CONFLUENTES MATHEMATICI

Fenfang XIE, Jinjin WANG, Jiayi XIA, and Guo ZHONG

Finite Groups with some s-Permutably Embedded and Weakly s-Permutable Subgroups

Tome 5, nº 1 (2013), p. 93-100.

<http://cml.cedram.org/item?id=CML_2013__5_1_93_0>

Confluentes Math.
5, 1 (2013) 93-100

FINITE GROUPS WITH SOME s-PERMUTABLY EMBEDDED AND WEAKLY s-PERMUTABLE SUBGROUPS

FENFANG XIE, JINJIN WANG, JIAYI XIA, AND GUO ZHONG

Abstract. Let G be a finite group, p the smallest prime dividing the order of G and P a Sylow p-subgroup of G with the smallest generator number d. There is a set $\mathcal{M}_d(P) = \{P_1, P_2, \cdots, P_d\}$ of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. In the present paper, we investigate the structure of a finite group under the assumption that every member of $\mathcal{M}_d(P)$ is either s-permutably embedded or weakly s-permutable in G to give criteria for a group to be p-supersolvable or p-nilpotent.

1. Introduction

All groups considered in this paper are finite. Terminology and notation employed agree with standard usage, as in Robinson [15].

In this paper, we let $\mathcal{M}(G)$ be the set of all maximal subgroups of a group G. An interesting problem in group theory is to study the influence of the elements of $\mathcal{M}(G)$ on the structure of G. A classical result in this orientation is attributed to Srinivasan [19]. Srinivasan obtained that G is supersolvable provided that every member of $\mathcal{M}(G)$ is normal in G. This result has been extensively generalized.

Two subgroups H and K of a group G are said to be permutable if $HK = KH$. H is said to be s-permutable in G if H permutes with every Sylow subgroup of G, i.e., $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by O. H. Kegel in [9] and has been studied widely by many authors, such as [5, 17]. Recently, Ballester-Bolinches and Pedraza-Aquilera [3] generalized s-permutable subgroups to s-permutably embedded subgroups. H is said to be s-permutably embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-permutable subgroup of G. On the other hand, Wang [22] introduced the concept of c-normal subgroups. Applying the c-normality of subgroups, Wang obtained new criteria for supersolvability of groups. More recently, Skiba [19] introduced the concept of weakly s-permutable subgroups. H is called a weakly s-permutable subgroup of G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leqslant H_{sG}$ the subgroup of H generated by all those subgroups of H which are s-permutable in G. Weakly s-permutability covers both s-permutability and c-normality. Skiba applied weakly s-permutability to unify viewpoint for a series of similar problems. Let P be a Sylow p-subgroup of G. Many authors have studied the influence of the members of $\mathcal{M}_d(P)$ (see the Definition 2.1) on the structure of G, such as [8, 12, 16, 18]. Now, in this paper we continue these work. Speaking more precisely, the structure of a finite group under some assumptions on the s-permutably embedded or weakly s-permutable subgroups in $\mathcal{M}_d(P)$, for each prime p, is studied and obtain some sufficient conditions for a p-supersolvable group or a p-nilpotent group.

2. Preliminaries

Definition 2.1 ([10, Definition 1.1]). — *Let d be the smallest generator number of a p-group P. Let $\mathcal{M}_d(P) = \{P_1, P_2, \cdots, P_d\}$ be a subset of $\mathcal{M}(P)$ such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, the Frattini subgroup of P.*

Math. classification: 20D10, 20D20.

Keywords: weakly s-permutable subgoups; s-permutably embedded subgroups; p-nilpotent groups.

Since $|\mathcal{M}(P)| = (p^d - 1)/(p-1), |\mathcal{M}_d(P)| = d$ and when $d \to \infty$,

$$((p^d - 1)/(p-1))/d \to \infty.$$

Hence $|\mathcal{M}(P)| >> |\mathcal{M}_d(P)|$.

Lemma 2.2 ([19, Lemma 2.10]). — *Let U be a weakly s-permutable subgroup of G and N a normal subgroup of G. Then:*

(1) If $U \leqslant H \leqslant G$, then U is weakly s-permutable in H;

(2) If $N \leqslant U$, then U/N is weakly s-permutable in G/N;

(3) Let π be a set of primes, U a π'-subgroup and N a π-subgroup. Then UN/N is weakly s-permutable in G/N;

Lemma 2.3 ([3, Lemma 1]). — *Suppose that H is an s-permutably embedded subgroup of G, $K \leqslant G$ and N is a normal subgroup of G. Then we have the following:*

(1) If $H \leqslant K$, then H is an s-permutably embedded subgroup of K.

