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QUASI-LINEAR FUNCTIONALS ON LOCALLY COMPACT SPACES

SVETLANA V. BUTLER

Abstract. This paper combines new and known results in a single convenient source for anyone interested in learning about quasi-linear functionals on locally compact spaces. We define singly generated subalgebras in different settings and study signed and positive quasi-linear functionals. Quasi-linear functionals are, in general, nonlinear, but linear on singly generated subalgebras. The paper gives representation theorems for quasi-linear functionals on $C_c(X)$, for bounded quasi-linear functionals on $C_0(X)$ on a locally compact space, and for quasi-linear functionals on $C(X)$ on a compact space. There is an order-preserving bijection between quasi-linear functionals and compact-finite topological measures, which is also "isometric" when topological measures are finite. We present many properties of quasi-linear functionals and give an explicit example of a quasi-linear functional on \mathbb{R}^2 . Results of the paper will be helpful for further study and application of quasi-linear functionals in different areas of mathematics, including symplectic geometry.

INTRODUCTION

The origins of the theory of quasi-linear functionals and topological measures are connected to quantum physics and have a fascinating story. Mathematical interpretations of quantum physics by G. W. Mackey and R. V. Kadison ([40], [41], [38]) led to very interesting mathematical problems. Let R be a von Neumann algebra, and let P be the lattice of orthogonal projections in R . A nonnegative function μ on P is a measure if $\mu(0) = 0$ and $\mu(\sum_{i \in I} e_i) = \sum_{i \in I} \mu(e_i)$ for any family $\{e_i : i \in I\}$ in P . The extension problem asks whether there exists a positive state ρ on R such that $\rho|_P = \mu$. The first important affirmative answer was obtained by Gleason ([33]) for the case where R is the family of all bounded linear operators on a separable Hilbert space H with $\dim H \geq 3$. The extension problem can be viewed as a special case of the following linearity problem for quasi-states (see [1], [2]). Let \mathcal{A} be a C^* -algebra with identity 1. A quasi-state is a function $\rho : \mathcal{A} \rightarrow \mathbb{C}$ which is a state on each C^* -subalgebra of \mathcal{A} generated by a single self-adjoint $a \in \mathcal{A}$ and 1, and which satisfies $\rho(a + ib) = \rho(a) + i\rho(b)$ for self-adjoint $a, b \in \mathcal{A}$. The problem is to determine whether ρ is linear. In [1, Theorem 1] Aarnes claimed that any positive quasi-linear functional ρ on an abelian C^* -algebra is linear. However, C. Akemann and M. Newberger found a gap in the proof (see [6]). It turned out that the gap was unbridgeable due to the existence of nonlinear quasi-linear functionals, which J. Aarnes demonstrated almost twenty years later in [3]. Any abelian unital C^* -algebra is isomorphic to $C(X)$ for some compact Hausdorff space X , and Aarnes introduced quasi-linear functionals on $C(X)$ for a compact Hausdorff space. He proved a representation theorem connecting quasi-linear functionals to set functions generalizing measures (initially called quasi-measures, now topological measures). Aarnes's quasi-linear

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functionals are functionals that are linear on singly generated subalgebras, but (in general) not linear.

Quasi-linear functionals are also related to the mathematical model of quantum mechanics of von Neumann ([46]). Let \mathcal{A} be the algebra of observables in quantum mechanics. In a simple form, \mathcal{A} is a space of Hermitian operators on a finite dimensional Hilbert space. In von Neumann's definition, a quantum state is a linear positive normalized functional on \mathcal{A} . A number of physicists disagreed with the additivity axiom $\rho(A + B) = \rho(A) + \rho(B)$, arguing that it makes sense only if observables A and B are simultaneously measurable, i.e commute. Two Hermitian operators on a finite dimensional Hilbert space commute iff they can be written as the polynomials of the same Hermitian operator. This justifies the following modifications of the additivity axiom: (*) $\rho(A + B) = \rho(A) + \rho(B)$ if A, B belong to a singly generated subalgebra of \mathcal{A} . (**) $\rho(A + B) = \rho(A) + \rho(B)$ if $[A, B]_{\hbar} = 0$, where the bracket $[A, B]_{\hbar} = -\frac{i}{\hbar}(AB - BA)$, and \hbar is the Planck constant. A positive homogeneous functional with additivity as in (*) is a positive quasi-linear functional introduced by Aarnes. The requirement of additivity in (**) leads to the notion of a Lie quasi-state (see [28], [43, Sect. 5.6]). For more information about the physical interpretation of quasi-linear functionals see [1], [2], [3], [24], [26], [31], [32], [43].

M. Entov and L. Polterovich first linked the theory of quasi-linear functionals and topological measures to symplectic geometry. Their seminal paper [26] was followed by extensive research in which quasi-linear functionals and topological measures are used in connection with rigidity phenomenon in symplectic geometry. The objects and theory from the area of quasi-linear functionals became a prominent part of of function theory on symplectic manifolds, which is the subject of an excellent monograph [43]. We also note that a quasi-linear functional can be viewed as a Choquet integral. We discuss more the connections of the theory of quasi-linear functionals to symplectic geometry and Choquet theory in the last section of the paper.

The vast majority of papers dealing with quasi-linear functionals and topological measures (including papers in symplectic geometry) consider a compact underlying space, finite topological measures, and bounded quasi-linear functionals. In this paper we consider locally compact spaces, infinite topological measures, and unbounded quasi-linear functionals. Transitioning from the compact setting to the locally compact one requires certain adjustments. On a compact space we work with open sets and closed sets, while on a locally compact space the focus is on compact sets and open sets, which are no longer complements of each other. If a space is compact, constant functions are in singly generated subalgebras, which makes proof of many results much easier. This is no longer the case for locally compact noncompact spaces. Still, many results for topological measures and quasi-linear functionals remain valid for locally compact spaces. Also, results transfer nicely from real-valued maps in a compact setting to maps into extended real numbers in a locally compact setting.

This paper, influenced by various works, including [3], [34], [44], combines existing results, improved versions of known results, and new results in a single source for anyone interested in (1) learning about quasi-linear functionals on locally compact

noncompact spaces or on compact spaces; (2) further study of quasi-linear functionals, signed quasi-linear functionals, and other related nonlinear functionals; (3) applying quasi-linear functionals in other areas of mathematics.

Definitions of a topological measure and a quasi-linear functional in this paper have an equivalent but simpler form than other definitions encountered previously in the literature. Eliminating monotonicity and Lipschitz continuity (which we prove as properties) from definitions used in some papers and in symplectic geometry simplifies the task of checking whether a particular functional is a quasi-linear functional (in particular, a symplectic quasi-state) or whether a particular set function is a topological measure.

Function theory on symplectic manifolds would not be possible without the theory of quasi-linear functionals and rigidity phenomenon. C^0 -rigidity property holds for open or closed manifolds, and it allows one to extend the notion of Poisson commutativity from smooth to continuous functions (see [29], [43, Sect. 2.1]). At the moment, function theory on symplectic manifolds is mostly developed for closed manifolds, perhaps because until very recently the theory of topological measures and quasi-linear functionals dealt almost exclusively with the compact case. We believe that properties of quasi-linear functionals, representation theorems, and other parts of this paper (perhaps in conjunction with other recent papers by the author devoted to the theory of topological measures, deficient topological measures and corresponding nonlinear functionals on locally compact spaces) may allow extension of the fascinating function theory on symplectic manifolds to nonclosed manifolds and lead the way to new contributions.

The paper is structured as follows. In Section 1 we define signed and positive quasi-linear functionals on locally compact spaces, singly generated subalgebras, and topological measures. We give the explicit form of a singly generated subalgebra in different settings. In Section 2 we present various properties of quasi-linear functionals. Some of them are new while others generalize properties known in the compact case. We describe situations where signed and positive quasi-linear functionals possess some linearity. Then we show how to construct quasi-linear functionals from topological measures and discuss some properties of quasi-linear functionals in relation to the topological measures from which they are built. Our main focus is on two situations: quasi-linear functionals on $C_0(X)$ when the topological measure is finite, and on $C_c(X)$ when the topological measure is compact-finite. In Section 3 we build a topological measure from a quasi-linear functional and give properties of the resulting topological measure in terms of the generating quasi-linear functional. We then prove a representation theorem. We show that there is an order-preserving bijection between quasi-linear functionals and compact-finite topological measures, which is also "isometric" when topological measures are finite. Our approach allows us to obtain representation theorems for quasi-linear functionals on functions with compact support on a locally compact space, for bounded quasi-linear functionals on functions vanishing at infinity on a locally compact space, and for quasi-linear functionals on continuous functions on a compact space. In Section 4 we further study properties of quasi-linear functionals, including monotonicity, uniform continuity, continuity with respect to topology of uniform convergence on compacta, and Lipschitz continuity. We give an explicit example of a quasi-linear functional on

\mathbb{R}^2 . In Section 5 we discuss connections of quasi-linear functionals with symplectic geometry and Choquet theory.

1. PRELIMINARIES

In this paper X is a locally compact, connected space.

By $C(X)$ we denote the set of all real-valued continuous functions on X with the extended uniform norm $\|f\| = \sup\{|f(x)| : x \in X\}$, by $C_b(X)$ the set of bounded continuous functions on X , by $C_0(X)$ the set of continuous functions on X vanishing at infinity, and by $C_c(X)$ the set of continuous functions with compact support.

We denote by \overline{E} the closure of a set E , and by \sqcup a union of disjoint sets. When we consider set functions into extended real numbers, they are not identically ∞ .

We denote by 1 the constant function $1(x) = 1$, by id the identity function $id(x) = x$, and by 1_E the characteristic function of a set E . By $\text{supp } f$ we mean $\{x : f(x) \neq 0\}$.

Several collections of sets will be used often. They include: $\mathcal{O}(X)$, the collection of open subsets of X ; $\mathcal{C}(X)$ the collection of closed subsets of X ; $\mathcal{K}(X)$ the collection of compact subsets of X ; $\mathcal{A}(X) = \mathcal{C}(X) \cup \mathcal{O}(X)$.

A set function μ on $\mathcal{C}(X) \cup \mathcal{O}(X)$ with values in $[0, \infty]$ is monotone if $A \subseteq B$ implies $\mu(A) \leq \mu(B)$; a nonnegative μ is compact-finite if $\mu(K) < \infty$ for any $K \in \mathcal{K}(X)$.

A measure on X is a countably additive set function on a σ -algebra of subsets of X with values in $[0, \infty]$. A Borel measure on X is a measure on the Borel σ -algebra on X . A measure m is inner regular on open sets if $m(U) = \sup\{m(K) : K \subseteq U, K \text{ is compact}\}$ for every open set U ; a measure is outer regular if $m(E) = \inf\{m(U) : E \subseteq U, U \text{ is open}\}$ for every set E . A measure is regular if it is inner and outer regular on all sets. For a Borel measure m that is inner regular on open sets we define $\text{supp } m$, the support of m , to be the complement of the largest open set W such that $m(W) = 0$.

Given a Borel measure m on X and a continuous map $\phi : X \rightarrow Y$ we denote by ϕ^*m the Borel measure on Y defined by $\phi^*m = m \circ \phi^{-1}$ on Borel subsets of Y . In this case $\int_Y g d\phi^*m = \int_X g \circ \phi dm$ for any $g \in C(Y)$.

We recall the following fact (see, for example, [23, Ch. XI, 6.2]):

LEMMA 1.1. — *Let $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$ in a locally compact space X . Then there is a set $V \in \mathcal{O}(X)$ with compact closure such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Remark 1.2. — The space X is connected, so for a bounded continuous function f we have $\overline{f(X)} = [a, b]$ for some real numbers a and b . $\overline{f(X)} = f(X)$ when X is compact. Let X be locally compact but not compact. For $f \in C_0(X)$ we have $0 \in \overline{f(X)} = [a, b]$. If $f \in C_0(X, [a, b])$ and $\phi \in C([a, b])$ then $\phi \circ f \in C_0(X)$ iff $\phi(0) = 0$.

DEFINITION 1.3. — Let X be locally compact.

- (a) Let $f \in C_b(X)$. Define $A(f)$ to be the smallest closed subalgebra of $C_b(X)$ containing f and 1 . Hence, when X is compact, $A(f)$ is the smallest closed subalgebra of $C(X)$ containing f and 1 . We call $A(f)$ the singly generated subalgebra of $C(X)$ generated by f .

- (b) Let \mathcal{B} be a subalgebra of $C_b(X)$, for example, $C_c(X)$ or $C_0(X)$. For $f \in \mathcal{B}$, define $B(f)$ to be the smallest closed subalgebra of \mathcal{B} containing f . We call $B(f)$ the singly generated subalgebra of \mathcal{B} generated by f .

