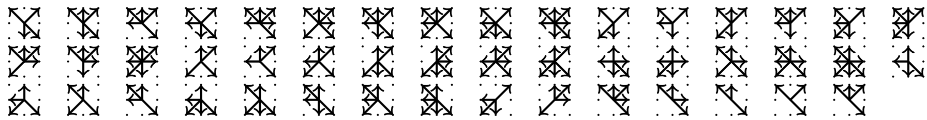


FIGURE 2. Classification of the 79 models with respect to the x and y-variables.

these two variables. For instance, the function $t\Gamma(x)$ is holonomic with respect to t but differentially transcendental with respect to x, thanks to Hölder's result. In that case, the fibration induced by t is "isotrivial". The main difficulty in our case is to show that such a situation does not happen and that the x- and t-algebraic behavior are intrinsically connected.

Our main result is as follows:

THEOREM 1 (Theorem 3.1 and Corollary 4.16 below). — *For any of the* 79 *models of Figure 2, the complete generating function is* $\frac{d}{dt}$*-differentially algebraic over* $\mathbb{Q}$ *if and only if it is* $\frac{d}{dx}$*-differentially algebraic over* $\mathbb{Q}$.

Theorem 1 is the corollary of the following proposition proved in the more general setting of weighted walks that are walks whose directions are weighted (see § 2). To any such a walk, one attaches an algebraic curve of genus zero or one called the *kernel curve* and a group of automorphisms of that curve called *the group of the walk* (see § 2). The following holds.

THEOREM 2 (Theorems 3.1 and 4.14 below). — *For a genus zero kernel curve attached to the models* (G0)*, the generating series is* $\frac{d}{dt}$*-differentially transcendental over* $\mathbb{Q}$*. For a genus one kernel curve with infinite group of the walk, if the generating series is* $\frac{d}{dt}$*-differentially algebraic over* $\mathbb{Q}$*, then it is* $\frac{d}{dx}$*-differentially algebraic over* $\mathbb{Q}$.

In [11], the authors proved that for a genus zero kernel curve attached to the models (G0), the generating series was $\frac{d}{dx}$-differentially transcendental over $\mathbb{Q}$. The authors of [2] proved that the nine nonholomic $\frac{d}{dx}$-differentially algebraic models of Figure 2 were also $\frac{d}{dt}$-differentially algebraic over $\mathbb{Q}$ by giving an explicit description of the series in terms of analytic invariants. In § 4.4, we will discuss how the construction of [2] and the results of [18] have been used in [8] to show that the second statement of Theorem 2 is in fact an equivalence.

Strategy of the proof. The classification results of Figure 2 come from many approaches: probabilistic methods, combinatorial classification, computer algebra and “Guess and Prove”, analysis and boundary value problems, and more recently difference Galois theory and algebraic geometry. The analytic approach consists in studying the asymptotic growth of the coefficients of the generating function, or else showing that it has an infinite number of singularities, in order to prove its nonholonomicity. This approach also allows for the study of some important specializations of the complete generating function as for instance $Q(1,1,t)$ the generating function for the number of nearest-neighbor walks in the first quadrant with steps from $\{\nwarrow, \uparrow, \nearrow, \rightarrow, \searrow\}$ (see [26, 27]). Though very powerful, these analytic techniques are unable to detect the differentially algebraic generating functions among the nonholonomic ones. For instance, the generating function $\prod_{k=1}^{\infty} \frac{1}{(1-x^k)}$ counting the number of partitions has an infinite number of singularities, and yet is $\frac{d}{dx}$-differentially algebraic.

In order to detect these more subtle kinds of functional dependencies it is necessary to use new arguments that focus on the functional equation satisfied by the complete generating function. Indeed, the combinatorial decomposition of a walk into a shorter walk followed by an admissible step translates into a functional equation for the generating function. Following the ideas of Fayolle, Iasnogorodski and Malyshev [14], the authors of [23] and [11] specialized this functional equation to the so-called *kernel curve* to find a linear discrete equation: a linear $\mathbf{q}$-difference equation in the genus zero case and a shift difference equation in genus one. Difference Galois theory allowed then to characterize the differentially transcendental complete generating function ([9, 11]) whereas the clever use of Tutte invariants produces explicit differential algebraic relations for the 9 nonholomic but differentially algebraic cases ([2]). Unfortunately, all the above methods for proving the differential transcendence are only valid for a fixed value of the parameter t in the field of complex numbers. This allowed the authors to consider the kernel curve as a complex algebraic curve but prevented them to study the variations of the parameter t.

Our work relies on a nonarchimedean uniformization of the kernel curve, which we consider as an algebraic curve over $\mathbb{Q}(t)$. We use here the formalism of Tate curves over $\mathbb{Q}(t)$ as in [30] to show that for both situations, genus one and zero, the differential algebraic properties of the complete generating functions are encoded by the differential algebraic properties of a solution of a rank one nonhomogeneous linear $\mathbf{q}$-difference equation which unifies the genus zero and the genus one cases. Then, we generalize some Galoisian criterias for $\mathbf{q}$-difference equations of [17] to prove Theorem 3.1 and Theorem 4.14 below.

Organization of the paper. The paper is organized as follows. In Section 1, we introduce the weighted walks in the quarter plane and their generating series. In Section 2 we present some reminders and notations for walks in the quarter plane. In Section 3 we consider walks with genus zero kernel curve while Section 4 deals with the genus one case. Since this paper combines many different fields, nonarchimedian uniformization, combinatorics, and Galois theory, we choose to postpone many technical intermediate results to the appendices. This should allow the reader to understand the articulation of our proofs in Sections 3 and 4 in three steps without being lost in too many technicalities. These three steps are the uniformization of the kernel and the construction of a linear **q**-difference equation, the Galoisian criteria, and finally, the resolution of telescoping problems. Appendix A is devoted to the nonarchimedean estimates that we used in the uniformization procedure. Appendix B contains some reminders on special functions on Tate curves and their normal forms. Appendix C proves the Galoisian criteria mentioned above. Finally, Appendix D studies the transcendence properties of special functions on Tate curves which will be used for the descent of our telescoping equations.

1. The walks in the quadrant

Let us introduce the generating function $Q(x,y,t)$ of a walk confined in the quarter plane.

The cardinal directions of the plane $\{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$ are identified with pairs of integers $(i,j) \in \{0,\pm 1\}^2 \backslash \{(0,0)\}$. A walk $\mathcal{W}$ in the quarter plane $\mathbb{Z}^2_{\geqslant 0}$ is a sequence of points $(M_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ such that

- it starts at $(0,0)$, that is, $M_0 = (0,0)$;
- for all $n \in \mathbb{Z}_{\geqslant 0}$, the point M_n belongs to the quadrant $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$;
- for all $n \in \mathbb{Z}_{\geqslant 0}$, the vector $M_{n+1} - M_n$ belongs to a given subset $\mathcal{D}$ of the set of cardinal directions.

Fixing a family of elements $(d_{i,j})_{(i,j)\in\{0,\pm 1\}^2}$ of $\mathbb{Q} \cap [0,1]$ such that $\sum_{i,j} d_{i,j} = 1$, one can choose to weight the model of the walk in order to add a probabilistic flavor to our study. For $(i,j) \in \{0,\pm 1\}^2 \backslash \{(0,0)\}$ (resp. $(0,0)$), the element $d_{i,j}$ can be viewed as the probability for the walk to go in the direction (i,j) (resp. to stay at the same position). In that case, the $d_{i,j}$ are called the weights and the model is called a weighted model.

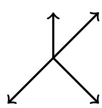
Remark 1.1. — For simplicity, we assume that the weights $d_{i,j}$ belong to $\mathbb{Q}$. However, we would like to mention that any of the arguments and statements below will hold with arbitrary real weights in $[0,1]$. One just needs to replace the field $\mathbb{Q}$ with the field $\mathbb{Q}(d_{i,j})$.

The set of steps $\mathcal{D}$ of the walk is the set of cardinal directions with nonzero weight, that is,

$$\mathcal{D} = \{(i,j) \in \{0,\pm 1\}^2 | d_{i,j} \neq 0\}.$$

A model is *unweighted* if $d_{0,0} = 0$ and if the nonzero $d_{i,j}$'s all have the same value.

Remark 1.2. — In what follows we will represent model of walks with arrows. For instance, the family of models represented by

 or $\left\{\nearrow, \searrow, \uparrow, \swarrow\right\}$,

correspond to models with $d_{1,1}, d_{1,-1}, d_{0,1}, d_{-1,-1} \neq 0$, $d_{1,0} = d_{0,-1} = d_{-1,1} = d_{-1,0} = 0$, and where nothing is assumed on the value of $d_{0,0}$. In the following results, the behavior of the kernel curve never depends on $d_{0,0}$. This is the reason why, to reduce the amount of notations, we have decided not to mention $d_{0,0}$ in the graphical representation of the model.

The *weight of the walk* is defined to be the product of the weights of its component steps. For any $(i,j) \in \mathbb{Z}^2_{\geqslant 0}$ and any $k \in \mathbb{Z}_{\geqslant 0}$, we let $q_{i,j,k}$ be the sum of the weights of all walks reaching the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating function

$$Q(x,y,t) := \sum_{i,j,k \geqslant 0} q_{i,j,k} x^i y^j t^k.$$

Note that the generating function is not exactly the same as the one that we defined in the introduction. To recover the latter, we should take $d_{i,j} \in \{0,1\}$ and $d_{i,j} = 1$ if and only if the corresponding direction belongs to $\mathcal{D}$. Fortunately, the assumption $\sum_{i,j} d_{i,j} = 1$ can be relaxed by rescaling the t-variable, and the results of the present paper stay valid for the generating function of the introduction since both generating functions have the same nature.

The *kernel* polynomial of a weighted model $(d_{i,j})_{i,j \in \{0,\pm 1\}^2}$ is defined by

$$K(x,y,t) := xy(1 - tS(x,y)) \tag{1.1}$$

where

$$\begin{aligned} S(x,y) &= \textstyle\sum_{(i,j) \in \{0,\pm 1\}^2} d_{i,j} x^i y^j \\ &= A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y \\ &= B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x, \end{aligned} \tag{1.2}$$

and $A_i(x) \in x^{-1}\mathbb{Q}[x]$, $B_i(y) \in y^{-1}\mathbb{Q}[y]$.

By [11, Lemma 1.1], see also [4, Lemma 4], the generating function $Q(x,y,t)$ satisfies the following functional equation:

$$K(x,y,t)Q(x,y,t) = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t), \tag{1.3}$$

where

$$F^1(x,t) := K(x,0,t)Q(x,0,t), \quad \text{and} \quad F^2(y,t) := K(0,y,t)Q(0,y,t).$$

Remark 1.3. — We shall often use the following symmetry argument between x and y. Exchanging x and y in the kernel polynomial amounts to consider the kernel polynomial of a weighted model $\mathcal{D}' := \{(i,j) \text{ such that } (j,i) \in \mathcal{D}\}$ with weights $d'_{i,j} := d_{j,i}$.

We need to discard some degenerate cases. Following [14], we have the following definition.

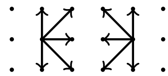
Definition 1.4. — A weighted model is called *degenerate* if one of the following holds:

- $K(x,y,t)$ is reducible as an element of the polynomial ring $\mathbb{C}[x,y]$,
- $K(x,y,t)$ has x-degree less than or equal to 1,
- $K(x,y,t)$ has y-degree less than or equal to 1.

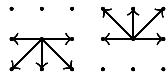
The following proposition gives very simple conditions on $\mathcal{D}$ to decide whether a weighted model is degenerate or not.

Proposition 1.5 (Lemma 2.3.2 in [14]). — *A weighted model is degenerate if and only if at least one of the following holds:*

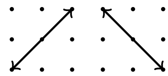
(1) *There exists* $i \in \{-1,1\}$ *such that* $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. *This corresponds to walks with steps supported in one of the following configurations*



(2) *There exists* $j \in \{-1,1\}$ *such that* $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. *This corresponds to walks with steps supported in one of the following configurations*



(3) *All the weights are zero, with the possible exception of* $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$ *or* $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. *This corresponds to walks with steps supported in one of the following configurations*



Note that we only discard trivial cases, walks in the half plane, and one dimensional problems as explained in [4]. For all the degenerate cases, the generating function $Q(x,y,t)$ is algebraic.

From now on, we shall always assume that the weighted model under consideration is nondegenerate.

2. Notations and preliminaries

The goal of this section is to introduce some basic properties of walks in the quarter plane. In § 2.1, we attach to any walk a *kernel curve*, which is an algebraic curve defined over $\mathbb{Q}[t]$. Associated to this curve, one introduces in § 2.2 a subgroup, called the *group of the walk*, of its group of automorphisms. This curve has been intensively studied as an algebraic curve over $\mathbb{C}$ by fixing a morphism from $\mathbb{Q}[t]$ to $\mathbb{C}$. For instance, [14] is concerned with $t = 1$ whereas the papers [9] and [12] focus respectively on $t \in \mathbb{C}$ transcendental over $\mathbb{Q}$ and $t \in]0,1[$. Unfortunately, specializing t even generically does not allow to study the t-dependencies of the generating function. In this paper, we do not work with a specialization of t. This forces us to move away from the archimedean framework of the field of complex numbers and to consider the kernel curve over a suitable valued field extension of $\mathbb{Q}(t)$ endowed with the valuation at 0.

2.1. **The kernel curve.**

The *kernel* polynomial may be seen as a bivariate polynomial in x, y with coefficients in $\mathbb{Q}(t)$. The latter is a valued field endowed with the valuation at zero. It is neither algebraically closed nor complete. In order to use the theory of Tate curves, one needs to consider a complete algebraically closed field extension of $\mathbb{Q}(t)$. The field of Puiseux series with coefficients in $\overline{\mathbb{Q}}$ is algebraically closed but not complete.

Therefore, we consider here the field C of Hahn series or Malcev-Neumann series with coefficients in $\overline{\mathbb{Q}}$, and monomials from $\mathbb{Q}$. We recall that a Hahn series f is a formal power series $\sum_{\gamma \in \mathbb{Q}} c_\gamma t^\gamma$ with coefficients c_γ in $\overline{\mathbb{Q}}$ and such that the subset $\{\gamma | c_\gamma \neq 0\}$ is a well ordered subset of $\mathbb{Q}$. The valuation $v_0(f)$ of the Hahn series $f = \sum_{\gamma \in \mathbb{Q}} c_\gamma t^\gamma \in C$ is the smallest element of the subset $\{\gamma | c_\gamma \neq 0\}$. The field C is algebraically closed by [25, Theorem 1] and spherically complete with respect to the valuation at zero and thereby complete (see [1, Corollaries 2.2.7 and 3.2.9]). Let us fix once for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. For any $f \in C$, we define the norm of f as $|f| = \alpha^{v_0(f)}$.

To any weighted model, we attach a curve E, called the *kernel curve*, that is defined as the zero set in $\mathbf{P}^1(C) \times \mathbf{P}^1(C)$ of the following homogeneous polynomial

$$\widetilde{K}(x_0, x_1, y_0, y_1, t) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j} = x_1^2 y_1^2 K\Big(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\Big).$$

Remark 2.1. — In [11], the authors specialize the variable t as a transcendental complex number. Then, they study the kernel curve as a complex algebraic curve in $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$. In this work, we shall use any algebraic geometric result of [11] by appealing to Lefschetz Principle: every true statement about an algebraic variety defined over $\mathbb{C}$ remains true when $\mathbb{C}$ is replaced by an algebraically closed field of characteristic zero.

Put $\widetilde{K}(x_0, x_1, y_0, y_1, t) = \sum_{i,j=0}^{2} A_{i,j} x_0^i x_1^{2-i} y_0^j y_1^{2-j}$, where $A_{i,j} = -t d_{i-1,j-1}$ if $(i,j) \neq (1,1)$ and $A_{1,1} = 1 - t d_{0,0}$. The partial discriminants of $\widetilde{K}(x_0, x_1, y_0, y_1, t)$ are defined as the discriminants of the second degree homogeneous polynomials $y \mapsto \widetilde{K}(x_0, x_1, y, 1, t)$ and $x \mapsto \widetilde{K}(x, 1, y_0, y_1, t)$, respectively, i.e.

$$\Delta_x(x_0, x_1) = \Big(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,1}\Big)^2 - 4\Big(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,0}\Big) \times \Big(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,2}\Big)$$
$$\Delta_y(y_0, y_1) = \Big(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{1,j}\Big)^2 - 4\Big(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{0,j}\Big) \times \Big(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{2,j}\Big).$$

Introduce

$$\mathfrak{D}(x) := \Delta_x(x, 1) = \sum_{j=0}^{4} \alpha_j x^j \quad \text{and} \quad \mathfrak{E}(y) := \Delta_y(y, 1) = \sum_{j=0}^{4} \beta_j y^j, \tag{2.1}$$

where

$$\begin{aligned}
\alpha_4 &= \big(d_{1,0}^2 - 4 d_{1,1} d_{1,-1}\big) t^2 \\
\alpha_3 &= 2t^2 d_{1,0} d_{0,0} - 2t d_{1,0} - 4t^2 (d_{0,1} d_{1,-1} + d_{1,1} d_{0,-1}) \\
\alpha_2 &= 1 + t^2 d_{0,0}^2 + 2t^2 d_{-1,0} d_{1,0} - 4t^2 (d_{-1,1} d_{1,-1} + d_{0,1} d_{0,-1} + d_{1,1} d_{-1,-1}) - 2t d_{0,0} \\
\alpha_1 &= 2t^2 d_{-1,0} d_{0,0} - 2t d_{-1,0} - 4t^2 (d_{-1,1} d_{0,-1} + d_{0,1} d_{-1,-1}) \\
\alpha_0 &= \big(d_{-1,0}^2 - 4 d_{-1,1} d_{-1,-1}\big) t^2 \\
\\
\beta_4 &= \big(d_{0,1}^2 - 4 d_{1,1} d_{-1,1}\big) t^2 \\
\beta_3 &= 2t^2 d_{0,1} d_{0,0} - 2t d_{0,1} - 4t^2 (d_{1,0} d_{-1,1} + d_{1,1} d_{-1,0}) \\
\beta_2 &= 1 + t^2 d_{0,0}^2 + 2t^2 d_{0,-1} d_{0,1} - 4t^2 (d_{1,-1} d_{-1,1} + d_{1,0} d_{-1,0} + d_{1,1} d_{-1,-1}) - 2t d_{0,0} \\
\beta_1 &= 2t^2 d_{0,-1} d_{0,0} - 2t d_{0,-1} - 4t^2 (d_{1,-1} d_{-1,0} + d_{1,0} d_{-1,-1}) \\
\beta_0 &= \big(d_{0,-1}^2 - 4 d_{1,-1} d_{-1,-1}\big) t^2.
\end{aligned} \tag{2.2}$$

The discriminants $\Delta_x(x_0, x_1), \Delta_y(y_0, y_1)$ are homogeneous polynomials of degree 4. Their Eisenstein invariants can be defined as follows:

Definition 2.2 (§2.3.5 in [13]). — For any homogeneous polynomial of the form

$$f(x_0, x_1) = a_0 x_1^4 + 4a_1 x_0 x_1^3 + 6a_2 x_0^2 x_1^2 + 4a_3 x_0^3 x_1 + a_4 x_0^4 \in C[x_0, x_1],$$

we define the Eisenstein invariants of $f(x_0, x_1)$ as

- $D(f) = a_0 a_4 + 3a_2^2 - 4a_1 a_3$
- $E(f) = a_0 a_3^2 + a_1^2 a_4 - a_0 a_2 a_4 - 2a_1 a_2 a_3 + a_2^3$
- $F(f) = 27E(f)^2 - D(f)^3$.

Since C is algebraically closed of characteristic zero, we can apply [13, §2.4] to the kernel curve. The following proposition characterizes the smoothness of the kernel curve in terms of the invariants $F(\Delta_x)$, $F(\Delta_y)$.

Proposition 2.3 (Proposition 2.4.3 in [13] and Proposition 2.1 in [10]). — *The following statements are equivalent:*

- *The kernel curve E is smooth, i.e. it has no singular point;*
- $F(\Delta_x) \neq 0$*;*
- $F(\Delta_y) \neq 0$.

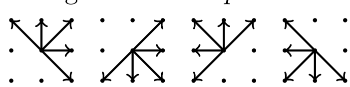
Furthermore, if E is smooth then it is an elliptic curve with J-invariant given by the element $J(E) \in C$ such that

$$J(E) = \frac{D(\Delta_y)^3}{-F(\Delta_y)}.$$

Otherwise, if E is nondegenerate and singular, E has a unique singular point and is a genus zero curve.

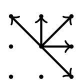
We define the genus of a weighted model as the genus of the associated kernel curve E. We recall the results obtained in [14, Theorem 6.1.1] and [10, Corollary 2.6], that classify all the weighted models attached to a genus zero kernel.

Theorem 2.4. — *Any nondegenerate weighted model of genus zero has steps included in one of the following 4 sets of steps:*



Otherwise, for any other nondegenerate weighted model, the kernel curve E is an elliptic curve.

Remark 2.5. — The walks corresponding to the fourth configuration never enter the quarter-plane. As described in [4, §2.1], if we consider walks corresponding to the second and third configurations we are in the situation where one of the quarter plane constraints implies the other. In the last three configurations, the generating function is algebraic. So the only interesting nondegenerate genus zero weighted models have steps included in



Note that due to Proposition 1.5, the anti-diagonal steps have nonzero attached weights.

Moreover, by Theorem 2.4, combined with Proposition 1.5, the nondegenerate weighted models of genus one are the walks where there are no three consecutive cardinal directions with weight zero. Or equivalently, this corresponds to the situation where the set of steps is not included in any half plane.

Thanks to Theorem 2.4, one can reduce our study to two cases depending on the genus of the kernel curve attached to a nondegenerate weighted model. The following lemma proves that when the kernel curve is of genus one, its J-invariant has modulus strictly greater than 1. This property allows us to use the theory of Tate curves in order to analytically uniformize the kernel curve.

Lemma 2.6. — *When E is smooth, the invariant $J(E)$ belongs to $\mathbb{Q}(t)$ and is such that $|J(E)| > 1$, where $|\ |$ denotes the norm of $(C, |\ |)$.*

Proof. — At $t = 0$, $\Delta_y(y_0, y_1)$ reduces to $y_0^2 y_1^2$. This proves that the reduction of $D(\Delta_y)$ (resp. $E(\Delta_y)$) at $t = 0$ is $\frac{1}{12}$ (resp. $\frac{1}{6^3}$). One concludes that $F(\Delta_y)$ vanishes for $t = 0$. By Proposition 2.3, $J(E) \in \mathbb{Q}(t)$ has a strictly negative valuation at $t = 0$. Thus, $|J(E)| > 1$. □

2.2. **The automorphism of the walk.** Following [4, §3] or [20, §3], we introduce the involutive birational transformations of $\mathbf{P}^1(C) \times \mathbf{P}^1(C)$ given by

$$i_1(x, y) = \Big(x, \frac{A_{-1}(x)}{A_1(x)y}\Big) \text{ and } i_2(x, y) = \Big(\frac{B_{-1}(y)}{B_1(y)x}, y\Big),$$

(see § 1 for the significance of the A_i, B_i's).

They induce two involutive automorphisms $\iota_1, \iota_2 : E \dashrightarrow E$ given by

$$\begin{aligned} \iota_1([x_0 : x_1], [y_0 : y_1]) &= \left([x_0 : x_1], \left[\frac{A_{-1}(\frac{x_0}{x_1})}{A_1(\frac{x_0}{x_1})\frac{y_0}{y_1}} : 1\right]\right), \\ \text{and} \quad \iota_2([x_0 : x_1], [y_0 : y_1]) &= \left(\left[\frac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}} : 1\right], [y_0 : y_1]\right). \end{aligned}$$

Note that ι_1 and ι_2 are nothing but the vertical and horizontal switches of E, see Figure 3. That is, for any $P = (x, y) \in E$, we have

$$\{P, \iota_1(P)\} = E \cap (\{x\} \times \mathbf{P}^1(C)) \text{ and } \{P, \iota_2(P)\} = E \cap (\mathbf{P}^1(C) \times \{y\}).$$

The automorphism of the walk σ is defined by

$$\sigma = \iota_2 \circ \iota_1.$$

The following holds.

