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CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS

MICKAËL DOS SANTOS

Abstract. Given a smooth bounded planar domain Ω, we construct a compact set on the boundary such that its characteristic function is not the trace of a least gradient function. This generalizes the construction of Spradlin and Tamasan [3] when Ω is a disc.

1. Introduction

We let Ω be a bounded C^2 domain of $\mathbb{R}^2$. For a function $h \in L^1(\partial\Omega, \mathbb{R})$, the least gradient problem with boundary datum h consists in deciding whether

$$\inf \left\{ \int_\Omega |Dw| \; ; \; w \in BV(\Omega) \text{ and } \mathrm{tr}_{\partial\Omega} w = h \right\} \tag{1.1}$$

is achieved or not.

In the above minimization problem, $BV(\Omega)$ is the space of functions of bounded variation. It is the space of functions $w \in L^1(\Omega)$ having a distributional gradient Dw which is a bounded Radon measure.

If the *infimum* in (1.1) is achieved, minimal functions are called *functions of least gradient*.

Sternberg, Williams and Ziemmer proved in [4] that if $h : \partial\Omega \to \mathbb{R}$ is a continuous map and if $\partial\Omega$ satisfies a geometric properties then there exists a (unique) function of least gradient. For further use, we note that the geometric property is satisfied by Euclidean balls.

On the other hand, Spradlin and Tamasan [3] proved that, for the disc $\Omega = \{x \in \mathbb{R}^2 \; : \; |x| < 1\}$, we may find a function $h_0 \in L^1(\partial\Omega)$ which is not continuous such that the *infimum* in (1.1) is not achieved. The function h_0 is the characteristic function of a Cantor type set $\mathcal{K} \subset \mathbb{S}^1 = \{x \in \mathbb{R}^2 \; : \; |x| = 1\}$

The goal of this article is to extend the main result of [3] to a general C^2 bounded open set $\Omega \subset \mathbb{R}^2$.

We prove the following theorem.

Theorem 1.1. — *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 open set. Then there exists a measurable set $\mathcal{K} \subset \partial\Omega$ such that the infimum*

$$\inf \left\{ \int_\Omega |Dw| \; ; \; w \in BV(\Omega) \text{ and } \mathrm{tr}_{\partial\Omega} w = 1\!\!1_{\mathcal{K}} \right\} \tag{1.2}$$

is not achieved.

The calculations in [3] are specific to the case $\Omega = \mathbb{D}$. The proof of Theorem 1.1 relies on new arguments for the construction of the Cantor set $\mathcal{K}$ and the strategy of the proof.

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2. Strategy of the proof

2.1. **The model problem.** We illustrate the strategy developed to prove Theorem 1.1 on the model case $\mathcal{Q} = (0,1)^2$. Clearly, this model case does not satisfy the C^2 assumption.

Nevertheless, the flatness of $\partial\mathcal{Q}$ allows to get a more general counterpart of Theorem 1.1. Namely, the counterpart of Theorem 1.1 (see Proposition 2.1 below) is no more an existence result of a set $\mathcal{K} \subset \partial\mathcal{Q}$ such that Problem (1.2) is not achieved. It is a non existence result of a least gradient function for $h = 1\!\!1_{\mathcal{M}}$ for *any* measurable domain $\mathcal{M} \subset [0,1] \times \{0\} \subset \partial\mathcal{Q}$ with positive Lebesgue measure.

We thus prove the following result whose strategy of the proof is due to Petru Mironescu.

Proposition 2.1 (P. Mironescu). — *Let $\tilde{\mathcal{M}} \subset [0,1]$ be a measurable set with positive Lebesgue measure. Then the infimum in*

$$\inf\left\{\int_{\mathcal{Q}} |Dw| \ ;\ w \in BV(\mathcal{Q}) \text{ and } \mathrm{tr}_{\partial\mathcal{Q}} w = 1\!\!1_{\tilde{\mathcal{M}}\times\{0\}}\right\} \tag{2.1}$$

is not achieved.