LEMMA 1.4. — (a) When X is locally compact $A(f)$ for $f \in C_b(X)$ has the form:

$$A(f) = \{\phi \circ f : \phi \in C(\overline{f(X)})\}.$$

In particular, when X is compact $A(f)$ for $f \in C(X)$ has the form:

$$A(f) = \{\phi \circ f : \phi \in C(f(X))\}.$$

- (b) When X is locally compact, noncompact, and $\mathcal{B} = C_0(X)$, for $f \in C_0(X)$ its singly generated subalgebra has the form:

$$B(f) = \{\phi \circ f : \phi(0) = 0, \phi \in C(\overline{f(X)})\}.$$

- (c) When X is locally compact, noncompact, and $\mathcal{B} = C_c(X)$, for $f \in C_c(X)$ its singly generated subalgebra has the form:

$$B(f) = \{\phi \circ f : \phi(0) = 0, \phi \in C(f(X))\}.$$

Proof. —

- (a) Let $[a, b] = \overline{f(X)}$. Consider $Y(f) = \{\phi \circ f : \phi \in C([a, b])\}$. It is clear that $Y(f)$ is a subalgebra of $C_b(X)$ containing $f = id \circ f$ and $1 = 1 \circ f$, and

$$\|\phi \circ f\|_{C_b(X)} = \sup_{x \in X} |\phi(f(x))| = \|\phi\|_{C([a, b])}.$$

The map $T : C([a, b]) \rightarrow C_b(X)$ with $T(\phi) = \phi \circ f$ is a linear norm preserving map, so its image $Y(f)$ is closed in $C_b(X)$. Now suppose $Y \subseteq C(X)$ is any closed subalgebra of $C_b(X)$ containing f and 1 . If $p \in C([a, b])$ is a polynomial then $p \circ f \in Y$. Since polynomials on $[a, b]$ are dense in $C([a, b])$,

$$Y(f) = T(C([a, b])) \subseteq \overline{Y} = Y,$$

showing that $Y(f)$ is the smallest closed subalgebra of $C_b(X)$ containing f and 1 , i.e. $Y(f) = A(f)$.

- (b) We assume that X is locally compact noncompact and $f \in C_0(X)$. As in Remark 1.2 let $[a, b] = \overline{f(X)}$ and $\phi \in C([a, b])$. The proof of part (b) is similar to the one in part (a) using Remark 1.2 and the observation that polynomials on $[a, b]$ that are 0 at 0 are dense in $\{\phi : \phi \in C([a, b]), \phi(0) = 0\}$.

- (c) Follows from part (b). □

We consider functionals on various subalgebras of $C(X)$, such as $C_c(X)$, $C_0(X)$, $C_b(X)$ or $C(X)$.

DEFINITION 1.5. — Let X be locally compact, and let \mathcal{B} be a subalgebra of $C(X)$ containing $C_c(X)$. A real-valued map ρ on \mathcal{B} is called a signed quasi-linear functional on \mathcal{B} if

(QI1) ρ is homogeneous, i.e. $\rho(af) = a\rho(f)$ for $a \in \mathbb{R}$.

(QI2) for each $h \in \mathcal{B}$ we have: $\rho(f + g) = \rho(f) + \rho(g)$ for f, g in the singly generated subalgebra $B(h)$ generated by h .

We say that ρ is a quasi-linear functional (or a positive quasi-linear functional) if, in addition,

(QI3) $f \geq 0 \implies \rho(f) \geq 0$.

When X is compact, we call ρ a quasi-state if $\rho(1) = 1$.

Remark 1.6. — If X is compact, each singly generated subalgebra contains constants. Suppose ρ is a quasi-linear functional on $C(X)$. Then

$$\rho(f + c) = \rho(f) + \rho(c) = \rho(f) + c\rho(1)$$

for every constant c and every $f \in C(X)$. If ρ is a quasi-state then $\rho(f + c) = \rho(f) + c$.

Remark 1.7. — A quasi-linear functional is also called a quasi-integral for reasons that will be apparent later; see Definition 2.13 and Theorem 3.9 below.

Remark 1.8. — There are situations when the whole algebra $C_b(X)$ is generated by one function, in which case every quasi-linear functional is linear. This happens, for example, if $X = [a, b] \subset \mathbb{R}$: the whole algebra $C_b(X)$ is singly generated by the identity function:

$$C_b(X) = C(X) = \{\phi \circ id : \phi \in C(X)\} = A(id(X)).$$

Example 1.9. — The following example is due to D. Grubb ([35]). Let $X = S^1 \subseteq \mathbb{R}^2$. We shall show that $C(S^1)$ is not singly generated by any $f \in C(S^1)$. Suppose to the contrary that $C(S^1)$ is singly generated by some function $f \in C(S^1)$. Let π_1 and π_2 be the projections of X onto the first and the second coordinates. Then $\pi_1, \pi_2 \in C(S^1)$, and so $\pi_1 = \phi \circ f, \pi_2 = \psi \circ f$ for some functions $\phi, \psi \in C(f(S^1))$. Choose $x \in S^1$ such that $f(x) = f(-x)$. If $x \neq \pm \frac{\pi}{2}$ then $\pi_1(x) \neq \pi_1(-x)$, while also

$$\pi_1(x) = \phi(f(x)) = \phi(f(-x)) = \pi_1(-x).$$

If $x = \pm \frac{\pi}{2}$ then $\pi_2(x) \neq \pi_2(-x)$, but also $\pi_2(x) = \psi(f(x)) = \psi(f(-x)) = \pi_2(-x)$. In either case we get a contradiction. Therefore, $C(S^1)$ is not singly generated by any function $f \in C(S^1)$.

Even though $C(S^1)$ is not singly generated, every quasi-linear functional on $C(S^1)$ is linear. This is because every topological measure on a compact space with the covering dimension ≤ 1 is a measure (see [47], [36], and [45]).

DEFINITION 1.10. — If X is locally compact and ρ is a quasi-linear functional on $C_0(X)$ we define

$$\|\rho\| = \sup\{\rho(f) : f \in C_0(X), 0 \leq f \leq 1\}.$$

Similarly, if ρ is a quasi-linear functional on $C_c(X)$ we define

$$\|\rho\| = \sup\{\rho(f) : f \in C_c(X), 0 \leq f \leq 1\}.$$

If $\|\rho\| < \infty$ we say that ρ is bounded.

Remark 1.11. — $\|\rho\|$ satisfies the following properties: $\|\alpha\rho\| = \alpha\|\rho\|$ for $\alpha > 0$, $\|\rho\| = 0$ iff $\rho = 0$, and $\|\rho + \eta\| \leq \|\rho\| + \|\eta\|$. Thus, $\|\rho\|$ has properties similar to properties of an extended norm, but it is defined on a positive cone of nonnegative functions.

DEFINITION 1.12. — A topological measure on X is a (not identically ∞) set function $\mu : \mathcal{C}(X) \cup \mathcal{O}(X) \rightarrow [0, \infty]$ which is finitely additive, inner regular and outer regular, i.e.

- (TM1) if $A, B, A \sqcup B \in \mathcal{K}(X) \cup \mathcal{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$;
 (TM2) $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq U\}$ for $U \in \mathcal{O}(X)$;
 (TM3) $\mu(F) = \inf\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\}$ for $F \in \mathcal{C}(X)$.

Clearly, for a closed set F , $\mu(F) = \infty$ iff $\mu(U) = \infty$ for every open set U containing F .

Remark 1.13. — It is easy to check that a topological measure μ is monotone on $\mathcal{O}(X) \cup \mathcal{C}(X)$ and that $\mu(\emptyset) = 0$. If ν and μ are topological measures that agree on $\mathcal{K}(X)$ (or on $\mathcal{O}(X)$) then $\nu = \mu$. If ν and μ are topological measures such that $\nu \leq \mu$ on $\mathcal{K}(X)$ (or on $\mathcal{O}(X)$) then $\nu \leq \mu$.

Remark 1.14. — If X is locally compact, (TM1) of Definition 1.12 is equivalent to the following two conditions:

$$\begin{aligned} \mu(K \sqcup C) &= \mu(K) + \mu(C), \quad C, K \in \mathcal{K}(X), \\ \mu(U) &\leq \nu(C) + \mu(U \setminus C), \quad C \subseteq U, \quad C \in \mathcal{K}(X), \quad U \in \mathcal{O}(X). \end{aligned}$$

This result follows from [17, Theorem 4.4] or [19, Prop. 5.4], but it was first observed for compact-finite topological measures in a slightly different form in [44, Prop. 2.2].

DEFINITION 1.15. — We denote $\|\mu\| = \mu(X)$. If $\mu(X) < \infty$ we say that μ is finite. If μ assumes only values 0 and 1, we say μ is simple.

Remark 1.16. — $\|\mu\|$ has the following properties: $\|\mu\| = 0$ iff $\mu = 0$; $\|\alpha\mu + \nu\| = \alpha\|\mu\| + \|\nu\|$ for $\alpha > 0$. Again, $\|\mu\|$, defined on a positive cone of all topological measures, has properties similar to properties of an extended norm.

We denote by $TM(X)$ the collection of all topological measures on X , and by $M(X)$ the collection of all Borel measures on X that are inner regular on open sets and outer regular (restricted to $\mathcal{O}(X) \cup \mathcal{C}(X)$).

Remark 1.17. — Let X be locally compact. We have:

$$M(X) \subsetneq TM(X). \tag{1.1}$$

For more information on proper inclusion, criteria for a topological measure to be a measure from $M(X)$, and examples of finite, compact-finite, and infinite topological measures, see [17, Sect. 4, Sect. 5], [19, Sect. 5, Sect. 15], and [13]. When X is compact the proper inclusion in (1.1) was first demonstrated in [3]; in fact, $M(X)$ is nowhere dense in $TM(X)$ (see [10]).

Remark 1.18. — If X is compact $\mathcal{C}(X) = \mathcal{K}(X)$, and a real-valued topological measure is a set function on $\mathcal{O}(X) \cup \mathcal{C}(X)$ satisfying (TM1) and either one of (TM2), (TM3). Since every topological measure is monotone, this gives an equivalent but simpler definition of a topological measure compared to the one originally used in [3], many subsequent papers, and almost all the literature in symplectic geometry.

Remark 1.19. — If X is compact and μ is finite, condition (TM1) of Definition 1.12 is equivalent, as was noticed in [34], to the following three conditions:

- (i) $\mu(U \sqcup V) = \mu(U) + \mu(V)$ for any two disjoint open sets U, V .
- (ii) If $X = U \cup V$ for $U, V \in \mathcal{O}(X)$ then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.
- (iii) $\mu(X \setminus U) = \mu(X) - \mu(U)$ for any open set U .

The same equivalence holds if we replace open sets by closed sets. Thus, when X is compact, a finite topological measure (and more generally, a bounded signed topological measure) can be defined by its actions on open (respectively, on closed) sets. The idea of determining a topological measure on a closed manifold by its values on closed submanifolds with boundary is in [48, Sect. 2].

The following properties of topological measures are proved (for a wider class of set functions) in [17, Sect. 3].

LEMMA 1.20. — *Let X be a locally compact space.*

- (a) *A topological measure is τ -smooth on open sets, i.e. if $U_\alpha \nearrow U, U_\alpha, U \in \mathcal{O}(X)$ then $\mu(U_\alpha) \rightarrow \mu(U)$. In particular, a topological measure is additive on any collection of disjoint open sets.*
- (b) *If $F \in \mathcal{C}(X), C_1, \dots, C_n \in \mathcal{K}(X)$ are disjoint then $\mu(F \sqcup C_1 \sqcup \dots \sqcup C_n) = \mu(F) + \mu(C_1) + \dots + \mu(C_n)$.*
- (c) *A topological measure μ is superadditive, i.e. if $\bigsqcup_{t \in T} A_t \subseteq A$, where $A_t, A \in \mathcal{O}(X) \cup \mathcal{C}(X)$, and at most one of the closed sets is not compact, then $\mu(A) \geq \sum_{t \in T} \mu(A_t)$.*

The following result is an immediate consequence of superadditivity of a topological measure.

LEMMA 1.21. — *Suppose μ is a finite topological measure on a locally compact space X , and $A_1, \dots, A_n \in \mathcal{O}(X) \cup \mathcal{C}(X), n \geq 2$, where at most one of the closed sets (if there are any) is not compact. If $\mu(A_1) > \frac{\mu(X)}{n}$ and $\mu(A_i) \geq \frac{\mu(X)}{n}$ for $i = 2, \dots, n$ then A_1, \dots, A_n are not disjoint.*

2. QUASI-LINEAR FUNCTIONALS

If ρ is a quasi-linear functional we can not say that $\rho(f + g) = \rho(f) + \rho(g)$ for arbitrary functions f and g . However, we have the following two lemmas. When X is compact we take $\mathcal{B} = C(X)$. If X is locally compact we may take \mathcal{B} to be $C_b(X)$, or $C_c(X)$, or $C_0(X)$. By singly generated subalgebra we mean $A(f)$ if X is compact and $B(f)$ if X is locally compact noncompact, as in Definition 1.3 and Lemma 1.4.

LEMMA 2.1. — *Let X be locally compact.*

- (s1) *For any $f \geq 0$ and any const $\delta > 0$ the function $f_\delta = \inf\{f, \delta\}$ is in the subalgebra generated by f .*
- (s2) *If $f \cdot g = 0, f, g \geq 0, f, g \in C_b(X)$ then f, g are in the subalgebra generated by $f - g$, and if $f \cdot g = 0, f \geq 0, g \leq 0$ then f, g are in the subalgebra generated by $f + g$. In particular, for any $f \in C_b(X)$ the functions f^+, f^- and $|f|$ are in the subalgebra generated by f .*
- (s3) *(Approximation property). If $f \in C_0(X)$ and $\epsilon > 0$ then there is $h \in C_c(X)$ such that h is in the subalgebra generated by f and $\|f - h\| \leq \epsilon$.*
- (s4) *If $0 \leq g(x) \leq f(x) \leq c$ and $f = c$ on $\{x : g(x) > 0\}$ then g, f belong to the same subalgebra generated by $f + g$.*

Proof. —

- (s1) Note that $f_\delta = (id \wedge \delta) \circ f$.