Lemma 2.7 (Lemma 3.3 in [10]). — *Let $P \in E$. The following statements are equivalent:*

- *P is fixed by σ;*
- *P is fixed by ι_1 and ι_2;*
- *P is the only singular point of E, and E is of genus zero.*

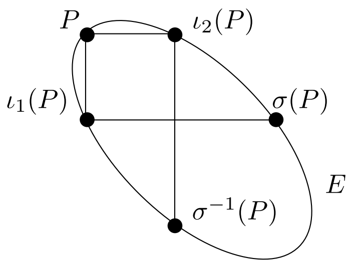
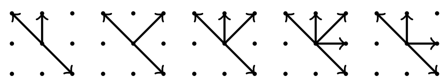


FIGURE 3. The maps ι_1, ι_2 restricted to the kernel curve E

3. Generating functions for walks, genus zero case

In this section, we fix a nondegenerate weighted model of genus zero. Following Remark 2.5, after eliminating duplications of trivial cases and the interchange of x and y, we should focus on walks $\mathcal{W}$ arising from the following 5 sets of steps:


(G0)

A function $f(x,y,t) \in \mathbb{Q}[[x,y,t]]$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$ if there exists a nonzero polynomial P with coefficients in $\mathbb{Q}$ such that

$$P(f(x,y,t), \frac{d}{dx} f(x,y,t), \frac{d}{dt} f(x,y,t), \dots) = 0.$$

The function $f(x,y,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially transcendental over $\mathbb{Q}$ otherwise. Note that if f is $\frac{d}{dt}$-differentially algebraic over $\mathbb{Q}$ then it is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$. We define similarly the notion of $(\frac{d}{dy}, \frac{d}{dt})$-differential algebraicity.

In this section, we prove the following theorem:

THEOREM 3.1. — *For any weighted model listed in* (G0)*, the generating function $Q(x,0,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially transcendental over $\mathbb{Q}$.*

For any weighted model listed in (G0)*, the generating function $Q(0,y,t)$ is $(\frac{d}{dy}, \frac{d}{dt})$-differentially transcendental over $\mathbb{Q}$.*

Theorem 3.1 implies the $\frac{d}{dt}$-differential transcendence of the complete generating function.

COROLLARY 3.2. — *For any weighted model listed in* (G0)*, the generating function $Q(x,y,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$ and $(\frac{d}{dy}, \frac{d}{dt})$-differentially transcendental over $\mathbb{Q}$. Therefore, $Q(x,y,t)$ is $\frac{d}{dt}$-differentially transcendental over $\mathbb{Q}$.*

Proof of Corollary 3.2. — Suppose to the contrary that $Q(x,y,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-algebraic over $\mathbb{Q}$. Let P be a nonzero polynomial with coefficients in $\mathbb{Q}$ such that $P(Q(x,y,t), \frac{d}{dx}Q(x,y,t), \frac{d}{dt}Q(x,y,t), \dots) = 0$. Specializing at $y=0$ this relation and noting that $\frac{d^i}{dx^i}\frac{d^j}{dt^j}(Q(x,0,t))$ is the specialization of $\frac{d^i}{dx^i}\frac{d^j}{dt^j}(Q(x,y,t))$, one finds a nontrivial differential algebraic relation for $Q(x,0,t)$ in the derivatives $\frac{d}{dx}$ and $\frac{d}{dt}$.

This contradicts Theorem 3.1. The proof for the $(\frac{d}{dy}, \frac{d}{dt})$-differential transcendence is similar. □

As detailed in the introduction, our proof of Theorem 3.1 has three major steps:

Step 1: we attach to the incomplete generating functions $Q(x,0,t)$ and $Q(0,y,t)$ some auxiliary functions which share the same differential behavior than the generating series but satisfy simple $\mathbf{q}$-difference equations. This is done via the uniformization of the kernel curve (see § 3.1 and § 3.2).

Step 2: we apply difference Galois theory to the $\mathbf{q}$-difference equations satisfied by the auxiliary functions in order to relate the differential algebraicity of the incomplete generating functions to the existence of *telescoping relations*. These telescoping relations are of the form (3.7) below.

Step 3: we prove that there is no such telescoping relation. This allows us to conclude that the generating series is $\frac{d}{dt}$-differentially transcendental over $\mathbb{Q}$ (see § 3.3).

3.1. **Uniformization of the kernel curve.** With the notation of §2, especially (2.2), any weighted model listed in (G0) satisfies $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 0$. Moreover, since the weighted model is nondegenerate, one finds that the product $d_{1,-1}d_{-1,1}$ is nonzero. Furthermore,

$$-1 + d_{0,0}t \pm \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2} \neq 0.$$

The uniformization of the kernel curve of a weighted model listed in (G0) is given by the following proposition.

Proposition 3.3 (Propositions 1.5 in [11]). — *Let us consider a weighted model listed in* (G0) *and let E be its kernel curve. There exist $\lambda \in C^*$ and a parametrization $\phi : \mathbf{P}^1(C) \to E$ with*

$$\phi(s) = (x(s), y(s)) = \left(\frac{4\alpha_2}{\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}(s + \frac{1}{s}) - 2\alpha_3}, \frac{4\beta_2}{\sqrt{\beta_3^2 - 4\beta_2\beta_4}(\frac{s}{\lambda} + \frac{\lambda}{s}) - 2\beta_3} \right),$$

such that

- *$\phi : \mathbf{P}^1(C) \setminus \{0,\infty\} \to E \setminus \{(0,0)\}$ is a bijection and $\phi^{-1}((0,0)) = \{0,\infty\}$;*
- *The automorphisms ι_1, ι_2, σ of E induce automorphisms $\tilde{\iota}_1, \tilde{\iota}_2, \sigma_{\mathbf{q}}$ of $\mathbf{P}^1(C)$ via ϕ that satisfy $\tilde{\iota}_1(s) = \frac{1}{s}$, $\tilde{\iota}_2(s) = \frac{\mathbf{q}}{s}$, $\sigma_{\mathbf{q}}(s) = \mathbf{q}s$, with $\lambda^2 = \mathbf{q} \in \{\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}}^{-1}\}$ and*

$$\widetilde{\mathbf{q}} = \frac{-1 + d_{0,0}t - \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}} \in C^*.$$

Thus, we have the commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{\iota_k} & E \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\phi} \\ \mathbb{P}^1(C) & \xrightarrow[\tilde{\iota}_k]{} & \mathbb{P}^1(C) \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\phi} \\ \mathbb{P}^1(C) & \xrightarrow[\sigma_{\mathbf{q}}]{} & \mathbb{P}^1(C) \end{array}$$

The following estimate on the norm of $\widetilde{\mathbf{q}}$ holds:

Lemma 3.4. — *We have $|\widetilde{\mathbf{q}}| > 1$.*

Proof. — We consider the expansion as a Puiseux series of $\widetilde{\mathbf{q}}$. Since the numerator of $\widetilde{\mathbf{q}}$ goes to -2 when t goes to 0 and the denominator tends to 0 when t tends to 0, we find that the valuation of $\widetilde{\mathbf{q}}$ at t equal zero is negative, which gives $|\widetilde{\mathbf{q}}| > 1$. □

Example 3.5. — When $d_{0,0} = 0$, the Puiseux expansion of $\widetilde{\mathbf{q}}$ at 0 is particularly simple. For instance we find, with $\mathfrak{d} := d_{1,-1}d_{-1,1}$, the following expansion:

$$\widetilde{\mathbf{q}} = \frac{1+\sqrt{1-4\mathfrak{d}t^2}}{1-\sqrt{1-4\mathfrak{d}t^2}} = \frac{1}{\mathfrak{d}t^2} - 2 - \mathfrak{d}t^2 - 2\mathfrak{d}^2t^4 - 5\mathfrak{d}^3t^6 - 14\mathfrak{d}^4t^8 + O(t^{10}).$$

3.2. **Meromorphic continuation of the generating functions.** In this paragraph, we combine the functional equation (1.3) with the uniformization of the kernel curve obtained above to meromorphically continue the generating function.

We define the norm of an element $b = [b_0 : b_1] \in \mathbf{P}^1(C)$ as follows: if $b_1 \neq 0$, we set $|b| = |\frac{b_0}{b_1}|$ and $|[1:0]| = \infty$ by convention. Since $|t| < 1$, the generating function $Q(x,y,t)$ as well as $F^1(x,t), F^2(y,t)$ converge for any $(x,y) \in \mathbf{P}^1(C) \times \mathbf{P}^1(C)$ such that $|x|$ and $|y|$ are smaller than or equal to 1. On that domain, they satisfy

$$K(x,y,t)Q(x,y,t) = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t). \tag{3.1}$$

We claim that there exist two positive real numbers c_0, c_∞ such that ϕ maps the disks $U_0 = \{s \in \mathbf{P}^1(C) | |s| < c_0\}$ and $U_\infty = \{s \in \mathbf{P}^1(C) | |s| > c_\infty\}$ into the domain $\mathcal{U}$ defined by $\{(x,y) \in E$ such that $|x| \leqslant 1$ and $|y| \leqslant 1\}$. Indeed, the α_i and β_i are of norm smaller than or equal to 1 and $|\alpha_2| = 1$ (see (2.2)). Thus, if $|s| < min(1, |\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|)$, then

$$|x(s)| = \left| \frac{4\alpha_2 s}{\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}(s^2+1) - 2\alpha_3 s} \right| = \frac{|4\alpha_2 s|}{|\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|} < 1.$$

An analogous reasoning for $y(s)$ shows that when $|s|$ is sufficiently small, we find $|x(s)|, |y(s)| \leqslant 1$. Similarly, one can prove that, when $|s|$ is sufficiently big, one has $|x(s)|, |y(s)| \leqslant 1$. This proves our claim.

We set $\breve{F}^1(s) = F^1(x(s),t)$ and $\breve{F}^2(s) = F^2(y(s),t)$. Based on the above, these functions are well defined on $U_0 \cup U_\infty$. Evaluating (3.1) for $(x,y) = (x(s),y(s))$, one finds

$$0 = x(s)y(s) + \breve{F}^1(s) + \breve{F}^2(s) + td_{-1,-1}Q(0,0,t). \tag{3.2}$$

The following lemma shows that one can use the above equation to meromorphically continue the functions $\breve{F}^i(s)$ so that they satisfy a $\mathbf{q}$-difference equation.

Lemma 3.6. — *For $i = 1,2$, the restriction of the function $\breve{F}^i(s)$ to U_0 can be continued to a meromorphic function $\widetilde{\breve{F}}^i(s)$ on C such that*

$$\widetilde{\breve{F}}^1(\mathbf{q}s) - \widetilde{\breve{F}}^1(s) = b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$$

and

$$\widetilde{\breve{F}}^2(\mathbf{q}s) - \widetilde{\breve{F}}^2(s) = b_2 = (y(\mathbf{q}s) - y(s))x(s).$$

Proof. — We just give a sketch of a proof since the arguments are the exact analogue in our ultrametric context of those employed in [11, §2.1]. Since $\tilde{\imath}_1(s) = \frac{1}{s}$ and $\tilde{\imath}_2(s) = \frac{\mathbf{q}}{s}$, we can assume without loss of generality that $\tilde{\imath}_1(U_0) \subset U_\infty$ and $\tilde{\imath}_2(U_\infty) \subset U_0$. Then one can evaluate (3.2) at any $s \in U_0$. We obtain

$$0 = x(s)y(s) + \breve{F}^1(s) + \breve{F}^2(s) + td_{-1,-1}Q(0,0,t).$$

Evaluating (3.2) at $\tilde{\iota}_1(s) \in U_\infty$, we find

$$0 = x(\tilde{\iota}_1(s))y(\tilde{\iota}_1(s)) + \breve{F}^1(\tilde{\iota}_1(s)) + \breve{F}^2(\tilde{\iota}_1(s)) + td_{-1,-1}Q(0,0,t).$$

Using the invariance of $x(s)$ (resp. $y(s)$) with respect to $\tilde{\iota}_1$ (resp. $\tilde{\iota}_2$), the second equation is

$$0 = x(s)y(\mathbf{q}s) + \breve{F}^1(s) + \breve{F}^2(\mathbf{q}s) + td_{-1,-1}Q(0,0,t).$$

Subtracting this last equation to the first, we find that, for any $s \in U_0$, we have

$$\breve{F}^2(\mathbf{q}s) - \breve{F}^2(s) = (y(\mathbf{q}s) - y(s))x(s). \tag{3.3}$$

By Lemma 3.4, the norm of $\widetilde{\mathbf{q}}$ is strictly greater than one and therefore the norm of $|\mathbf{q}|$ is distinct from 1. This allows us to use (3.3) to meromorphically continue $\breve{F}^2$ to C so that it satisfies (3.3) everywhere. The proof for $\breve{F}^1$ is similar. □

Note that, for $i = 1, 2$, the function $\widetilde{F}^i(s)$ does not coincide a priori with $\breve{F}^i(s)$ in the neighborhood of infinity.

3.3. **Differential transcendence in the genus zero case.** We recall that any holomorphic function f on C^* can be represented as an everywhere convergent Laurent series with coefficients in C, see [24, Theorem 2.1, Chapter 5]. Moreover any nonzero meromorphic function on C^* can be written as the quotient of two holomorphic functions on C^* with no common zeros. We denote by $\mathcal{M}er(C^*)$ the field of meromorphic functions over C^* and by $\sigma_\mathbf{q}$ the $\mathbf{q}$-difference operator that maps a meromorphic function $g(s)$ onto $g(\mathbf{q}s)$. Finally, let $C_\mathbf{q}$ be the field formed by the meromorphic functions over C^* fixed by $\sigma_\mathbf{q}$.

One can endow C with a derivation ∂_t as follows

$$\partial_t\Big(\sum_{\gamma\in\mathbb{Q}} c_\gamma t^\gamma\Big) = \sum_{\gamma\in\mathbb{Q}} c_\gamma \gamma t^\gamma.$$

Then, ∂_t extends the derivation $t\frac{d}{dt}$ of $\mathbb{Q}(t)$, see [1, Example (2), §4.4]. For any Hahn series f such that $|f| < 1$, we have $|\partial_t(f)| < 1$. This is not true when ∂_t is replaced by $\frac{d}{dt}$. In order to use the machinery of the parametrized Galois theory of linear difference equations developed in [17], we need to consider derivations that commute with the automorphism $\sigma_\mathbf{q}$. Unfortunately, the derivation ∂_t of C does not commute with $\sigma_\mathbf{q}$ since $\partial_t \circ \sigma_\mathbf{q} = \sigma_\mathbf{q} \circ (\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\partial_s + \partial_t)$ where $\partial_s := s\frac{d}{ds}$. Following [6, §2], one looks for a $\mathcal{M}er(C^*)$-linear combination $\Delta = \alpha\partial_t + \beta\partial_s$ of ∂_s and ∂_t that will commute with $\sigma_\mathbf{q}$. Using the commutation rules for ∂_t and $\partial_s \circ \sigma_\mathbf{q} = \sigma_\mathbf{q} \circ \partial_s$, we obtain

$$\Delta \circ \sigma_\mathbf{q} = (\alpha\partial_t + \beta\partial_s)\sigma_\mathbf{q} = \alpha\sigma_\mathbf{q}(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\partial_s + \partial_t) + \beta\sigma_\mathbf{q}\partial_s$$
$$\sigma_\mathbf{q} \circ \Delta = \sigma_\mathbf{q}(\alpha\partial_t + \beta\partial_s) = \sigma_\mathbf{q}(\alpha)\sigma_\mathbf{q}\partial_t + \sigma_\mathbf{q}(\beta)\sigma_\mathbf{q}\partial_s.$$

Then α and β must satisfy the q-difference equations $\sigma_\mathbf{q}(\alpha) = \alpha$ and $\sigma_\mathbf{q}(\beta) = \beta + \alpha\frac{\partial_t(\mathbf{q})}{\mathbf{q}}$. Fixing α equal to 1, one remarks that β must be equal to $\frac{\partial_t(\mathbf{q})}{\mathbf{q}}z$, where z is a solution of $\sigma_\mathbf{q}y = y + 1$. The latter can be constructed with the help of the Jacobi Theta function that we introduce now. If $|\mathbf{q}| > 1$, the Jacobi Theta function is the meromorphic function defined by $\theta_\mathbf{q}(s) = \sum_{n\in\mathbb{Z}} \mathbf{q}^{-n(n+1)/2}s^n \in \mathcal{M}er(C^*)$. It satisfies the $\mathbf{q}$-difference equation

$$\theta_\mathbf{q}(\mathbf{q}s) = s\theta_\mathbf{q}(s).$$

The Jacobi Theta function is the building block of the construction of meromorphic functions on the Tate curve $C^*/\mathbf{q}^{\mathbb{Z}}$. Its logarithmic derivative $\ell_{\mathbf{q}}(s) = \frac{\partial_s(\theta_{\mathbf{q}})}{\theta_{\mathbf{q}}} \in \mathcal{M}er(C^*)$ satisfies $\ell_{\mathbf{q}}(\mathbf{q}s) = \ell_{\mathbf{q}}(s) + 1$. If $|\mathbf{q}| < 1$ then the meromorphic function $-\ell_{1/\mathbf{q}}$ is solution of $\sigma_{\mathbf{q}}(-\ell_{1/\mathbf{q}}) = -\ell_{1/\mathbf{q}} + 1$. Abusing the notation, we still denote by $\ell_{\mathbf{q}}$ the function $-\ell_{1/\mathbf{q}}$ when $|\mathbf{q}| < 1$.

Since we want to use the $\mathbf{q}$-difference equations of Lemma 3.6 as a constraint for the form of the differential algebraic relations satisfied by the functions $\widetilde{F}^i(s)$, we need to consider derivations that are compatible with $\sigma_{\mathbf{q}}$ in the sense that they commute with $\sigma_{\mathbf{q}}$. This is not the case for the derivation $\partial_t = t\frac{d}{dt}$. From the above discussion, we conclude that the derivations $\partial_s = s\frac{d}{ds}$ and $\Delta_{t,\mathbf{q}} = \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}(s)\partial_s + \partial_t$ commute with $\sigma_{\mathbf{q}}$. The following lemma relates the differential transcendence of the incomplete generating functions $Q(x,0,t)$ and $Q(0,y,t)$ to the differential transcendence of the auxiliary functions $\widetilde{F}^i(s)$. We refer to Definition C.5 for the notion of $(\partial_s, \Delta_{t,\mathbf{q}})$-differential algebraicity over a field.

Lemma 3.7. — *If the generating function $Q(x,0,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$, then $\widetilde{F}^1(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K} = C_{\mathbf{q}}(s, \ell_{\mathbf{q}}(s))$.*

If the generating function $Q(0,y,t)$ is $(\frac{d}{dy}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$, then $\widetilde{F}^2(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K} = C_{\mathbf{q}}(s, \ell_{\mathbf{q}}(s))$.

Proof. — The statement being symmetrical in x and y, we prove it only for $Q(x,0,t)$. Assume that the generating function $Q(x,0,t)$ is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. Since $F^1(x,t)$ is the product of $Q(x,0,t)$ by the polynomial $K(x,0,t) \in \mathbb{Q}[x,t]$, the function $F^1(x,t)$ is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. It is therefore $\left(\frac{d}{dx}, \partial_t\right)$-differentially algebraic over $\mathbb{Q}(t)$, and finally $\left(\frac{d}{dx}, \partial_t\right)$-differentially algebraic over $\mathbb{Q}$, since t is ∂_t-differentially algebraic over $\mathbb{Q}$. Remember that $\widetilde{F}^1(s)$ coincides with $F^1(x(s),t)$ for $s \in U_0$ where $x(s)$ is defined thanks to Proposition 3.3. Therefore, we need to understand the relations between the x- and t derivatives of $F^1(x,t)$ and the derivatives of $F^1(x(s),t)$ with respect to ∂_s and $\Delta_{t,\mathbf{q}}$.

Let us study these relations for an arbitrary bivariate function $G(x,t)$ which converges on $|x|, |t| \leqslant 1$. Denote by δ_x the derivation $\frac{d}{dx}$ and by $\widetilde{G}(s) = G(x(s),t)$ for $s \in U_0$. From the equality $\partial_s\widetilde{G}(s) = \partial_s(x(s))(\delta_x G)(x(s),t)$, we conclude that

$$\partial_t(\widetilde{G}(s)) = (\partial_t G)(x(s),t) + \partial_t(x(s))(\delta_x G)(x(s),t) = \partial_t G(x(s),t) + c\partial_s(\widetilde{G}(s)),$$

where $c = \frac{\partial_t(x(s))}{\partial_s(x(s))}$. The element c belongs to $\widetilde{K}$ because $x(s) \in \widetilde{K}$ and $\widetilde{K}$ is stable by $\partial_s, \Delta_{t,\mathbf{q}}$ see Lemma D.5, and thereby by $\partial_t = \Delta_{t,\mathbf{q}} - \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}(s)\partial_s$. An easy induction shows that,

$$(\partial_t^n G)(x(s),t) = \partial_t^n(\widetilde{G}(s)) + \sum_{i \leqslant n, j < n} b_{i,j}\partial_t^j\partial_s^i(\widetilde{G}(s)), \tag{3.4}$$

where the $b_{i,j}$'s belong to $\widetilde{K}$. By Lemma D.2, we have $\partial_s\Delta_{t,\mathbf{q}} - \Delta_{t,\mathbf{q}}\partial_s = f\partial_s$, where $f = \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\partial_s(\ell_{\mathbf{q}}) \in \widetilde{K}$. Combining (3.4) with $\partial_t = \Delta_{t,\mathbf{q}} - \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}(s)\partial_s$, we find that

$$(\partial_t^n G)(x(s),t) = \Delta_{t,\mathbf{q}}^n(\widetilde{G}(s)) + \sum_{i \leqslant 2n, j < n} d_{i,j}\Delta_{t,\mathbf{q}}^j\partial_s^i(\widetilde{G}(s)), \tag{3.5}$$

for some $d_{i,j}$'s in $\widetilde{K}$. Moreover, an easy induction shows that, for any $m \in \mathbb{N}^*$, we have

$$(\delta_x^m G)(x(s),t) = \frac{1}{\partial_s(x(s))^m}\partial_s^m(\widetilde{G}(s)) + \sum_{i=1}^{m-1} a_i \partial_s^i(\widetilde{G}(s)), \tag{3.6}$$

where $a_i \in \widetilde{K}$. Applying (3.5) with G replaced by $\delta_x^m G$, we find that for every $m, n \in \mathbb{N}$,

$$(\partial_t^n \delta_x^m G)(x(s),t) = \Delta_{t,\mathbf{q}}^n((\delta_x^m G)(x(s),t)) + \sum_{i \leqslant 2n, j<n} d_{i,j}\Delta_{t,\mathbf{q}}^j \partial_s^i((\delta_x^m G)(x(s),t)).$$

Combining this equation with (3.6), we conclude that

$$(\partial_t^n \delta_x^m G)(x(s),t) = \frac{1}{\partial_s(x(s))^m}\Delta_{t,\mathbf{q}}^n \partial_s^m(\widetilde{G}(s)) + \sum_{i \leqslant 2n+m, j<n} r_{i,j}\Delta_{t,\mathbf{q}}^j \partial_s^i(\widetilde{G}(s)),$$

where the $r_{i,j}$'s are elements of $\widetilde{K}$.