(2) HN/N is an s-permutably embedded subgroup of G/N.

Lemma 2.4 ([4, 9, 17]). — *(1) If $H \leqslant K \leqslant G$ and H is s-permutable in G, then H is s-permutable in K.*

(2) If both H and K are s-permutable subgroups of G, then both $H \cap K$ and $< H, K >$ are s-permutable in G.

(3) If H is s-permutable subgroups of G and $N \trianglelefteq G$, then HN is s-permutable subgroups of G and HN/N is s-permutable subgroups of G/N.

(4) A p-subgroup H of G is s-permutable in G if and only if $N_G(H) \geqslant O^p(G)$ for some prime $p \in \pi(G)$.

(5) If H is s-permutable in G, then H is subnormal in G.

Lemma 2.5 ([24, Lemma 2.8]). — *Let G be a group and let p be a prime number dividing $|G|$ with $(|G|, p-1) = 1$.Then*

(1) If N is normal in G of order p, then N lies in $Z(G)$;

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent;

(3) If M is a subgroup of G with index p, then M is normal in G.

Lemma 2.6 ([7, IV, Satz 4.7]). — *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \cap N \leqslant \Phi(P)$, then N is p-nilpotent.*

Lemma 2.7 ([7, III, Satz 3.3]). — *Let G be a group, and let N be a normal subgroup of G and $H \leqslant G$. If $N \leqslant \Phi(H)$, then $N \leqslant \Phi(G)$.*

Lemma 2.8 ([23, Lemma 2.6]). — *Let N be a normal subgroup of a group $G(N \neq 1)$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G that are contained in $F(N)$.*

Lemma 2.9 ([14, Lemma 2.1]). — *Let G be a group and $H \leqslant G$. Then H_{sG} is the uniquely determined largest s-permutable subgroup of G contained in H. In particular, $N_G(H) \leqslant N_G(H_{sG})$.*

Lemma 2.10 ([25]). — *(1) If A is subnormal in G and the index $|G : A|$ is a p'-number, then A contains all Sylow p-subgroups of G.*

(2) If A is a subnormal Hall subgroup of G , then A is normal in G

Lemma 2.11 ([5]). — *If H is an s-permutable subgroup of a group G, then H/H_G is nilpotent.*

Lemma 2.12 ([17]). — *For a nilpotent subgroup H of G, the following two statements are equivalent:*

(1) H is s-permutable in G.

(2) The Sylow subgroups of H are s-permutable in G.

LEMMA 2.13 ([2]). — *Let P be a Sylow p-subgroup of G, and P_1 a maximal subgroup of P. Then the following two statements are equivalent:*
(1) P_1 is normal in G.
(2) P_1 is s-permutable in G.

3. Main Results

THEOREM 3.1. — *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is either weakly s-permutable or s-permutably embedded in G, Then G is p-nilpotent.*

Proof. — Assume G is not p-nilpotent and let the theorem is false and G a counter-example of minimal order. We write $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$. Then, each P_i is either weakly s-permutable or s-permutably embedded in G. Without loss of generality, suppose that $1 \leqslant k \leqslant d$ such that (i) every $P_i (1 \leqslant i \leqslant k)$ is weakly s-permutable in G. Then there exists a subnormal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i \leqslant (P_i)_{sG}$. (ii)each $P_j (k+1 \leqslant j \leqslant d)$ is s-permutably embedded in G. Then there exists an s-permutable subgroup $M_j \leqslant G$ such that P_j is a Sylow p-subgroup of M_j.

Now we prove the theorem by the following several steps.

(1) $O_{p'}(G) = 1$.

Consider the quotient group $G/O_{p'}(G)$. Since $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$, which is isomorphic to P, so $PO_{p'}(G)/O_{p'}(G)$ has the same smallest generator number d as P. Set

$$\mathcal{M}_d(PO_{p'}(G)/O_{p'}(G)) = \{P_1O_{p'}(G)/O_{p'}(G), \cdots, P_dO_{p'}(G)/O_{p'}(G)\}.$$

Also, each $P_sO_{p'}(G)/O_{p'}(G)$ for $s \in \{1, \cdots, d\}$ is either s-permutably embedded or weakly s-permutable in $G/O_{p'}(G)$ by Lemmas 2.2 and 2.3. Thus, $G/O_{p'}(G)$ satisfies the conditions of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G itself is p-nilpotent, a contradiction.