- (s2) Assume that $f, g \geq 0$, $f \cdot g = 0$, $f, g \in C_b(X)$. With $h = f - g$ we have $f = (id \vee 0) \circ h$ and $g = ((-id) \vee 0) \circ h$. Thus f, g belong to the subalgebra singly generated by h . If $f \geq 0, g \leq 0, f \cdot g = 0$, then f, g are in the subalgebra generated by $h = f + g$, since $f = (id \vee 0) \circ h, g = (id \wedge 0) \circ h$.
- (s3) Assume first that $f \in C_0(X)$ and $f \geq 0$. For $\epsilon > 0$ by part (s1) the function f_ϵ belongs to the subalgebra generated by f , and then so does $h = f - f_\epsilon$. Note that h is supported on the compact set $\{x : f(x) \geq \epsilon\}$, and $\|f - h\| = \|f_\epsilon\| \leq \epsilon$.
Now take any $f \in C_0(X)$ and $\epsilon > 0$. Choose $h^+ \in C_c(X)$ such that $\|f^+ - h^+\| \leq \frac{\epsilon}{2}$ and h^+ is in the subalgebra generated by f^+ , and hence, by part (s2), is in the subalgebra generated by f . Similarly, choose $h^- \in C_c(X)$ such that $\|f^- - h^-\| \leq \frac{\epsilon}{2}$ and h^- is in the subalgebra generated by f . Let $h = h^+ - h^-$. Then $h \in C_c(X)$, h is in the subalgebra generated by f , and $\|f - h\| = \|f^+ - f^- - h^+ + h^-\| \leq \|f^+ - h^+\| + \|f^- - h^-\| \leq \epsilon$.
- (s4) Note that $c \geq 0$ and $f = (id \wedge c) \circ (f + g), g = (0 \vee (id - c)) \circ (f + g)$. \square

LEMMA 2.2. — *Let X be locally compact. Let ρ be a signed quasi-linear functional on a subalgebra of $C_b(X)$.*

- (q1) *If $\phi_i \in C(\overline{f(X)})$, $i = 1, \dots, n$ (if X is locally compact noncompact we also require $\phi_i(0) = 0$) and $\sum_{i=1}^n \phi_i = id$ then $\sum_{i=1}^n \rho(\phi_i \circ f) = \rho(f)$.*
- (q2) *If $f \cdot g = 0, f, g \in C_b(X)$ then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$. In particular, $\rho(f) = \rho(f^+) - \rho(f^-)$ for any $f \in C_b(X)$.*
- (q3) *If $sign\ g = const$ and $f = \|f\|$ (or $f = -\|f\|$) on $\{x : g(x) \neq 0\}$ then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$.*
- (q4) *Suppose each singly generated subalgebra contains constants, and $\rho(1) \in \mathbb{R}$. Suppose $f, g \in C_b(X)$ and $f = c$ on the set $\{x : g(x) \neq 0\}$. Then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$.*

If ρ is a positive quasi-linear functional then we have:

- (i) *If f and g are from the same singly generated subalgebra and $f \geq g$ then $\rho(f) \geq \rho(g)$.*
- (ii) *If $0 \leq g(x) \leq f(x) \leq c$ and $f = c$ on $\{x : g(x) > 0\}$ then $\rho(f) \geq \rho(g)$ and $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$.*
- (iii) *If $f \geq g, f, g \in C_c(X)$ then $\rho(f) \geq \rho(g)$.*
- (iv) *Suppose each singly generated subalgebra contains constants, and $\rho(1) \in \mathbb{R}$. Suppose $f, g \in C_b(X)$ and $f = c$ on the set $\{x : g(x) \neq 0\}$. Then $\rho(af + bg) = a\rho(f) + b\rho(g)$ for any $a, b \in \mathbb{R}$. If $f \geq g$ then $\rho(f) \geq \rho(g)$. If $f \leq g$ then $\rho(f) \leq \rho(g)$.*

Proof. —

- (q1) Let $f \in \mathcal{B}$ and $\phi_i \in C(\overline{f(X)})$, $i = 1, \dots, n$ with $\sum_{i=1}^n \phi_i = id$. (If X is locally compact noncompact, we also require $\phi_i(0) = 0$.) Since $\phi_i \circ f$ for $i = 1, \dots, n$ belong to the singly generated subalgebra generated by f , we have: $\sum_{i=1}^n \rho(\phi_i \circ f) = \rho(\sum_{i=1}^n \phi_i \circ f) = \rho(id \circ f) = \rho(f)$.
- (q2) Since a quasi-linear functional is homogeneous, it is enough to show that $\rho(f + g) = \rho(f) + \rho(g)$. From part (s2) of Lemma 2.1 it follows that $\rho(f + g) = \rho(f) + \rho(g)$ when $f, g \geq 0, f \cdot g = 0, f, g \in C_b(X)$. In the

general case, if $f \cdot g = 0$ then $(f + g)^+ = f^+ + g^+$, $(f + g)^- = f^- + g^-$, and $f^+ \cdot g^+ = 0$, $f^- \cdot g^- = 0$. Thus, for example, $\rho(f^+ + g^+) = \rho(f^+) + \rho(g^+)$. Then

$$\begin{aligned} \rho(f + g) &= \rho((f + g)^+) - \rho((f + g)^-) = \rho(f^+ + g^+) - \rho(f^- + g^-) \\ &= \rho(f^+) + \rho(g^+) - \rho(f^-) - \rho(g^-) = \rho(f) + \rho(g). \end{aligned}$$

- (q3) It is enough to prove the statement for $g \geq 0$ and $\|g\|, \|f\| > 0$. Suppose first that $f \geq 0$ and $f = \|f\|$ on $\{x : g(x) > 0\}$. Since $0 \leq g \leq \frac{f\|g\|}{\|f\|} \leq \|g\|$, by part (s4) of Lemma 2.1 functions g and $f_1 = \frac{f\|g\|}{\|f\|}$, and, hence, f belong to the same singly generated subalgebra. The statement follows. For general f , if $f = \|f\|$ on $\{x : g(x) > 0\}$ then $f^+ = \|f\|$ on $\{x : g(x) > 0\}$, and by the above argument $\rho(af^+ + bg) = a\rho(f^+) + b\rho(g)$. Since $(af^+ + bg)(af^-) = 0$, using part (q2) we have: $a\rho(f) + b\rho(g) = a\rho(f^+) - a\rho(f^-) + b\rho(g) = \rho(af^+ + bg) - \rho(af^-) = \rho(af^+ + bg - af^-) = \rho(af + bg)$. If $f = -\|f\|$ on $\{x : g(x) > 0\}$ use $-f$ instead of f .

- (q4) It is enough to show that $\rho(f + g) = \rho(f) + \rho(g)$. Note that $(f - c) \cdot g = 0$. Using Remark 1.6 and part (q2) we get:

$$\rho(f + g) = \rho((f - c) + g) + \rho(c) = \rho(f - c) + \rho(g) + \rho(c) = \rho(f) + \rho(g).$$

- (i) Using additivity of ρ on singly generated subalgebras and the positivity of ρ we have $\rho(f) - \rho(g) = \rho(f - g) \geq 0$.
(ii) Follows from part (i) and Lemma 2.1, part (s4).
(iii) Given $\delta > 0$, suppose first that $g \geq 0$ and that $f(x) \geq g(x) + \delta$ when $g(x) > 0$. Choose $n \in \mathbb{N}$ such that $n\delta > \|f\|$ and define functions ϕ_i , $i = 1, \dots, n$ by

$$\phi_i(x) = \begin{cases} 0 & \text{if } x \leq (i-1)\delta \\ x - (i-1)\delta & \text{if } (i-1)\delta < x < i\delta \\ \delta & \text{if } x \geq i\delta. \end{cases}$$

Then $\phi_i(g(x)) > 0$ implies $\phi_i(f(x)) = \delta$, so by part (ii) we have $\rho(\phi_i(g)) \leq \rho(\phi_i(f))$. Since $\sum_{i=1}^n \phi_i = id$, by part (q1) we have:

$$\rho(g) = \sum_{i=1}^n \rho(\phi_i(g)) \leq \sum_{i=1}^n \rho(\phi_i(f)) = \rho(f).$$

Suppose now that $0 \leq g \leq f$. Choose $h \in C_c(X)$ such that $0 \leq h(x) \leq \|f\|$ and $h(x) = \|f\|$ when $f(x) > 0$. Given $\epsilon > 0$ choose δ such that $\delta\rho(h) < \epsilon$. By part (q3) and the argument above we have:

$$\rho(g) \leq \rho(f + \delta h) = \rho(f) + \delta\rho(h) < \rho(f) + \epsilon,$$

so $\rho(g) \leq \rho(f)$.

Now suppose that $g \leq f$. Since $g^+ \leq f^+$, $f^- \leq g^-$, using part (q2) we have:

$$\rho(g) = \rho(g^+) - \rho(g^-) \leq \rho(f^+) - \rho(f^-) = \rho(f).$$

- (iv) By part (q4) we have $\rho(af + bg) = a\rho(f) + b\rho(g)$. Now assume that $f \geq g$. Since ρ is positive, we have $\rho(f) - \rho(g) = \rho(f - g) \geq 0$, i.e. $\rho(f) \geq \rho(g)$. \square

Remark 2.3. — Some of the properties in the last two lemmas are new (for example, parts (s1) and (s3) of Lemma 2.1 and part (q1) of Lemma 2.2); some are known or generalize known properties. Part (q3) generalizes [44, Lemma 2.4] and is related to [3, Lemma 3.4]. Parts (q2) and (q4) of Lemma 2.2 basically appeared in [3, Lemma 3.3, Lemma 3.4] and [34, p.1081]. The proofs of part (s4) of Lemma 2.1 and part (iii) of Lemma 2.2 follow [44, Lemma 2.4, Proposition 3.4]. In Lemma 4.2 below we will improve part (iii) of Lemma 2.2.

Let μ be a topological measure on X . Our goal is to construct a quasi-linear functional on X using μ .

DEFINITION 2.4. — Let X be locally compact and μ be a topological measure on X . Define F , a distribution function of f with respect to μ , as follows:

(A) If $\mu(X) < \infty$ and $f \in C(X)$, let

$$F(a) = \mu(f^{-1}(a, \infty)).$$

(B) If μ is compact-finite and $f \in C_c(X)$, let

$$F(a) = \mu(f^{-1}((a, \infty) \setminus \{0\})).$$

LEMMA 2.5. — *The function F on \mathbb{R} in Definition 2.4 has the following properties:*

- (i) F is real-valued, and in case (A) $0 \leq F \leq \mu(X)$, while in case (B) $0 \leq F \leq \mu(\text{supp } f)$.
- (ii) If f is bounded then $F(a) = 0$ for all $a \geq \|f\|$.
- (iii) F is nonincreasing.
- (iv) F is right-continuous.

Proof. — The right continuity of F follows from Lemma 1.20. The rest is easy. \square

LEMMA 2.6. — *Let μ be a topological measure on a locally compact space X .*

(A) *If $\mu(X) < \infty$ and $f \in C_0(X)$ then there exists a finite Borel measure m_f on \mathbb{R} such that*

$$m_f(W) = \mu(f^{-1}(W)) \text{ for every open set } W \in \mathbb{R}.$$

(B) *If μ is compact-finite and $f \in C_c(X)$ then there exists a finite Borel measure m_f on \mathbb{R} such that*

$$m_f(W) = \mu(f^{-1}(W \setminus \{0\})) \text{ for every open set } W \in \mathbb{R}.$$

In particular, $m_f(W) = \mu(f^{-1}(W))$ for every open set $W \in \mathbb{R} \setminus \{0\}$.

In either case, m_f is the Stieltjes measure on \mathbb{R} associated with F given by Definition 2.4, and $\text{supp } m_f \subseteq \overline{f(X)}$.

Proof. — We will give the proof for case (B). The argument for case (A) is similar but simpler. Let $f \in C_c(X)$. Let the function F on \mathbb{R} be as in Definition 2.4, and let m_f be the Stieltjes measure on \mathbb{R} associated with F . We shall show that m_f is the desired measure. First, consider open subsets of \mathbb{R} of the form (a, b) . Since $m_f((a, b)) = F(a) - F(b^-)$, we shall show that

$$F(a) - F(b^-) = \mu(f^{-1}((a, b) \setminus \{0\})).$$

For any $t \in (a, b)$ we have by Lemma 1.20:

$$\mu(f^{-1}((a, \infty) \setminus \{0\})) \geq \mu(f^{-1}((a, t) \setminus \{0\})) + \mu(f^{-1}((t, \infty) \setminus \{0\})),$$

i.e.

$$F(a) \geq \mu(f^{-1}((a, t) \setminus \{0\})) + F(t).$$

As $t \rightarrow b^-$, by Lemma 1.20 we have $\mu(f^{-1}((a, t) \setminus \{0\})) \rightarrow \mu(f^{-1}((a, b) \setminus \{0\}))$, so

$$F(a) - F(b^-) \geq \mu(f^{-1}((a, b) \setminus \{0\})).$$

Now we shall show the opposite inequality. Note that in

$$f^{-1}((a, \infty) \setminus \{0\}) = f^{-1}((a, b) \setminus \{0\}) \sqcup f^{-1}(\{b\} \setminus \{0\}) \sqcup f^{-1}((b, \infty) \setminus \{0\})$$

all the sets are open except for the middle set on the right hand side, which is compact since $f \in C_0(X)$. Applying μ we obtain by (TM1) of Definition 1.12

$$F(a) = \mu(f^{-1}((a, b) \setminus \{0\})) + \mu(f^{-1}(\{b\} \setminus \{0\})) + \mu(f^{-1}((b, \infty) \setminus \{0\})). \quad (2.1)$$

Since for any $t < b$

$$f^{-1}((b, \infty) \setminus \{0\}) \sqcup f^{-1}(\{b\} \setminus \{0\}) \subseteq f^{-1}((t, \infty) \setminus \{0\}),$$

by Lemma 1.20 we have:

$$\mu(f^{-1}(\{b\} \setminus \{0\})) + \mu(f^{-1}((b, \infty) \setminus \{0\})) \leq \mu(f^{-1}((t, \infty) \setminus \{0\})) = F(t).$$

Thus, from (2.1) we see that

$$F(a) \leq \mu(f^{-1}((a, b) \setminus \{0\})) + F(t).$$

As $t \rightarrow b^-$ we obtain:

$$F(a) - F(b^-) \leq \mu(f^{-1}((a, b) \setminus \{0\})).$$

Therefore, the result is true for finite open intervals in \mathbb{R} . Since both μ and m_f are τ -smooth and additive on open sets (see Lemma 1.20), the result holds for any open set in \mathbb{R} .