Applying the computations above to $G = F^1(x,t)$, we find that any nontrivial polynomial equation in the derivatives $\delta_x^m \partial_t^n F^1(x,t)$ over $\mathbb{Q}$ yields a nontrivial polynomial equation over $\widetilde{K}$ between the derivatives $\Delta_{t,\mathbf{q}}^j \partial_s^i(\widetilde{F}^1(s))$. □

Thus, we have reduced the proof of Theorem 3.1 to the following proposition:

Proposition 3.8. — *The functions $\widetilde{F}^1(s)$ and $\widetilde{F}^2(s)$ are $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially transcendental over $\widetilde{K}$.*

Proof. — Suppose to the contrary that $\widetilde{F}^1(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K}$. By Lemma 3.6, the meromorphic function $\widetilde{F}^1(s)$ satisfies $\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$ with $b_1 \in C(s) \subset C_{\mathbf{q}}(s)$. We now apply difference Galois theory to this $\mathbf{q}$-difference equation. More precisely, by Proposition D.6 and Corollary D.13 with $K = C_{\mathbf{q}}(s)$, there exist $m \in \mathbb{N}$, $d_0, \dots, d_m \in C_{\mathbf{q}}$ not all zero and $h \in C_{\mathbf{q}}(s)$ such that

$$d_0 b_1 + d_1\partial_s(b_1) + \cdots + d_m\partial_s^m(b_1) = \sigma_{\mathbf{q}}(h) - h. \tag{3.7}$$

We need now to use a descent argument to show that if (3.7) holds with the d_i's in $C_{\mathbf{q}}$ and $h \in C_{\mathbf{q}}(s)$ then it holds for some d_i's in C not all zero and $h \in C(s)$. Such reasoning is classical and can be found for instance in [17, Corollary 3.2]. Let us write $\partial_s^k(b_1) = P_k/Q_k$, where $P_k, Q_k \in C[s]$ and $h = A/B$, where $A, B \in C_{\mathbf{q}}[s]$. After clearing the denominators, we find that (3.7) is equivalent to

$$B\sigma_{\mathbf{q}}(B)\Big(\sum_{k=0}^{m} d_k P_k\Big) = \big(\sigma_{\mathbf{q}}(A)B - A\sigma_{\mathbf{q}}(B)\big)\prod_{k=0}^{m} Q_k. \tag{3.8}$$

Let a_k be the s^k-coefficients of A. Since $\mathbf{q} \in C$, and a_k is $\sigma_{\mathbf{q}}$-invariant, we find that the s^k-coefficients of $\sigma_{\mathbf{q}}(A)$ is $\mathbf{q}^k a_k$. A similar statement holds for B. Equating like powers of s on both sides of (3.8) allows to find that the d_k, and the s-coefficients of A and B are solutions of a collection of polynomial equations with coefficients in C. Since this collection of polynomial equations has a nonzero solution in $C_{\mathbf{q}}$, we can conclude that it has a nonzero solution in C because C is algebraically closed. Therefore, there exist $c_k \in C$ not all zero and $g \in C(s)$ such that

$$\sum_{k=0}^{m} c_k \partial_s^k(b_1) = \sigma_{\mathbf{q}}(g) - g.$$

By [17, Lemma 6.4] there exist $f \in C(s)$ and $c \in C$, such that

$$\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1 = \sigma_{\mathbf{q}}(f) - f + c.$$

Since $\widetilde{F}^1$ is meromorphic at $s = 0$, the function $f_0 := \widetilde{F}^1 - f$ is also meromorphic at $s = 0$. Since it satisfies $\sigma_{\mathbf{q}}(f_0) = f_0 + c$, we conclude that c must be equal to zero. Finally, we have shown that there exists $f \in C(s)$ such that

$$b_1 = \sigma_{\mathbf{q}}(f) - f. \tag{3.9}$$

By duality, the morphism $\phi : \mathbf{P}^1 \to E$ gives rise to a field isomorphism ϕ^* from the field $C(E) = C(x, y)$[1] of rational functions on E and the field $C(s)$ of rational functions on $\mathbf{P}^1$. Moreover, one has $\sigma_{\mathbf{q}}\phi^* = \phi^*\sigma^*$, where σ^* is the action induced by the automorphism of the walk on $C(E)$. Then, it is easily seen that the equation (3.9) is equivalent to

$$(\sigma(x) - x)\sigma(y) = \sigma(\tilde{f}) - \tilde{f}, \tag{3.10}$$

where $\tilde{f} \in C(x, y)$ is the rational function corresponding to f via ϕ^*. The coefficients of $\tilde{f}$ as a rational function over E belong to a finitely generated extension F of $\mathbb{Q}(t)$.

There exists a $\mathbb{Q}$-embedding ψ of F into $\mathbb{C}$ that maps t onto a transcendental complex number. Since σ and E are defined over $\mathbb{Q}(t)$, we apply ψ to (3.10) and we find

$$(\overline{\sigma}(x) - x)\overline{\sigma}(y) = \overline{\sigma}(\overline{f}) - \overline{f},$$

where $\overline{f}$ belongs to $\mathbb{C}(\overline{E})$ the field of rational functions on the complex algebraic curve $\overline{E}$ defined by the kernel polynomial $K(x, y, \psi(t))$ and where $\overline{\sigma}$ is the automorphism of $\mathbb{C}(\overline{E})$ induced by the automorphism of the walk corresponding to $\overline{E}$. In [11, §3.2], the authors proved that such equation has no solutions. This concludes the proof by contradiction. □

4. Generating functions of walks, genus one case

In this section we consider the situation where the kernel curve E is an elliptic curve. By Remark 2.5, this corresponds to the case where the set of steps is not included in a half plane. Unlike the genus zero cases of (G0), the group of the walk might be finite for genus one walks. For unweighted walks of genus one with finite group, it was proved in [3, 4] that the series was holonomic with respect to the three variables. More recently, the authors of [12] studied weighted walks of genus one with finite group. They proved that the lifting of the generating series was a product of zeta functions and elliptic functions over a curve isogenous to the kernel curve. This allowed them to conclude that the generating series was holonomic with respect to the variables x and y.

We shall focus on the weighted walks of genus one with infinite group and we will prove analogously to the genus zero case that the $(\frac{d}{dx}, \frac{d}{dt})$-differential algebraicity of the series implies its $\frac{d}{dx}$-differential algebraicity. This result combined to [2] shows that, for unweighted walks of genus one with infinite group, the series is $\frac{d}{dx}$-differentiallly algebraic if and only if it is $\frac{d}{dt}$-differentially algebraic (see Corollary 4.16 below).

[1] Here x and y denote the coordinate functions on the curve E.

Our strategy follows the basic lines of the ones employed in the genus zero situation. However, the uniformization procedure in the genus one case is more delicate and differs from previous works such as [12, 14, 23] which relied on the uniformization of elliptic curves over $\mathbb{C}$ by a fundamental parallelogram of periods. Over a nonarchimedean field C, there might be a lack of nontrivial lattices. For instance, if $C = \mathbb{Q}_p$ the field of p-adic numbers and Λ is any nonzero subgroup of $\mathbb{Q}_p$, we easily see that, for any nonzero element λ in Λ and any integer $n \geqslant 0$, the element $p^n\lambda$ belongs to the lattice Λ. Then, 0 is an accumulation point of Λ. If C is the completion of an algebraic closure of $\mathbf{F}_p((\frac{1}{t}))$, the Lattice $\mathbf{F}_p[t]$ is an infinite discrete subgroup of C but the quotient $C/\mathbf{F}_p[t]$ is not an abelian variety (see [34, p.147]). Therefore, in the nonarchimedean framework, one prefers to consider multiplicative subgroups rather than lattices. When an element q in C is such that $0 < |q| < 1$, the set $q^{\mathbb{Z}}$ is a discrete subgroup of C^*. Rigid analytic geometry gives a geometric meaning to the quotient $C^*/q^{\mathbb{Z}}$. This geometric quotient is called a Tate curve (see [30] for more details). For simplicity of exposition, we will not give here many details on this nonarchimedean geometry. The multiplicative uniformization of the kernel curve allows us as in § 3.2 to attach to the incomplete generating functions $Q(x, 0, t)$ and $Q(0, y, t)$ some meromorphic functions $\widetilde{F}^i(s)$ satisfying

$$\widetilde{F}^i(\mathbf{q}s) - \widetilde{F}^i(s) = b_i(s),$$

for some $\mathbf{q} \in C^*$ and $b_i(s) \in C_q$, the field of q-periodic meromorphic functions over C^*. This equation is the multiplicative and nonarchimedean analogue of [23, Theorem 4]. Here C_q is the function field of the elliptic curve $C^*/q^{\mathbb{Z}}$. The analogue of the Tate curve $C^*/q^{\mathbb{Z}}$ in the additive and archimedean setting of [23] is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ of $\mathbb{C}$ by a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of $\mathbb{C}$. Via this analogy, the operator $\sigma_{\mathbf{q}}$ is the multiplicative analogue of the translation by a point ω_3 ([30, Corollary Vb]). In the archimedean setting of [23] one needs two difference equations, corresponding to two translations, in order to continue the generating series. Our multiplicative version only requires one single difference equation. This is essentially due to the fact that, in the multiplicative uniformization, only the loop around zero matters.

The multiplicative uniformization detailed in § 4.1, 4.2 and 4.3 has many advantages. Though technical, it is simpler than the uniformization by a fundamental parallelogram of periods since we only have to deal with one generator of the fundamental group of the elliptic curve, precisely the loop around the origin in C^*. Moreover, it gives a unified framework to study the genus zero and one case, namely, the Galois theory of $\mathbf{q}$-difference equations. This is the content of § 4.4 where we apply the Galoisian criteria of Appendix C to translate the differential algebraicity of the generating function in terms of the existence of a telescoper.

4.1. **Uniformization of the kernel curve.** Let us fix a weighted model of genus one. By Lemma 2.6, the norm of the J-invariant $J(E)$ of the kernel curve is such that $|J(E)| > 1$. By Proposition B.2, there exists one and only one $q \in C$ such that $0 < |q| < 1$ and $J(E) = J(E_q)$, where E_q is the elliptic curve attached to the Tate curve $C^*/q^{\mathbb{Z}}$. Moreover $|J(E)| = \frac{1}{|q|}$ (see Proposition 4.2, Lemmas B.5 and B.7).

Remark 4.1. — In the nonarchimedean framework, the element q is obtained from $J(E)$ by inverting the function $J(E_q) = \frac{1}{q} + R(q)$ where $R(q) = 744 + 196884q + \dots$ is a universal power series with integral coefficients (see [30, Lemma 1 p.30]). In the archimedean framework of [12, Proposition 2.1] where t is a fixed real number in $]0,1[$, the authors gave an explicit Weierstrass normal form $y^2 = 4x^3 - g_2x - g_3$ for the kernel curve, viewed as an algebraic curve in $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$. The invariants g_2, g_3 are real and the discriminant of $4x^3 - g_2x - g_3$ is strictly positive (see [12, p. 60]). Then by [36, §20.32], the polynomial $4x^3 - g_2x - g_3$ has three distinct real roots $e_1 = \wp(\frac{\omega_1}{2}) > e_2 = \wp(\frac{\omega_1+\omega_2}{2}) > e_3 = \wp(\frac{\omega_2}{2})$ where ω_1, ω_2 are two fundamental periods for the kernel curve and $\wp$ is the corresponding Weierstrass function. An element q such that $J(E_q) = 12^3 \frac{g_2^3}{g_2^3 - 27g_3^2}$ is given by the following expression

$$q = exp(2i\pi\frac{\omega_1}{\omega_2}) = exp\Big(-2\pi \times \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \lambda)}\Big),$$

where ${}_2F_1$ is the Gauss hypergeometric function, $\lambda = \frac{e_2 - e_1}{e_3 - e_1}$ is the value of the Lambda modular function for the elliptic curve $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \sim \mathbb{C}^*/q^{\mathbb{Z}}$ (see [19, §9, Theorem 6.1]).

The curve E_q can be analytically uniformized by C^* thanks to special functions, which have their origins in the theory of Jacobi q-theta functions (see Proposition 4.2 below). Finally, since E and E_q have the same J-invariant, there exists an algebraic isomorphism between these two elliptic curves. In order to describe the uniformization of the kernel curve E, one needs to make explicit this algebraic isomorphism. This is not completely obvious since E_q is given by its Tate normal form in $\mathbf{P}^2$, i.e. by an equation of the form

$$Y^2 + XY = X^3 + BX + \widetilde{C}.$$

Therefore, many intermediate technical results are postponed to the appendix B. The following proposition describes the multiplicative uniformization of an elliptic curve given by a Tate normal form.

Following [30, Page 28], we set $s_k = \sum_{n>0} \frac{n^k q^n}{1-q^n} \in C$ for $k \geqslant 1$.

Proposition 4.2. — *The series*

- $X(s) = \sum_{n\in\mathbb{Z}} \frac{q^n s}{(1-q^n s)^2} - 2s_1;$
- $Y(s) = \sum_{n\in\mathbb{Z}} \frac{(q^n s)^2}{(1-q^n s)^3} + s_1;$

are q-periodic meromorphic functions over C^*. Furthermore $X(s) = X(1/s)$, and $X(s)$ has a pole of order 2 at any element of the form $q^{\mathbb{Z}}$. Moreover, the analytic map

$$\begin{array}{rccl} \pi: & C^* & \rightarrow & \mathbf{P}^2(C), \\ & s & \mapsto & [X(s):Y(s):1] \end{array}$$

is onto and its image is E_q, the elliptic curve defined by the following Tate normal form

$$Y^2 + XY = X^3 + BX + \widetilde{C}, \tag{4.1}$$

where $B = -5s_3$ and $\widetilde{C} = -\frac{1}{12}(5s_3 + 7s_5)$. Moreover, $\pi(s_1) = \pi(s_2)$ if and only if $s_1 \in s_2 q^{\mathbb{Z}}$.

Proof. — This is [16, Theorem 5.1.4, Corollary 5.1.5, and Theorem 5.1.10]. ☐

In the notation of Section 2.1, set $\mathfrak{D}(x) := \Delta_x(x,1)$. Let us write the kernel polynomial

$$K(x,y,t) = \widetilde{A}_0(x) + \widetilde{A}_1(x)y + \widetilde{A}_2(x)y^2 = \widetilde{B}_0(y) + \widetilde{B}_1(y)x + \widetilde{B}_2(y)x^2$$

with $\widetilde{A}_i(x) \in C[x]$ and $\widetilde{B}_i(y) \in C[y]$. For $i \geqslant 1$, let $\mathfrak{D}^{(i)}(x)$ denotes the i-th derivative with respect to x of $\mathfrak{D}(x)$. The analytic uniformization of the kernel curve is given by the following theorem. As in [12, Proposition 2.1], the uniformization procedure involves the choice of a "small branch point", that is, a root of the discriminant with a small modulus.

Theorem 4.3. — *There exists a root a of $\mathfrak{D}(x)$ in C such that*

$$|q|^{1/2} < \begin{array}{ll} |a| & <1 \\ |\mathfrak{D}^{(1)}(a)| & <1 \\ |\mathfrak{D}^{(2)}(a)-2| & <1 \\ |\mathfrak{D}^{(3)}(a)| & <1 \\ |\mathfrak{D}^{(4)}(a)| & <1. \end{array}$$

For any such a, there exists $u \in C^*$ with $|u|=1$ such that the map ϕ given by

$$\begin{array}{rccl} \phi: & C^* & \to & E, \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$

is surjective, where

$$\begin{aligned} \overline{x}(s) &= a + \frac{\mathfrak{D}^{(1)}(a)}{u^2X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \\ \overline{y}(s) &= \frac{\frac{\mathfrak{D}^{(1)}(a)(2u^3Y(s)+u^3X(s))}{2\left(u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(1)}(a)}{6}\right)^2} - \widetilde{A}_1\left(a + \frac{\mathfrak{D}^{(1)}(a)}{u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}{2\widetilde{A}_2\left(a+\frac{\mathfrak{D}^{(1)}(a)}{u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}. \end{aligned} \tag{4.2}$$

Proof. — Lemma A.1 and Lemma B.7 guaranty the existence of a. The element a allows us to write down the isomorphism between the kernel curve E and one of its Weierstrass normal forms E_1. More precisely, by Proposition B.4, the application w_E

$$\begin{array}{lcl} E_1 & \to & E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) \\ [x_1 : y_1 : 1] & \mapsto & (\overline{x}, \overline{y}) \end{array}$$

where

$$\overline{x} = a + \frac{\mathfrak{D}^{(1)}(a)}{x_1 - \frac{\mathfrak{D}^{(2)}(a)}{6}} \text{ and } \overline{y} = \frac{\frac{\mathfrak{D}^{(1)}(a)y_1}{2\left(x_1-\frac{\mathfrak{D}^{(1)}(a)}{6}\right)^2} - \widetilde{A}_1\left(a+\frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}{2\widetilde{A}_2\left(a+\frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)},$$

is an isomorphism between the elliptic curve $E_1 \subset \mathbf{P}^2(C)$ given by the equation $y_1^2 = 4x_1^3 - g_2x_1 - g_3$ and the kernel curve E. Now, it remains to make explicit the isomorphism between E_q and one of its Weierstrass normal form $\widetilde{E}_1$. By Lemma B.5, the application

$$\begin{array}{rlcl} w_T: & E_q & \to & \widetilde{E}_1, \\ & [X:Y:1] & \mapsto & [X+\frac{1}{12} : 2Y+X : 1] \end{array}$$

induces an isomorphism between E_q and the curve $\widetilde{E}_1$ given by $y^2 = 4x^3 - h_2x - h_3$. Since E and E_q have the same J-invariants and are therefore isomorphic, the same holds for their Weierstrass normal forms. Thus, there exists $u \in C^*$ such that

$$\begin{array}{rlcl} \psi : & \widetilde{E}_1 & \to & E_1, \\ & [x:y:1] & \mapsto & [u^2x : u^3y : 1] \end{array}$$

induces an isomorphism of elliptic curves (see Lemma B.6). To conclude, we set $\phi = w_E \circ \psi \circ w_T \circ \pi$ where π is the uniformization of E_q by C^* given in Proposition 4.2. The norm estimate on u is Lemma B.7. □

Remark 4.4. —

- Note that by construction $\phi(s_1) = \phi(s_2)$ if and only if $s_1 \in s_2 q^{\mathbb{Z}}$ (see Proposition 4.2).
- Via ϕ, the field of rational functions over E can be identified with the field of q-periodic meromorphic functions over C^*.
- The conditions on a are crucial to guaranty the meromorphic continuation of the generating function (see the proof of Lemma 4.8).
- The symmetry arguments between x and y of Remark 1.3 can be pushed further and one can construct another uniformization of E as follows. Denote by $\mathfrak{E}(y)$ the polynomial $\Delta_y(y,1)$. One can prove that there exists a root $b \in C^*$ of $\mathfrak{E}$ such that $|b|, |\mathfrak{E}^{(2)}(b) - 2|, |\mathfrak{E}^{(i)}(b)| < 1$ for $i = 3, 4$ and $|q|^{1/2} < |\mathfrak{E}^{(1)}(b)| < 1$ and $v \in C^*$ with $|v| = 1$ such that the analytic map ϕ_y given by
$$\begin{array}{rlcl} \phi_y : & C^* & \to & E, \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$
is surjective with $\overline{y}(s) = b + \frac{\mathfrak{E}^{(1)}(b)}{v^2 X(s) + \frac{v^2}{12} - \frac{\mathfrak{E}^{(2)}(b)}{6}}$ (see [12, (2.16)] for similar arguments).

4.2. **The group of the walk.** The following proposition gives an explicit form for the automorphisms of C^* induced via ϕ by the automorphisms σ, ι_1, ι_2 of E.

Proposition 4.5. — *There exists* $\mathbf{q}$ *in* C^* *such that the automorphism of* C^* *defined by* $\sigma_{\mathbf{q}} : s \mapsto \mathbf{q}s$ *induces via* ϕ *the automorphism* σ*, that is* $\sigma \circ \phi = \phi \circ \sigma_{\mathbf{q}}$*. Similarly, the involutions* $\tilde{\iota}_1, \tilde{\iota}_2$ *of* C^**, that are defined by* $\tilde{\iota}_1(s) = 1/s$ *and* $\tilde{\iota}_2(s) = \mathbf{q}/s$*, induce via* ϕ *the automorphisms* ι_1, ι_2*.*

Proof. — By [13, Proposition 2.5.2], the automorphism σ corresponds to the addition by a prescribed point Ω of E. Let $\pi : C^* \to E_q$ be the surjective map defined in Proposition 4.2. By [16, Exercise 5.1.9], the map π is a group isomorphism between the multiplicative group $(C^*, *)$ and the Mordell-Weil group of E_q[2]. Moreover, since E_q and E are elliptic curves, any isomorphism between E_q and E is a group morphism between their respective Mordell-Weil groups. This proves that ϕ is a group morphism. Then, there exists $\mathbf{q} \in C^*$ such that $\sigma \circ \phi = \phi \circ \sigma_{\mathbf{q}}$. Since ϕ is q-invariant, the element $\mathbf{q}$ is determined modulo $q^{\mathbb{Z}}$ (see Remark 4.4). This proves the first statement.

Let us denote by $\tilde{\iota}_1, \tilde{\iota}_2$ some automorphisms of C^*, obtained by pulling back to C^* via ϕ the automorphisms ι_1, ι_2 of E. The automorphisms $\tilde{\iota}_1, \tilde{\iota}_2$ are uniquely

[2] This is the group whose underlying set is the set of points of E_q and whose group law is given by the addition on the elliptic curve E.

determined up to multiplication by some power of q. The automorphisms of C^* are of the form $s \mapsto ls^{\pm 1}$ with $l \in C^*$. Note that $\overline{x}(q^{\mathbb{Z}}) = a$, and $(a, \frac{-B(a)}{2A(a)}) \in E$ is fixed by ι_1. Indeed, by construction $\mathfrak{D}(a) = 0$. This proves that $\tilde{\iota}_1(1)$ belongs to $q^{\mathbb{Z}}$. Since ι_1 is not the identity, we can modify $\tilde{\iota}_1$ by a suitable power of q to get $\tilde{\iota}_1(s) = 1/s$. The expression of $\tilde{\iota}_2$ follows with $\sigma = \iota_2 \circ \iota_1$. □

Remark 4.6. —

- The choice of the element $\mathbf{q}$ is unique up to multiplication by $q^{\mathbb{Z}}$. Since $|q| \neq 1$, we can choose $\mathbf{q}$ such that $|q|^{1/2} \leqslant |\mathbf{q}| < |q|^{-1/2}$.
- Pursuing the symmetry arguments of Remark 4.4, we easily note that Proposition 4.5 has a straightforward analogue when one replaces ϕ by ϕ_y and one exchanges $\tilde{\iota}_1$ and $\tilde{\iota}_2$.

The proof of the following lemma is straightforward.

Lemma 4.7. — *The automorphism σ has infinite order if and only if $\mathbf{q}$ and q are* multiplicatively independent[3], *that is, there is no $(r, l) \in \mathbb{Z}^2 \setminus (0, 0)$ such that $q^r = \mathbf{q}^l$.*

4.3. **Meromorphic continuation.** In this section, we prove that the functions

$$F^1(x,t) := K(x,0,t)Q(x,0,t) \text{ and } \quad F^2(y,t) := K(0,y,t)Q(0,y,t)$$

can be meromorphically continued to C^*. We follow some of the ideas initiated in [14]. We note that, since $|t| < 1$, the series $F^1(x,t)$ and $F^2(y,t)$ converge on the affinoid subset $U = \{(x,y) \in E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) | |x| \leqslant 1, |y| \leqslant 1\}$ of E. With Lemma A.3, U is not empty. For $(x,y) \in U$, we have

$$0 = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t).$$

Set $U_x = \{(x,y) \in E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) | |x| \leqslant 1\}$. Note that $F^1(x,t)$ is analytic on U_x. We continue $F^2(y,t)$ on U_x by setting

$$F^2(y,t) = -xy - F^1(x,t) - td_{-1,-1}Q(0,0,t).$$

Composing $F^i(x,t)$ with the surjective map

$$\begin{array}{rccc} \phi: & C^* & \to & E \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$

we define the functions $\breve{F}^1(s) = F^1(\overline{x}(s), t)$ and $\breve{F}^2(s) = F^2(\overline{y}(s), t)$ for any s in the set

$$\mathcal{U}_x := \phi^{-1}(U_x) \cap \{s \in C^* | |s| \in [|q|^{1/2}, |q|^{-1/2}[\}.$$

The goal of the following lemma is to prove that $\mathcal{U}_x$ is an annulus whose size is large enough in order to continue the functions $\breve{F}^1, \breve{F}^2$, to the whole C^* (see Figure 4).

Lemma 4.8. — *Let $|s| \in [|q|^{1/2}, |q|^{-1/2}[$. The following statements hold:*

- *if $|s| \in]|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}[$, then $|\overline{x}(s)| < 1$;*
- *if $|s| = |\mathfrak{D}^{(1)}(a)|^{\pm 1}$, then $|\overline{x}(s)| = 1$;*
- *otherwise $|\overline{x}(s)| > 1$.*

In conclusion, $\mathcal{U}_x = [|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}]$.

[3] Note that multiplicatively independent is sometimes replaced in the literature by noncommensurable (see [30, §6]).

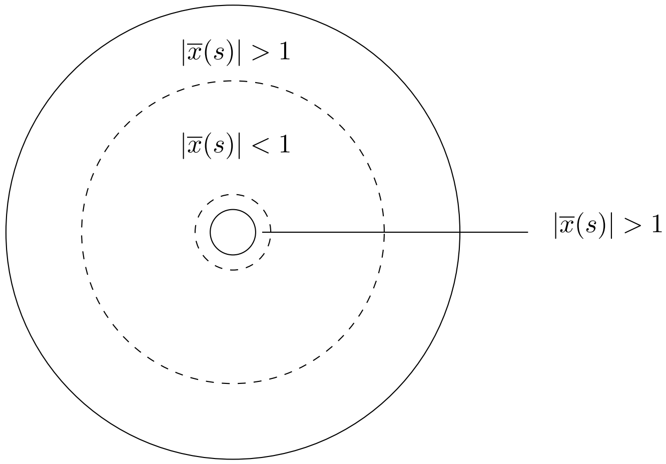


FIGURE 4. The plain circles correspond to $|s| = |q|^{\pm 1/2}$. The dashed circles correspond to $|\overline{x}(s)| = 1$.