(2) $(P_i)_{sG} \lhd G$ and the quotient group $G/(P_i)_{sG}$ is p-nilpotent for every $i \in \{1, 2, \cdots, k\}$.

Since $P_i \lhd P$, by Lemma 2.9 we have $P \leqslant N_G(P_i) \leqslant N_G((P_i)_{sG})$, that is, $(P_i)_{sG}$ is normalized by P. Clearly, $(P_i)_{sG}$ is an s-permutable p-group and so $O^p(G) \leqslant N_G((P_i)_{sG})$ by Lemma 2.4. Now we can get that $(P_i)_{sG} \lhd PO^p(G) = G$. Since $G = P_iK_i$ and $P_i \cap K_i \leqslant (P_i)_{sG}$. If $P_i \cap K_i < (P_i)_{sG}$, let T_i denote the subnormal subgroup $K_i(P_i)_{sG}$. It follows that

$$P_iT_i = P_iK_i(P_i)_{sG} = P_i(P_i)_{sG}K_i = P_iK_i = G$$

and

$$P_i \cap T_i = P_i \cap K_i(P_i)_{sG} = (P_i \cap K_i)(P_i)_{sG} = (P_i)_{sG}.$$

Now we can assume $G = P_iK_i$ and $P_i \cap K_i = (P_i)_{sG}$. Then

$$G/(P_i)_{sG} = P_i/(P_i)_{sG} \cdot K_i/(P_i)_{sG}.$$

Therefore,

$$|K_i/(P_i)_{sG}|_p = |G : P_i|_p = |P : P_i| = p,$$

i.e., the factor group $K_i/(P_i)_{sG}$ possesses a cyclic Sylow subgroup of order p. By Lemma 2.5, we have that $K_i/(P_i)_{sG}$ is p-nilpotent. So $K_i/(P_i)_{sG}$ has a Hall normal p'-subgroup $H/(P_i)_{sG}$. Then

$$H/(P_i)_{sG} \lhd\lhd G/(P_i)_{sG} \quad \text{and} \quad H/(P_i)_{sG} \in Hall(G/(P_i)_{sG}).$$

It follows from Lemma 2.10 that $H/(P_i)_{sG}$ is a normal p-complement of $G/(P_i)_{sG}$. Consequently, $G/(P_i)_{sG}$ is p-nilpotent, as desired.

(3) For every $j \in \{k+1, k+2, \cdots, d\}$, the factor group $G/(M_j)_G$ is p-nilpotent.

By the definition of an s-permutably embedded subgroup, P_j is a Sylow p-subgroup of the s-permutable subgroup M_j of G. It follows that $M_j/(M_j)_G$ is

s-permutable in $G/(M_j)_G$ and $M_j/(M_j)_G$ is nilpotent by Lemma 2.11. Hence, we may apply Lemma 2.12 to see that every Sylow subgroup of $M_j/(M_j)_G$ is s-permutable in $G/(M_j)_G$. Thus, $P_j(M_j)_G/(M_j)_G$ is s-permutable in $G/(M_j)_G$ because $P_j(M_j)_G/(M_j)_G$ is a Sylow p-subgroup of $M_j/(M_j)_G$. It follows by Lemma 2.13 that $P_j(M_j)_G/(M_j)_G$ is normal in $G/(M_j)_G$. So the core $(M_j)_G$ of M_j contains the Sylow p-subgroup P_j of M_j and we have $|G/(M_j)_G|_p = p$. We conclude that $G/(M_j)_G$ is p-nilpotent by Lemma 2.5. We have that (3) holds.

(4) Let $N = (\bigcap_{i=1}^{k}(P_i)_{sG})\cap(\bigcap_{j=k+1}^{d}(M_j)_G)$. We have $N \trianglelefteq G$. Now, we can obtain that N is p-nilpotent. Consider the subgroup $P\cap N$. Recall that $P_j \in Syl_p((M_j)_G)$ and P_j is a maximal subgroup of P. We have

$$P\cap N = (\bigcap_{i=1}^{k}(P_i)_{sG})\cap(\bigcap_{j=k+1}^{d}((M_j)_G\cap P)) = \bigcap_{i=1}^{k}(P_i)_{sG}\cap(\bigcap_{j=k+1}^{d}P_j) \leqslant \bigcap_{s=1}^{d}P_s = \Phi(P).$$

Thus $P\cap N \leqslant \Phi(P)$ and $N \trianglelefteq PN$. It is easy to see that N is p-nilpotent by Lemma 2.6.