Let $V = \mathbb{R} \setminus \overline{f(X)}$. By Remark 1.2 $\overline{f(X)}$ is compact and $0 \in \overline{f(X)}$. Then $m_f(V) = \mu(f^{-1}(V)) = \mu(\emptyset) = 0$. Thus, $\text{supp } m_f \subseteq \overline{f(X)}$. By part (i) of Lemma 2.5, for any $[a, b] \subseteq \mathbb{R}$ we have $|m_f([a, b])| = F(a^-) - F(b) \leq F(a^-) \leq \mu(\text{supp } f) < \infty$, so m_f is compact-finite. Being compactly supported, m_f is finite. \square

Remark 2.7. — Our proof of part (B) is very similar to one in [44, Proposition 3.1], even though in [44] a different distribution function (which is left-continuous) is used. In particular, it follows that whether one defines a distribution function using right semi-infinite intervals (as our function F) or using left semi-infinite intervals (as in [44]), one obtains the same measure m_f on \mathbb{R} . In [15] we explore more the question of when different distribution functions with respect to a topological measure (and more generally, a deficient topological measure) produce the same measure on \mathbb{R} .

DEFINITION 2.8. — Let μ be a topological measure on a locally compact space X . Define a functional ρ_μ

- (A) on $C_0(X)$, if $\mu(X) < \infty$, or
- (B) on $C_c(X)$, if μ is compact-finite,

by:

$$\rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f. \quad (2.2)$$

Here measure m_f is as in Lemma 2.6. If X is compact, ρ_μ is a functional on $C(X)$.

Remark 2.9. — If μ is a measure then $\rho_\mu(f) = \int_X f \, d\mu$ in the usual sense.

We will need the following fact.

Remark 2.10. — Suppose Borel measures m_1, m_2 on \mathbb{R} are compact-finite (hence, regular, see [37, p.329 and (12.55)]) and satisfy $m_1(W) = m_2(W)$ for any open interval $W \in \mathbb{R} \setminus \{0\}$. If a function g with $g(0) = 0$ is integrable with respect to m_1 then g is a integrable with respect to m_2 (use [7, Prop. 2.6.2]) and

$$\int_{\mathbb{R}} g \, dm_1 = \int_{\mathbb{R}} g \, dm_2.$$

PROPOSITION 2.11. — *Let μ be a topological measure on a locally compact space X . If*

- (A) $\mu(X) < \infty$ and $f \in C_0(X)$, or
- (B) μ is compact-finite and $f \in C_c(X)$

then for every $\phi \in C(\overline{f(X)})$ (with $\phi(0) = 0$ if X is noncompact) we have:

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} \phi \, dm_f = \int_{[a,b]} \phi \, dm_f, \quad (2.3)$$

where $[a, b] = \overline{f(X)}$.

Proof. — Assume first that $\mu(X) < \infty$ and $f \in C_0(X)$. Let $\phi \in C(\overline{f(X)})$ and $\phi(0) = 0$. By Remark 1.2 $\phi \circ f \in C_0(X)$. Consider measures $m_{\phi \circ f}$ and $\phi^* m_f$ defined as in Lemma 2.6 and in Section 1. For an open set U in \mathbb{R} , by Lemma 2.6 we have:

$$m_{\phi \circ f}(U) = \mu((\phi \circ f)^{-1}(U)) = \mu(f^{-1}\phi^{-1})(U) = m_f(\phi^{-1}(U)) = \phi^* m_f(U). \quad (2.4)$$

Then $m_{\phi \circ f} = \phi^* m_f$ as measures on \mathbb{R} and

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} id \, dm_{\phi \circ f} = \int_{\mathbb{R}} id \, d\phi^* m_f = \int_{\mathbb{R}} \phi \, dm_f = \int_{[a,b]} \phi \, dm_f.$$

Now let μ be compact-finite and $f \in C_c(X)$. From Lemma 2.6 and reasoning as in (2.4) it follows that $m_{\phi \circ f} = \phi^* m_f$ on open sets in $\mathbb{R} \setminus \{0\}$. The Borel measure $m_{\phi \circ f}$ is finite and has a compact support, so by Remark (2.10) we have:

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} id \, dm_{\phi \circ f} = \int_{\mathbb{R}} id \, d\phi^* m_f = \int_{\mathbb{R}} \phi \, dm_f = \int_{[a,b]} \phi \, dm_f. \quad \square$$

THEOREM 2.12. — *Let μ be a topological measure on a locally compact space X .*

- (A) *If $\mu(X) < \infty$ then ρ_μ defined in Definition 2.8 is a quasi-linear functional on $C_0(X)$ with $\|\rho_\mu\| \leq \mu(X)$.*
- (B) *If μ is compact-finite then ρ_μ defined in Definition 2.8 is a quasi-linear functional on $C_c(X)$ such that $|\rho_\mu(f)| \leq \|f\| m_f(\text{supp } m_f)$.*

Proof. — Let μ be a topological measure on X . The proof (which is close to the compact case proof in [3, Corollary 3.1]) is similar for both cases, and we will demonstrate it for case (B). If $f \geq 0$ then by Lemma 2.6 $m_f(-\infty, 0) = \mu(f^{-1}(-\infty, 0)) = \mu(\emptyset) = 0$. Then

$$\rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f = \int_0^\infty id \, dm_f \geq 0.$$

Thus, (QI3) of Definition 1.5 holds. To show (QI2), let $f \in C_c(X)$. If $\phi \circ f, \psi \circ f \in B(f)$ as in Lemma 1.4, using formula (2.3) and the fact that m_f is a measure on \mathbb{R} we have:

$$\begin{aligned} \rho_\mu(\phi \circ f + \psi \circ f) &= \rho_\mu((\phi + \psi) \circ f) = \int_{\mathbb{R}} (\phi + \psi) dm_f \\ &= \int_{\mathbb{R}} \phi \, dm_f + \int_{\mathbb{R}} \psi \, dm_f = \rho_\mu(\phi \circ f) + \rho_\mu(\psi \circ f). \end{aligned}$$

For any constant $c \in \mathbb{R}$ we also have $\rho_\mu(cf) = \rho_\mu((c \, id) \circ f) = \int c \, id \, dm_f = c\rho_\mu(f)$, so (QI1) holds.

In case (A) for any $0 \leq f \leq 1$ we see that $\rho_\mu(f) \leq m_f(\text{supp } m_f) \leq m_f(\mathbb{R}) = \mu(X)$, so $\|\rho_\mu\| \leq \mu(X)$. In case (B) $|\rho_\mu(f)| \leq \|f\| m_f(\text{supp } m_f)$. \square

DEFINITION 2.13. — We call a quasi-linear functional ρ_μ as in Definition 2.8 and Theorem 2.12 a quasi-integral and write

$$\int_X f \, d\mu = \rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f.$$

It is understood that ρ_μ is a quasi-linear functional on $C(X)$ when X is compact; ρ_μ is a quasi-linear functional on $C_0(X)$ when X is locally compact and $\mu(X) < \infty$; ρ_μ is a quasi-linear functional on $C_c(X)$ when X is locally compact and μ is compact-finite.

LEMMA 2.14. — For the functional ρ_μ we have:

- (i) If $U \in \mathcal{O}(X)$ and $f \in C_c(X)$ is such that $\text{supp } f \subseteq U$, $0 \leq f \leq 1$ then $\rho_\mu(f) \leq \mu(U)$.
- (ii) If $K \in \mathcal{K}(X)$ and f is such that $0 \leq f \leq 1$, $f = 1$ on K , then $\rho_\mu(f) \geq \mu(K)$.

Proof. —

- (a) Using Lemma 2.6 we have: $\rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f \leq 1 \cdot m_f(\{t : t > 0\}) = \mu(f^{-1}(0, \infty)) \leq \mu(U)$.
- (b) We have:

$$\begin{aligned} \rho_\mu(f) &= \int_{\mathbb{R}} id \, dm_f \geq 1 \cdot m_f(\{t : t = 1\}) = m_f(\{t : t \geq 1\}) \\ &= \lim_{\alpha \rightarrow 0} m_f((1 - \alpha, \infty)) = \lim_{\alpha \rightarrow 0} \mu(f^{-1}((1 - \alpha, \infty))) \geq \mu(K). \quad \square \end{aligned}$$

3. REPRESENTATION THEOREM FOR A LOCALLY COMPACT SPACE

We shall establish a correspondence between topological measures and quasi-linear functionals.

DEFINITION 3.1. — Let X be locally compact, and let ρ be a quasi-linear functional on $C_0(X)$ or $C_c(X)$. Define a set function $\mu_\rho : \mathcal{O}(X) \cup \mathcal{C}(X) \rightarrow [0, \infty]$ as follows: for an open set $U \subseteq X$ let

$$\mu_\rho(U) = \sup\{\rho(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U\},$$

and for a closed set $F \subseteq X$ let

$$\mu_\rho(F) = \inf\{\mu_\rho(U) : F \subseteq U, U \in \mathcal{O}(X)\}.$$

LEMMA 3.2. — For the set function μ_ρ from Definition 3.1 the following holds:

- p1. μ_ρ is nonnegative and is not identically ∞ .
- p2. μ_ρ is monotone.
- p3. Given an open set U , for any compact $K \subseteq U$

$$\mu_\rho(U) = \sup\{\rho(g) : 1_K \leq g \leq 1, g \in C_c(X), \text{supp } g \subseteq U\}.$$

- p4. For any $K \in \mathcal{K}(X)$

$$\mu_\rho(K) = \inf\{\rho(g) : g \in C_c(X), g \geq 1_K\}.$$

- p5. For any $K \in \mathcal{K}(X)$

$$\mu_\rho(K) = \inf\{\rho(g) : g \in C_c(X), 0 \leq g \leq 1, g = 1 \text{ on } K\}.$$

- p6. μ_ρ is compact-finite.

- p7. Given $K \in \mathcal{K}(X)$, for any open U such that $K \subseteq U$

$$\mu_\rho(K) = \inf\{\mu_\rho(V) : V \in \mathcal{O}(X), K \subseteq V \subseteq \bar{V} \subseteq U\}.$$

- p8. For any $U \in \mathcal{O}(X)$

$$\mu_\rho(U) = \sup\{\mu_\rho(K) : K \in \mathcal{K}(X), K \subseteq U\}.$$

- p9. For any disjoint compact sets K and C

$$\mu_\rho(K \sqcup C) = \mu_\rho(K) + \mu_\rho(C).$$

- p10. If $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$ then $\mu_\rho(U) = \mu_\rho(K) + \mu_\rho(U \setminus K)$.

- p11. If $K \in \mathcal{K}(X)$, and $f \in C_c(X)$, $0 \leq f \leq 1$, $\text{supp } f \subseteq K$, then $\rho(f) \leq \mu_\rho(K)$.

- p12. If $K \in \mathcal{K}(X)$ and $f_i \in C_c(X)$, $0 \leq f_i \leq 1$, $\text{supp } f_i \subseteq K$ for $i = 1, 2$ then $|\rho(f_1) - \rho(f_2)| \leq \mu_\rho(K)$.

Proof. —

- (p1) μ_ρ is nonnegative since ρ is positive. μ_ρ is not identically ∞ since for a nonempty open set U with compact closure, by part (iii) of Lemma 2.2 $\mu_\rho(U) \leq \rho(h) < \infty$, where h is a Urysohn function such that $h = 1$ on \bar{U} .
- (p2) From Definition 3.1 we have monotonicity on $\mathcal{O}(X)$; then monotonicity on $\mathcal{O}(X) \cup \mathcal{C}(X)$ easily follows.
- (p3) From Definition 3.1 we see that

$$\sup\{\rho(g) : 1_K \leq g \leq 1, g \in C_c(X), \text{supp } g \subseteq U\} \leq \mu_\rho(U).$$

To show the opposite inequality, assume first that $\mu_\rho(U) < \infty$. For $\epsilon > 0$ choose $f \in C_c(X)$ such that $0 \leq f \leq 1$, $\text{supp } f \subseteq U$, and $\rho(f) > \mu_\rho(U) - \epsilon$. Choose Urysohn function $g \in C_c(X)$ such that $g = 1$ on the compact set $K \cup \text{supp } f$ and $\text{supp } g \subseteq U$. By part (iii) of Lemma 2.2

$$\rho(g) \geq \rho(f) > \mu_\rho(U) - \epsilon.$$

Therefore,

$$\mu_\rho(U) = \sup\{\rho(g) : 1_K \leq g \leq 1, g \in C_c(X), \text{supp } g \subseteq U\}.$$

When $\mu_\rho(U) = \infty$ we replace f by functions $f_n \in C_c(X)$ such that $\rho(f_n) \geq n$, $\text{supp } f_n \subseteq U$, and use a similar argument to show that $\sup\{\rho(g) : 1_K \leq g \leq 1, g \in C_c(X), \text{supp } g \subseteq U\} = \infty$.