Proof. — From the definition of $X(s)$, we have $X(s) = X(1/s)$ so that $\overline{x}(s) = \overline{x}(1/s)$. Using this symmetry, we just have to prove Lemma 4.8 for $|s| \in [|q|^{1/2}, 1]$. We have

$$|\overline{x}(s)| = \left| a + \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| \leqslant \max\left(|a|, \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| \right), \quad (4.3)$$

with equality if $|a| \neq \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right|$. Remember that $|u| = 1$, $|a| < 1$, and $|q|^{1/2} < |\mathfrak{D}^{(1)}(a)| < 1$, see Theorem 4.3. Let us first assume that $|s| \in [|\mathfrak{D}^{(1)}(a)|, 1[$. By Lemma B.3, $|u^2 X(s)| = |s|$ and by Lemma B.8, $\left| \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6} \right| < |\mathfrak{D}^{(1)}(a)|$. Therefore

$$\left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| = \left| \frac{\mathfrak{D}^{(1)}(a)}{s} \right|.$$

Combining this equality with (4.3) and $|a| < 1$ yields $|\overline{x}(s)| < 1$ if $|s| \in]|\mathfrak{D}^{(1)}(a)|, 1[$, and $|\overline{x}(s)| = 1$ if $|s| = |\mathfrak{D}^{(1)}(a)|$.

Assume now that $|s| = 1$. By construction, $|\overline{x}(1)| = |a| < 1$. So let us assume that $s \neq 1$. Since $\left| \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6} \right| < |\mathfrak{D}^{(1)}(a)| < 1$ and $|u^2 X(s)| \geqslant 1$ by Lemma B.3, we find

$$\left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| = \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s)} \right| \leqslant |\mathfrak{D}^{(1)}(a)| < 1.$$

This concludes the proof of the first two points.

Assume that $|s| \in]|q|^{1/2}, |\mathfrak{D}^{(1)}(a)|[$. By Lemma B.3, $|u^2 X(s)| = |X(s)| = |s|$. Since

$$\left| \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6} \right| < |\mathfrak{D}^{(1)}(a)| < 1,$$

we find that $\left|u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}}{6}\right|<|\mathfrak{D}^{(1)}(a)|$ and therefore, $|\overline{x}(s)|>1$. If we have $|s|=|q|^{1/2}<|\mathfrak{D}^{(1)}(a)|$, then Lemma B.3 implies that

$$|u^2X(s)|=|X(s)|\leqslant|s|<|\mathfrak{D}^{(1)}(a)|.$$

Since $\left|\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}\right|<|\mathfrak{D}^{(1)}(a)|$, we deduce that $\left|u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}\right|<|\mathfrak{D}^{(1)}(a)|$ and therefore, $|\overline{x}(s)|>1$. This concludes the proof. □

Remark 4.9. — By symmetry between x and y, one could have defined $U_y=\{(x,y)\in E\subset\mathbf{P}^1(C)\times\mathbf{P}^1(C)||y|\leqslant 1\}$ and continue $F^1(x,t)$ on U_y by setting

$$F^1(x,t)=-xy-F^2(y,t)-td_{-1,-1}Q(0,0,t).$$

Then, the composition of the F^i with the surjective map ϕ_y defined in Remark 4.4 yields functions $\breve{F}^i$ that are defined on

$$\mathcal{U}_y:=\phi_y^{-1}(U_y)\cap\{s\in C^*||s|\in[|q|^{1/2},|q|^{-1/2}[\}.$$

The analogue of Lemma 4.8 is as follows. For $|s|\in[|q|^{1/2},|q|^{-1/2}[$, the following statements hold:

- if $|s|\in]|\mathfrak{E}^{(1)}(b)|,|\mathfrak{E}^{(1)}(b)|^{-1}[$, then $|\overline{y}(s)|<1$;
- if $|s|=|\mathfrak{E}^{(1)}(b)|^{\pm1}$ then $|\overline{y}(s)|=1$;
- otherwise $|\overline{y}(s)|>1$.

By Proposition 4.5, the automorphism of the walk corresponds to the $\mathbf{q}$-dilatation on C^*. The following lemma shows that one can cover C^* either with the $\mathbf{q}$-orbit of the set $\mathcal{U}_x$ or with the $\mathbf{q}$-orbit of $\mathcal{U}_y$. This property is crucial in order to continue the series as a meromorphic function over C^*.

Lemma 4.10. — *The following statements hold:*

- $|\mathbf{q}|\neq 1$*;*
- *moreover, up to replace* $\mathbf{q}$ *by some convenient* $q^{\mathbb{Z}}$*-multiple, the following holds:*
  - *if either* $d_{-1,1}=0$ *or* $d_{1,-1}\neq 0$*, then,*
  $$\bigcup_{\ell\in\mathbb{Z}}\sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x)=C^*;$$
  - *if either* $d_{-1,1}\neq 0$ *or* $d_{1,-1}=0$ *then,*
  $$\bigcup_{\ell\in\mathbb{Z}}\sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_y)=C^*.$$

Remark 4.11. — For all the genus one walks, we find that

$$\bigcup_{\ell\in\mathbb{Z}}\sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x\cup\mathcal{U}_y)=C^*.$$

A similar statement may be found in the archimedean context in [14, §3].

Proof. — Let us first prove that $|\mathbf{q}|\neq 1$. By Remark 4.6, one can choose $\mathbf{q}$ so that we have $|q|^{1/2}\leqslant|\mathbf{q}|<|q|^{-1/2}$. By construction, $\overline{x}(1)=a$. Let $b\in\mathbf{P}^1(C)$ such that $(a,b)\in E$. Since $\iota_1(a,b)=(a,b)$ we have $\iota_2(a,b)\neq(a,b)$ by Lemma 2.7. So let $a'\in\mathbf{P}^1(C)$ distinct from a such that $\sigma(a,b)=(a',b)$. Then, $\overline{x}(\mathbf{q})=a'$. By Lemma 4.8, $|\overline{x}(s)|<1$ for $|s|=1$. Thus, it suffices to prove that $|\overline{x}(\mathbf{q})|=|a'|\geqslant 1$ to conclude that $|\mathbf{q}|\neq 1$.

Remember that $K(x,y,t) = \widetilde{A}_{-1}(x) + \widetilde{A}_0(x)y + \widetilde{A}_1(x)y^2 = \widetilde{B}_{-1}(y) + \widetilde{B}_0(y)x + \widetilde{B}_1(y)x^2$ with $\widetilde{A}_i(x) \in C[x]$ and $\widetilde{B}_i(y) \in C[y]$. With $\iota_1(a,b) = (a,b)$ and the formulas in § 2.2, one finds that

$$b^2 = \frac{A_{-1}(a)}{A_1(a)} = \frac{\widetilde{A}_{-1}(a)}{\widetilde{A}_1(a)}.$$

Let ν be the valuation at $X=0$ of $\frac{\widetilde{A}_{-1}(X)}{\widetilde{A}_1(X)}$. Lemma A.2 with $|a|<1$ gives $|b|^2 = |a|^\nu$. Note that $\widetilde{A}_1$ and $\widetilde{A}_{-1}$ are polynomial of degree at most two in X, so the integer ν belongs to $\{-2,-1,0,1,2\}$. We have

$$a' = \frac{\widetilde{B}_{-1}(b)}{\widetilde{B}_1(b)a}. \tag{4.4}$$

We will prove that $|a'| \geqslant 1$ with a case by case study of the values of ν.

Remember that

$$\begin{array}{lcl} \widetilde{A}_{-1} & = & d_{-1,-1} + d_{0,-1}x + d_{1,-1}x^2 \\ \widetilde{A}_1 & = & d_{-1,1} + d_{0,1}x + d_{1,1}x^2 \\ \widetilde{B}_{-1} & = & d_{-1,-1} + d_{-1,0}y + d_{-1,1}y^2 \\ \widetilde{B}_1 & = & d_{1,-1} + d_{1,0}y + d_{1,1}y^2. \end{array} \tag{4.5}$$

Case $\nu \geqslant 1$. Then, $|b| = |a|^{\nu/2} < 1$. Combining (4.4) and Lemma A.2, we find $|a||a'| = |b|^l$ where l is the valuation at $X=0$ of $\frac{\widetilde{B}_{-1}(X)}{\widetilde{B}_1(X)}$. This gives $|a'| = |a|^{l\nu/2-1}$. Since l belongs to $\{-2,\dots,2\}$ and ν is in $\{1,2\}$, we get $-3 \leqslant l\nu/2-1 \leqslant 1$. If $l\nu/2-1$ equals 1 then ν must be equal to 2 and by (4.5), we must have $d_{-1,-1} = d_{0,-1} = 0$ and $d_{-1,1} \neq 0$. By Remark 2.5, we must have $d_{-1,0}d_{1,-1} \neq 0$ so that $l=1$ and $l\nu/2-1=0$. A contradiction. Then, $l\nu/2-1 \leqslant 0$ and $|a'| \geqslant 1$.

Case $\nu = 0$. Then, $|b|=1$. With Lemma A.3 and $|a|<1$, we obtain $|a'|>1$.

Case $\nu \leqslant -1$. Then $|b| = |a|^{\nu/2} > 1$. Combining (4.4) and Lemma A.2, we find $|a'| = |a|^{l\nu/2-1}$ where $l \in \{-2,\dots,2\}$ is the degree in X of $\frac{\widetilde{B}_{-1}(X)}{\widetilde{B}_1(X)}$. Since l belongs to $\{-2,\dots,2\}$ and ν is in $\{-1,-2\}$, we get $1 \geqslant l\nu/2-1 \geqslant -3$. If $l\nu/2-1=1$ then $\nu=-2$ and by (4.5), we must have $d_{-1,1} = d_{0,1} = 0$ and $d_{-1,-1} \neq 0$. By Remark 2.5, we must have $d_{-1,0}d_{1,1} \neq 0$ so that $l=-1$ and $l\nu/2-1=0$. A contradiction. Then, $l\nu/2-1 \leqslant 0$ and $|a'| \geqslant 1$.

Assume that either $d_{-1,1}=0$ or $d_{1,-1} \neq 0$ and let us prove that

$$\bigcup_{\ell\in\mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x) = C^*.$$

By Lemma A.4, there exists $(a_0,b_0) \in E$ such that $|a_0|=1$ and $\sigma(a_0,b_0) = (a_1,b_1)$ with $|a_1| \leqslant 1$. By Lemma 4.8, there exists $s_0 \in C^*$ with $|s_0| = |\mathfrak{D}^{(1)}(a)|^{\pm 1}$ such that $\overline{x}(s_0) = a_0$. Since $|q|^{1/2} \leqslant |\mathbf{q}| < |q|^{-1/2}$ and $|q|^{1/2} < |\mathfrak{D}^{(1)}(a)| < 1$, we find that $|q| < |\mathbf{q}s_0| < |q|^{-1/2}$. Since $|\overline{x}(\mathbf{q}s_0)| = |a_1| \leqslant 1$, we conclude using Lemma 4.8 that

- either $|\mathbf{q}s_0| \in \mathcal{U}_x$. This proves that

$$\mathcal{U}_x \cap \sigma_{\mathbf{q}}(\mathcal{U}_x) = [|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}] \cap \sigma_{\mathbf{q}}([|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}]) \neq \varnothing.$$

Since $|\mathbf{q}| \neq 1$, we deduce that

$$\bigcup_{\ell \in \mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x) = C^*.$$

- or $|\mathbf{q}s_0| \in [|q||\mathfrak{D}^{(1)}(a)|, |q||\mathfrak{D}^{(1)}(a)|^{-1}]$. Replacing $\mathbf{q}$ by $\mathbf{q}/q$ allows to conclude.
- or $|\mathbf{q}s_0| \in [|q|^{-1}|\mathfrak{D}^{(1)}(a)|, |q|^{-1}|\mathfrak{D}^{(1)}(a)|^{-1}]$. Replacing $\mathbf{q}$ by $q\mathbf{q}$ allows to conclude.

The proof for $\mathcal{U}_y$ is obtained by a symmetry argument using Lemma A.4 and Remark 4.9. □

According to Lemma 4.10, we define some auxiliary functions as follows:

- if $d_{-1,1} = 0$, we define, for $i = 1, 2$, the function $\widetilde{F}^i(s)$ on $\mathcal{U}_x$ as $F^i(\phi(s), t)$;
- if $d_{-1,1} \neq 0$, the function $\widetilde{F}^i(s)$ is defined on $\mathcal{U}_y$ as $F^i(\phi_y(s), t)$.

A priori the auxiliary functions $\widetilde{F}^1(s), \widetilde{F}^2(s)$ are defined on $\mathcal{U}_x$ if $d_{-1,1} = 0$ and on $\mathcal{U}_y$ otherwise. Theorem 4.12 below shows that one can meromorphically continue the functions $\widetilde{F}^i(s)$ on C^* so that they satisfy some nonhomogeneous rank 1 linear $\mathbf{q}$-difference equations.

Theorem 4.12. — *The auxiliary functions $\widetilde{F}^1(s), \widetilde{F}^2(s)$ can be continued meromorphically on C^* so that they satisfy*

$$\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1$$

and

$$\widetilde{F}^2(\mathbf{q}s) - \widetilde{F}^2(s) = b_2,$$

where $b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$ and $b_2 = (y(\mathbf{q}s) - y(s))x(s)$ are two q-periodic meromorphic functions over C^*.

Proof. — The proof is completely similar to the proof of Lemma 3.6 and relies on the fact that either the $\mathbf{q}$-orbit of $\mathcal{U}_x$ or the $\mathbf{q}$-orbit of $\mathcal{U}_y$ covers C^*. □

Note that by Remark 4.4, the coefficients b_1, b_2 of the $\mathbf{q}$-difference equation can be identified with rational functions on the algebraic curve E. A direct corollary of Theorem 4.12 is that the C_q-algebra $C_q[\widetilde{F}^1(s), \widetilde{F}^2(s)]$, generated by the solutions $\widetilde{F}^i(s)$, is contained in the field of meromorphic functions over C^*. In that field, the elements fixed by $\sigma_{\mathbf{q}}$ are precisely the elements of $C_{\mathbf{q}}$. These elliptic functions can be easily handled by the differential Galois theory of the appendix D. If the $\mathbf{q}$-orbit of the neighborhoods $\mathcal{U}_x$ and $\mathcal{U}_y$ was only a proper open subset $\mathcal{U}$ of C^*, then one would have to take into consideration "$\sigma_{\mathbf{q}}$-constants " whose algebraic complexity could not be controlled a priori. These $\sigma_{\mathbf{q}}$-constants are the meromorphic functions over $\mathcal{U}$ fixed by $\sigma_{\mathbf{q}}$. If, for instance, $|\mathbf{q}| = 3$ and $\mathcal{U} = \bigcup_{l \in \mathbb{Z}} \sigma_{\mathbf{q}}^l(\{s | \frac{1}{2} < |s| < 1\})$ then the ring of meromorphic functions over the disconnected set $\mathcal{U}$ which are fixed by $\sigma_{\mathbf{q}}$ can be identified with the ring of meromorphic functions on the fundamental annulus $\{s | \frac{1}{2} < |s| < 1\}$. The latter ring contains highly differentially transcendental functions such as restricted Gamma functions. These differentially transcendental $\sigma_{\mathbf{q}}$-constants render impossible the use of difference Galois theory to obtain applicable results.

4.4. **Differential transcendence.** The strategy to study the differential transcendence of generating functions of nondegenerate weighted models of genus one with infinite group is similar to the one employed in § 3. One first relates the differential behavior of the incomplete generating functions to the differential algebraic properties of their associated auxiliary functions. Then, one applies to these auxiliary functions the Galois theory of $\mathbf{q}$-difference equations. However, since the coefficients of the $\mathbf{q}$-difference equations satisfied by the auxiliary functions are no longer rational but elliptic, the Galoisian criteria as well as the descent method to obtain some "simple telescopers" are quite technical and postponed to Appendix C. Theorem 4.13 below gives a first criteria to guaranty the differential transcendence of the incomplete generating function.

Theorem 4.13. — *Assume that the weighted model is nondegenerate, of genus one, and that the group of the walk is infinite. If $Q(x,0,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$ then there exist $c_0,\dots,c_n\in C$ not all zero and $h\in C_q$ such that*

$$c_0 b_1 + c_1\partial_s(b_1)+\dots+c_n\partial_s^n(b_1)=\sigma_{\mathbf{q}}(h)-h. \tag{4.6}$$

A symmetrical result holds for $Q(0,y,t)$ replacing b_1 by b_2.

Proof. — Since the group of the walk is of infinite order, the automorphism σ is of infinite order. Therefore by Lemma 4.7 the elements $\mathbf{q}$ and q defined in Proposition 4.5 are multiplicatively independent. Assume that $Q(x,0,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. Let $\widetilde{F}^1(s)$ be the auxiliary function defined above.

We denote by $C_{\mathbf{q}}.C_q$ the compositum of the fields C_q and $C_{\mathbf{q}}$ inside the field of meromorphic functions over C^*, that is, the smallest subfield of $\mathcal{M}er(C^*)$ that contains C_q and $C_{\mathbf{q}}$. We claim that $\widetilde{F}^1(s)$ is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. Let us prove this claim when $d_{-1,1}=0$, the proof when $d_{-1,1}\neq 0$ being similar. Reasoning as in Lemma 3.7, one can show that, for $n,m\in\mathbb{N}$ and $s\in\mathcal{U}_x$, one has

$$(\partial_t^n\partial_x^m F^1)(\overline{x}(s),t)=\frac{1}{\partial_s(\overline{x}(s))^m}\Delta_{t,q}^n\partial_s^m(\widetilde{F}^1(s))+\sum_{i\leqslant 2n+m,j<n} r_{i,j}\Delta_{t,q}^j\partial_s^i(\widetilde{F}^1(s)),$$

where $r_{i,j}\in C_{\mathbf{q}}(\ell_{\mathbf{q}})(\overline{x}(s),\partial_s^l\partial_t^k(\overline{x}(s)),\dots)$. By construction, $\overline{x}(s)$ is in C_q so that Lemma D.5 implies that $\partial_s^l\partial_t^k(\overline{x}(s))\in C_q(\ell_q)$ for any positive integers k,l. Then, the field $C_{\mathbf{q}}(\ell_{\mathbf{q}})(\overline{x}(s),\partial_s^l\partial_t^k(\overline{x}(s)),\dots)$ generated by $\overline{x}$ and its derivatives with respect to ∂_s and ∂_t is contained in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. Thus, any nontrivial polynomial relation between the x and t-derivatives of $Q(x,0,t)$ yields a nontrivial polynomial relation between the derivatives of $\widetilde{F}^1(s)$ with respect to ∂_s and $\Delta_{t,\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. This proves the claim.

By Theorem 4.12, the function $\widetilde{F}^1(s)$ satisfies $\widetilde{F}^1(\mathbf{q}s)-\widetilde{F}^1(s)=b_1(s)$ with $b_1(s)\in C_q\subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. Since $\widetilde{F}^1(s)$ is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$, Proposition D.6 and Corollary D.13 imply that there exist $m\in\mathbb{N}$ and $d_0,\dots,d_m\in C_{\mathbf{q}}$ not all zero and $g\in C_{\mathbf{q}}.C_q(\ell_q)$ such that

$$d_0b_1+d_1\partial_s(b_1)+\dots+d_m\partial_s^m(b_1)=\sigma_{\mathbf{q}}(g)-g.$$

Since b_1 is in C_q, Lemma D.14 allows to perform a descent on the coefficients of the telescoping relation above. Thus, there exist $c_0,\dots,c_n\in C$ not all zero and $h\in C_q$ such that

$$c_0b_1+c_1\partial_s(b_1)+\dots+c_n\partial_s^n(b_1)=\sigma_{\mathbf{q}}(h)-h.$$

This concludes the proof. The symmetry argument between x and y gives the proof for $Q(0,y,t)$. □

Theorem 4.13 has an easy corollary concerning the differential transcendence of the complete generating function for weighted models of genus one with infinite group.

Theorem 4.14. — *For any nondegenerate weighted model of genus one with infinite group, the following statements are equivalent:*

(1) *the series $Q(x,0,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$;*
(2) *the series $Q(x,0,t)$ is $\frac{d}{dx}$-differentially algebraic over $\mathbb{Q}$.*

Remark 4.15. — An analogous result holds for $Q(0,y,t)$ replacing the derivation $\frac{d}{dx}$ by $\frac{d}{dy}$.

Proof. — Since the group is infinite, the automorphism σ is of infinite order. Therefore by Lemma 4.7 the elements $\mathbf{q}$ and q defined in Proposition 4.5 are multiplicatively independent.

Assume that (1) holds. By Theorem 4.13, there exist $c_0,\dots,c_n\in C$ not all zero and $h\in C_q$ such that

$$c_0 b_1 + c_1\partial_s(b_1)+\cdots+c_n\partial_s^n(b_1)=\sigma_{\mathbf{q}}(h)-h. \tag{4.7}$$

Combining (4.7) with the functional equation satisfied by $\widetilde{F}^1(s)$ and using the commutativity of $\sigma_{\mathbf{q}}$ and ∂_s, one finds that

$$\sigma_{\mathbf{q}}\big[c_0\widetilde{F}^1(s)+\cdots+c_n\partial_s^n(\widetilde{F}^1(s))-h\big]=c_0\widetilde{F}^1(s)+\cdots+c_n\partial_s^n(\widetilde{F}^1(s))-h. \tag{4.8}$$

Then, there exists $g\in C_{\mathbf{q}}$ such that

$$c_0\widetilde{F}^1(s)+\cdots+c_n\partial_s^n(\widetilde{F}^1(s))-h=g.$$

Therefore, $\widetilde{F}^1(s)$ is ∂_s-differentially algebraic over $C_{\mathbf{q}}.C_q$. We claim that any element of C_q is ∂_s-differentially algebraic over C. Since σ_q and ∂_s commute, the field C_q is stable under ∂_s. Since C_q is of transcendence degree one over C, any element f in C_q is algebraically dependent with its derivative $\partial_s(f)$ over C. Similarly, any element of $C_{\mathbf{q}}$ is ∂_s-differentially algebraic over C and by Remark C.7, we obtain that $\widetilde{F}^1(s)$ is ∂_s-differentially algebraic over C. We assume that $d_{-1,1}=0$ so that $\widetilde{F}^1(s)$ coincides with $F^1(\overline{x}(s),t)$ for any $s\in\mathcal{U}_x$. The rest of the proof is entirely similar in the case $d_{-1,1}\neq 0$ if one replaces $\mathcal{U}_x$ with $\mathcal{U}_y$ and ϕ with ϕ_y. For δ_x the derivation $\frac{d}{dx}$, one finds that $(\delta_x F^1)(\overline{x}(s),t)=\frac{1}{\partial_s(\overline{x}(s))}\partial_s(\widetilde{F}^1(s))$ for any $s\in\mathcal{U}_x$. An easy induction shows that for any integer m and $s\in\mathcal{U}_x$, one has

$$(\delta_x^m F^1)(\overline{x}(s),t)=\sum_{i=1}^{m} a_{i,m}\partial_s^i(\widetilde{F}^1(s)),$$

where $a_{i,m}\in C(\partial_s^k\overline{x}(s))_{k\in\mathbb{N}}$ and $a_{m,m}=\frac{1}{\partial_s(\overline{x}(s))^m}$. Since $\overline{x}(s)$ and $\widetilde{F}^1(s)$ are ∂_s-differentially algebraic over C, we find that the transcendence degree of the field $C\big(a_{i,j},\partial_s^i(\widetilde{F}^1(s)),i,j\in\mathbb{N}\big)$ over C is finite. Since

$$C\big((\delta_x^m F^1)(\overline{x}(s),t),m\in\mathbb{N}\big)\subset C\big(a_{i,j},\partial_s^i(\widetilde{F}^1(s)),i,j\in\mathbb{N}\big)\subset \mathcal{M}er(\mathcal{U}_x),$$

there is a nontrivial algebraic relation with coefficients in C between the elements $F^1(\overline{x}(s),t),\dots,(\delta_x^m F^1)(\overline{x}(s),t),\dots$. By principle of isolated zeroes, we conclude that $F^1(x,t) = K(x,0,t)Q(x,0,t)$ is $\frac{d}{dx}$-differentially algebraic over C and therefore over $\mathbb{Q}$ by Remark C.7. This proves that (1) $\Rightarrow$ (2). Statement (2) implies obviously (1). $\square$

A corollary of Theorem 4.13 is that the $\frac{d}{dt}$-differential algebraicity of the series implies the $\frac{d}{dx}$-algebraicity of the series. One of the major breakthroughs of [2] is to show that for unweighted walks, the series was $\frac{d}{dx}$-differentially algebraic over $\mathbb{Q}$ if and only if the model was decoupled, that is, there exist $f, g \in \mathbb{Q}(t)(X)$ such that

$$xy = f(x) + g(y) \text{ modulo } K(x,y,t). \tag{4.9}$$

The authors of [2] used boundary value problems and the notion of analytic invariants to deduce from (4.9) a closed form of the generating series allowing them to conclude that the series was also $\frac{d}{dt}$-algebraic (see [2, §6]). Combining our result to [2], one finds the following corollary:

Corollary 4.16. — *If the walk is unweighted of genus one with infinite group, the following statements are equivalent:*

- *the generating series is* $\frac{d}{dx}$*-differentially algebraic over* $\mathbb{Q}$*;*
- *the generating series is* $\frac{d}{dt}$*-differentially algebraic over* $\mathbb{Q}$*.*

In a recent publication [18], M.F. Singer and the second author generalized the results of [2] and proved that a weighted model of genus one with infinite group was decoupled if and only if the series was $\frac{d}{dx}$-differentially algebraic. If a model is decoupled then there exists $h_i \in C_q$ such that $b_i(s) = \sigma_{\mathbf{q}}(h_i) - h_i$ for $i = 1, 2$. Thanks to this characterization, the first author was able to prove the converse implication of Theorem 1: the x-differential algebraicity of the generating series of a weighted model with genus one kernel curve and infinite group implies its t-differential algebraicity (see [8]). The first author proved that the same holds for the generating series associated to models with genus one kernel curve and finite group of the walk by using the explicit description of the generating series obtained in [12]. However, in that case the question of the holonomy with respect to the variable t is still open.