This section is devoted to the proof of Proposition 2.1. We fix a measurable set $\tilde{\mathcal{M}} \subset [0,1]$ with positive measure and we let $h = 1\!\!1_{\tilde{\mathcal{M}}\times\{0\}}$. We argue by contradiction: we assume that there exists a minimizer u_0 of (2.1). We obtain a contradiction in 3 steps.

Step 1. Upper bound and lower bound

This first step consists in obtaining two estimates. The first estimate is the upper bound

$$\int_{\mathcal{Q}} |Du_0| \leqslant \|1\!\!1_{\tilde{\mathcal{M}}\times\{0\}}\|_{L^1(\partial\mathcal{Q})} = \mathscr{H}^1(\tilde{\mathcal{M}}). \tag{2.2}$$

Here, $\mathscr{H}^1(\tilde{\mathcal{M}})$ is the length of $\tilde{\mathcal{M}}$.

Estimate (2.2) follows from Theorem 2.16 and Remark 2.17 in [2]. Indeed, by combining Theorem 2.16 and Remark 2.17 in [2] we may prove that for $h \in L^1(\partial\Omega)$ and for all $\varepsilon > 0$ there exists a map $u_\varepsilon \in BV(\Omega)$ such that

$$\int_{\Omega} |Du_\varepsilon| \leqslant (1+\varepsilon)\|h\|_{L^1(\partial\Omega)} \text{ and } \mathrm{tr}_{\partial\Omega} u_\varepsilon = h.$$

The proof of this inequality when Ω is a half space is presented in [2]. It is easy to adapt the argument when $\Omega = \mathcal{Q} = (0,1)^2$. The extension for a C^2 set Ω is presented in Appendix E.

Step 2. Optimality of (2.2) (see (2.3))

The optimality of (2.2) is obtained *via* the following lemma.

Lemma 2.2. — *For $u \in BV(\mathcal{Q})$ we have*

$$\int_{\mathcal{Q}} |D_2 u| \geqslant \int_0^1 |\mathrm{tr}_{\partial\mathcal{Q}} u(\cdot,0) - \mathrm{tr}_{\partial\mathcal{Q}} u(\cdot,1)|.$$

Here, for $k \in \{1,2\}$ we denoted

$$\int_{\mathcal{Q}} |D_k u| = \sup\left\{\int_{\mathcal{Q}} u\partial_k\xi \ :\ \xi \in C^1_c(\mathcal{Q}) \text{ and } |\xi| \leqslant 1\right\}$$

where $C^1_c(\mathcal{Q})$ are the set of real valued C^1-functions with compact support included in $\mathcal{Q}$.

Lemma 2.2 is proved in Appendix B.1.

From Lemma 2.2 we get

$$\int_{\mathcal{Q}} |D_2 u_0| \geqslant \int_0^1 |\mathrm{tr}_{\partial\mathcal{Q}} u_0(\cdot,0) - \mathrm{tr}_{\partial\mathcal{Q}} u_0(\cdot,1)| = \int_0^1 \mathbb{1}_{\tilde{\mathcal{M}}\times\{0\}} = \mathscr{H}^1(\tilde{\mathcal{M}}).$$

Since we have

$$\begin{aligned}\int_{\mathcal{Q}} |Du_0| &:= \sup\left\{\int_{\mathcal{Q}} u\mathrm{div}(\xi) \,:\, \xi = (\xi_1,\xi_2) \in C^1_c(\mathcal{Q},\mathbb{R}^2) \text{ and } \xi_1^2+\xi_2^2 \leqslant 1\right\} \\ &\geqslant \int_{\mathcal{Q}} |D_2 u_0| \geqslant \mathscr{H}^1(\tilde{\mathcal{M}}),\end{aligned} \tag{2.3}$$

we get the optimality of (2.2).