- (p4) Take any $U \in \mathcal{O}(X)$ such that $K \subseteq U$. By part (p3) we see that $\inf\{\rho(g) : g \in C_c(X), g \geq 1_K\} \leq \mu_\rho(U)$. Taking infimum over all open sets containing K we have:

$$\inf\{\rho(g) : g \in C_c(X), g \geq 1_K\} \leq \mu_\rho(K).$$

To prove the opposite inequality, take $g \geq 1_K, g \in C_c(X)$. Let $0 < \delta < 1$. Let $U = \{x : g(x) > 1 - \delta\}$. Then U is open and $K \subseteq U$. By part (s1) of Lemma 2.1 and part (i) of Lemma 2.2 for $h = \inf\{g, 1 - \delta\}$ we have $\rho(h) \leq \rho(g)$. Since $\frac{h}{1 - \delta} = 1$ on U , for any function $f \in C_c(X), 0 \leq f \leq 1$, $\text{supp } f \subseteq U$ we have $f \leq \frac{h}{1 - \delta}$ and so by part (ii) of Lemma 2.2

$$\rho(f) \leq \rho\left(\frac{h}{1 - \delta}\right) = \frac{\rho(h)}{1 - \delta}.$$

Then $(1 - \delta)\rho(f) \leq \rho(h) \leq \rho(g)$, and so

$$\begin{aligned} (1 - \delta)\mu_\rho(K) &\leq (1 - \delta)\mu_\rho(U) \\ &= (1 - \delta) \sup\{\rho(f) : 0 \leq f \leq 1_U, \text{supp } f \subseteq U\} \leq \rho(g). \end{aligned}$$

Thus, for any $g \in C_c(X)$ such that $g \geq 1_K$ and any $0 < \delta < 1$

$$(1 - \delta)\mu_\rho(K) \leq \rho(g).$$

Therefore,

$$\mu_\rho(K) \leq \inf\{\rho(g) : g \in C_b(X), g \geq 1_K\}.$$

- (p5) Essentially identical to the proof for part (p4).
 (p6) Follows from part (p4).
 (p7) The proof uses Lemma 1.1 and is left to the reader.
 (p8) From Definition 3.1 we see that $\mu_\rho(K) \leq \mu_\rho(U)$ for any $K \subseteq U, K \in \mathcal{K}(X)$, hence,

$$\sup\{\mu_\rho(K) : K \in \mathcal{K}(X), K \subseteq U\} \leq \mu_\rho(U).$$

For the opposite equality, assume first that $\mu_\rho(U) < \infty$. For $\epsilon > 0$ find a function $f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U$ for which $\mu_\rho(U) - \epsilon < \rho(f)$. Let $K = \text{supp } f$. Choose $V \in \mathcal{O}(X)$ such that $K \subseteq V \subseteq U, \mu_\rho(V) < \mu_\rho(K) + \epsilon$. Pick Urysohn function $g \in C_c(X)$ such that $g = 1$ on K and $\text{supp } g \subseteq V$, so $\rho(g) \leq \mu_\rho(V)$. Using part (iii) of Lemma 2.2 we have $\rho(f) \leq \rho(g)$, and

$$\mu_\rho(U) - \epsilon < \rho(f) \leq \rho(g) \leq \mu_\rho(V) < \mu_\rho(K) + \epsilon.$$

This gives us

$$\sup\{\mu_\rho(K) : K \in \mathcal{K}(X), K \subseteq U\} \geq \mu_\rho(U).$$

When $\mu_\rho(U) = \infty$ we replace f by functions $f_n \in C_c(X)$ such that $\rho(f_n) \geq n$, $K_n = \text{supp } f_n \subseteq U$, and use a similar argument to show that $\sup\{\mu_\rho(K) : K \in \mathcal{K}(X), K \subseteq U\} = \infty$.

- (p9) Let $K = K_1 \sqcup K_2$, $K_1, K_2 \in \mathcal{K}(X)$. It is enough to consider the case when both $\mu_\rho(K_1)$ and $\mu_\rho(K_2)$ are finite. There are disjoint open sets V_1, V_2 such that $K_i \subseteq V_i$. For $\epsilon > 0$ by part (p5) pick functions $g_1, g_2 \in C_c(X)$ such that $\text{supp } g_i \subseteq V_i$, $1_{K_i} \leq g_i \leq 1$ and $\rho(g_i) - \mu_\rho(K_i) < \epsilon$ for $i = 1, 2$. Since $g_1 + g_2 = 1$ on K and $g_1 g_2 = 0$, by part (p4) and Lemma 2.2, part (q2)

$$\mu_\rho(K) \leq \rho(g_1 + g_2) = \rho(g_1) + \rho(g_2) < \mu_\rho(K_1) + \mu_\rho(K_2) + 2\epsilon,$$

showing that $\mu_\rho(K) \leq \mu_\rho(K_1) + \mu_\rho(K_2)$. Now for $\epsilon > 0$ by part (p4) let $f \in C_c(X)$ be such that $f \geq 1_K$ and $\rho(f) - \mu_\rho(K) < \epsilon$. For the functions g_1, g_2 as above $g_1 + g_2 \leq 1$, $(g_1 f)(g_2 f) = 0$, $g_i f \geq 0$, and $g_i f \geq 1$ on K_i . Then by parts (q2) and (iii) of Lemma 2.2

$$\mu_\rho(K_1) + \mu_\rho(K_2) \leq \rho(g_1 f) + \rho(g_2 f) = \rho((g_1 + g_2)f) \leq \rho(f) \leq \mu_\rho(K) + \epsilon,$$

giving $\mu_\rho(K_1) + \mu_\rho(K_2) \leq \mu_\rho(K)$.

- (p10) Let $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$. First we shall show that

$$\mu_\rho(U \setminus K) + \mu_\rho(K) \geq \mu_\rho(U). \quad (3.1)$$

By Lemma 1.1 let V be an open set with compact closure such that

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$

If $\mu_\rho(K) = \infty$, inequality (3.1) trivially holds, so we assume that $\mu_\rho(K) < \infty$. For $\epsilon > 0$ choose $W_1 \in \mathcal{O}(X)$ such that $K \subseteq W_1 \subseteq V$ and $\mu_\rho(W_1) < \mu_\rho(K) + \epsilon$. Also, there exists an open set W with compact closure such that

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq V \subseteq \overline{V} \subseteq U.$$

Choose Urysohn function $g \in C_c(X)$ such that $1_{\overline{W}} \leq g \leq 1$, $\text{supp } g \subseteq W_1$. Then

$$\rho(g) \leq \mu_\rho(W_1) < \mu_\rho(K) + \epsilon.$$

First assume that $\mu_\rho(U) < \infty$. By part (p3) choose $f \in C_c(X)$ such that $1_{\overline{V}} \leq f \leq 1$, $\text{supp } f \subseteq U$, and

$$\rho(f) > \mu_\rho(U) - \epsilon.$$

Note that $0 \leq f - g \leq 1$, and, since $f - g = 0$ on \overline{W} , we have $\text{supp}(f - g) \subseteq U \setminus K$. Also, by part (q3) of Lemma 2.2 we have $\rho(f - g) = \rho(f) - \rho(g)$. So

$$\mu_\rho(U \setminus K) \geq \rho(f - g) = \rho(f) - \rho(g) \geq \mu_\rho(U) - \epsilon - \mu_\rho(K) - \epsilon,$$

which gives us inequality (3.1). If $\mu_\rho(U) = \infty$, use instead of f functions f_n with $1_{\overline{V}} \leq f_n \leq 1$, $\text{supp } f_n \subseteq U$, $\rho(f_n) \geq n$ in the above argument to show that $\mu_\rho(U \setminus K) = \infty$. Then inequality (3.1) holds.

Now we would like to show

$$\mu_\rho(U) \geq \mu_\rho(U \setminus K) + \mu_\rho(K). \quad (3.2)$$

When $\mu_\rho(U \setminus K) = \infty$, inequality (3.2) holds trivially, so we assume that $\mu_\rho(U \setminus K) < \infty$. Given $\epsilon > 0$, choose $g \in C_c(X)$, $0 \leq g \leq 1$ such that $C = \text{supp } g \subseteq U \setminus K$ and

$$\rho(g) > \mu_\rho(U \setminus K) - \epsilon.$$

If $\mu_\rho(U \setminus C) = \infty$, then $\mu_\rho(U) = \infty$, so (3.2) holds. So assume that $\mu_\rho(U \setminus C) < \infty$. Since $K \subseteq U \setminus C$, by part (p3) choose $f \in C_c(X)$ such that $1_K \leq f \leq 1$, $\text{supp } f \subseteq U \setminus C$, and $\rho(f) > \mu_\rho(U \setminus C) - \epsilon$. Then

$$\rho(f) > \mu_\rho(U \setminus C) - \epsilon \geq \mu_\rho(K) - \epsilon.$$

Since $fg = 0$, applying part (q2) of Lemma 2.2 we obtain $\rho(f + g) = \rho(f) + \rho(g)$. Since $f + g \in C_c(X)$ with $\text{supp}(f + g) \subseteq U$, we obtain:

$$\mu_\rho(U) \geq \rho(f + g) = \rho(f) + \rho(g) \geq \mu_\rho(K) + \mu_\rho(U \setminus K) - 2\epsilon.$$

Therefore, $\mu_\rho(U) \geq \mu_\rho(U \setminus K) + \mu_\rho(K)$.

(p11) For any open set U containing K we have $\rho(f) \leq \mu_\rho(U)$. Then $\rho(f) \leq \mu_\rho(K)$.

(p12) Using part (p11) we see for $i = 1, 2$ that $0 \leq \rho(f_i) \leq \mu_\rho(K)$, and the statement follows. \square

Remark 3.3. — In part (p4) we basically follow a proof given by D. Grubb in [35]. Proof of inequality (3.1) is essentially from [44].

THEOREM 3.4. — *Let X be locally compact. If ρ is a quasi-linear functional on $C_0(X)$ or on $C_c(X)$, then μ_ρ defined in Definition 3.1 is a compact-finite topological measure. If $\|\rho\| < \infty$ then μ_ρ is finite with $\mu_\rho(X) \leq \|\rho\|$.*

Proof. — By part (p1) of Lemma 3.2 μ_ρ is nonnegative and is not identically ∞ . Definition 3.1 and part (p8) of Lemma 3.2 give conditions (TM2) and (TM3) of Definition 1.12. Parts (p9) and (p10) of Lemma 3.2 and Remark 1.14 give condition (TM1). Thus, μ_ρ is a topological measure, and μ_ρ is compact-finite by part (p6) of Lemma 3.2. Clearly, $\mu_\rho(X) \leq \|\rho\|$. \square

Remark 3.5. — When X is compact for an open set U we may define $\mu_\rho(U) = \sup\{\rho(f) : f \in C(X), 0 \leq f \leq 1, \text{supp } f \subseteq U\}$ or $\mu_\rho(U) = \sup\{\rho(f) : f \in C(X), 0 \leq f \leq 1_U\}$, and for a closed set C define $\mu_\rho(C) = \mu_\rho(X) - \mu_\rho(X \setminus C)$. One may show that μ_ρ is a topological measure.

THEOREM 3.6 (Representation theorem). — *Let X be locally compact. If*

- (A) ρ is a quasi-linear functional on $C_0(X)$ such that $\|\rho\| < \infty$, or
- (B) ρ is a quasi-linear functional on $C_c(X)$

then there exists a unique topological measure μ on X such that $\rho = \rho_\mu$. In fact, $\mu = \mu_\rho$. In case (A) μ is finite with $\mu(X) = \|\rho\|$, and in case (B) μ is compact-finite.