Remark 4.17. — In this paper, we give a unified viewpoint for the genus zero and genus one models and we proved that, in both cases, the behavior of the generating series is controlled by a $\mathbf{q}$-difference equation. One can view any genus zero model as a limit of genus one models. In terms of the Lambda modular function introduced in Remark 4.1, this limit process is equivalent to let λ tend to $0, 1$. In that setting, one could try to find a suitable path for λ and prove the convergence along this path of the analytic continuation of the generating series for the genus one models to the analytic continuation of the generating series of the genus zero model. However, such a question is very delicate and technical. It is close to the studies of confluence for $\mathbf{q}$-difference equations where the convergence of the solutions is investigated when $\mathbf{q}$ which goes to 1 (see for instance [7, 31]). In our situation, note that it is not $\mathbf{q}$ which goes to 1 but q.

Appendix A. Nonarchimedean estimates

In this section, we give some nonarchimedean estimates, which will be crucial to uniformize the kernel curve. Remember that C denotes the field of Hahn series in the variable t endowed with the nonarchimedean norm $|\ |$ corresponding to the valuation at t equal zero (see § 2.1).

A.1. **Discriminants of the kernel equation.** Lemma A.1 relates the genus of the kernel curve to the simplicity of the roots of the discriminant of the kernel polynomial. It also ensures the existence of a root with convenient norm estimates. Let us remind, see (2.1), that we have defined $\mathfrak{D}(x) := \Delta_x(x,1)$, where $\Delta_x(x_0,x_1)$ is the discriminant of the second degree homogeneous polynomial $y \mapsto \widetilde{K}(x_0,x_1,y,1,t)$.

Lemma A.1. — *For any nondegenerate weighted model of genus one, the following holds:*

- *all the roots of $\Delta_x(x_0,x_1)$ in $\mathbf{P}^1(C)$ are simple;*
- *the discriminant $\mathfrak{D}(x) := \Delta_x(x,1)$ has a root $a \in C$ such that $|a| < 1$, $|\mathfrak{D}^{(2)}(a) - 2| < 1$, and $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1$ where $\mathfrak{D}^{(i)}$ denotes the i-th derivative with respect to x of $\mathfrak{D}(x)$.*

A symmetric statement holds for $\Delta_y(y_0,y_1)$ by replacing $\mathfrak{D}$ by $\mathfrak{E}$.

Proof. — The first assertion is [10, Proposition 2.1]. As in (2.1), let us denote by α_i the coefficients of $\mathfrak{D}(x)$, that is, $\mathfrak{D}(x) = \sum_{j=0}^{4} \alpha_j x^j$. First, let us prove the existence of a root $a \in C$ of $\mathfrak{D}(x)$ such that $|a| < 1$. Suppose to the contrary that all the roots of $\mathfrak{D}(x)$ have a norm greater than or equal to 1. If α_0 is zero then zero is a root: a contradiction. Thus, we can assume that $\alpha_0 = \big(d_{-1,0}^2 - 4d_{-1,1}d_{-1,-1}\big)t^2$ is nonzero.

Let us first assume that $\alpha_4 \neq 0$. The product of the roots of $\mathfrak{D}(x)$ equals

$$\frac{\alpha_0}{\alpha_4} = \frac{t^2(d_{-1,0}^2 - 4d_{-1,-1}d_{-1,1})}{t^2(d_{1,0}^2 - 4d_{1,-1}d_{1,1})}.$$

Then we conclude that $|\frac{\alpha_0}{\alpha_4}| = 1$ so that each of the roots must have norm 1. Then, considering the symmetric functions of the roots of $\mathfrak{D}(x)$, we conclude that, for any $i = 0,\dots,3$, the element $\frac{\alpha_i}{\alpha_4}$ should have norm smaller than or equal to 1. Since

$$\frac{\alpha_2}{\alpha_4} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,-1}d_{0,1}t^2 - 4d_{1,-1}d_{-1,1}t^2 + 2d_{-1,0}d_{1,0}t^2 + d_{0,0}^2t^2 - 2td_{0,0} + 1}{t^2(d_{1,0}^2 - 4d_{1,-1}d_{1,1})}$$

has norm strictly greater than 1, we find a contradiction.

Assume now that $\alpha_4 = 0$. Since the roots of $\Delta_x(x_0,x_1)$ in $\mathbf{P}^1(C)$ are simple, the coefficient α_3 is nonzero. The product of the roots of $\mathfrak{D}(x)$ equals

$$-\frac{\alpha_0}{\alpha_3} = \frac{-t^2(d_{-1,0}^2 - 4d_{-1,-1}d_{-1,1})}{2t^2d_{1,0}d_{0,0} - 2td_{1,0} - 4t^2(d_{0,1}d_{1,-1} + d_{1,1}d_{0,-1})}.$$

Then, it is clear that $|\frac{\alpha_0}{\alpha_3}| \leqslant 1$ and that each of the roots has norm 1. Thus, the symmetric function $\frac{\alpha_2}{\alpha_3}$ should also have norm smaller than or equal to 1. But

$$-\frac{\alpha_2}{\alpha_3} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,-1}d_{0,1}t^2 - 4d_{1,-1}d_{-1,1}t^2 + 2d_{-1,0}d_{1,0}t^2 + d_{0,0}^2t^2 - 2td_{0,0} + 1}{2t^2d_{1,0}d_{0,0} - 2td_{1,0} - 4t^2(d_{0,1}d_{1,-1} + d_{1,1}d_{0,-1})}$$

has norm strictly bigger than 1. We find a contradiction again.

Let a be a root of $\mathfrak{D}(x)$ in C with $|a| < 1$. Since $a, \alpha_1, \alpha_3, \alpha_4$ have norm smaller than 1, $|\alpha_2 - 1| < 1$, and

- $\mathfrak{D}^{(1)}(a) = \alpha_1 + 2\alpha_2 a + 3\alpha_3 a^2 + 4\alpha_4 a^3$;

- $\mathfrak{D}^{(2)}(a) = 2\alpha_2 + 6\alpha_3 a + 12\alpha_4 a^2$;
- $\mathfrak{D}^{(3)}(a) = 6\alpha_3 + 24\alpha_4 a$;
- $\mathfrak{D}^{(4)}(a) = 24\alpha_4$,

we have $|\mathfrak{D}^{(2)}(a) - 2| < 1$, and $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1$. The statement for $\Delta_y(y_0, y_1)$ is symmetrical and we omit its proof. □

A.2. **Automorphisms of the walk on the domain of convergence.** In this section, we study the action of the group of the walk on the product of the unit disks in $\mathbf{P}^1(C) \times \mathbf{P}^1(C)$. This product is the fundamental domain of convergence of the generating function.

We need a preliminary lemma that explains how one can compute the norm of the values of a rational function.

Lemma A.2. — *Let $f \in C(X)$ be a nonzero rational function and let $a \in \mathbf{P}^1(C)$. Let ν (resp. d) be the valuation at $X = 0$ (resp. ∞) of f with the convention that $\nu = +\infty$, $d = -\infty$ if $f = 0$. The following statements hold:*

- *if $|a| < 1$, then $|f(a)| = |a|^{\nu}$;*
- *if $|a| > 1$, then $|f(a)| = |a|^{d}$.*

Proof. — If $f = 0$ the result is clear. Assume that f is nonzero. Let us prove the first case, the second being completely symmetrical. Let us write $f(X)$ as $\frac{\sum_{i=\nu_1}^{r_1} c_i X^i}{\sum_{j=\nu_2}^{r_2} d_j X^j}$ with $c_{\nu_1} d_{\nu_2} \neq 0$. If $k > l$, we note that $|a^k| < |a^l|$. Then

$$|f(a)| = \frac{|\sum_{i=\nu_1}^{r_1} c_i a^i|}{|\sum_{j=\nu_2}^{r_2} d_j a^j|} = |a|^{\nu_1 - \nu_2} = |a|^{\nu}. \quad \square$$

The following lemma explains how the fundamental involutions permute the interior and the exterior of the fundamental domain of convergence.

Lemma A.3. — *For any nondegenerate weighted model, the following statements hold:*

1. *for any $a \in C$ with $|a| = 1$, there exist $b_{\pm} \in \mathbf{P}^1(C)$ with $|b_-| < 1$, and $|b_+| > 1$, such that $K(a, b_{\pm}, t) = 0$;*
2. *for any $b \in C$ with $|b| = 1$, there exist $a_{\pm} \in \mathbf{P}^1(C)$ with $|a_-| < 1$, and $|a_+| > 1$, such that $K(a_{\pm}, b, t) = 0$.*

Proof. — See [12, §1.3] for a similar result in the situation where C is replaced by $\mathbb{C}$.

The statements are symmetrical, so we only prove the first one. Since C is algebraically closed and the model is nondegenerate, Proposition 1.5 implies that $K(x, y, t)$ is of degree 2 in y. Then, for any $a \in C$, there are two elements $b_{\pm} \in \mathbf{P}^1(C)$ such that $K(a, b_{\pm}, t) = 0$. Let $a \in C$ with $|a| = 1$. We write

$$K(a, y, t) = t\alpha + \beta y + t\gamma y^2 \tag{A.1}$$

where

- $\alpha = -\sum_{i=-1}^{1} d_{i,-1} a^{i+1}$;
- $\beta = a - t \sum_{i=-1}^{1} d_{i,0} a^{i+1}$;
- $\gamma = -\sum_{i=-1}^{1} d_{i,1} a^{i+1}$.

Since $|a| = 1$, we find $|\beta| = 1$, $|\alpha|, |\gamma| \leqslant 1$. First let us prove that there is no point $(a_0, b_0) \in E$ such that $|a_0| = |b_0| = 1$. Indeed, suppose to the contrary that $|a_0| = |b_0| = 1$ and $K(a_0, b_0, t) = 0$. Then, $|\beta| = |a_0| = 1$ and $|\gamma|, |\alpha| \leqslant 1$ so that the equality $|\beta b_0| = |t(\alpha + \gamma b_0^2)|$ implies $|b_0| < 1$. We find a contradiction. From the equation $K(a, b, t) = 0$, we deduce that

$$\text{if } |b| < 1, \text{ then } |t\alpha| = |\beta b + t\gamma b^2| = |\beta b| \text{ which gives } |b| = |t\alpha|; \tag{A.2}$$

$$\text{if } |b| > 1, \text{ then } |\frac{1}{b}| < 1 \text{ and we find } |t\gamma| = \left|\frac{t\alpha}{b^2} + \frac{\beta}{b}\right| = \left|\frac{\beta}{b}\right| = \left|\frac{1}{b}\right|. \tag{A.3}$$

Using $K(a, b_\pm, t) = 0$, we find

$$b_- b_+ = \frac{\alpha}{\gamma}, \tag{A.4}$$

with the convention that b_+ is $[1 : 0]$ if $\gamma = 0$. If $\gamma = 0$ then $b_- = \frac{-t\alpha}{\beta}$ has norm smaller than 1, which concludes the proof in that case. Assume now that $\gamma \neq 0$. Since $|b_+|$ and $|b_-|$ cannot have norm 1, we just need to discard the cases "$|b_+| < 1$ and $|b_-| < 1$" or "$|b_+| > 1$ and $|b_-| > 1$". If $\alpha = 0$, then one of the roots is zero, say $b_- = 0$, and $|b_+| = \frac{|\beta|}{|t\gamma|} > 1$, which concludes the proof in that case. If $\alpha \neq 0$ then one can suppose to the contrary that $|b_+| < 1$ and $|b_-| < 1$. From (A.2), we obtain $|b_+| = |b_-| = |t\alpha|$, which gives

$$|b_+ b_-| = |t\alpha|^2 = \frac{|\alpha|}{|\gamma|}.$$

Then, $|t^2\alpha| = \frac{1}{|\gamma|} \geqslant 1$, which contradicts $|t^2\alpha| < 1$. Suppose to the contrary that $|b_+| > 1$ and $|b_-| > 1$. By (A.3), $|b_+| = |b_-| = \frac{1}{|t\gamma|}$ which gives

$$|b_+ b_-| = \frac{1}{|t\gamma|^2} = \frac{|\alpha|}{|\gamma|}.$$

Thus, $|t^2\alpha| = \frac{1}{|\gamma|} \geqslant 1$, and once again, we find a contradiction. □

Lemma A.4 explains how the intersection of the fundamental domain of convergence of the generating function and its image by σ is nonempty. This result is therefore crucial in order to continue the generating function to the whole C^*.

Lemma A.4. — *For any nondegenerate weighted model, the following statements hold:*

- *if $d_{-1,1} = 0$ or $d_{1,-1} \neq 0$ there exists $(a, b) \in E$ with $|a| = 1$ such that $\sigma(a, b) = (a', b')$ with $|a'| \leqslant 1$;*
- *if $d_{-1,1} \neq 0$ or $d_{1,-1} = 0$ there exists $(a, b) \in E$ with $|b| = 1$ such that $\sigma(a, b) = (a', b')$ with $|b'| \leqslant 1$.*

Proof. — Using the symmetry between x and y mentioned in Remark 1.3, we only prove the first statement of Lemma A.4.

Let $a \in \mathbf{P}^1(C)$ such that $|a| = 1$. By Lemma A.3, there exist $b_+ \in \mathbf{P}^1(C)$ with $|b_+| > 1$ and $b_- \in C$ with $|b_-| < 1$ such that $(a, b_\pm) \in E$. Let B_i as in (1.2) and note that by Proposition 1.5, B_1 is not identically zero. Let ν (resp. d) be the valuation at 0 (resp. ∞) of the rational fraction $\frac{B_{-1}(y)}{B_1(y)} = \frac{\sum_{j=-1}^{1} d_{-1,j} y^j}{\sum_{j=-1}^{1} d_{1,j} y^j} \in C(y)$. We claim that either $\nu \geqslant 0$ or $d \leqslant 0$. If $d_{1,-1} \neq 0$ then $\nu \geqslant 0$. If $d_{-1,1} = 0$ then either $d \leqslant 0$ or $d = 1$. In the latter situation, we must have $d_{1,1} = d_{1,0} = 0$ and $d_{-1,0} \neq 0$. Since the model is nondegenerate, we must have $d_{1,-1} \neq 0$ by Proposition 1.5. In that case, $\nu \geqslant 0$. This proves the claim.

Let $a_+, a_- \in \mathbf{P}^1(C)$ such that $\iota_2(a, b_+) = (a_+, b_+)$ and $\iota_2(a, b_-) = (a_-, b_-)$. This gives

$$a_+ = \frac{B_{-1}(b_+)}{B_1(b_+)a} \text{ and } a_- = \frac{B_{-1}(b_-)}{B_1(b_-)a}. \tag{A.5}$$

Since $\sigma(a, b_-) = (a_+, b_+)$ (resp. $\sigma(a, b_+) = (a_-, b_-)$), it is enough to prove that either a_+ or a_- has norm smaller or equal to 1. If $d \leqslant 0$, we combine (A.5), Lemma A.2 and $|b+| > 1$ to find $|aa_+| = |a_+| = |b_+|^d \leqslant 1$. If $\nu \geqslant 0$, we combine (A.5), Lemma A.2 and $|b_-| < 1$ to find $|aa_-| = |a_-| = |b_-|^\nu \leqslant 1$. This ends the proof. □

Appendix B. Tate curves and their normal forms

Let $(C, | \ |)$ be a complete nonarchimedean algebraically closed valued field of zero characteristic and let $q \in C$ such that $0 < |q| < 1$. In this section, we recall some of the basic properties of elliptic curves over nonarchimedean fields. The period lattice is here replaced by a discrete multiplicative group of the form $q^{\mathbb{Z}}$. Then, the quotient of $\mathbb{C}$ by a period lattice is replaced by the so called *Tate curve*, which corresponds to the naive quotient of the multiplicative group C^* by $q^{\mathbb{Z}}$. However, in the nonarchimedean context, only elliptic curves with J-invariant of norm greater than or equal to one can be analytically uniformized by Tate curves (see Proposition B.2). The analytic geometry behind is the rigid analytic geometry as developed in [16]. We will not introduce this theory here but we just recall briefly the algebraic geometrical and special functions aspects of Tate curves.

B.1. Special functions on a Tate curve.

We recall that any holomorphic function f on C^* can be represented by an everywhere convergent Laurent series $\sum_{n\in\mathbb{Z}} a_n s^n$ with $a_n \in C$. Moreover any nonzero meromorphic function on C^* can be written as $\frac{g}{h}$ such that the holomorphic functions g and h have no common zeros. We shall denote by $\mathcal{M}er(C^*)$ the field of meromorphic functions over C^*.

Remark B.1. — If k is a complete nonarchimedian sub-valued field of C and q belongs to k, every result quoted above still holds over k.

The analytification of the elliptic curve E_q is isomorphic to the Tate curve, which is the rigid analytic space corresponding to the naive quotient of $C^*/q^{\mathbb{Z}}$. The curve E_q is therefore a“canonical” elliptic curve. A natural question is "Given an elliptic curve E defined over C, is there a q such that E is isomorphic to E_q?" The answer is positive under certain assumption on the J-invariant $J(E)$ of E.

Proposition B.2 (Theorem 5.1.18 in [16] and VII p. 31 in [30]). — *Let E be an elliptic curve over C such that $|J(E)| > 1$. Then, there exists one and only one $q \in C$ such that $0 < |q| < 1$ and $J(E) = J(E_q)$, that is, E is isomorphic to the elliptic curve E_q.*

Remind that we have defined $s_k = \sum_{n>0} \frac{n^k q^n}{1-q^n} \in C$ for $k \geqslant 1$, and

$$X(s) = \sum_{n\in\mathbb{Z}} \frac{q^n s}{(1-q^n s)^2} - 2s_1, \quad Y(s) = \sum_{n\in\mathbb{Z}} \frac{(q^n s)^2}{(1-q^n s)^3} + s_1.$$

They are q-periodic meromorphic functions over C^*. By Proposition 4.2, the field C_q of q-periodic meromorphic functions over C^* coincides with the field generated over C by $X(s)$ and $Y(s)$.

Since we need to understand what is the pullback of the fundamental domain of convergence of the generating function via this uniformization, we prove some basic properties on the norm of $X(s)$. Remind that $X(s) = X(1/s)$ and $X(qs) = X(s)$. Thus it suffices to study $|X(s)|$ for $|q|^{1/2} \leqslant |s| \leqslant 1$. The following study follows the arguments of [32, §V.4].

Lemma B.3. — *Let $s \in C^*$. The following holds:*

- *If $|q|^{1/2} < |s| < 1$, then $|X(s)| = |s|$;*
- *If $|s| = 1$, then $|X(s)| \geqslant 1$;*
- *If $|s| = |q|^{1/2}$, then $|X(s)| \leqslant |s|$.*

Proof. — Since $X(s)$ has a pole in $s = 1$ we may further assume that $s \neq 1$. Let us rewrite $X(s)$:

$$X(s) = \frac{s}{(1-s)^2} + \sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}.$$

This means that we have

$$|X(s)| \leqslant \max\left(\Big|\frac{s}{(1-s)^2}\Big|, \Big|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\Big|\right), \tag{B.1}$$

with equality when $|\frac{s}{(1-s)^2}| \neq |\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}|$. Let us consider $s \in C^* \setminus \{1\}$ with $|q|^{1/2} \leqslant |s| \leqslant 1$. Using $|q| < 1$ we find that $|q^n s| \leqslant |qs| < 1$ for every $n \geqslant 1$. This shows that the norm of $q^n s$ is strictly smaller than 1. Then, $\big|\frac{q^n s}{(1-q^n s)^2}\big| = |q^n s| < |s|$. On the other hand, $|q^n| \leqslant |q| < |s|$ and $|\frac{q^n}{1-q^n}| < |s|$. Finally, when $|q|^{1/2} < |s|$, we have $|q^n s^{-1}| \leqslant |qs^{-1}| < |qq^{-1/2}| < |s|$ and therefore $\big|\frac{q^n s^{-1}}{(1-q^n s^{-1})^2}\big| = |q^n s^{-1}| < |s|$. This proves that, for any $s \in \mathbf{P}^1(C)$ such that $|q|^{1/2} < |s| \leqslant 1$, we have

$$\Big|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\Big| < |s|. \tag{B.2}$$

When $|q|^{1/2} = |s|$ and $n \geqslant 2$, we have $|q^n s^{-1}| \leqslant |q^2 s^{-1}| = |q^2 q^{-1/2}| < |s|$, whence $\big|\frac{q^n s^{-1}}{(1-q^n s^{-1})^2}\big| = |q^n s^{-1}| < |s|$. Moreover, if $|q|^{1/2} = |s|$ then $|qs^{-1}| = |qq^{-1/2}| = |s|$. Therefore $\big|\frac{qs^{-1}}{(1-qs^{-1})^2}\big| = |qs^{-1}| = |s|$. We conclude that when $|q|^{1/2} = |s|$,

$$\Big|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\Big| = |s|. \tag{B.3}$$

It remains to consider the term $\frac{s}{(1-s)^2}$. If $|s| < 1$ then we have $\big|\frac{s}{(1-s)^2}\big| = |s|$. Combining with (B.1), (B.2) and (B.3) respectively, we obtain the result when $|q|^{1/2} < |s| < 1$ and $|q|^{1/2} = |s| < 1$ respectively. If $|s| = 1$ and $s \neq 1$ then $|1 - s| \leqslant 1$. Thus, $\big|\frac{s}{(1-s)^2}\big| \geqslant |s| = 1$, which, combined with (B.1) concludes the proof. □

B.2. **Tate and Weierstrass normal forms.** In [12], the authors generalize the results of [23] and attach a Weierstrass normal form to the kernel curve. The following proposition proves that, with some care, their result passes to a nonarchimedean framework.

Let us consider a nondegenerate weighted model of genus one and let us write its kernel polynomial as follows: $K(x,y,t) = \widetilde{A}_0(x) + \widetilde{A}_1(x)y + \widetilde{A}_2(x)y^2 = \widetilde{B}_0(y) + \widetilde{B}_1(y)x + \widetilde{B}_2(y)x^2$ with $\widetilde{A}_i(x) \in C[x]$ and $\widetilde{B}_i(y) \in C[y]$. The following proposition gives a Weierstrass normal form for the kernel curve.

Proposition B.4. — *Let $a \in C$ be as in Lemma A.1. Let E_1 be the elliptic curve defined by the Weierstrass equation*

$$y_1^2 = 4x_1^3 - g_2x_1 - g_3, \tag{B.4}$$

with

$$\begin{aligned} g_2 &= \frac{\mathfrak{D}^{(2)}(a)^2}{3} - 2\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(3)}(a)}{3} \\ g_3 &= -\frac{\mathfrak{D}^{(2)}(a)^3}{27} + \frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)}{9} - \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(4)}(a)}{6}. \end{aligned} \tag{B.5}$$

Then, the rational map

$$\begin{array}{ccc} E_1 & \to & E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) \\ [x_1 : y_1 : 1] & \mapsto & (\overline{x}, \overline{y}) \end{array}$$

where

$$\overline{x} = a + \frac{\mathfrak{D}^{(1)}(a)}{x_1 - \frac{\mathfrak{D}^{(2)}(a)}{6}} \text{ and } \overline{y} = \frac{\frac{\mathfrak{D}^{(1)}(a)y_1}{2(x_1 - \frac{\mathfrak{D}^{(1)}(a)}{6})^2} - \widetilde{A}_1\Big(a + \frac{\mathfrak{D}^{(1)}(a)}{x_1 - \frac{\mathfrak{D}^{(2)}(a)}{6}}\Big)}{2\widetilde{A}_2\Big(a + \frac{\mathfrak{D}^{(1)}(a)}{x_1 - \frac{\mathfrak{D}^{(2)}(a)}{6}}\Big)},$$

is an isomorphism of elliptic curves that sends the point $\mathcal{O} = [1:0:0]$ in E_1 to the point $\left(a, \dfrac{-\widetilde{A}_1(a)}{2\widetilde{A}_2(a)}\right) \in E$.