(5) $N \leqslant \Phi(G)$.

We know that N possesses a normal Hall p'-subgroup U such that $N = N_pU$, where $N_p \in Syl_p(N)$. Then U is normal in G and $U \leqslant O_{p'}(G) = 1$, so $U = 1$. Therefore, N is a normal p-subgroup of G. Now, $N \leqslant P\cap N \leqslant \Phi(P)$. We see that $N \leqslant \Phi(G)$ by Lemma 2.7.

(6) The final contradiction.

By (2) and (3), $G/(P_i)_{sG}$ and $G/(M_j)_G$ are p-nilpotent. Hence, G/N is a p-nilpotent. Since $N \leqslant \Phi(G)$, it is easy to see that G is p-nilpotent, the final contradiction. The proof of Theorem 3.1 is now complete. □

Corollary 3.2 ([1, Theorem 3.5]). — *Let p be the smallest prime dividing $|G|$. If P is a Sylow p-subgroup of G such that every member of $\mathcal{M}(P)$ is s-permutable in G, then G has a normal p-complement.*

Corollary 3.3 ([11, Theorem 3.1]). — *Suppose that $p \in \pi(G)$ is such that $(|G|, p-1) = 1$. Let P be a Sylow p-subgroup of a group G. Assume that every member of $\mathcal{M}(P)$ is either c-normal or s-permutably embedded in G. Then G is p-nilpotent.*

Corollary 3.4 ([14, Theorem 3.2]). — *Let G be a group and P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that every member of $\mathcal{M}(P)$ is weakly s-permutable in G, then G is p-nilpotent.*

Theorem 3.5. — *Let G be a group and let P be a Sylowp p-subgroup of G such that $N_G(P)$ is p-nilpotent, where p is a prime divisor of $|G|$. If every member in some fixed $\mathcal{M}_d(P)$ is either weakly s-permutable or s-permutably embedded in G, then G is p-nilpotent.*

Proof. — By Theorem 3.1, it is easy to see that the theorem holds when $p = 2$. Assume that the theorem is false and let G be a counter-example of minimal order. By the hypotheses of the theorem, denote $\mathcal{M}_d(P) = \{P_1, P_2, \cdots, P_d\}$. Then, each P_i is either weakly s-permutable or s-permutably embedded in G. Furthermore, we have

(1) $O_{p'}(G) = 1$.

Consider the quotient group $G/O_{p'}(G)$. Since $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$, which is isomorphic to P, so $PO_{p'}(G)/O_{p'}(G)$ has the same smallest generator number d as P. Set

$$\mathcal{M}_d(PO_{p'}(G)/O_{p'}(G)) = \{P_1O_{p'}(G)/O_{p'}(G), \cdots, P_dO_{p'}(G)/O_{p'}(G)\}.$$

Also, each $P_sO_{p'}(G)/O_{p'}(G)$ for $s \in \{1, \cdots, d\}$ is either s-permutably embedded or weakly s-permutable in $G/O_{p'}(G)$ by Lemmas 2.2 and 2.3. Thus, $G/O_{p'}(G)$

satisfies the conditions of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G itself is p-nilpotent, a contradiction. Also, $N_{G/N}(PN/N) = N_G(P)N/N$, hence it is p-nilpotent because $N_G(P)$ is p-nilpotent. Thus $G/O_{p'}(G)$ satisfies the hypothesis of our theorem. By the choice of G, $G/O_{p'}(G)$ is p-nilpotent and it follows that G is p-nilpotent, a contradiction.

(2) If $P \leqslant H < G$, then H is p-nilpotent.

Since $N_H(P) \leqslant N_G(P)$, we have that $N_H(P)$ is p-nilpotent. By Lemmas 2.2 and 2.3, H satisfies the hypotheses of the theorem. By the choice of G, H is p-nilpotent, as desired.

(3) $G = PQ$, where Q is a Sylow q-subgroup of G with $p \neq q$.

Since G is not p-nilpotent, by a result of Thompson [21, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leqslant P$. Now, T char $P \lhd N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leqslant N_G(T)$. By (2), we have that $N_G(T) = G$ and $T = O_p(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that PQ is a subgroup of G [6,Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Therefore, $PQ = G$, as desired.