Proof. — The proof is similar in both cases, and we will provide it for case (A). Given a quasi-linear functional ρ on $C_0(X)$, by Theorem 3.4 construct a finite topological measure $\mu = \mu_\rho$ with $\mu(X) \leq \|\rho\|$. By Theorem 2.12 obtain from μ a quasi-linear functional ρ_μ with $\|\rho_\mu\| \leq \mu(X)$. We shall show that $\rho = \rho_\mu$. (This will also imply that $\|\rho_\mu\| = \mu(X)$.) Fix $f \in C_0(X)$. Recall from Definition 2.8 that

$$\rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f$$

where m_f is a measure on \mathbb{R} (supported on $\overline{f(X)}$) such that $m_f(W) = \mu(f^{-1}(W))$ for every open set $W \subseteq \mathbb{R}$ by Lemma 2.6. For a continuous function ϕ on $\overline{f(X)}$ let $\tilde{\phi}(x) = \phi(x) - \phi(0)$. Consider the map $L : C(\overline{f(X)}) \rightarrow \mathbb{R}$ given by $L(\phi) = \rho(\tilde{\phi} \circ f)$ for each $\phi \in C(\overline{f(X)})$. From linearity of ρ on the subalgebra generated by f it follows

that L is a bounded linear functional. Therefore, there exists a signed measure m on the compact $\overline{f(X)} \subseteq \mathbb{R}$ of the form $m = m_1 - m_2$, where m_1, m_2 are finite Borel measures on $\overline{f(X)}$, such that

$$L(\phi) = \rho(\tilde{\phi} \circ f) = \int_{\overline{f(X)}} \phi \, dm$$

for each $\phi \in C(\overline{f(X)})$. Thus, for each $\phi \in C(\overline{f(X)})$ with $\phi(0) = 0$ we have:

$$\rho(\phi \circ f) = \int_{\overline{f(X)}} \phi \, dm. \quad (3.3)$$

We shall show now that $m_f = m$ on open intervals in $\overline{f(X)} \setminus \{0\}$. Let $W = (\alpha, \beta)$ be such an interval. Choose a sequence of compact sets $\{D_n\}_{n=1}^\infty$ such that $D_n \subseteq D_{n+1} \subseteq W$ and $\bigcup_{n=1}^\infty D_n = W$ (here D^0 denotes the interior of D). For each n choose an Urysohn function ϕ_n such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on D_n and $\text{supp } \phi_n \subseteq W$. Note that $0 \leq \phi_n \circ f \leq 1$, and $\phi_n \circ f$ has compact support contained in $f^{-1}(W)$. By Definition 3.1 applied to $\mu = \mu_\rho$ and Lemma 2.6 applied to an open set $f^{-1}(W)$ we see that

$$\rho(\phi_n \circ f) \leq \mu(f^{-1}(W)) = m_f(W). \quad (3.4)$$

For measures m_1, m_2 we have $m_i(W) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \phi_n \, dm_i$ (for $i = 1, 2$), so for m , using (3.3) and (3.4) we have:

$$m(W) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \phi_n \, dm = \lim_{n \rightarrow \infty} \rho(\phi_n \circ f) \leq m_f(W). \quad (3.5)$$

On the other hand, given $\epsilon > 0$, choose compact set $K \subseteq f^{-1}(W)$ such that $\mu(K) > \mu(f^{-1}(W)) - \epsilon$. The set $f(K)$ is compact, and by choice of $\{D_n\}_{n=1}^\infty$ there exists $n' \in \mathbb{N}$ such that $f(K) \subseteq D_n^0 \subseteq D_n$ for all $n \geq n'$. Then for all $n \geq n'$ we have $1_K \leq \phi_n \circ f$ and using part (p4) of Lemma 3.2 we see that

$$\mu(f^{-1}(W)) - \epsilon < \mu(K) \leq \rho(\phi_n \circ f).$$

Then as in (3.5)

$$m(W) = \lim_{n \rightarrow \infty} \rho(\phi_n \circ f) \geq \mu(f^{-1}(W)) - \epsilon = m_f(W) - \epsilon.$$

Therefore, $m_f(W) = m(W)$. Then $m_f = m$ on all open sets in $\overline{f(X)} \setminus \{0\}$. Let $m'_i = m_i - m_i(\{0\})\delta_0$ (here δ_0 is the point mass at 0) for $i = 1, 2$. By Remark (2.10) for $i = 1, 2$

$$\int_{\overline{f(X)}} id \, dm_i = \int_{\overline{f(X)}} id \, dm'_i.$$

Then (positive) Borel measure $m' = m'_1 - m'_2$ coincides with m_f and m on all open sets in $\overline{f(X)} \setminus \{0\}$. Since $m' = m'_1 - m'_2$ and $m = m_1 - m_2$, using Remark (2.10) we have:

$$\rho(f) = \rho(id \circ f) = \int_{\overline{f(X)}} id \, dm = \int_{\overline{f(X)}} id \, dm' = \int_{\overline{f(X)}} id \, dm_f = \rho_\mu(f),$$

so $\rho = \rho_\mu$.

Now we need to show uniqueness of μ . Suppose there are topological measures μ and ν such that $\rho_\mu = \rho_\nu = \rho$, and $\mu \neq \nu$. Then there exists $U \in \mathcal{O}(X)$ with

$\mu(U) < \nu(U)$. Pick a compact set K such that $K \subseteq U$, $\mu(U) < \nu(K)$. Let $f \in C_c(X)$ be a function such that $1_K \leq f \leq 1_U$. Then using Lemma 3.2

$$\rho_\mu(f) \leq \mu(U) < \nu(K) \leq \rho_\nu(f),$$

i.e. $\rho_\mu \neq \rho_\nu$. This contradiction shows the uniqueness of μ , and the proof is complete. \square

Remark 3.7. — Our proof of Theorem 3.6 uses techniques from [3], [34], [35], and [44].

Remark 3.8. — Theorem 3.6 and existence of topological measures that are not measures (see Remark 1.17) indicate that there exist quasi-linear functionals that are not linear. Also, in Example 4.13 below we construct a quasi-linear functional on \mathbb{R}^2 which is not linear.

Let X be locally compact. Let $TM_c(X)$ be the collection of compact-finite topological measures on X , $\mathbf{TM}(X)$ the collection of finite topological measures on X .

Let $\mathbf{QL}_0(X)$ denote the collection of all bounded quasi-linear functionals on $C_0(X)$, and let $QI_c(X)$ be the collection of all quasi-linear functionals on $C_c(X)$.

- THEOREM 3.9.** — (I) *The map $\Pi : TM_c(X) \rightarrow QI_c(X)$ where $\Pi(\mu) = \rho_\mu$, is an order-preserving bijection with Π^{-1} given by $\Pi^{-1}(\rho) = \mu_\rho$, and μ is a measure iff $\Pi(\mu)$ is a linear functional.*
- (II) *The map $\Pi : \mathbf{TM}(X) \rightarrow \mathbf{QL}_0(X)$ (where $\Pi(\mu) = \rho_\mu$, and $\Pi^{-1}(\rho) = \mu_\rho$) is an order-preserving bijection such that $\|\rho\| = \|\mu\|$, and μ is a measure iff $\Pi(\mu)$ is a linear functional.*

Proof. — By Theorem 3.6 the map Π is a bijection, and $\|\rho\| = \mu(X) = \|\mu\|$ in case (II). From formula (2.2), Remark 2.10, and the relationship between m_f and μ in Lemma 2.6 it is easy to see that Π is order-preserving. The rest follows from Remark 2.9. \square

Remark 3.10. — Quasi-linear functionals are obtained by integration with respect to a topological measure. For more information on such integration see [16].

Remark 3.11. — In [3] Aarnes proved the first representation theorem for quasi-integrals and studied their properties (for a compact space). A much simplified proof of Aarnes's representation theorem in the compact case was given by D. Grubb in a series of lectures, but it was never published. Grubb's approach was different from the one presented in this paper because it used the symmetry between closed and open sets and the presence of constant functions in subalgebras. A. Rustad in [44] first gave a proof of a representation theorem for positive quasi-integrals on functions with compact support when X is locally compact. For a compact space, D. Grubb proved a representation theorem for bounded signed topological measures in [34]. At the time of [34] it was not known whether a signed topological measure is a difference of two topological measures; for such a decomposition on a locally compact space and equivalence of different definitions of signed topological measures see [14].

4. PROPERTIES OF QUASI-INTEGRALS

Remark 4.1. — From Theorem 3.6 it follows that on a locally compact space X any quasi-linear functional ρ on $C_0(X)$ with $\|\rho\| < \infty$ or any quasi-linear functional ρ on $C_c(X)$ is given by

$$\rho(f) = \int f d\mu = \int_{\mathbb{R}} id dm_f = \int_{[a,b]} id dm_f,$$

where m_f is a measure obtained from topological measure μ and the function f , and supported on $[a, b] = \overline{f(X)}$ as in Lemma 2.6. We may also take $[a, b]$ to be any closed interval containing $\overline{f(X)}$. The integral

$$\int_{[a,b]} id dm_f = \int_{[a,b]} id dF$$

is the Riemann-Stieltjes integral over $[a, b]$ with respect to the function F given by Definition 2.4. It is easy to see (apply, for example, [37, Theorem (21.67)]) that

$$\rho(f) = \int_{[a,b]} id dm_f = \int_a^b F(t) dt + aF(a^-).$$

If $\|\rho\| < \infty$ then $\mu(X) < \infty$, and we have

$$\rho(f) = \int_{[a,b]} id dm_f = \int_a^b F(t) dt + a\mu(X). \tag{4.1}$$

If $\|\rho\| < \infty$ and $f \geq 0$ then we may take $a = 0$ and

$$\rho(f) = \int_{[0,b]} id dm_f = \int_0^b F(t) dt. \tag{4.2}$$

LEMMA 4.2. — *Let X be locally compact. Suppose*

- (A) ρ is a quasi-linear functional on $C_0(X)$ such that $\|\rho\| < \infty$ or
- (B) ρ is a quasi-linear functional on $C_c(X)$.

If $f \geq g$ then $\rho(f) \geq \rho(g)$.

Proof. — Case (B) is proved in Lemma 2.2, part (iii). For case (A), define distribution functions F and G for f and g as in Definition 2.4. Let $[a, b]$ contain both $\overline{f(X)}$ and $\overline{g(X)}$. Since $\mu(X) < \infty$ and $F \geq G$, the assertion follows from formula (4.1). □

DEFINITION 4.3. — We say a functional ρ is monotone if $f \leq g \implies \rho(f) \leq \rho(g)$.

Remark 4.4. — Since $f \leq |f|$, by Lemma 4.2 we see that if $\rho \in \mathbf{QI}_0(X)$ then

$$\|\rho\| = \sup\{\rho(f) : f \in C_0(X), 0 \leq f \leq 1\} = \sup\{|\rho(f)| : f \in C_0(X), \|f\| \leq 1\},$$

and for $\rho \in \mathbf{QI}_c(X)$

$$\|\rho\| = \sup\{\rho(f) : f \in C_c(X), 0 \leq f \leq 1\} = \sup\{|\rho(f)| : f \in C_c(X), \|f\| \leq 1\}.$$

The right hand side is a norm on the space of signed quasi-linear functionals, and $\|\rho\|$ from Definition 1.10 is the restriction of this norm to positive quasi-linear functionals.

THEOREM 4.5. — *Suppose X is locally compact and ρ is a quasi-linear functional on $C_c(X)$ or on $C_0(X)$.*

- (i) Suppose μ is compact-finite. If $f, g \in C_c(X)$, $f, g \geq 0$, $\text{supp } f, \text{supp } g \subseteq K$, where K is compact, then

$$|\rho(f) - \rho(g)| \leq \|f - g\| \mu(K).$$

In particular, for any $f \in C_c(X)$

$$|\rho(f)| \leq \|f\| \mu(\text{supp } f).$$

If $f, g \in C_c(X)$, $\text{supp } f, \text{supp } g \subseteq K$, where K is compact, then

$$|\rho(f) - \rho(g)| \leq 2\|f - g\| \mu(K).$$

Thus, ρ is continuous with respect to the topology of uniform convergence on compact sets.

- (ii) Suppose $\mu(X) < \infty$ (i.e. $\|\rho\| < \infty$.) If $f \in C_c(X)$ then

$$|\rho(f)| \leq \|f\| \mu(X) = \|f\| \|\rho\|. \quad (4.3)$$

If $f, g \in C_c(X)$, $f, g \geq 0$ then

$$|\rho(f) - \rho(g)| \leq \|f - g\| \mu(X) = \|f - g\| \|\rho\|.$$

For arbitrary $f, g \in C_c(X)$

$$|\rho(f) - \rho(g)| \leq 2\|f - g\| \mu(X) = 2\|f - g\| \|\rho\|.$$

Thus, ρ is uniformly continuous.

- (iii) Let X be compact. Let ρ be a quasi-linear functional on $C(X)$. Then for any $f, g \in C(X)$

$$|\rho(f) - \rho(g)| \leq \rho(\|f - g\|) = \|f - g\| \rho(1).$$

In particular, ρ is uniformly continuous. If $f + g = c$, where c is a constant, then $\rho(f) + \rho(g) = \rho(c)$.