Proof. — This is the same proof as in [12, Proposition 2.1]. Note that there is only one configuration here since we have chosen a root of the discriminant $|a| < 1$ which can not be infinity. ☐

We recall that the J-invariant $J(E_1)$ of the elliptic curve E_1 given in a Weierstrass form $y_1^2 = 4x_1^3 - g_2x_1 - g_3$ equals $J(E_1) = 12^3 \frac{g_2^3}{g_2^3 - 27g_3^2}$. For a weighted model of genus one, the J-invariant $J(E)$ of the kernel curve has modulus strictly greater than 1 by Lemma 2.6. Since $J(E) = J(E_1)$, Proposition B.2 shows that there exists $q \in C^*$ such that $0 < |q| < 1$ and E_1 is isomorphic to E_q. In order to make explicit this isomorphism, we need to understand how one passes from a Tate normal form to a Weierstrass normal form. This is the content of the following lemmas.

Lemma B.5. — *[§6, p. 29 in [30]] In the notation of Proposition 4.2, the change of variable $X = x - \frac{1}{12}$ and $Y = \frac{1}{2}(y - x + \frac{1}{12})$ maps the Tate equation*

$$Y^2 + XY = X^3 + BX + \widetilde{C}$$

onto the Weierstrass equation

$$y^2 = 4x^3 - h_2x - h_3,$$

where $h_2 = \frac{1}{12} + 20s_3$ *and* $h_3 = \frac{-1}{6^3} + \frac{7}{3}s_5$.

As detailed above, the elliptic curves E_1 and E_q are isomorphic. The following lemma gives the form of an explicit isomorphism between theses two curves.

Lemma B.6. — *Let* $y^2 = 4x^3 - h_2x - h_3$ *be the Weierstrass normal form (resp.* $Y^2 + XY = X^3 + BX + \widetilde{C}$ *its Tate normal form) of* E_q *as in Lemma B.5 and let* $y_1^2 = 4x_1^3 - g_2x_1 - g_3$ *be the Weierstrass normal form of* E_1 *as in Proposition B.4.*
There exists $u \in C^*$ *such that the following map*

$$\begin{array}{lcl} E_q & \to & E_1, \\ (X,Y) & \mapsto & (u^2(X + \frac{1}{12}), u^3(2Y + X)) \end{array}$$

is an isomorphism of elliptic curves. Moreover, the following holds

- $h_2 = \frac{g_2}{u^4}$ *and* $h_3 = \frac{g_3}{u^6}$;
- $\Delta_q = \frac{\Delta_1}{u^{12}}$ *where* Δ_1 *and* Δ_q *denote the discriminants of the Weierstrass equations of* E_1 *and* E_q *respectively.*

Proof. — From [33, Proposition 3.1, Chapter III], we deduce that any isomorphism between the elliptic curves E_1 and E_q is given by $x_1 = u^2x + \alpha$ and $y_1 = u^3y + \beta u^2x + \gamma$ with $u \in C^*$, $\alpha, \beta, \gamma \in C$. Since both equations are in Weierstrass normal form, we necessarily have $\alpha = \beta = \gamma = 0$. This proves the first point. From $y_1^2 = 4x_1^3 - g_2x_1 - g_3$, we substitute x_1, y_1 by x, y to find

$$u^6y^2 = 4u^6x^3 - g_2u^2x - g_3.$$

Dividing the both sides by u^6 we find $h_2 = \frac{g_2}{u^4}$ and $h_3 = \frac{g_3}{u^6}$. The assertion on the discriminants follows from $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$. □

The lemma below gives some precise estimate for the norms of $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$, the discriminants of the elliptic curves E_q, E_1, and the element u defined in Lemma B.6.

Lemma B.7. — *The following statements hold:*

- $|\Delta_q| = |q|$, *with* $|h_2 - \frac{1}{12}| = |q|$ *and* $|h_3 - (-\frac{1}{6^3})| = |q|$;
- $|\Delta_1| = |q|$ *with* $|g_2 - \frac{4}{3}| < 1$, $|g_3 - (-\frac{8}{27})| < 1$;
- $|u| = 1$;
- $|\mathfrak{D}^{(1)}(a)| \in]\,|q|^{1/2}, 1[$.

Proof. — Following [30, p. 29-30], we find $|\Delta_q| = |q|, |s_3| = |q| = |s_5|$. Combining the latter norm estimates with Lemma B.5, we find $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - (-\frac{1}{6^3})| = |q|$.

Let us prove the second point. It follows from (2.2) that $|1-\alpha_2| < 1$ and $|\alpha_i| < 1$ for $i = 0, 1, 3, 4$. By Lemma A.1, $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1, |\mathfrak{D}^{(2)}(a) - 2| < 1$. Combining these norm estimates with (B.5), we find $|g_2 - \frac{4}{3}| < 1$, $|g_3 - (-\frac{8}{27})| < 1$. Since $|J(E_1)| = |J(E_q)| = |\frac{12^3 g_2^3}{\Delta_1}| = |\frac{12^3 h_2^3}{\Delta_q}|$ and $|g_2| = |h_2| = 1$, we find $|\Delta_q| = |\Delta_1| = |q|$. By Lemma B.6, $\Delta_q = \frac{\Delta_1}{u^{12}}$, and then $|u| = 1$.

Let us prove the last point. Let us expand $\Delta_1 = g_2^3 - 27g_3^2$ with the expression of g_2, g_3 given in (B.5):

$$\begin{aligned}\Delta_1 &= \left(\frac{\mathfrak{D}^{(2)}(a)^2}{3} - 2\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(3)}(a)}{3}\right)^3 \\ &\quad - 27\left(\frac{-\mathfrak{D}^{(2)}(a)^3}{27} + \frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)}{9} - \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(4)}(a)}{6}\right)^2 \\ &= \frac{\mathfrak{D}^{(2)}(a)^6}{27} - \frac{2\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)^4\mathfrak{D}^{(3)}(a)}{9} + \frac{4\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{9} \\ &\quad - \frac{8\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(3)}(a)^3}{27} - \frac{\mathfrak{D}^{(2)}(a)^6}{27} - \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{3} \\ &\quad - \frac{3\mathfrak{D}^{(1)}(a)^4\mathfrak{D}^{(4)}(a)^2}{4} + \frac{2\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)^4\mathfrak{D}^{(3)}(a)}{9} - \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^3\mathfrak{D}^{(4)}(a)}{3} \\ &\quad + \mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)\mathfrak{D}^{(4)}(a) \\ &= \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{9} - \frac{8\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(3)}(a)^3}{27} - \frac{3\mathfrak{D}^{(1)}(a)^4\mathfrak{D}^{(4)}(a)^2}{4} \\ &\quad - \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^3\mathfrak{D}^{(4)}(a)}{3} + \mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)\mathfrak{D}^{(4)}(a).\end{aligned}$$

Since $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1, |\mathfrak{D}^{(2)} - 2| < 1$, the previous expression is a sum of terms that are all strictly smaller in norm than $|\mathfrak{D}^{(1)}(a)|^2$. This proves that $|\Delta_1| = |q| < |\mathfrak{D}^{(1)}(a)|^2$. □

The following estimate will be required to uniformize the generating function.

Lemma B.8. — *In the notation of Theorem 4.3, we have* $|\frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}| < |\mathfrak{D}^{(1)}(a)|$.

Proof. — Using (B.5) and the norm estimate on the $\mathfrak{D}^{(i)}(a)$'s, we get

$$g_2 = \frac{\mathfrak{D}^{(2)}(a)^2}{3} + \mathfrak{D}^{(1)}(a)\omega, \quad g_3 = \frac{-\mathfrak{D}^{(2)}(a)^3}{27} + \mathfrak{D}^{(1)}(a)\omega', \tag{B.6}$$

where $|\omega|, |\omega'| < 1$. This proves that

$$\frac{g_3}{g_2} = \frac{-\mathfrak{D}^{(2)}(a)}{9} + \mathfrak{D}^{(1)}(a)\omega''$$

with $|\omega''| < 1$. Then, we find

$$\left|\frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}\right| = \left|\frac{u^2}{12} + \frac{3g_3}{2g_2} - \frac{3g_3}{2g_2} - \frac{\mathfrak{D}^{(2)}(a)}{6}\right| \leqslant \max\left(\left|\frac{u^2}{12} + \frac{3g_3}{2g_2}\right|, \left|\frac{3}{2}\mathfrak{D}^{(1)}(a)\omega''\right|\right).$$

Finally, with the norm estimate of Lemma B.7, it is sufficient to show that $|\frac{u^2}{12} + \frac{3g_3}{2g_2}| \leqslant |q| < \mathfrak{D}^{(1)}(a)$. By Lemma B.6, we have $\frac{u^2}{12} = \frac{g_3 h_2}{12 g_2 h_3}$. By Lemma B.7,

$|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - (-\frac{1}{6^3})| = |q|$. Then, by Lemma B.7 again, we find

$$\begin{aligned}\Big|\frac{u^2}{12} + \frac{3g_3}{2g_2}\Big| &= \Big|\frac{g_3 h_2}{12 g_2 h_3} + \frac{3g_3}{2g_2}\Big| = \Big|\frac{g_3}{g_2}\Big|\Big|\frac{h_2}{12h_3} + \frac{3}{2}\Big| = \Big|\frac{h_2 + 18h_3}{12h_3}\Big| \\ &= |h_2 + 18h_3| = \Big|\Big(h_2 - \frac{1}{12}\Big) + 18\Big(h_3 - \Big(-\frac{1}{6^3}\Big)\Big)\Big| \\ &\leqslant \max\Big(\Big|h_2 - \frac{1}{12}\Big|, \Big|h_3 - \Big(-\frac{1}{6^3}\Big)\Big|\Big) \leqslant |q|. \quad \square\end{aligned}$$

Appendix C. Difference Galois theory

In this section, we establish some criteria to guaranty the transcendence of functions satisfying a difference equation of order 1. This criteria is based on the Galois theory of difference fields as developed in [35] but generalizes some of the existing results in the literature, for instance the assumption that the field of constants is algebraically closed (see for instance Theorem C.9).

The algebraic framework of this section is difference algebra and more precisely the notion of difference fields. A difference field is a pair (K, σ) where K is a field and σ is an automorphism of K. The field σ-constants K^σ of (K, σ) is formed by the elements $f \in K$ such that $\sigma(f) = f$. An extension $(K, \sigma_K) \subset (L, \sigma_L)$ of difference fields is a field extension $K \subset L$ such that σ_L coincides with σ_K on K. If there is no confusion, we shall denote by σ the automorphisms σ_K and σ_L. For a complete introduction on difference algebra, we shall refer to [5].

C.1. **Rank one difference equations.** In this section, we focus on rank one difference equations.

Lemma C.1. — *Let $(K, \sigma) \subset (L, \sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $x \in L$. The following statements are equivalent*

1. *x is algebraic over K^σ;*
2. *there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$.*

Proof. — Assume that x is algebraic over K^σ. Then, σ induces a permutation on the set of roots of the minimal polynomial of x over K^σ. Thus, there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$. Conversely, if there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$, the polynomial $P(X) = \prod_{i=0}^{r-1}(X - \sigma^i(x)) \in L[X]$ is fixed by σ and thereby $P(X) \in L^\sigma[X] = K^\sigma[X]$. Since $P(x) = 0$, we have proved that x is algebraic over K^σ. $\square$

Lemma C.2. — *Let $(K, \sigma) \subset (L, \sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $f \in L$ and $0 \neq c \in K$, such that $\sigma(f) = f + c$. The following statements are equivalent*

1. *$f \in K$;*
2. *f is algebraic over K;*
3. *There exists $\alpha \in K$ such that $\sigma(\alpha) = \alpha + c$.*

Moreover, let $\overline{K}$ be the algebraic closure of K endowed with a structure of σ-field extension of K. For all $\alpha \in \overline{K}$, $i \in \mathbb{Z}$ we denote by α_i the element of $\overline{K}$ such that $\sigma^i(f - \alpha) = f - \alpha_i$. If f is transcendental over K then for $i, j \in \mathbb{Z}$ such that $i \neq j$, the elements α_j and α_i are distinct.

Proof. — Let us prove the first part of the proposition. The first statement implies trivially the second one. Assume that f is algebraic over K and let $P(X) = X^n + a_{n-1}X^{n-1} + \dots a_0 \in K[X] \setminus K$ be its minimal polynomial over K. Note that $n \neq 0$. Using $\sigma(f) - f = c$ and $P(f) = 0$, we find that $\sigma(P(f)) - P(f) = 0 = (nc + \sigma(a_{n-1}) - a_{n-1})f^{n-1} + b_{n-2}f^{n-2} + \dots + b_0$ with $b_i \in K$ for $i = 0, \dots, n-2$. By minimality of $P(X)$, we find that $\sigma(a_{n-1}) - a_{n-1} = -nc$ with $a_{n-1} \in K$. Then, $\sigma(\alpha) - \alpha = c$ with $\alpha = \frac{a_{n-1}}{-n} \in K$. We have shown that the second statement implies the third. Finally, assume that there exists $\alpha \in K$ such that $\sigma(\alpha) = \alpha + c$. With $\sigma(f) - f = c$, we find that $\sigma(\alpha - f) = \alpha - f$. This gives that $\alpha - f \in L^\sigma = K^\sigma$ and the element f belongs to K.

Now, let us assume that f is transcendental over K. Suppose to the contrary that there exist $\alpha \in \overline{K}$ and $i > j \in \mathbb{Z}$ such that

$$\alpha_i = (\sigma^i(\alpha) - c - \sigma(c) - \dots - \sigma^{i-1}(c)) = \alpha_j = (\sigma^j(\alpha) - c - \sigma(c) - \dots - \sigma^{j-1}(c)).$$

The latter equality gives $\sigma^r(\beta) - \beta = \gamma$ where $r = i - j > 0$, $\beta = \sigma^j(\alpha)$ and $\gamma = \sigma^{i-1}(c) + \dots + \sigma^j(c)$. Since α is algebraic over K, the same holds for β. Let $P(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in K[X] \setminus K$ be the minimal polynomial of β over K. Using the fact that $\sigma^r(\beta) - \beta = \gamma$ and the minimality of P, we conclude, as above, that $\sigma^r(a_{n-1}) - a_{n-1} = -n\gamma$, that is $\sigma^r(\tilde{\beta}) - \tilde{\beta} = \gamma$ where $\tilde{\beta} = \frac{a_{n-1}}{-n} \in K$. Combining this equality with $\sigma^r(\sigma^j(f)) - \sigma^j(f) = \gamma$, we find that $\tilde{\beta} - \sigma^j(f) \in L$ is fixed by σ^r. By Lemma C.1, this means that $\tilde{\beta} - \sigma^j(f)$ is algebraic over K^σ, which yields f algebraic over K. We find a contradiction. □

Lemma C.3. — *Let $(K, \sigma) \subset (L, \sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $f \in L$ and $0 \neq c \in K$, such that $\sigma(f) = f + c$. Assume that f is transcendental over K. If there exists $g \in K(f)$ such that $\sigma(g) - g \in K[f]$, then $g \in K[f]$.*

Proof. — Let $\overline{K}$ be an algebraic closure of K, endowed with a structure of σ-field extension of K. Since f is transcendental over K, we can write a partial fraction decomposition of $g \in \overline{K}(f)$. Let R be the largest integer such that there exists $\alpha \in \overline{K}$ so that the element $\frac{1}{(f-\alpha)^R}$ appears in the partial fraction decomposition of g. Suppose to the contrary that $R > 0$ and let $\alpha \in \overline{K}$ such that $\frac{1}{(f-\alpha)^R}$ appears in the partial fraction decomposition of g. We deduce from Lemma C.2 applied to K and f, that the elements $\{\alpha_i, i \in \mathbb{Z}\}$ are all distinct. Then, there exists N, the largest integer such that $\sigma^N(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of g. The element $\sigma^{N+1}(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of $\sigma(g)$. This proves that $\sigma^{N+1}(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of $\sigma(g) - g$. A contradiction with $\sigma(g) - g \in K[f]$. This proves that $g \in K[f]$. □

C.2. **Differential transcendence criteria.** In this section, a $(\sigma, \partial, \Delta)$-field K is a difference field (K, σ) endowed with two derivations ∂, Δ commuting with σ such that $\partial\Delta - \Delta\partial = c_K\partial$ with $c_K \in K^\sigma$. We assume that ∂ is nontrivial on K, that is, it is not the zero derivation. The element c_K has to be considered as part of the data of the notion of $(\sigma, \partial, \Delta)$-field. An extension of $(\sigma, \partial, \Delta)$-fields is an inclusion of two $(\sigma, \partial, \Delta)$-fields $(K, \sigma_K, \partial_K, \Delta_K) \subset (L, \sigma_L, \partial_L, \Delta_L)$ such that

- $K \subset L$ is a field extension;
- $\sigma_K, \partial_K, \Delta_K$ are the restrictions of $\sigma_L, \partial_L, \Delta_L$ to K;

- $c_K = c_L$.

If there is no confusion, we shall omit the subscripts ${}_K, {}_L$. If σ is the identity, we shall speak of (∂,Δ)-fields, (∂,Δ)-field extension for short.

Example C.4. — As we will see in § D, the following fields are (σ,∂,Δ)-fields, that correspond respectively to the framework of the genus zero and genus one kernel curve. Remind that $\sigma_{\mathbf{q}}$ denotes the automorphism of $\mathcal{M}er(C^*)$ defined by $f(s) \mapsto f(\mathbf{q}s)$ and $C_{\mathbf{q}}$ denotes the field of meromorphic functions fixed by $\sigma_{\mathbf{q}}$. In the two examples, we have $\Delta_{\mathbf{q},t} = \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}(s)\partial_s + \partial_t$ where $\ell_{\mathbf{q}}$ is the so called $\mathbf{q}$-logarithm. That is, an element of $\mathcal{M}er(C^*)$ satisfying $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$, and $c_K = \frac{\partial_t(\mathbf{q})}{\mathbf{q}}\partial_s(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$.

- Let $\mathbf{q} \in C^*$ with $|\mathbf{q}| \neq 1$. Then, the inclusion
$$(C_{\mathbf{q}}(s,\ell_{\mathbf{q}}),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}})$$
is an extension of (σ,∂,Δ)-fields.
- Let $\mathbf{q}$ and q two elements of C^* such that $|q|,|\mathbf{q}| \neq 1$, that are *multiplicatively independent*, that is, there are no $(r,l) \in \mathbb{Z}^2 \setminus (0,0)$ such that $q^r = \mathbf{q}^l$. Since $C_{\mathbf{q}} \subset \mathcal{M}er(C^*)$ and $C_q \subset \mathcal{M}er(C^*)$, we consider $C_{\mathbf{q}}.C_q \subset \mathcal{M}er(C^*)$, the field compositum of $C_{\mathbf{q}}$ and C_q inside $\mathcal{M}er(C^*)$. Then, the inclusion
$$(C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}})$$
is an extension of (σ,∂,Δ)-fields.

Definition C.5. — Let $(K,\partial,\Delta) \subset (L,\partial,\Delta)$. An element f in L is said to be (∂,Δ)-differentially algebraic over K if there exists $N \in \mathbb{N}$, such that the elements

- $\partial^i(f)$ for $i \leqslant N$ are algebraically dependent over K if Δ is a K-multiple of ∂;
- $\partial^i\Delta^j(f)$ for $i,j \leqslant N$ are algebraically dependent over K otherwise.

Otherwise, we will say that f is (∂,Δ)-transcendental over K.

Remark C.6. — Note that since $\partial\Delta - \Delta\partial = c\partial$ with $c \in K^\sigma \subset K$, the (∂,Δ)-field extension of K generated by some element $f \in L$ coincides with the field extension of K generated by the set $\{\partial^i\Delta^j(f),\ \text{for } i,j \in \mathbb{N}\}$.

Let us make a remark concerning the field of definition of the coefficients of the differential polynomials.

Remark C.7. — Let $(K,\partial,\Delta) \subset (K',\partial,\Delta) \subset (L,\partial,\Delta)$ and assume that K' is a field generated over K by elements that are (∂,Δ)-differentially algebraic over K. By [22, Proposition 8, p. 101], an element f in L is (∂,Δ)-differentially transcendental over K if and only if it is (∂,Δ)-differentially transcendental over K'.

The following lemma will be crucial in many arguments:

Lemma C.8. — *If $K \subset M$ is a σ-field extension such that $M^\sigma = K$ and $K \subset L$ is a σ-field extension with $L^\sigma = L$. Then M and L are linearly disjoint over K.*

Proof. — Let $c_1,\dots,c_r \in L$ be K-linearly independent elements, that become dependent over M. Up to a permutation of the c_i's, a minimal linear relation among these elements over M has the following form

$$c_1 + \sum_{i=2}^{r} \lambda_i c_i = 0, \tag{C.1}$$

with $\lambda_i \in M$ for $i = 2, \dots, r$. Computing $\sigma((\mathrm{C.1})) - (\mathrm{C.1})$, we find

$$\sum_{i=2}^{r} (\sigma(\lambda_i) - \lambda_i) c_i = 0.$$

By minimality, $\sigma(\lambda_i) = \lambda_i$ and $\lambda_i \in M^\sigma = K$. By K-linear independence of the c_i, we find that $\lambda_i = 0$ for $i = 2, \dots, r$ and then $c_1 = 0$. A contradiction. ☐

The following statement, whose proof is due to Michael Singer, is a version of an old theorem of Ostrowski [28, 21] and its proof follows the lines of the proof of [9, Proposition 3.6]. In this last paper, it was assumed that K^σ is algebraically closed, which is not the case in this article. One could use the powerful scheme-theoretic tools developed in [29] to prove the result in our more general setting. Instead we will argue in a more elementary way to reduce Theorem C.9 to the case where K^σ is algebraically closed.

Theorem C.9. — *Let $(K, \sigma, \partial, \Delta)$ be a $(\sigma, \partial, \Delta)$-field such that K^σ is relatively algebraically closed in K, that is there are no proper algebraic extension of K^σ inside K. Let $(L, \sigma, \partial, \Delta)$ be a $(\sigma, \partial, \Delta)$-ring extension of $(K, \sigma, \partial, \Delta)$. Let $f \in L$ and $b \in K$ such that $\sigma(f) = f + b$. If f is (∂, Δ)-differentially algebraic over K then there exist $\ell_1, \ell_2 \in \mathbb{N}$, $c_{i,j} \in K^\sigma$ not all zero and $g \in K$ such that*

$$\sum_{\substack{0 \leqslant i \leqslant \ell_1, \\ 0 \leqslant j \leqslant \ell_2}} c_{i,j} \partial^i \Delta^j (b) = \sigma(g) - g. \tag{C.2}$$

Furthermore, we may take $\ell_2 = 0$ in the case where ∂ and Δ are K-linearly dependent. We call (C.2) *a telescoping relation for b.*

The proof of this result depends on results from the Galois theory of linear difference equations and we will refer to [9, Appendix A] and the references given there for relevant facts from this theory. Let (K, σ) be a difference field and consider the system of difference equations

$$\sigma(y_0) - y_0 = b_0, \dots, \sigma(y_n) - y_n = b_n, \quad \text{with } b_0, \dots, b_n \in K. \tag{C.3}$$

Let us see (C.3) as a system $\sigma(Y) = AY$, where $A \in \mathrm{GL}_{2(n+1)}(K)$ is a diagonal bloc matrix $A = \mathrm{Diag}(A_0, \dots, A_n)$ with $A_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}$ which corresponds to the equation $\sigma(y_i) - y_i = b_i$. A Picard-Vessiot extension for $\sigma(Y) = AY$ is a difference ring extension (R, σ) of (K, σ) such that:

- there exists $U \in \mathrm{GL}_{2(n+1)}(R)$ such that $\sigma(U) = AU$;
- R is generated as a K-algebra by the entries of U and $\det(U)^{-1}$;
- R is a simple difference ring, that is, the σ-ideals of R are $\{0\}$ and R.

We will need the following result.

Lemma C.10 (Proposition A.9 in [9]). — *Assume that (K, σ) is a difference field with K^σ algebraically closed. Let R be a Picard-Vessiot extension for the system*

(C.3) and $z_0, \ldots, z_n \in R$ be solutions of this system. If $z_0, \ldots, z_n$ are algebraically dependent over K, then there exist $c_i \in K^\sigma$, not all zero, and $g \in K$ such that

$$c_0 b_0 + \ldots + c_n b_n = \sigma(g) - g.$$

Before proving Theorem C.9, we give a slight generalization of Lemma C.10.