(4) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p.

As $O_{p'}(G) = 1$, we get that $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leqslant \Phi(P)$, by Lemma 2.7, then $N \leqslant \Phi(G)$, and G/N satisfies the hypotheses of the theorem. By the choice of G, G/N is p-nilpotent. So $G/\Phi(G)$ is p-nilpotent, it follows that G is p-nilpotent, a contradiction. Thus $N \nleqslant \Phi(P)$. Since $\bigcap_{i=1}^{d} P_i = \Phi(P)$, where $P_i \in \mathcal{M}_d(P)$, we can assume $N \nleqslant P_1$ without loss of generality. By the conditions of the theorem, P_1 is weakly s-permutable in G or s-permutably embedded in G. We claim that $|N| = p$.

(i) We first consider the case that P_1 is weakly s-permutable in G. Then there exists $K_1 \lhd\lhd G$ such that $G = P_1K_1$ and $P_1 \cap K_1 \leqslant (P_1)_{sG}$. Since $P_1 \lhd P$, by Lemma 2.9 we have $P \leqslant N_G(P_1) \leqslant N_G((P_1)_{sG})$, that is, $(P_1)_{sG}$ is normalized by P. Clearly, $(P_1)_{sG}$ is a s-permutable p-group and so $O^p(G) \leqslant N_G((P_1)_{sG})$ by Lemma 2.4. Now we can get that $(P_1)_{sG} \lhd PO^p(G) = G$. Then $(P_1)_{sG} \cap N = 1$ or N. If $(P_1)_{sG} \cap N = N$, then $N \leqslant (P_1)_{sG} \leqslant P_1$, a contradiction. So we have that $(P_1)_{sG} \cap N = 1$, then $P_1 \cap K_1 \cap N = 1$. From the minimal normality of N, we know that $(K_1)_G \cap N = 1$ or N. If $(K_1)_G \cap N = 1$, then

$$N \cong N(K_1)_G/(K_1)_G \lhd G/(K_1)_G,$$

where $G/(K_1)_G$ is a p-group since all Sylow q-subgroups of G is contained in K_1 by Lemma 2.10. Thus we have that $|N| = p$. If $(K_1)_G \cap N \neq 1$, we get that $N \leqslant (K_1)_G \leqslant K_1$. Then

$$1 = P_1 \cap K_1 \cap N = P_1 \cap N$$

and so $NP_1 = P$. We also get $|N| = p$.

(ii) Next, we consider that case that P_1 is s-permutably embedded in G. If P_1 is s-permutably embedded in G, then there exists an s-permutable subgroup H such that $P_1 \in Syl_p(H)$. Hence, HQ is a subgroup of G. Since $N \lhd G$, we have that $N_1 = N \cap HQ \lhd HQ$. It follows that $N_1 \lhd \langle HQ, N\rangle = G$. Moreover, by the minimality normality of N, we get that $N_1 = 1$ and so $|N| = p$.

Now, we know that $N \cap P_1 = 1$. By [7, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Certainly, $N \nleqslant \Phi(G)$. From Lemma 2.8, we conclude

$$O_p(G) = R_1 \times R_2 \times \cdots \times R_t,$$

where $R_i(i = 1, \cdots, t)$ is a normal subgroup of order p. It follows that

$$P \leqslant \bigcap_{i=1}^{t} C_G(R_i) = C_G(O_p(G)).$$

Furthermore, according to [15, Theorem 9.31] and (3), we have that $C_G(O_p(G)) \leqslant O_p(G)$ and so $P = O_p(G)$. Thus $G = N_G(P)$. Now, by the hypotheses that $N_G(P)$ is p-nilpotent, we conclude that G is p-nilpotent. This is the final contradiction and the proof is complete. □

Corollary 3.6 ([14, Theorem 3.1]). — *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of $\mathcal{M}(P)$ is weakly s-permutable in G, then G is p-nilpotent.*

Theorem 3.7. — *Let G be a p-solvable group and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. If every member in some fixed $\mathcal{M}_d(P)$ is either weakly s-permutable or s-permutably embedded in G, then G is p-supersolvable.*

Proof. — Assume that the theorem is false and let G be a counter-example of minimal order. We write $\mathcal{M}_d(P) = \{P_1, \cdots, P_d\}$. Then, each P_i is either weakly s-permutable or s-permutably embedded in G.