Proof. —

- (i) It is enough to consider $K = \text{supp } f \cup \text{supp } g$. Suppose first that $f, g \in C_c(X)$, $f, g \geq 0$. Let $h \in C_c(X)$ be such that $0 \leq h \leq 1$, $h = 1$ on K . Since $f - g \leq \|f - g\| h$, i.e. $f \leq g + \|f - g\| h$, by Lemma 4.2 $\rho(f) \leq \rho(g + \|f - g\| h)$. Using part (q3) of Lemma 2.2 we have:

$$\rho(f) \leq \rho(g + \|f - g\| h) = \rho(g) + \|f - g\| \rho(h).$$

Similarly, $\rho(g) \leq \rho(f) + \|f - g\| \rho(h)$, so $|\rho(f) - \rho(g)| \leq \|f - g\| \rho(h)$. Using part (p5) of Lemma 3.2 we have:

$$|\rho(f) - \rho(g)| \leq \|f - g\| \mu(K).$$

For arbitrary $f, g \in C_c(X)$ we have:

$$\begin{aligned} |\rho(f) - \rho(g)| &= |\rho(f^+) - \rho(f^-) - \rho(g^+) + \rho(g^-)| \\ &\leq |\rho(f^+) - \rho(g^+)| + |\rho(g^-) - \rho(f^-)| \\ &\leq \|f^+ - g^+\| \mu(K) + \|g^- - f^-\| \mu(K) \leq 2\|f - g\| \mu(K). \end{aligned}$$

- (ii) Apply part (i) and equality $\|\rho\| = \mu(X)$ from Theorem 3.9.
 (iii) Since $f \leq g + \|f - g\|$ and $g \leq f + \|f - g\|$ by Lemma 4.2 and Remark 1.6 we have $\rho(f) \leq \rho(g) + \|f - g\| \rho(1)$ and $\rho(g) \leq \rho(f) + \|f - g\| \rho(1)$, which gives the first assertion. The last assertion follows from Remark 1.6 since $f = -g + c$. \square

Remark 4.6. — Let X be locally compact. Suppose $\rho \in \mathbf{QI}_0(X)$, $f \in C_0(X)$. Let $\|f\| = b$. From Remark 4.1 $\rho(f) = \int_{[-b,b]} id \, dm_f$, so using Lemma 2.6 and Theorem 3.9 we obtain inequality (4.3) for functions from $C_0(X)$:

$$|\rho(f)| \leq bm_f(\mathbb{R}) = b\mu(X) = \|f\|\mu(X) = \|f\|\|\rho\|.$$

PROPOSITION 4.7. — *Suppose X is locally compact. A quasi-linear functional on $C_0(X)$ is monotone iff it is bounded.*

Proof. — (\Leftarrow) is part (A) of Lemma 4.2. (\Rightarrow): The proof follows that of [44, Lemma 2.3]. Suppose to the contrary that $\rho(f) \in \mathbb{R}$ for every $f \in C_0(X)$ but $\|\rho\| = \infty$. There are functions $f_k \in C_0(X)$, $0 \leq f_k \leq 1$ such that $\rho(f_k) \geq 2^{2k}$. Consider $f = \sum_{k=1}^{\infty} 2^{-k} f_k$, so $f \in C_0(X)$ and $0 \leq f \leq 1$. For each k we have $f \geq 2^{-k} f_k$, and so $\rho(f) \geq \rho(2^{-k} f_k) = 2^{-k} \rho(f_k) \geq 2^k$, i.e. $\rho(f) = \infty$. This gives a contradiction. \square

THEOREM 4.8. — *Suppose X is locally compact.*

- (I) *A bounded quasi-integral on $C_c(X)$ extends uniquely to a bounded quasi-integral on $C_0(X)$ with the same norm.*
- (II) *A bounded quasi-integral on $C_0(X)$ is the unique extension of a bounded quasi-integral on $C_c(X)$.*

Proof. —

- (I) Let ρ be a quasi-linear functional on $C_c(X)$ with $\|\rho\| < \infty$. Let $f \in C_0(X)$. Choose a sequence of function $f_n \in C_c(X)$ converging uniformly to f . Since by Theorem 4.5

$$|\rho(f_n) - \rho(f_m)| \leq 2\|f_n - f_m\| \|\rho\|,$$

the sequence $\rho(f_n)$ is Cauchy. Let $L = \lim_{n \rightarrow \infty} \rho(f_n)$. If g_n is another sequence of functions from $C_c(X)$ converging to f , by Theorem 4.5 $|\rho(f_n) - \rho(g_n)| \leq 2\|\rho\| \|f_n - g_n\| \rightarrow 0$, so the limit L is well defined. We extend ρ from $C_c(X)$ to $C_0(X)$ by defining $\rho(f) = L$.

Let $\phi \circ f \in B(f)$, $\phi(0) = 0$. Since $\phi \circ f_n$ converges uniformly to $\phi \circ f$, it is easy to see that ρ is a quasi-linear functional on $C_0(X)$. Using part (s1) of Lemma 2.1 we see that the norm of an extended functional stays the same.

- (II) Let ρ be a bounded quasi-integral on $C_0(X)$. The restriction of ρ to $C_c(X)$ is a bounded quasi-integral. Now let $f \in C_0(X)$, and let $f_n \in C_c(X)$ converge to f . By part (s3) of Lemma 2.1 we may assume that f_n and f are in the subalgebra generated by f and $\|f - f_n\| \leq \frac{1}{n}$. Let $L = \lim \rho(f_n)$ as in part (I). We only need to show that $L = \rho(f)$. But since f_n and f are in the same subalgebra, using Remark 4.6 we have:

$$|\rho(f) - \rho(f_n)| = |\rho(f - f_n)| \leq \|f - f_n\| \|\rho\| \leq \frac{1}{n} \|\rho\|,$$

so $\lim_{n \rightarrow \infty} \rho(f_n) = \rho(f) = L$. \square

Remark 4.9. — Part (I) of Theorem 4.8 was first proved (in a different way) in [44, Corollary 3.10]. The first assertion of part (i) in Theorem 4.5 is closely related to [44, Corollary 3.5].

Using Proposition 4.7, Remark 4.6, and Theorem 4.8 we may extend part (ii) of Theorem 4.5 to functions vanishing at infinity:

COROLLARY 4.10. — Suppose ρ is a bounded quasi-linear functional on $C_0(X)$. If $f \in C_0(X)$ then

$$|\rho(f)| \leq \|f\| \mu(X) = \|f\| \|\rho\|.$$

If $f, g \in C_0(X)$, $f, g \geq 0$ then

$$|\rho(f) - \rho(g)| \leq \|f - g\| \mu(X) = \|f - g\| \|\rho\|.$$

For arbitrary $f, g \in C_0(X)$

$$|\rho(f) - \rho(g)| \leq 2\|f - g\| \mu(X) = 2\|f - g\| \|\rho\|.$$

Remark 4.11. — In symplectic geometry, a functional η on $C_c(X)$ is Lipschitz continuous if for every compact $K \subseteq X$ there is a number $N_K \geq 0$ such that $|\eta(f) - \eta(g)| \leq N_K \|f - g\|$ for all f, g with support contained in K . Theorem 4.5 and Corollary 4.10 say that ρ is Lipschitz continuous. Lipschitz continuity is used in many results in symplectic geometry; it is also used in the proof of results involving the Kantorovich-Rubinstein metric for (deficient) topological measures (see [22], [18]).

Remark 4.12. — Although we do not always state this explicitly, all results in this paper remain valid (with algebras $B(f)$ replaced by $A(f)$ and simpler proofs) for quasi-linear functionals on a compact space. Thus we obtain all known properties and the representation theorem for quasi-linear functionals on $C(X)$ when X is compact, in particular, from [3].

We conclude with an example of a quasi-integral on \mathbb{R}^2 .

Example 4.13. — Let $X = \mathbb{R}^2$. Let K be the closed rectangle with vertices $(1, 5)$, $(1, 7)$, $(7, 7)$, $(7, 5)$, and C be the closed rectangle with vertices $(5, 1)$, $(5, 7)$, $(7, 7)$, $(7, 1)$. Choose five points as follows: three points in the interior of the square $K \cap C$, one point in the interior of $K \setminus C$, and one point in the interior of $C \setminus K$. Let μ be the topological measure as in [13, Example 18] based on the five chosen points, i.e. for a compact solid or a bounded open solid set A we have $\mu(A) = 0$ if A contains no more than 1 of the chosen points, $\mu(A) = 1/2$ if A contains 2 or 3 of the chosen points, and $\mu(A) = 1$ if A contains at least 4 of the chosen points. (A set is called solid, if it is connected and its complement has only unbounded connected components.) Let ρ be the quasi-linear functional on $C_0(X)$ corresponding to μ according to Theorem 3.6. Let $b > 0$ and let $f \in C_c(X)$ be the function such that $f = b$ on K and $\text{supp } f \subseteq U$, where U is an open rectangle containing K but not the chosen point in $C \setminus K$. Similarly, let $g \in C_c(X)$ be the function such that $g = b$ on C and $\text{supp } g \subseteq V$, where V is an open rectangle containing C but not the chosen point in $K \setminus C$. Let $h = f + g$. Let F and H be the distribution functions of f and h respectively as in Definition 2.4. Since $f(X) = [0, b]$ and $h(X) = [0, 2b]$, from formula (4.2) we have:

$$\rho(f) = \int_0^b F(t) dt, \quad \rho(h) = \int_0^{2b} H(t) dt.$$

Observe that $F(t) = 1$ for $t \in (0, b)$. Then $\rho(f) = b$. In the same way, $\rho(g) = b$. Since $H(t) = 1$ for $t \in (0, b)$ and $H(t) = 0.5$ for $t \in [b, 2b)$, we have $\rho(f + g) = \rho(h) = 1.5b$. Thus, $\rho(f) + \rho(g) \neq \rho(f + g)$, and the functional ρ is not linear.

Remark 4.14. — In symplectic geometry one often sees quasi-linear functionals corresponding to topological measures that assume values 0,1 and are obtained by composition of a measure with a q-function that takes two values (see [5], [11] for this method on compact spaces). Example 4.13 shows integration with topological measures that assume more than 2 values. Various methods (including q-functions) for obtaining topological measures on locally compact spaces are presented in [13].

5. QUASI-LINEAR FUNCTIONALS, SYMPLECTIC GEOMETRY, AND CHOQUET THEORY

The kind of quasi-linear functionals that are used in symplectic geometry are symplectic (partial symplectic) quasi-states. They exist on a variety of manifolds, including $\mathbb{C}P^n$, complex Grassmanian, S^2 , $S^2 \times S^2$ (see [24], [26], [27], [43, Ch. 5]). We shall begin by giving some background and outlining a method for obtaining symplectic quasi-states.

Let M^{2n} be a smooth connected manifold, and let ω be a closed 2-form on M such that $\omega^n = \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} \neq 0$. Then ω is called a symplectic form and (M, ω) a symplectic manifold. In a classical mechanics framework, M can represent the phase space of a mechanical system. Smooth functions on M (which may also depend smoothly on an additional parameter t , viewed as time) are called Hamiltonians. For a Hamiltonian $H : M \times I \rightarrow \mathbb{R}$ (where, for example, $I = [0, 1]$) by $H_t(x)$ we mean $H(x, t)$. A diffeomorphism ϕ on M is called a symplectomorphism if it preserves ω , i.e. $\phi^*\omega = \omega$. Given a Hamiltonian H , there exists a unique vector field ξ on M such that pointwise

$$\omega(\eta, \xi_t) = dH_t(\eta)$$

for any vector field η on M . The vector field ξ is called the the symplectic gradient of H and is denoted $sgradH$; it is also called Hamiltonian vector field of H and is denoted X_H . The ordinary differential equation on M

$$\dot{x}(t) = sgradH_t(x(t)) \tag{5.1}$$

generates a one-parameter family of diffeomorphisms $\phi_H^t : M \rightarrow M$, defined by $\phi_H^t(x_0) = x(t)$, where $x(t)$ is the unique solution of (5.1) with the initial condition $x(0) = x_0$. The family $\{\phi_H^t\}$ is called the Hamiltonian flow of H .

Let M be a closed manifold. A Hamiltonian $H(x, t)$ on M is called normalized, if $\int_M H_t \omega^n = 0$ for all $t \in I$. The maps forming the Hamiltonian flow of a normalized Hamiltonian are called Hamiltonian diffeomorphisms. (Such maps are, in fact, symplectomorphisms as they preserve the symplectic form.) We denote by $Ham(M, \omega)$ the set of all Hamiltonian diffeomorphisms; it is a group with respect to composition. We denote by $\widetilde{Ham}(M, \omega)$ the universal cover of $Ham(M, \omega)$. Elements of $\widetilde{Ham}(M, \omega)$ are smooth paths in $Ham(M, \omega)$ based at the identity, considered up to homotopy with fixed end-points. The universal cover $\widetilde{Ham}(M, \omega)$ is a group under composition. Interestingly, every smooth path in $Ham(M, \omega)$ based at identity is the Hamiltonian flow generated by some normalized Hamiltonian. Thus, we may denote an element $\tilde{\phi}$ of $\widetilde{Ham}(M, \omega)$ by $\tilde{\phi}_H$ (the homotopy class with fixed end-points of the path $\{\phi_H^t\}_{t \in [0,1]}$ in $Ham(M, \omega)$ corresponding to

normalized Hamiltonian H .) (Note that in some literature $\widetilde{\phi}_H$ is denoted by ϕ_H .) For more information see [43, Ch. 1], [24].

For smooth functions H, G on M define the Poisson bracket by $\{H, G\} = \omega(X_G, X_H)$. In appropriate local coordinates ω is given by $\omega = \sum_{j=1}^n dp_j \wedge dq_j$, and the Poisson bracket of smooth compactly supported functions H, G on M is a canonical operation given by

$$\{H, G\} = \sum_j \left(\frac{\partial H}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial H}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

(Note that different authors may use different signs in definitions of Hamiltonian vector fields and Poisson brackets.)

Smooth functions H, G on a closed symplectic manifold are called Poisson commuting if the Poisson bracket $\{H, G\} = 0$.

DEFINITION 5.1. — A symplectic quasi-state on M is a functional $\rho : C(M) \rightarrow \mathbb{R}$ such that

- (1) $\rho(H) \geq \rho(G)$ if $H \geq G$;
- (2) $\rho(aH) = a\rho(H)$ for any $a \in \mathbb{R}$;
- (3) if H and G are Poisson commuting, then $\rho(H + G) = \rho(H) + \rho(G)$;
- (4) $\rho(1) = 1$.