Step 3. A transverse argument

From (2.2) and (2.3) we may prove

$$\int_{\mathcal{Q}} |D_1 u_0| = 0. \tag{2.4}$$

Equality (2.4) is a direct consequence of the following lemma.

Lemma 2.3. — *Let Ω be a planar open set. If $u \in BV(\Omega)$ is such that*

$$\int_\Omega |Du| = \int_\Omega |D_2 u|,$$

then $\int_\Omega |D_1 u| = 0.$

Lemma 2.3 is proved in Appendix B.2.

In order to conclude we state an easy lemma.

Lemma 2.4 (Poincaré inequality). — *For $u \in BV(\mathcal{Q})$ satisfying $\mathrm{tr}_{\partial\mathcal{Q}} u = 0$ in $\{0\}\times[0,1]$ we have*

$$\int_{\mathcal{Q}} |u| \leqslant \int_{\mathcal{Q}} |D_1 u|.$$

Lemma 2.4 is proved in Appendix B.3.

Hence, from (2.4) and Lemma 2.4 we have $u_0 = 0$ which is in contradiction with $\mathrm{tr}_{\partial\mathcal{Q}} u_0 = \mathbb{1}_{\tilde{\mathcal{M}}\times\{0\}}$ with $\mathscr{H}^1(\tilde{\mathcal{M}}) > 0$.

2.2. **Outline of the proof of Theorem 1.1.** The idea is to adapt the above construction and argument to the case of a general C^2 domain Ω. If Ω has a flat or concave part Γ of the boundary $\partial\Omega$, then a rather straightforward variant of the above proof shows that $\mathbb{1}_{\mathcal{M}}$, where $\mathcal{M}$ is a non trivial part of Γ, is not the trace of a least gradient function.

Remark 2.5. — Things are more involved when Ω is convex. For simplicity we illustrate this fact when $\Omega = \mathbb{D} = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}$. Let $\mathcal{M} \subset \mathbb{S}^1 \cap \{(x,y) \in \mathbb{R}^2 \,:\, x < 0\}$ be an arc whose endpoints are symmetric with respect to the x-axis. We let $(x_0,-y_0)$ and (x_0,y_0) be the endpoints of $\mathcal{M}$ (here $x_0 \leqslant 0$ and $y_0 > 0$).

We let $\mathscr{C}$ be the chord of $\mathcal{M}$. On the one hand, if $u \in C^1(\mathbb{D}) \cap W^{1,1}(\mathbb{D})$ is such that $\mathrm{tr}_{\mathbb{S}^1} u = 1\!\!1_{\mathcal{M}}$ then, using the Fundamental Theorem of calculus, we have for $-y_0 < y < y_0$

$$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x,y)| \geqslant 1.$$

Thus we easily get

$$\int_{\mathbb{D}} |\nabla u| \geqslant \int_{\mathbb{D}} |\partial_x u| \geqslant \int_{-y_0}^{y_0} \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x,y)| \geqslant 2y_0 = \mathscr{H}^1(\mathscr{C}).$$

Consequently, with the help of a density argument (*e.g.* Lemma A.1 in Appendix A) we obtain

$$\inf \left\{ \int_{\mathbb{D}} |Du| \; ; \; u \in BV(\mathbb{D}) \text{ and } \mathrm{tr}_{\mathbb{S}^1} u = 1\!\!1_{\mathcal{M}} \right\} \geqslant \mathscr{H}^1(\mathscr{C}).$$

On the other hand we let $\omega := \{(x,y) \in \mathbb{R}^2 \; : \; x < x_0\}$. It is clear that $u_0 = 1\!\!1_\omega \in BV(\mathbb{D})$ and $\mathrm{tr}_{\mathbb{S}^1} u_0 = 1\!\!1_{\mathcal{M}}$. Moreover

$$\int_{\mathbb{D}} |Du_0| = \mathscr{H}^1(\mathscr{C}).$$

Consequently u_0 is a function of least gradient. We may do the same argument for a domain Ω as soon as we have a chord entirely contained in Ω. This example suggest that for a convex set Ω, the construction of a set $\mathcal{K} \subset \partial\Omega$ such that (1.2) is not achieved has to be "sophisticated".