(1) $O_{p'}(G) = 1$.

With an argument similar to that above, (1) holds.

(2) $\Phi(P)_G = 1$, in particular, $\Phi(O_p(G)) = 1$.

Otherwise, then let $N = \Phi(P)_G > 1$. We consider the factor group G/N. Obviously, $\mathcal{M}_d(P/N) = \{P_1/N, \cdots, P_d/N\}$. By Lemmas 2.2 and 2.3, P_i/N is either weakly s-permutable or s-permutably embedded in G/N for any $i \in \{1, \cdots, d\}$. Therefore, G/N satisfies the hypotheses of the theorem and consequently, G/N is p-supersolvable by the minimality of G. Since $N \leqslant \Phi(P), N \leqslant \Phi(G)$ by Lemma 2.7, it follows from G/N being p-supersolvable that G is p-supersolvable, which is contrary to the choice of G.

(3) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p.

As $O_{p'}(G) = 1$, we get that $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leqslant \Phi(P)$, by Lemma 2.7, then $N \leqslant \Phi(G)$, and G/N satisfies the hypotheses of the theorem. By the choice of G, G/N is p-supersolvable. Since the class of p-supersolvable groups is a saturated formation, we have G is p-supersolvable, a contradiction. Thus $N \nleqslant \Phi(P)$. Since $\bigcap_{i=1}^{d} P_i = \Phi(P)$, where $P_i \in \mathcal{M}_d(P)$, we can assume $N \nleqslant P_1$ without loss of generality. By the conditions of the theorem, P_1 is weakly s-permutable in G or s-permutably embedded in G. We claim that $|N| = p$.

(i) We first consider the case that P_1 is weakly s-permutable in G. Then there exists $K_1 \lhd\lhd G$ such that $G = P_1 K_1$ and $P_1 \cap K_1 \leqslant (P_1)_{sG}$. Since $P_1 \lhd P$, by Lemma 2.9 we have $P \leqslant N_G(P_1) \leqslant N_G((P_1)_{sG})$, that is, $(P_1)_{sG}$ is normalized by P. Clearly, $(P_1)_{sG}$ is a s-permutable p-group and so $O^p(G) \leqslant N_G((P_1)_{sG})$ by Lemma 2.4. Now we can get that $(P_1)_{sG} \lhd PO^p(G) = G$. Then $(P_1)_{sG} \cap N = 1$ or N. If $(P_1)_{sG} \cap N = N$, then $N \leqslant (P_1)_{sG} \leqslant P_1$, a contradiction. So we have that $(P_1)_{sG} \cap N = 1$, then $P_1 \cap K_1 \cap N = 1$. We consider $(K_1)_G \cap N$. By the minimality of N, we know that $(K_1)_G \cap N = 1$ or N. If $(K_1)_G \cap N = 1$, then

$$N \cong N(K_1)_G/(K_1)_G \lhd G/(K_1)_G,$$

where $G/(K_1)_G$ is a p-group since all Sylow q-subgroups of G is contained in K_1 by Lemma 2.10. Thus we have that $|N| = p$. If $(K_1)_G \cap N \neq 1$, we get that $N \leqslant (K_1)_G \leqslant K_1$. Then

$$1 = P_1 \cap K_1 \cap N = P_1 \cap N$$

and so $NP_1 = P$. We also get $|N| = p$.

(ii) Next, we consider the case that P_1 is s-permutably embedded in G. If P_1 is s-permutably embedded in G, then there exists an s-permutable subgroup H such that $P_1 \in Syl_p(H)$. Hence, HQ is a subgroup of G. Since $N \lhd G$, we have that $N_1 = N \cap HQ \lhd HQ$. It follows that $N_1 \lhd \langle HQ, N\rangle = G$. Moreover, by the minimality normality of N, we get that $N_1 = 1$ and so $|N| = p$.

Therefore, $N \cap P_1 = 1$. By [7, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Certainly, $N \nleqslant \Phi(G)$. Now, we can use Lemma 2.8 to derive that $O_p(G)$ is a direct product of normal subgroups of G of order p.

(4) The counter-example does not exist.