In [43, Sect.5.1]) this definition is generalized: monotonicity is replaced by positivity (i.e., $\rho(H) \geq 0$ for $H \geq 0$), and two continuous functions H, G on a closed symplectic manifold are called Poisson commuting if there are sequences (H_k) and (G_k) of smooth functions such that uniformly $H_k \rightarrow H$, $G_k \rightarrow G$, and $\{H_k, G_k\} \rightarrow 0$.

Remark 5.2. — We would like to point out a few facts. A symplectic quasi-state is linear on the subspace spanned by Poisson commuting H and G . The restriction of a symplectic quasi-state to $C^\infty(M)$ is always a Lie quasi-state. Any symplectic quasi-state on a closed symplectic manifold is a quasi-linear functional. On a closed oriented surface any positive quasi-linear functional is a symplectic quasi-state. See [43, Ch. 5], [24].

DEFINITION 5.3. — A quasi-morphism on a group G is a function $r : G \rightarrow \mathbb{R}$ such that there exists a constant C such that $|r(gh) - r(g) - r(h)| \leq C$ for any $g, h \in G$; a quasi-morphism is homogeneous if $r(g^n) = nr(g)$ for each $g \in G$ and each $n \in \mathbb{Z}$.

In the absence of nontrivial homomorphisms homogeneous quasi-morphisms are, in a sense, the closest approximation to homomorphisms. Quasi-morphisms originated and are studied in a number of mathematical disciplines, including group theory, geometry and dynamics. For more information on quasi-morphisms and their use, with references to literature on the subject, see [30], [43, Sect. 3.1].

A homogeneous quasi-morphism μ on $\widetilde{Ham}(M, \omega)$ is called stable if for time-dependent Hamiltonians H and G

$$\int_0^1 \min_M (H_t - G_t) dt \leq \mu(\widetilde{\phi}_G) - \mu(\widetilde{\phi}_H) \leq \int_0^1 \max_M (H_t - G_t) dt.$$

Remark 5.4. — Any stable homogeneous quasi-morphism μ on $\widetilde{Ham}(M, \omega)$ induces a symplectic quasi-state: assuming for simplicity that the symplectic volume $vol(M) = \int_M \omega^n = 1$, define for a smooth function H on M

$$\rho(H) = \int_M H \omega^n - \mu(\phi_H).$$

The functional ρ then can be extended to $C(M)$.

Existence of a stable homogeneous quasi-morphism on $\widetilde{Ham}(M, \omega)$ is not trivial. However, such quasi-morphisms can be constructed using Floer homology theory. The construction involves spectral numbers $c(a, H)$ obtained by a certain minimax procedure in Floer theory for a nonzero quantum homology class a and a Hamiltonian H . If normalized Hamiltonians H, G are such that $\widetilde{\phi}_H = \widetilde{\phi}_G$, then their spectral numbers coincide. Therefore, for each element $\tilde{\phi} = \phi_H \in \widetilde{Ham}(M, \omega)$ the number $c(a, \tilde{\phi}) := c(a, H)$ is well defined. A stable homogeneous quasi-morphism μ on $\widetilde{Ham}(M, \omega)$ now can be defined by

$$\mu(\tilde{\phi}) = - \lim_{k \rightarrow +\infty} \frac{c(a, \tilde{\phi}^k)}{k}.$$

For details of this construction consult [24], [25], [26], [27], [43]. One can also obtain quasi-morphisms and symplectic quasi-states from existing examples by a method in [8].

An important example of a symplectic quasi-state is the median quasi-state. On S^2 , let ν be the normalized Lebesgue measure that comes from the standard round area form. For a Morse function f on S^2 there is a unique set $E_\nu(f)$ (which is a component of a level set of f and is called the median of f) such that $\nu(D) \leq \frac{1}{2}$ for any component D of $S^2 \setminus E_\nu(f)$. Define $\eta(f) = \nu(E_\nu(f))$, and, using density of Morse functions in $C(S^2)$, extend η to a functional on $C(S^2)$. The functional η is a nonlinear symplectic quasi-state on S^2 . Details of this example involve evaluating a measure on a Reeb graph of a function. From the point of view of the theory of topological measures, this is related to linearity of a quasi-linear map on singly generated analytical subalgebras (see [36]). It is interesting that the functional η also arises from quasi-morphisms on the group of area- and orientation preserving diffeomorphisms of (S^2, ν) . The median quasi-state is the unique quasi-state on S^2 which is invariant under the group of area-preserving diffeomorphisms and which vanishes on functions supported on disks of area at most $\frac{1}{2}$. For more information see [43, Ch. 5], [22], [32].

A set $E \subseteq M$ is called displaceable if there is a Hamilton diffeomorphism $\phi \in Ham(M, \omega)$ such that $\phi(E) \cap \overline{E} = \emptyset$. Let \mathcal{A} be a finite-dimensional subspace of smooth functions on M such that $\{H, G\} = 0$ for all $H, G \in \mathcal{A}$. Define the moment map $\Phi : M \rightarrow \mathcal{A}^*$, where \mathcal{A}^* is the dual space of \mathcal{A} , by $\Phi(x)(H) = H(x)$ for each $H \in \mathcal{A}$. Let $\Delta \subseteq \mathcal{A}^*$ be the image of Φ . For $p \in \Delta$, the set $\Phi^{-1}(p)$ is called the fiber of Φ over p . A fiber $\Phi^{-1}(p)$ is called a stem if all other fibers of Φ are displaceable. A subset of M is a stem if it is a stem of some subspace \mathcal{A} . (see [43, Sect. 6.1]).

Suppose a bounded (partial) quasi-state on a closed symplectic manifold M corresponds to a finite topological measure μ . For a stem Y we have $\mu(Y) = \mu(M)$ (see [24, Sect. 4.1] and [43, Prop. 5.4.22]). Then any two stems intersect by Lemma 1.21 (which provides an alternative proof to one used in symplectic

geometry literature; see, for example, [24, Remark 4.5] or [43, Sect. 6.1]). This fact has several very important consequences in symplectic geometry:

- (A) Every finite-dimensional Poisson commutative subspace of smooth functions has a nondisplaceable fiber.
- (B) Stems are not displaceable by compactly supported symplectomorphisms.
- (C) No partition of unity subordinated to a finite cover of a closed symplectic manifold M by displaceable open sets can consist of Poisson commuting functions.

For an illuminating discussion of these results, as well as a quantitative version of (A), see [43, Sect. 6.1.1, Sect. 9.2].

For a (symplectic) quasi-state ρ define $\pi(h, g) = |\rho(h + g) - \rho(h) - \rho(g)|$. The functional π is nontrivial iff ρ is not linear; and π satisfies the Lipschitz property, since ρ does. The functional π is important in a number of interesting results in symplectic geometry. The Poisson bracket $\{H, G\}$ of functions H, G involves first derivatives of the functions, and at first glance there is no restriction on change in the uniform norm of $\{H, G\}$ resulting from perturbations in H, G . For symplectic quasi-state ρ there is an estimate $\pi(H, G) \leq \text{const} \sqrt{\|\{H, G\}\|}$, and, therefore, there is such a restriction (see [30], [31]). As an illustration, on $M = S^2 \times S^2$ consider $H = x_1^2, G = y_1^2$, where (x_1, y_1, z_1) are the Euclidean coordinate functions of the first factor of M . Then

$$\inf\{\|\{H', G'\}\| : \|H - H'\| + \|G - G'\| < \epsilon\} \geq \frac{(1 - 2\epsilon)^2}{12}$$

for all $\epsilon \in (0, \frac{1}{2})$ (see [31, p.1043]). In [31] one can find more information on this phenomenon and qualitative expressions of it, as well as a discussion of how $\pi(H, G)$ appears in the context of simultaneous measurement of noncommuting observables H, G and provides a lower bound for the error.

We would like to add a few comments regarding topological measures in symplectic geometry.

- (1) One can determine that a quasi-morphism is not a homomorphism by showing that a certain quasi-state is not linear (see [39], for example).
- (2) In this paper we do not discuss partial symplectic quasi-states and partial quasi-morphisms whose definitions are more technical. For information and results involving partial symplectic quasi-states and partial quasi-morphisms one could consult a variety of sources, including [24], [26], [39], [43]. We will add that, roughly speaking, homogeneous quasi-morphisms correspond to symplectic quasi-states, while partial quasi-morphisms correspond to partial symplectic quasi-states ([39]).
- (3) Topological measures can be used to distinguish Lagrangian knots that have identical classical invariants ([43, Sect. 6.2, Sect. 12.6]).
- (4) Symplectic quasi-states, unlike Langrangian Floer theory, allow one to prove results about nondisplaceability for singular sets ([24, Sect. 4.1, Sect. 4.5]).
- (5) Symplectic quasi-states help to determine how well a pair of functions can be approximated by a pair of Poisson commuting functions, and, more generally, provide bounds for the profile function. See [9], [24, Sect. 4.3], [31], [43, Sect. 8.3]. The profile function $r_{H,G} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined

by $r_{H,G}(s) = d((H, G), \mathcal{K}_s)$, where the metric d on $C^\infty \times C^\infty$ is given by $d((H, G), (K, L)) = \|H - K\| + \|G - L\|$, and $\mathcal{K}_s = \{(H, G) : \|\{H, G\}\| \leq s\}$. The profile function is essential in symplectic approximation theory, one of the main objectives of which is to find the best approximation (with respect to the uniform norm) of a pair of functions on M by a pair of functions with a "small" Poisson bracket.

For more information one can consult a nice account of applications of topological measures and symplectic quasi-states to symplectic geometry in [43, Ch.6].

In [22] the authors present a wide class of quasi-states that can not be approximated by specific quasi-states. Symplectic quasi-states constructed via Floer homology on symplectic manifolds of dimension at least 4 can not be approximated by quasi-states corresponding to simple topological measures obtained as compositions of measures with a q-function that takes only two values. (A quasi-linear functional corresponding to a simple topological measure is multiplicative on singly generated subalgebras, see [4], [20]). From the point of view of the theory of topological measures, the explanation of the "non-approximation" results from [22] is suggested by [10, Cor. 4.13] or by the fact that approximation of a quasi-state, in general, requires a much larger collection than just quasi-integrals corresponding to simple topological measures. (see [12]).

In symplectic geometry there have been papers ([39], [42]) that considered quasi-linear functionals on locally compact spaces. The authors (apparently, unfamiliar with [44]) used techniques strongly connected to the compact case: in [42] the one-point compactification of the space was employed, and in [39] the functionals were considered on the subalgebra of $C_b(X)$ containing constants that consists of functions that differ by a constant from functions whose compact support is contained in the interior of a connected symplectic manifold. The approach in the present paper allows for a more general consideration.

We would also like to point out the connection of the theory of quasi-linear functionals with the theory of Choquet integrals. If μ is a topological measure, the quasi-linear functional $\rho(f) = \int_X f d\mu$ is a symmetric Choquet integral (see Lemma 2.2 and [21, Ch. 7]).

DEFINITION 5.5. — Let \mathcal{S} be a collection of subsets of X . A set function ν on \mathcal{S} is called supermodular (or 2-monotone) if $A, B \in \mathcal{S}$ are such that $A \cup B, A \cap B \in \mathcal{S}$ implies

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

Similarly, ν is n -monotone if $\mu(\bigcup_{i=1}^n A_n) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \nu(\bigcap_{i \in I} A_i)$ and ν is totally monotone (or ∞ -monotone) if it is monotone for all n .

DEFINITION 5.6. — Functions f, g on X are comonotonic if there is no $x_1, x_2 \in X$ such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$.

There are several equivalent definitions for comonotonic functions. One of them says that real-valued functions f, g are comonotonic if there is a real-valued function ϕ on X and increasing functions u, v on \mathbb{R} such that $f = u(\phi)$ and $g = v(\phi)$ (see [21, pp. 54, 55]).

Many results about Choquet integrals are proved for a supermodular, n -monotone or totally monotone set function, and/or for a set function whose domain is a σ -algebra, an algebra, or is closed under intersection and union; the proofs often use

step or simple functions. None of this is applicable for quasi-linear functionals. This is one reason why one can not say that results of Choquet theory automatically hold for quasi-linear functionals. Nevertheless, as this paper shows, we do have some of the typical and important results of Choquet integrals, such as properties of being

- monotone, i.e. $f \leq g$ implies $\int f \leq \int g$,
- homogeneous, i.e. $\int(cf) = c \int f$ for any constant c , and
- additive on comonotonic functions, i.e. f, g are comonotonic implies $\int(f + g) = \int(f) + \int(g)$.

(Here we informally denote by $\int f$ either the Choquet integral of f or quasi-linear functional $\rho(f)$.) We proved these properties with techniques different from techniques in Choquet theory, and they hold sometimes under weaker conditions. Sometimes the results for quasi-linear functionals are stronger than those for Choquet integrals. For instance, integrals with respect to topological measures are additive on a wider class of functions than comonotonic functions (see Lemma 2.2 and [21, Prop. 4.5]). Also, examples of integration with respect to topological measures show that some conditions can not be weakened in Choquet theory results. The deep interconnections of the two theories deserve further investigation.

These are just some examples to illustrate the point that the theory of quasi-linear functionals and topological measures is beneficial in symplectic geometry and Choquet theory as it leads to new results and provides deeper understanding, additional insight, alternative proofs, and new methods.

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