The strategy to prove Theorem 1.1 consists of constructing a special set $\mathcal{K} \subset \partial\Omega$ (of Cantor type) and to associate to $\mathcal{K}$ a set B_∞ (the analog of $\tilde{\mathcal{M}} \times (0,1)$ in the model problem) which "projects" onto $\mathcal{K}$ and such that, if u_0 is a minimizer of (1.1), then

$$\int_{B_\infty} |\vec{X} \cdot Du_0| \geqslant \mathscr{H}^1(\mathcal{K}). \tag{2.5}$$

Here, $\vec{X}$ is a vector field satisfying $|\vec{X}| \leqslant 1$. It is the curved analog of $\vec{X} = \mathbf{e}_2$ used in the above proof.

By (2.5) (and Proposition E.1 in Appendix E), if u_0 is a minimizer, then

$$\int_{\Omega \setminus B_\infty} |Du_0| + \int_{B_\infty} (|Du_0| - |\vec{X} \cdot Du_0|) = 0. \tag{2.6}$$

We next establish a Poincaré type inequality implying that any u_0 satisfying (2.6) and $\mathrm{tr}_{\partial\Omega \setminus \mathcal{K}} u = 0$ is 0, which is not possible.

The heart of the proof consists of constructing $\mathcal{K}, B_\infty$ and $\vec{X}$ (see Sections 4 and 5).

3. Notation, definitions

The ambient space is the Euclidean plan $\mathbb{R}^2$. We let $\mathcal{B}_{\mathrm{can}}$ be the canonical basis of $\mathbb{R}^2$.

a) The open ball centered at $A \in \mathbb{R}^2$ with radius $r > 0$ is denoted $B(A,r)$.

b) A vector may be denoted by an arrow when it is defined by its endpoints (*e.g.* $\overrightarrow{AB}$). It may be also denoted by a letter in bold font (*e.g.* $\mathbf{u}$) or more simply by a Greek letter in normal font (*e.g.* ν).
We let also $|\mathbf{u}|$ be the Euclidean norm of the vector $\mathbf{u}$.

c) For a vector $\mathbf{u}$ we let $\mathbf{u}^\perp$ be the direct orthogonal vector to $\mathbf{u}$, *i.e.*, if $\mathbf{u} = (x_1, x_2)$ then $\mathbf{u}^\perp = (-x_2, x_1)$.

d) For $A, B \in \mathbb{R}^2$, the segment of endpoints A and B is denoted $[AB] = \{A + t\overrightarrow{AB} : t \in [0,1]\}$ and $\operatorname{dist}(A,B) = |\overrightarrow{AB}|$ is the Euclidean distance.

e) For a set $U \subset \mathbb{R}^2$, the topological interior of U is denoted by $\overset{\circ}{U}$ and its topological closure is $\overline{U}$.

f) For $k \geqslant 1$, a C^k-curve is the range of a C^k injective map from $(0,1)$ to $\mathbb{R}^2$. Note that, in this article, C^k-curves are not closed sets of $\mathbb{R}^2$.

g) For Γ a C^1-curve, $\mathscr{H}^1(\Gamma)$ is the 1-dimensional Hausdorff measure of Γ.

h) For $k \geqslant 1$, a C^k-Jordan curve is the range of a C^k injective map from the unit circle $\mathbb{S}^1$ to $\mathbb{R}^2$.

i) For Γ a C^1-curve or a C^1-Jordan curve, $\mathscr{C} = [AB]$ is a chord of Γ when $A, B \in \overline{\Gamma}$ with $A \neq B$.

j) If Γ is a C^1-Jordan curve then, for $A, B \in \Gamma$ with $A \neq B$, the set $\Gamma \setminus \{A, B\}$ admits exactly two connected components: Γ_1 and Γ_2. These connected components are C^1-curves.