Lemma C.11. — *Let (K, σ) be a difference field with K^σ relatively algebraically closed in K and let $b_0, \ldots, b_n$ be some elements in K. Let (L, σ) be a σ-ring extension of (K, σ). Let $z_0, \ldots, z_n \in L$ be solutions of $\sigma(z_i) - z_i = b_i$. If $z_0, \ldots, z_n$ are algebraically dependent over K, then there exist $c_i \in K^\sigma$, not all zero, and $g \in K$ such that*

$$c_0 b_0 + \ldots + c_n b_n = \sigma(g) - g.$$

Proof. — Let $\mathbf{k}$ be the algebraic closure of K^σ. We extend σ to be the identity on $\mathbf{k}$[4]. Under the assumption that K^σ is relatively algebraically closed, the ring $\widetilde{K} = K \otimes_{K^\sigma} \mathbf{k}$ is an integral domain and in fact is a field. We have $\widetilde{K}^\sigma = \mathbf{k}$. Let $\widetilde{L} = L \otimes_{K^\sigma} \mathbf{k}$. We then have a natural inclusion of $\widetilde{K} \subset \widetilde{L}$. Let $S = \widetilde{K}[z_0, \ldots, z_n] \subset \widetilde{L}$. It is easily seen that S is a σ-ring extension of $\widetilde{K}$. Let I be a maximal difference ideal in S and let $R = S/I$. For each $r = 0, \ldots, n$, let u_r be the image of z_r in R. Since $\widetilde{K}^\sigma = \mathbf{k}$ is algebraically closed and R is a simple difference ring, we have that R is a Picard-Vessiot ring for the system associated to $\sigma(y_r) - y_r = b_r$, $r = 0, \ldots, n$, over $\widetilde{K}$. The elements $u_0, \ldots, u_n$ are algebraically dependent over K and solutions of $\sigma(y_r) - y_r = b_r, r = 0, \ldots, n$. Lemma C.10 proves that there exist $c_i \in \mathbf{k}$, not all zero, and $g \in \widetilde{K}$ such that

$$\sum_{0 \leqslant i \leqslant n} c_i b_i = \sigma(g) - g.$$

Let $\{d_r\} \subset \mathbf{k}$ be a K^σ-basis of $\mathbf{k}$. By Lemma C.8, applied with $K = K^\sigma$, $L = \mathbf{k}$ and $M = K$, it is also a K-basis of $\widetilde{K}$. We may write each c_i and g as

$$c_i = \sum_r c_{i,r} d_r \text{ and } g = \sum_r g_r d_r$$

for some $c_{i,r} \in K^\sigma$ and $g_r \in K$. Since not all the c_i are zero, there exists r such that $c_{i,r}$ are not all zero. For this r, we have

$$\sum_{i \leqslant n} c_{i,r} b_i = \sigma(g_r) - g_r.$$

This yields the conclusion of the proof. □

Proof of Theorem C.9. — Assuming that f is (∂, Δ)-differentially algebraic over K, there is some finite set $\{\partial^{i_0} \Delta^{j_0}(f), \ldots, \partial^{i_n} \Delta^{j_n}(f)\} \subset L$ of elements that are algebraically dependent over K. Note that $j_k = 0$ for all k if Δ is K-linearly dependent from ∂. Since σ commutes with Δ and ∂, we have for all $r = 0, \ldots, n$,

$$\sigma(\partial^{i_r} \Delta^{j_r}(f)) - \partial^{i_r} \Delta^{j_r}(f) = \partial^{i_r} \Delta^{j_r}(b).$$

To conclude it remains to apply Lemma C.11 with $z_r = \partial^{i_r} \Delta^{j_r}(f)$ and $b_r = \partial^{i_r} \Delta^{j_r}(b)$ for $r = 0, \ldots, n$. □

[4] On the other hand, there is no unique procedure to extend a field automorphism of K^σ to the algebraic closure $\mathbf{k}$. Indeed, these extensions are controlled by the Galois group of the field $\mathbf{k}$ over K^σ.

Appendix D. Meromorphic functions on a Tate curve and their derivations

In this section we translate the galoisian criteria of Theorem C.9 in the context of elliptic function field. We start by defining the derivations. Studying the transcendence properties of the $\mathbf{q}$-logarithm, we then perform a descent on the field of coefficients and on the number of derivations involved in the telescoping relation.

D.1. **Derivation on nonarchimedean elliptic function fields.** Let $\mathbf{q} \in C^*$ such that $|\mathbf{q}| \neq 1$ and let $\sigma_{\mathbf{q}}$ denote the automorphism of $\mathcal{M}er(C^*)$ defined by $\sigma_{\mathbf{q}}(f(s)) = f(\mathbf{q}s)$. We denote by $C_{\mathbf{q}}$ the field of meromorphic functions fixed by $\sigma_{\mathbf{q}}$. By Proposition 4.2, it is the field of rational functions on the Tate curve $E_{\mathbf{q}}$ or $E_{1/\mathbf{q}}$, depending whether $|\mathbf{q}| < 1$ or $|\mathbf{q}| > 1$. In this section, we construct, as in [6, §2] a derivation of these functions that encode their t-dependencies and commute with $\sigma_{\mathbf{q}}$.

The fact that $\partial_s = s\frac{d}{ds}$ acts on $\mathcal{M}er(C^*)$, and its commutation with $\sigma_{\mathbf{q}}$ is straightforward. Unfortunately, the t-derivative of $\mathbf{q}$ may be nontrivial, implying a more complicated commutation rule between $\partial_t = t\frac{d}{dt}$ and $\sigma_{\mathbf{q}}$. More precisely, we have

$$\begin{aligned} &\partial_s \circ \sigma_{\mathbf{q}} = \sigma_{\mathbf{q}} \circ \partial_s; \\ &\partial_t \circ \sigma_{\mathbf{q}} = \tfrac{\partial_t(\mathbf{q})}{\mathbf{q}} \sigma_{\mathbf{q}} \circ \partial_s + \sigma_{\mathbf{q}} \circ \partial_t. \end{aligned}$$

The following statement holds.

Lemma D.1. — *The ∂_s-constants $\mathcal{M}er(C^*)^{\partial_s} = \{f \in \mathcal{M}er(C^*) | \partial_s(f) = 0\}$ of $\mathcal{M}er(C^*)$ are precisely the constant functions C.*

The next lemma introduces a twisted t-derivation that commutes with $\sigma_{\mathbf{q}}$. Remind that the q-logarithm $\ell_{\mathbf{q}}$ has been defined in § 3.3.

Lemma D.2 (Lemma 2.1 in [6]). — *The following derivations of $\mathcal{M}er(C^*)$*

$$\begin{cases} \partial_s \\ \Delta_{t,\mathbf{q}} = \frac{\partial_t(\mathbf{q})}{\mathbf{q}} \ell_{\mathbf{q}}(s) \partial_s + \partial_t, \end{cases}$$

commute with $\sigma_{\mathbf{q}}$. Moreover, we have

$$\partial_s \Delta_{t,q} - \Delta_{t,q} \partial_s = \frac{\partial_t(\mathbf{q})}{\mathbf{q}} \partial_s(\ell_{\mathbf{q}}) \partial_s,$$

where $\frac{\partial_t(\mathbf{q})}{\mathbf{q}} \partial_s(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$.

Remark D.3. — Note that since $\partial_s, \Delta_{t,\mathbf{q}}$ commute with $\sigma_{\mathbf{q}}$, we can differentiate the equation $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ to find $\sigma_{\mathbf{q}}(\partial_s(\ell_{\mathbf{q}})) = \partial_s(\ell_{\mathbf{q}})$ and $\sigma_{\mathbf{q}}(\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})) = \Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$. We then conclude that $\partial_s(\ell_{\mathbf{q}}), \Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$ belong to $C_{\mathbf{q}}$.

The link with the iterates of $\Delta_{t,\mathbf{q}}$ and the derivatives ∂_s, ∂_t is now made in the following lemma.

Lemma D.4. — *For any $i \in \mathbb{N}$, there exist $c_{j,k,l} \in C_{\mathbf{q}}$ such that*

$$\Delta_{t,\mathbf{q}}^i = \Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}} \ell_{\mathbf{q}}\Big)^i \partial_s^i + \sum_{k=0}^{i-1} \sum_{j=0}^{k} \sum_{l=0}^{i} c_{j,k,l} \ell_{\mathbf{q}}^j \partial_s^k \partial_t^l.$$

Proof. — Let us prove the result by induction on i. For $i=1$, this comes from the fact that $\Delta_{t,\mathbf{q}}=\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\partial_s+\partial_t$. Let us fix $i\in\mathbb{N}$ and assume that the result holds for i. We find

$$\Delta_{t,\mathbf{q}}^{i+1}=\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\partial_s+\partial_t\Big)\bigg(\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\Big)^i\partial_s^i+\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i}c_{j,k,l}\ell_{\mathbf{q}}^j\partial_s^k\partial_t^l\bigg),$$

that is

$$\begin{aligned}\Delta_{t,\mathbf{q}}^{i+1}&=\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\Big)^{i+1}\partial_s^{i+1}+\Delta_{t,\mathbf{q}}\bigg(\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\Big)^i\bigg)\partial_s^i+\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\Big)^i\partial_t\partial_s^i\\&\quad+\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i}\Delta_{t,\mathbf{q}}(c_{j,k,l}\ell_{\mathbf{q}}^j)\partial_s^k\partial_t^l+\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i}c_{j,k,l}\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}^{j+1}\partial_s^{k+1}\partial_t^l\\&\quad+\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i}c_{j,k,l}\ell_{\mathbf{q}}^j\partial_s^k\partial_t^{l+1}.\end{aligned}$$

Note that the commutation of $\sigma_{\mathbf{q}}$ with $\Delta_{t,\mathbf{q}}$ implies that $C_{\mathbf{q}}$ is stabilized by $\Delta_{t,\mathbf{q}}$. Since by Remark D.3, $\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$ belongs to $C_{\mathbf{q}}$, we get that, for any integer j, any $\tilde{c}\in C_{\mathbf{q}}$, we have $\Delta_{t,\mathbf{q}}(\tilde{c}(\ell_{\mathbf{q}})^j)=\Delta_{t,\mathbf{q}}(\tilde{c})(\ell_{\mathbf{q}})^j+\tilde{c}c(\ell_{\mathbf{q}})^{j-1}$ where $c=j\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})\in C_{\mathbf{q}}$. Therefore, with $\Delta_{t,\mathbf{q}}(\tilde{c})\in C_{\mathbf{q}}$, we find that $\Delta_{t,\mathbf{q}}(\tilde{c}(\ell_{\mathbf{q}})^j)\in C_{\mathbf{q}}[\ell_{\mathbf{q}}]$ is of degree at most j in $\ell_{\mathbf{q}}$. With $\frac{\partial_t(\mathbf{q})}{\mathbf{q}}, c_{j,k,l}\in C_{\mathbf{q}}$, this ends the proof. □

From now on, let us fix $q\in C^*$ with $|q|\neq 1$, that is multiplicatively independent of $\mathbf{q}$, that is there are no $(r,l)\in\mathbb{Z}^2\setminus(0,0)$ such that $q^r=\mathbf{q}^l$. Remind that $C_{\mathbf{q}}.C_q\subset\mathcal{M}er(C^*)$ is the compositum of fields and $\ell_{\mathbf{q}}\in\mathcal{M}er(C^*)$ is a solution of $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}})=\ell_{\mathbf{q}}+1$. We now give examples of difference differential fields for $\sigma_{\mathbf{q}},\partial_s$ and $\Delta_{t,\mathbf{q}}$.

Lemma D.5. — *The following statements hold.*

1. *The field $C_{\mathbf{q}}(s,\ell_{\mathbf{q}})$ is stabilized by $\sigma_{\mathbf{q}}$, ∂_s and $\Delta_{t,\mathbf{q}}$. The field $C_{\mathbf{q}}(s)$ is stabilized by $\sigma_{\mathbf{q}}$ and ∂_s. The field $C(s)$ is stabilized by ∂_s, ∂_t.*
2. *The field $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$, ∂_s and $\Delta_{t,\mathbf{q}}$. The field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$ and ∂_s. The field $C_q(\ell_q)$ is stabilized by ∂_s and ∂_t.*

Proof. —

1. Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}})=\ell_{\mathbf{q}}+1$, we easily see that $C_{\mathbf{q}}(s,\ell_{\mathbf{q}}),C_{\mathbf{q}}(s)$ are stabilized by $\sigma_{\mathbf{q}}$. Since $\sigma_{\mathbf{q}}$ commutes with ∂_s and $\Delta_{t,\mathbf{q}}$, the field $C_{\mathbf{q}}$ is stabilized by ∂_s and $\Delta_{t,\mathbf{q}}$. It is now clear that $C_{\mathbf{q}}(s)$ is stabilized by ∂_s and $\Delta_{t,\mathbf{q}}(C_{\mathbf{q}}(s))\subset C_{\mathbf{q}}(s,\ell_{\mathbf{q}})$. By Remark D.3, $\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}}),\partial_s(\ell_{\mathbf{q}})\in C_{\mathbf{q}}$. Combining the last assertions, we obtain the result for $C_{\mathbf{q}}(s,\ell_{\mathbf{q}})$. Finally, the field $C(s)$ is stable by ∂_s, ∂_t, since C is stable by ∂_s,∂_t, and $\partial_s(s)=s$, $\partial_t(s)=0$.
2. Let us prove that $C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$. Using $\sigma_q(\ell_q)=\ell_q+1$ and the commutation between $\sigma_{\mathbf{q}}$ and σ_q, we find that $\sigma_{\mathbf{q}}(\ell_q)-\ell_q\in C_q$. Similarly, $\sigma_{\mathbf{q}}(C_q)\subset C_q$, proving that $C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$. Using $\partial_s(C_{\mathbf{q}})\subset C_{\mathbf{q}}$, $\partial_s(C_q)\subset C_q$, and $\partial_s(\ell_q)\in C_q$, we find that the field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$ and ∂_s.

   We now consider the field $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. From what precedes, $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by ∂_s. Similarly, we deduce that $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}})$ is stabilized by ∂_s. Then, $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$ is stable by ∂_s. The field $C_{\mathbf{q}}(\ell_{\mathbf{q}})$ is clearly stable by $\sigma_{\mathbf{q}}$.

From what precedes, $C_q(\ell_q)$ is stable by $\sigma_{\mathbf{q}}$, and therefore, $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$ is stable by $\sigma_{\mathbf{q}}$. It now remains to show that $\Delta_{t,\mathbf{q}}(C_q(\ell_q)) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. The same arguments as used in (1) prove that $\Delta_{t,\mathbf{q}}(C_{\mathbf{q}}(\ell_{\mathbf{q}})) \subset C_{\mathbf{q}}(\ell_{\mathbf{q}})$, $\partial_s(C_q(\ell_q)) \subset C_q(\ell_q)$, and $\partial_s(C_{\mathbf{q}}(\ell_{\mathbf{q}})) \subset C_{\mathbf{q}}(\ell_{\mathbf{q}})$. It remains to prove that $\Delta_{t,\mathbf{q}}(C_q(\ell_q)) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. We note that $\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\partial_s + \partial_t = \Delta_{t,\mathbf{q}} = \Delta_{t,q} + (\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}} - \frac{\partial_t(q)}{q}\ell_q)\partial_s$. Since C_q is stabilized by $\Delta_{t,q}$ and ∂_s, we find that $\Delta_{t,\mathbf{q}}(C_q) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. Moreover, since $\partial_s(\ell_q), \Delta_{t,q}(\ell_q)$ belong to C_q, see Remark D.3, we find that $\Delta_{t,\mathbf{q}}(\ell_q) \in C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. We have shown the inclusion $\Delta_{t,\mathbf{q}}(C_q(\ell_q)) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$. This concludes the proof for $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$.

Let us now consider $C_q(\ell_q)$. By Remark D.3 and $\partial_t = \Delta_{t,q} - \frac{\partial_t(q)}{q}\ell_q\partial_s$, we find that the inclusion holds $\partial_s(\ell_q), \partial_t(\ell_q) \in C_q(\ell_q)$. Since $\partial_s, \Delta_{t,q}$ commute with σ_q, C_q is stable by $\partial_s, \Delta_{t,q}$. With $\partial_t = \Delta_{t,q} - \frac{\partial_t(q)}{q}\ell_q\partial_s$, it follows that $\partial_t(C_q) \subset C_q(\ell_q)$. Finally, we obtain that the field $C_q(\ell_q)$ is stable by ∂_s, ∂_t. □

D.2. **Difference Galois theory for elliptic function fields.** In this section, we apply the results of § C to the specific cases of elliptic function fields introduced in Lemma D.5. The latter yields that the following field extensions are (σ,∂,Δ)-field extensions.

- Let $\mathbf{q} \in C^*$ with $|\mathbf{q}| \neq 1$. Then, let us consider
$$(C_{\mathbf{q}}(s,\ell_{\mathbf{q}}),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}).$$
- Let $\mathbf{q}$ and q two elements of C^* such that $|q|,|\mathbf{q}| \neq 1$, that are *multiplicatively independent*. Let us consider
$$(C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*),\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}}).$$

In that framework, the criteria obtained in § C to guaranty the $(\partial_s,\Delta_{t,\mathbf{q}})$-differential transcendence of a solution of a rank one $\mathbf{q}$-difference equation can be simplified via some descent arguments. We will prove that the existence of a telescoping relation involving the two derivatives implies the existence of a telescoping relation involving only the derivation ∂_s. More precisely, we find the following proposition that will be applied to our two examples in Section refsec:trsnforellipticfunctions:

Proposition D.6. — *Let $K \subset \mathcal{M}er(C^*)$ be a $(\sigma_{\mathbf{q}},\partial_s)$-field and let us assume that*

(H1) *$L = K(\ell_{\mathbf{q}})$ is a $(\sigma_{\mathbf{q}},\partial_s,\Delta_{t,\mathbf{q}})$-field;*
(H2) *$K^{\sigma_{\mathbf{q}}} = L^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$ is relatively algebraically closed in L;*
(H3) *$\ell_{\mathbf{q}}$ is transcendental over K.*

Let $f \in \mathcal{M}er(C^*)$, that satisfies $\sigma_{\mathbf{q}}(f) = f + b$, for some b that belongs to a subfield of K stable by ∂_s,∂_t.

If f is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over L, then there exist $m \in \mathbb{N}$, $h \in K$ and $d_0,\dots,d_m \in C_{\mathbf{q}}$ not all zero, such that
$$d_0 b_1 + d_1\partial_s(b) + \cdots + d_m\partial_s^m(b) = \sigma_{\mathbf{q}}(h) - h.$$

Proof. — Since f is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over L and $K^{\sigma_{\mathbf{q}}}$ is relatively algebraically closed, Theorem C.9 yields that there exist $M \in \mathbb{N}$, $c_{i,j} \in L^{\sigma_{\mathbf{q}}}$ not all zero, and $g\in L$ such that

$$\sum_{i,j\leqslant M} c_{i,j}\partial_s^i \Delta_{t,\mathbf{q}}^j(b)=\sigma_{\mathbf{q}}(g)-g. \tag{D.1}$$

By Lemma D.4, for all $i\in\mathbb{N}$, there exist $c_{j,k,l}\in C_{\mathbf{q}}$ such that

$$\Delta_{t,\mathbf{q}}^i=\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\ell_{\mathbf{q}}\Big)^i\partial_s^i+\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} c_{j,k,l}\ell_{\mathbf{q}}^j\partial_s^k\partial_t^l. \tag{D.2}$$

Combining this equation with Remark D.3, yields that the left hand side of (D.1) is a polynomial in $\ell_{\mathbf{q}}$ with coefficients in K. By Lemma C.3 with **(H2)** and **(H3)**, we find that $g\in K[\ell_{\mathbf{q}}]$ as well.

Thus, let us write $g=\sum_{k=0}^R \alpha_k\ell_{\mathbf{q}}^k$ with $\alpha_k\in K$ and $\alpha_R\neq 0$. Let

$$N=\max\{j\in\mathbb{N}|\exists i \text{ such that } c_{i,j}\neq 0\}.$$

By (D.2), the coefficient of highest degree in $\ell_{\mathbf{q}}$ of the left hand side of (D.1) is

$$\Big(\sum_{i\leqslant M} c_{i,N}\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\Big)^N\partial_s^{N+i}(b)\Big)\ell_{\mathbf{q}}^N. \tag{D.3}$$

On the other hand, we have

$$\sigma_{\mathbf{q}}(g)-g=\ell_{\mathbf{q}}^R(\sigma_{\mathbf{q}}(\alpha_R)-\alpha_R))+\ell_{\mathbf{q}}^{R-1}(\sigma_{\mathbf{q}}(\alpha_{R-1})-\alpha_{R-1}+R\sigma_{\mathbf{q}}(\alpha_R))+P(\ell_{\mathbf{q}}), \tag{D.4}$$

where $P(X)\in K[X]$ is a polynomial of degree strictly smaller than $R-1$. Then, comparing (D.3) and (D.4), we find that

- either $R<N$ so that
$$\sum_{i\leqslant M} c_{i,N}\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\Big)^N\partial_s^{N+i}(b)=0, \tag{D.5}$$
- either $R=N$ so that
$$\sum_{i\leqslant M} c_{i,N}\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\Big)^N\partial_s^{N+i}(b)=\sigma_{\mathbf{q}}(\alpha_N)-\alpha_N, \tag{D.6}$$
- or $R>N$ so that $R>0$, $0\neq\alpha_R\in L^{\sigma_{\mathbf{q}}}$. We claim that $R=N-1$. Indeed, $R>N-1$ implies $\sigma_{\mathbf{q}}(\alpha_R)=\sigma_{\mathbf{q}}(\alpha_R)$, $\sigma_{\mathbf{q}}(\alpha_{R-1})-\alpha_{R-1}+R\alpha_R=0$ and then $\sigma_{\mathbf{q}}(\frac{\alpha_{R-1}}{\alpha_R})-\frac{\alpha_{R-1}}{\alpha_R}+R=0$ with $\frac{\alpha_{R-1}}{\alpha_R}\in K$ in contradiction with Lemma C.2 applied to $f=\ell_{\mathbf{q}}$. Thus, we get $R=N-1$ and
$$\sum_{i\leqslant M}\frac{c_{i,N}}{\alpha_R}\Big(\frac{\partial_t(\mathbf{q})}{\mathbf{q}}\Big)^N\partial_s^{N+i}(b)=\sigma_{\mathbf{q}}\Big(\frac{\alpha_{R-1}}{\alpha_R}\Big)-\frac{\alpha_{R-1}}{\alpha_R}+R. \tag{D.7}$$

For all these cases, note that there exists i_0 such that $c_{i_0,N}\neq 0$ by definition of N. Since ∂_s commutes with $\sigma_{\mathbf{q}}$, we can differentiate (D.7) with respect to ∂_s and obtain that in any case, there exist $d_k\in L^{\sigma_{\mathbf{q}}}=C_{\mathbf{q}}$ not all zero and $h\in K$ such that

$$\sum_{k\leqslant M+1} d_k\partial_s^k(b)=\sigma_{\mathbf{q}}(h)-h. \tag{D.8}$$

□

D.3. **Transcendence properties.** The goal of this subsection is to prove some transcendence properties of the $\mathbf{q}$-logarithm in order to perform some descent procedure on telescopers. More precisely, we need to prove that the assumptions **(H1)** to **(H3)** of Proposition D.6 are satisfied for the fields $C_{\mathbf{q}}(s)$ and $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$ for $\mathbf{q}$ and q two multiplicatively independent elements of C^* with $|q| \neq 1$, $|\mathbf{q}| \neq 1$. We recall that q and $\mathbf{q}$ are multiplicatively independent if there are no $(r,l) \in \mathbb{Z}^2 \setminus (0,0)$ such that $q^r = \mathbf{q}^l$. Remind that $C_{\mathbf{q}}.C_q \subset \mathcal{M}er(C^*)$ is the compositum of fields and $\ell_{\mathbf{q}} \in \mathcal{M}er(C^*)$ is a solution of $\sigma_{\mathbf{q}}(y) = y + 1$. With Lemma D.5, **(H1)** of Proposition D.6 is satisfied for $K = C_{\mathbf{q}}(s)$ and $K = C_{\mathbf{q}}.C_q(\ell_q)$.