Since $G/C_G(R_i)$ is a cyclic group of order $p-1$, certainly

$$G/\bigcap_{i=1}^{r} C_G(R_i) = G/C_G(O_p(G))$$

is p-supersolvable. On the other side, since G is p-solvable and $O_{p'}(G) = 1$, by [15, Theorem 9.3.1], $C_G(O_p(G)) \leqslant O_p(G)$. Hence, $G/O_p(G)$ is p-supersolvable. Now, claim (3) implies that G is p-supersolvable. We are done. □

Corollary 3.8 ([14, Theorem 3.3]). — *Let G be a p-solvable group and P a Sylow p-subgroup of G. If every member of $\mathcal{M}(P)$ is weakly s-permutable in G, then G is p-supersolvable.*

Acknowledgements

The authors are grateful to the referees who provided their valuable suggestions.

This research was supported by NSF of China (10961007, 11161006), NSF of Guangxi (0991101, 0991102), Higher school personnel subsidy scheme of Guangxi (5070).

References

[1] K. Al-Sharo, On some maximal S-quasinormal subgroups of finite groups, *Beiträge zur Algebra und Geometrie*, 49:227–232, 2008.
[2] M. Asaad, A. A. Heliel, On S-quasinormal embedded subgroups of finite groups, *J. Pure Appl. Algebra,* 165:129–135, 2001.
[3] A. Ballester-Bolinches, M. C. Pedraza-Aguilera, Sufficient conditions for supersolubility of finite groups, *J. Pure Appl. Algebra,* 127:113–118, 1998.
[4] K. Doerk, *Finite Soluble Groups,* Berlin, Walterde Gruyter, 1992.
[5] W. E. Deskins, On quasinormal subgroups of finite groups, *Mathematische Zeitschrift,* 82:125–132, 1963.
[6] D. Gorenstein, *Finite group,* Chelsea, New York, 1980.
[7] B. Huppert, *Endliche gruppen I,* Springer, Berlin, 1967.
[8] X. He, S. Li, X. Liu, On S-quasinormal and c-normal subgroups of prime power order in finite groups, *Algebra Colloq.,* **18** (2011), 685–692.
[9] O. H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.,* 78:205–221, 1962.
[10] S. Li, X. He, On normally embedded subgroups of prime power order in finite groups, *Comm. Algebra,* 36:2333–2340, 2008.
[11] S. Li, Y. Li, On S-quasinormal and c-normal subgroups of a finite group, *Czechoslovak Mathematical Journal* 58:1083–1095, 2008.
[12] S. Li, Z. Shen, J. Liu, et al, The influence of SS-quasinormality of some subgroups on the structure of finite groups, *J. Algebra,* 319:4275–4287, 2008.
[13] D. H. Mclain, The existence of subgroups of given order in finite groups, *Proc.Cambridge Philos.Soc,* 53:278–285, 1957.
[14] L. Miao, On weakly s-permutable subgroups of finite groups, *Bull Braz. Math. Soc. New Series*, 41:223–235, 2010.
[15] D. J. S. Robinson, *A course in the Theory of groups,* New York, Springer-Verlag, 1982.
[16] Z. Shen, W. Shi, Q. Zhang, S-quasinormality of finite groups, *Front. Math. China,* 5:329–339, 2010.
[17] P. Schmidt, Subgroups permutable with all Sylow subgroups, *J. Algebra,* 207:285–293, 1998.
[18] X. Shen, S. Li, W. Shi, Finite groups with normally embedded subgroups, *J. Group Theory,* 13:257–265, 2010.

[19] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra,* 315:192–209, 2007.
[20] S. Srinivasan, Two sufficient conditions for supersolvability of finite groups, *Israel J.Math,* 35:210–214, 1980.
[21] J. G. Thompson, Normal p-complements for finite groups, *J. Algebra,* 1:43–46, 1964.
[22] Y. Wang, c-normality of groups and its properties, *J. Algebra,* 180:954–965, 1996.
[23] Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *J. Algebra,* 224:464-478, 2000.
[24] H. Wei, Y. Wang, On c^*-normality and its properties, *J. Group Theory,* 10:211–223, 2007.
[25] H. Wielandt, *Subnormal subgroups and permutation groups*, lectures given at the Ohio State University, Columbus, Ohio, 1971.

Manuscript received October 5, 2012,
revised April 10, 2013,
accepted April 16, 2013.

Fenfang XIE, Jinjin WANG, Jiayi XIA & Guo ZHONG
School of Mathematical Sciences, Guangxi Teachers Education University, Nanning, Guangxi, 530023, P. R. China
zhg102003@163.com (Corresponding author)