Lemma D.7. — *Any element in a $\sigma_{\mathbf{q}}$-extension of C_q*[5] *that is algebraic over C_q and invariant by $\sigma_{\mathbf{q}}$ is in C. Any element in a σ_q-extension of $C_{\mathbf{q}}$ that is algebraic over $C_{\mathbf{q}}$ and invariant by σ_q is in C.*

Proof. — The two statements are symmetrical, so let us only prove the first one. First let us prove that $C_q \cap C_{\mathbf{q}} = C$. Let f be an element of C_q that is $\sigma_{\mathbf{q}}$-invariant. Suppose to the contrary that f is nonconstant. Then f has a nonzero pole c. Since $\sigma_{\mathbf{q}}(f) = f$, the multiplication by $\mathbf{q}$ induces a permutation of the poles of f modulo q. Since the set of poles modulo q is a finite set, there exists $m \in \mathbb{N}^*$ such that $\mathbf{q}^m c = q^d c$ for some $d \in \mathbb{Z}$. A contradiction with the fact that q and $\mathbf{q}$ are multiplicatively independent. Now, let f be in a $\sigma_{\mathbf{q}}$-extension of C_q algebraic over C_q and invariant by $\sigma_{\mathbf{q}}$. Let $\mu(X) \in C_q[X]$ be the monic minimal polynomial of f above C_q. Since $\sigma_{\mathbf{q}}(f) = f$, we easily see that the coefficients of μ must be fixed by $\sigma_{\mathbf{q}}$. Then, these coefficients belong to $C_q \cap C_{\mathbf{q}}$, which is equal to C. Then, f is algebraic over C. The latter field being algebraically closed, we conclude that $f \in C$. ☐

Lemma D.8. — *The following statements hold:*

1. *the fields $C_{\mathbf{q}}$ and C_q are linearly disjoint over C;*
2. *for all $\alpha \in C_{\mathbf{q}}.C_q$, $\sigma_q(\alpha) \neq \alpha + 1$ and $\sigma_{\mathbf{q}}(\alpha) \neq \alpha + 1$;*
3. *for all $\alpha \in C_{\mathbf{q}}(s)$, $\sigma_{\mathbf{q}}(\alpha) \neq \alpha + 1$.*

Proof. —

1. This is Lemmas D.7 and C.8 with $K = C$, $M = C_q$ and $L = C_{\mathbf{q}}$, $\sigma = \sigma_{\mathbf{q}}$.
2. Suppose to the contrary that there exists $\alpha \in C_{\mathbf{q}}.C_q$, such that $\sigma_q(\alpha) = \alpha + 1$. Since C_q is by Proposition 4.2, the field of meromorphic functions over a Tate curve, there exist $x, y \in C_q$ such that x is transcendental over C, y algebraic of degree 2 over $C(x)$ and $C_q = C(x,y)$. Since $C_{\mathbf{q}}$ is linearly disjoint from C_q over C, the field $C_{\mathbf{q}}.C_q$ equals $C_{\mathbf{q}}(x,y)$ and there are $P(X), Q(X) \in C_{\mathbf{q}}(X)$ such that $\alpha = P(x)y + Q(x)$. Since x, y are fixed by σ_q and y is of degree 2 over $C_{\mathbf{q}}(x)$, we deduce from $\sigma_q(\alpha) = \alpha + 1$ that $P^{\sigma_q}(x) = P(x)$ and $Q^{\sigma_q}(x) - Q(x) = 1$ where $P^{\sigma_q}(X)$ (resp. $Q^{\sigma_q}(X)$) denotes the fraction obtained from $P(X)$ (resp. $Q(X)$) by applying σ_q to the coefficients. Let $\overline{C_{\mathbf{q}}}$ be some algebraic closure of $C_{\mathbf{q}}$. We endow $\overline{C_{\mathbf{q}}}$ with a structure of σ_q-field extension of $C_{\mathbf{q}}$. Let us write $Q(X) = \frac{c_r}{X^r} + \cdots + \frac{c_1}{X} + R(X)$ with $R \in \overline{C_{\mathbf{q}}}(X)$ with no pole at $X = 0$. Then, since

[5] We recall that since $\sigma_{\mathbf{q}}$ and σ_q commute, the field C_q is a $\sigma_{\mathbf{q}}$-field.

x is transcendental over $\overline{C_{\mathbf{q}}}$ and fixed by σ_q

$$Q^{\sigma_q}(x) - Q(x) = 1 = \frac{\sigma_q(c_r) - c_r}{x^r} + \cdots + \frac{\sigma_q(c_1) - c_1}{x} + R^{\sigma_q}(x) - R(x).$$

Using the transcendence of x over $\overline{C_{\mathbf{q}}}$, we find that $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$ for $\tilde{\beta} = R(0) \in \overline{C_{\mathbf{q}}}$. There exists a unique derivation extending ∂_s to $\overline{C_{\mathbf{q}}}$ and this derivation commutes with σ_q. Denoting this derivation by ∂_s and differentiating $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$, we conclude that $\partial_s(\tilde{\beta}) \in C_q$. On the other hand, $\partial_s(\tilde{\beta}) \in \overline{C_{\mathbf{q}}}$. Then, $\sigma_{\mathbf{q}}$ induces a permutation on the set of roots of the minimal polynomial of $\partial_s(\tilde{\beta})$ over $C_{\mathbf{q}}$. Thus, there exists $r' \in \mathbb{N}^*$ such that $\sigma_{\mathbf{q}}^{r'}(\partial_s(\tilde{\beta})) = \partial_s(\tilde{\beta})$. Then, $\partial_s(\tilde{\beta}) \in C_q \cap C_{\mathbf{q}^{r'}}$. Note that q and $\mathbf{q}^{r'}$ are multiplicatively independent. By Lemma D.7, we find that $\partial_s(\tilde{\beta}) \in C$ which leads to $\tilde{\beta} = cs + d$ for some $c, d \in C$. A contradiction with $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$. The proof for $\mathbf{q}$ is similar.

(3) Let $\alpha \in C_{\mathbf{q}}(s)$. Using the partial fraction decomposition of α in $\overline{C_{\mathbf{q}}}(s)$, the fact that $\sigma_{\mathbf{q}}(s) = \mathbf{q}s$ and the transcendence of s over $C_{\mathbf{q}}$, one can easily see that $\sigma_{\mathbf{q}}(\alpha) - \alpha \neq 1$. □

Lemma D.9. — *The following statements hold:*

(1) *the function $\ell_{\mathbf{q}}$ (resp. ℓ_q) is transcendental over $C_{\mathbf{q}}.C_q$;*
(2) *the function $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}(s)$. In particular, **(H3)** of Proposition D.6 is satisfied for $K = C_{\mathbf{q}}(s)$.*

Proof. —

(1) Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ and $C_{\mathbf{q}} \subset (C_{\mathbf{q}}.C_q)^{\sigma_{\mathbf{q}}} \subset \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, we can apply Lemma C.2 and find that $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}.C_q$ if and only if there exists $\alpha \in C_{\mathbf{q}}.C_q$ such that $\sigma_{\mathbf{q}}(\alpha) = \alpha + 1$. We conclude by Lemma D.8. The proof for ℓ_q is symmetrical.
(2) Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ and $C_{\mathbf{q}} \subset (C_{\mathbf{q}}(s))^{\sigma_{\mathbf{q}}} \subset \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, we can apply Lemma C.2 and find that $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}(s)$ if and only if there exists $\alpha \in C_{\mathbf{q}}(s)$ such that $\sigma_{\mathbf{q}}(\alpha) = \alpha + 1$. We again conclude by Lemma D.8. □

Lemma D.10. — *The following statements hold:*

(1) *let $f \in C_q$. If there exists $\alpha \in C_{\mathbf{q}}.C_q$ satisfying $\sigma_{\mathbf{q}}(\alpha) - \alpha = f$, then there exists $\beta \in C_q$ such that $\sigma_{\mathbf{q}}(\beta) - \beta = f$;*
(2) *let $f \in C_{\mathbf{q}}.C_q$. If there exists $\alpha \in C_{\mathbf{q}}.C_q(\ell_q)$ satisfying $\sigma_{\mathbf{q}}(\alpha) - \alpha = f$, then, there exist $\tilde{a} \in C_{\mathbf{q}}, \tilde{b} \in C_{\mathbf{q}}.C_q$ such that $\sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b}) = f$.*

Proof. —

(1) Analogously to the proof of Lemma D.8, let us write $\alpha = P(x)y + Q(x)$ for $P(X), Q(X) \in C_q(X)$ and $C_{\mathbf{q}} = C(x, y)$. Reasoning as in the proof of Lemma D.8, we find that $Q^{\sigma_{\mathbf{q}}}(x) - Q(x) = f$. Since x is transcendental over C_q, we conclude as in Lemma D.8 that there is $\tilde{\beta} \in \overline{C_q}$, for some $\overline{C_q}$ algebraic closure of C_q such that $\sigma_{\mathbf{q}}(\tilde{\beta}) - \tilde{\beta} = f$. Since by Lemma D.7, $\overline{C_q}^{\sigma_{\mathbf{q}}} = C_q^{\sigma_{\mathbf{q}}} = C$, Lemma C.2 implies that there exists $\beta \in C_q$ such that $\sigma_{\mathbf{q}}(\beta) - \beta = f$.

(2) First of all, let us note that since $\sigma_{\mathbf{q}}$ and σ_q commute, there exists $d \in C_q$ such that

$$\sigma_{\mathbf{q}}(\ell_q) = \ell_q + d. \tag{D.9}$$

By Lemma D.9, the function ℓ_q is transcendental over $C_{\mathbf{q}}.C_q$. This implies that $\ell_q \notin C_{\mathbf{q}}$ and then $d \neq 0$. Since $C_{\mathbf{q}}.C_q(\ell_q)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}} = \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}.C_q^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, Lemma C.3, applied to $\sigma_{\mathbf{q}}(\ell_q) = \ell_q + d$, implies that there exists $P \in C_{\mathbf{q}}.C_q[X]$ such that

$$f = \sigma_{\mathbf{q}}(P(\ell_q)) - P(\ell_q).$$

Now, let us write $P(X) = \sum_{k=0}^{N} a_k X^k$ with $a_k \in C_{\mathbf{q}}.C_q$, and N minimal. We find

$$\begin{aligned} f =& (\sigma_{\mathbf{q}}(a_N) - a_N)\ell_q^N + (\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + N d \sigma_{\mathbf{q}}(a_N))\ell_q^{N-1} \\ &+ \text{terms of order less than } N-1. \end{aligned} \tag{D.10}$$

We conclude in view of (D.10) that if $N = 0$ we are done by setting $\tilde{a} = 0$ and $\tilde{b} = a_N$. Let us now assume that $N > 0$. Then, by minimality of N, $\sigma_{\mathbf{q}}(a_N) = a_N$. We claim that

$$\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + N d \sigma_{\mathbf{q}}(a_N) = \sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + N d a_N \neq 0.$$

Otherwise, $\sigma_{\mathbf{q}}(a_{N-1}) = a_{N-1} - N d a_N$ implies

$$\sigma_{\mathbf{q}}(\frac{a_{N-1}}{a_N} + N\ell_q) = \frac{a_{N-1}}{a_N} + N\ell_q \quad \text{and} \quad \frac{a_{N-1}}{a_N} + N\ell_q \in C_{\mathbf{q}},$$

contradicting the transcendence of ℓ_q over $C_{\mathbf{q}}.C_q$, see Lemma D.9. This proves the claim. If $N > 1$, then (D.10) with $\sigma_{\mathbf{q}}(a_N) = a_N$ and $\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + N d a_N \neq 0$, would give an equation of order $N-1$ which would contradict the transcendence of ℓ_q over $C_{\mathbf{q}}.C_q$. This proves that $N = 1$ and $f = \sigma_{\mathbf{q}}(a_1\ell_q + a_0) - (a_1\ell_q + a_0)$ for some $a_1 \in C_{\mathbf{q}}, a_0 \in C_{\mathbf{q}}.C_q$. □

LEMMA D.11. — *The function $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q(\ell_q)$. In particular, the assumption **(H3)** of Proposition D.6 holds for $K = C_{\mathbf{q}}.C_q(\ell_q)$.*

Proof. — By Lemma C.2, the function $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}.C_q(\ell_q)$ if and only if we have $\ell_{\mathbf{q}} \in C_{\mathbf{q}}.C_q(\ell_q)$. Suppose to the contrary that $\ell_{\mathbf{q}} \in C_{\mathbf{q}}.C_q(\ell_q)$. Since $1 = \sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}}$, we conclude by Lemma D.10 that there exist $\tilde{a} \in C_{\mathbf{q}}, \tilde{b} \in C_{\mathbf{q}}.C_q$ such that $1 = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$. Combining this equation with $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} = 1$, we find that $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$, proving that $\sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b} - \ell_{\mathbf{q}}) = \tilde{a}\ell_q + \tilde{b} - \ell_{\mathbf{q}} \in C_{\mathbf{q}}$. Then, there exists $\widetilde{b_1} \in C_{\mathbf{q}}.C_q$ such that

$$\ell_{\mathbf{q}} = \widetilde{a}\ell_q + \widetilde{b_1}. \tag{D.11}$$

Differentiating (D.11) with respect to ∂_s, we find

$$\partial_s(\ell_{\mathbf{q}}) = \partial_s(\widetilde{a})\ell_q + \widetilde{a}\partial_s(\ell_q) + \partial_s(\widetilde{b_1}).$$

By Remark D.3, $\partial_s(\ell_{\mathbf{q}}), \partial_s(\ell_q) \in C_{\mathbf{q}}.C_q$. In virtue of the commutation between ∂_s and $\sigma_{\mathbf{q}}, \sigma_q$, the fields $C_q, C_{\mathbf{q}}$ are stabilized by ∂_s, which implies $\partial_s(\widetilde{a}), \partial_s(\widetilde{b_1}) \in C_{\mathbf{q}}.C_q$. By Lemma D.9, the function ℓ_q is transcendental over the latter field, we conclude that $\partial_s(\widetilde{a}) = 0$ and therefore $\widetilde{a} \in C$. In particular it belongs to $C_{\mathbf{q}}$ and C_q. Using $1 = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$, we find

$$1 - \widetilde{a}d = \sigma_{\mathbf{q}}(\widetilde{b}) - \widetilde{b},$$

where $d = \sigma_{\mathbf{q}}(\ell_q) - \ell_q \in C_q$, see (D.9). Since $1 - \widetilde{a}d \in C_q$, we conclude by Lemma D.10, that there exists $\widetilde{b_2} \in C_q$ such that $1 - \widetilde{a}d = \sigma_{\mathbf{q}}(\widetilde{b_2}) - \widetilde{b_2}$. Replacing the left hand side gives

$$\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} - \sigma_{\mathbf{q}}(\widetilde{a}\ell_q) + \widetilde{a}\ell_q = \sigma_{\mathbf{q}}(\widetilde{b_2}) - \widetilde{b_2}.$$

This shows that $\ell_{\mathbf{q}} - \widetilde{a}\ell_q - \widetilde{b_2} \in C_{\mathbf{q}}$ and then, there exists $c \in C_{\mathbf{q}}$ such that $\ell_{\mathbf{q}} + c = \widetilde{a}\ell_q + \widetilde{b_2}$. Differentiating this equation with respect to ∂_s, we find (we use $\partial_s(\widetilde{a}) = 0$)

$$\partial_s(\ell_{\mathbf{q}}) + \partial_s(c) = \widetilde{a}\partial_s(\ell_q) + \partial_s(\widetilde{b_2}).$$

By Remark D.3, the left hand side of the equation belongs to $C_{\mathbf{q}}$ whereas the right hand side is in C_q. By Lemma D.7, we conclude that $\partial_s(\ell_{\mathbf{q}} + c) \in C$. This means that there exist $a_0, b_0 \in C$ such that $\ell_{\mathbf{q}} = a_0 s + b_0 - c$ in contradiction with $\ell_{\mathbf{q}}$ transcendental over $C_{\mathbf{q}}(s)$, see Lemma D.9. □

We can now prove that our fields satisfy the assumption **(H2)** of Proposition D.6.

LEMMA D.12. — *The following holds:*

(1) *$C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}(s, \ell_{\mathbf{q}})$;*
(2) *$C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$.*

*In particular, **(H2)** of Proposition D.6 holds for $K = C_{\mathbf{q}}(s)$ and $K = C_{\mathbf{q}}.C_q(\ell_q)$.*

Proof. —

(1) The first point is a consequence of transcendence of s over $C_{\mathbf{q}}$, and the transcendence of $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}(s)$, see Lemma D.9.
(2) Let us prove the second point. Let us start by proving that $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q$. As in the proof of Lemma D.8, we have $C_{\mathbf{q}} = C(x, y)$ and $C_{\mathbf{q}}.C_q = C_q(x, y)$ where y is of degree 2 over both $C(x)$ and $C_q(x)$. Let $f \in C_q(x, y)$. Then $f = P(x)y + Q(x)$ with $P(x), Q(x) \in C_q(x)$. If f is algebraic over $C_{\mathbf{q}}$ then Lemma C.1 implies that $\sigma_{\mathbf{q}}^r(f) = f$ for some $r \in \mathbb{Z}^*$ and therefore $\sigma_{\mathbf{q}}^r(P(x)) = P(x)$ and $\sigma_{\mathbf{q}}^r(Q(x)) = Q(x)$. We claim that $P(x)$ and $Q(x)$ are in $C(x)$, and therefore that $f \in C_{\mathbf{q}}$. Let $P(x) = P_1(x)/P_2(x)$ where $P_1(x), P_2(x) \in C_q[x]$ are relatively prime and $P_1(x)$ is monic. We then have that $\sigma_{\mathbf{q}}^r(P_1(x))P_2(x) = \sigma_{\mathbf{q}}^r(P_2(x))P_1(x)$ and consequently $P_1(x)$ divides $\sigma_{\mathbf{q}}^r(P_1(x))$ (resp. $\sigma_{\mathbf{q}}^r(P_1(x))$ divides $P_1(x)$). Since $P_1(x)$ is monic, $P_1(x) = \sigma_{\mathbf{q}}^r(P_1(x))$ and $P_2(x) = \sigma_{\mathbf{q}}^r(P_2(x))$. This implies that the coefficients of $P_1(x)$ and $P_2(x)$ are left fixed by $\sigma_{\mathbf{q}}^r$. Note that by assumption, q and $\mathbf{q}^r$ are multiplicatively independent. Therefore, by Lemma D.7, applied with $\mathbf{q}$ replaced by $\mathbf{q}^r$, $P_1, P_2 \in C[X]$. The proof for Q is similar. This proves our claim and shows that $f \in C_{\mathbf{q}}$. Then $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q$.

Note that Lemma D.9 implies that $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q$ and Lemma D.11 implies that ℓ_q is transcendental over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}})$. Therefore $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. □

The results of Appendix D.3 are summarized in the following crucial corollary.

COROLLARY D.13. — *The assumptions of Proposition D.6 are satisfied for*

- *Genus zero case: $K = C_{\mathbf{q}}(s)$ and $b \in C(s)$ with $\mathbf{q} \in C^*$ such that $|\mathbf{q}| \neq 1$;*

- *Genus one case:* $K = C_{\mathbf{q}}.C_q(\ell_q)$ *and* $b \in C_q(\ell_q)$ *with* $\mathbf{q}, q \in C^*$ *such that* $|\mathbf{q}|, |q| \neq 1$ *and* $\mathbf{q}$ *and* q *are multiplicatively independent.*

Proof. — The fact that the field K and b satisfy the assumptions **(Hi)** is Lemmas D.5, D.9, D.11, and D.12. ☐

Finally, we prove a lemma that will allow us to descend some *telescoping relations* on smaller base fields.

Lemma D.14. — *Let* $b \in C_q$ *such that there exist* $N \in \mathbb{N}$, $c_i \in C_{\mathbf{q}}$ *with* $c_N \neq 0$, *and* $g \in C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$ *that satisfy*

$$\sum_{i=0}^{N} c_i \partial_s^i(b) = \sigma_{\mathbf{q}}(g) - g. \tag{D.12}$$

Then, there exist $m \in \mathbb{N}$, $d_0, \ldots, d_m \in C$ *not all zero and* $h \in C_q$ *such that*

$$d_0 b + d_1 \partial_s(b) + \cdots + d_m \partial_s^m(b) = \sigma_{\mathbf{q}}(h) - h.$$

Proof. — First of all note that the left hand side of (D.12) belongs to $C_{\mathbf{q}}.C_q$. By Lemma D.11, the function $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q(\ell_q)$. By Lemma C.3, $g \in C_{\mathbf{q}}.C_q(\ell_q)[\ell_{\mathbf{q}}]$. So let us write $g = \sum_{k=0}^{R} \alpha_k \ell_{\mathbf{q}}^k$ with $\alpha_k \in C_{\mathbf{q}}.C_q(\ell_q)$, $\alpha_R \neq 0$.

Claim. There exist $m \in \mathbb{N}$, $c'_k \in C_{\mathbf{q}}$, $c'_m \neq 0$, and $\alpha \in C_{\mathbf{q}}.C_q(\ell_q)$ such that

$$\sum_{k=0}^{m} c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\alpha) - \alpha. \tag{D.13}$$

If $R = 0$ the claim is proved. Assume that $R > 0$. Then, we have

$$\sigma_{\mathbf{q}}(g) - g = \ell_{\mathbf{q}}^R(\sigma_{\mathbf{q}}(\alpha_R) - \alpha_R)) + \ell_{\mathbf{q}}^{R-1}(\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R) + P(\ell_{\mathbf{q}}), \tag{D.14}$$

where $P(X) \in C_{\mathbf{q}}.C_q(\ell_q)[X]$ is a polynomial of degree smaller than $R-1$. Then, comparing (D.14) and (D.12), we find, by transcendence of $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_q)$, see Lemma D.11, that $\sigma_{\mathbf{q}}(\alpha_R) = \alpha_R$. Let us prove that $\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R \neq 0$. Indeed if $\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R = 0$ then $\sigma_{\mathbf{q}}(\frac{\alpha_{R-1}}{\alpha_R}) - \frac{\alpha_{R-1}}{\alpha_R} + R = 0$ with $\frac{\alpha_{R-1}}{\alpha_R} \in C_{\mathbf{q}}.C_q$ in contradiction with Lemma D.9 and Lemma C.2. We then obtain that $R = 1$ since otherwise we would deduce from (D.14) an algebraic relation for $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_q)$, contradicting Lemma D.11. Thus,

$$\sum_{i=0}^{N} \frac{c_i}{\alpha_1} \partial_s^i(b) = \sigma_{\mathbf{q}}\Big(\frac{\alpha_0}{\alpha_1}\Big) - \frac{\alpha_0}{\alpha_1} + 1. \tag{D.15}$$

Remind that $\alpha_1 \in C_{\mathbf{q}}$ and the latter field is stable by ∂_s due to the commutation between ∂_s and $\sigma_{\mathbf{q}}$. By Lemma D.5, the field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by ∂_s. We can differentiate (D.15) with respect to ∂_s and using the commutation between $\sigma_{\mathbf{q}}$ and ∂_s, we obtain our claim.

Claim. There exist $M \in \mathbb{N}$, $d_k \in C_{\mathbf{q}}$, $d_M \neq 0$ and $\beta \in C_{\mathbf{q}}.C_q$ such that

$$\sum_{k=0}^{M} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta.$$

Indeed, by Lemma D.10, we can find $a \in C_{\mathbf{q}}, b \in C_{\mathbf{q}}.C_q$ such that

$$\sum_{k=0}^{m} c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(a\ell_q + b) - (a\ell_q + b). \tag{D.16}$$

If $a=0$, then $\sum_k c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(b) - b$ for some $b \in C_{\mathbf{q}}.C_q$. It remains to consider the case $a \neq 0$. Assume that $a \neq 0$. Dividing (D.16) by a and differentiating with respect to ∂_s, we find

$$\sum_{k=0}^{m+1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\partial_s(\ell_q) + \partial_s(b/a)) - (\partial_s(\ell_q) + \partial_s(b/a)),$$

where the d_k are in $C_{\mathbf{q}}$, $d_{m+1} = \frac{c'_m}{a} \neq 0$. Furthermore, by Remark D.3 and the fact that $C_{\mathbf{q}}$, C_q are stable by ∂_s, we find $\partial_s(\ell_q) + \partial_s(b/a) \in C_{\mathbf{q}}.C_q$. This proves the claim.

Now, let us consider an equation of the form

$$\sum_{k=0}^{M} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta,$$

with $\beta \in C_{\mathbf{q}}.C_q$, $d_k \in C_{\mathbf{q}}$ and $d_M \neq 0$, minimal with respect to the maximal order of derivation M of b. We can write this minimal equation as follows

$$d_M \partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta,$$

with $d_M \in C_{\mathbf{q}}^*$. Then dividing by d_M, we find

$$\partial_s^M(b) + \sum_{k=0}^{M-1} \frac{d_k}{d_M} \partial_s^k(b) = \sigma_{\mathbf{q}}\Big(\frac{\beta}{d_M}\Big) - \frac{\beta}{d_M}.$$

Therefore, we can without loss of generality assume that $d_M = 1$. Now, if we compute the element $\sigma_q(\sigma_{\mathbf{q}}(\beta) - \beta) - (\sigma_{\mathbf{q}}(\beta) - \beta)$ and use the fact that $b \in C_q$, we find

$$\sum_{k=0}^{M-1} (\sigma_q(d_k) - d_k) \partial_s^k(b) = \sigma_{\mathbf{q}}(\sigma_q(\beta) - \beta) - (\sigma_q(\beta) - \beta).$$

By minimality, we find that, for all k, the element $d_k \in C_{\mathbf{q}}$ is fixed by σ_q. This means that $d_k \in C$ by Lemma D.7.

Since $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) \in C_q$ and $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta$ with $\beta \in C_{\mathbf{q}}.C_q$, Lemma D.10 shows that we have the existence of $h \in C_q$ such that

$$\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(h) - h. \quad \square$$

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