By smoothness of Γ, it is clear that there is $\eta_\Gamma > 0$ such that for $0 < \operatorname{dist}(A,B) < \eta_\Gamma$ there always exists a unique smallest connected component: we have $\mathscr{H}^1(\Gamma_1) < \mathscr{H}^1(\Gamma_2)$ or $\mathscr{H}^1(\Gamma_2) < \mathscr{H}^1(\Gamma_1)$.

If $0 < \operatorname{dist}(A,B) < \eta_\Gamma$ we may define $\widehat{AB}$ by:

$$\widehat{AB} \text{ is the closure of the smallest curve between } \Gamma_1 \text{ and } \Gamma_2. \tag{3.1}$$

k) In this article $\Omega \subset \mathbb{R}^2$ is a C^2 bounded open set.

4. Construction of the Cantor set $\mathcal{K}$

It is clear that, in order to prove Theorem 1.1, we may assume that Ω is a connected set.

We fix $\Omega \subset \mathbb{R}^2$ a bounded C^2 open connected set. The set $\mathcal{K} \subset \partial\Omega$ is a Cantor type set we will construct below.

4.1. **First step: localization of $\partial\Omega$.**

From the regularity of Ω, there exist $\ell + 1$ C^2-open sets, $\omega_0, ..., \omega_\ell$, such that $\Omega = \omega_0 \setminus \overline{\omega_1 \cup \cdots \cup \omega_\ell}$ and

- ω_i is simply connected for $i = 0, ..., \ell$,
- $\overline{\omega_i} \subset \omega_0$ for $i = 1, ..., \ell$,
- $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ for $1 \leqslant i < j \leqslant \ell$.

We let $\Gamma = \partial\omega_0$. The Cantor type set $\mathcal{K}$ we construct "lives" on Γ. Note that Γ is a Jordan-curve.

Let $M_0 \in \Gamma$ be such that the inner curvature of Γ at M_0 is positive (the existence of M_0 follows from the Gauss-Bonnet formula). Then there exists $r_0 \in (0,1)$ such that $[AB] \subset \overline{\Omega}$ and $[AB] \cap \partial\Omega = \{A, B\}$, $\forall\, A, B \in B(M_0, r_0) \cap \Gamma$. Note that we may assume $2r_0 < \eta_\Gamma$ (where η_Γ is defined in Section 3.j).

We fix $A, B \in B(M_0, r_0) \cap \Gamma$ such that $A \neq B$. We have:

- By the definition of M_0 and r_0, the chord $\mathscr{C}_0 := [AB]$ is included in $\overline{\Omega}$.
- We let $\widehat{AB}$ be the closure of the smallest part of Γ which is delimited by A, B (see (3.1)). We may assume that $\widehat{AB}$ is the graph of $f \in C^2([0,\eta],\mathbb{R}^+)$ in the orthonormal frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$.
- The function f satisfies $f(x) > 0$ for $x \in (0,\eta)$ and $f''(x) < 0$ for $x \in [0,\eta]$.

For further use we note that the length of the chord $[AB]$ is η and that for intervals $I, J \subset [0,\eta]$, if $I \subset J$ then

$$\begin{cases} \|f'_{|I}\|_{L^\infty(I)} \leqslant \|f'_{|J}\|_{L^\infty(J)} \\ \|f''_{|I}\|_{L^\infty(I)} \leqslant \|f''_{|J}\|_{L^\infty(J)} \end{cases} \tag{4.1}$$

where $f_{|I}$ is the restriction of f to I.

Replacing the chord $\mathscr{C}_0 = [AB]$ with a smaller chord of $\widehat{AB}$ parallel to $\mathscr{C}_0$, we may assume that

$$0 < \eta < \min\left\{\frac{1}{2} \,:\, \frac{1}{16\|f''\|^2_{L^\infty([0,\eta])}} \,:\, \frac{1}{2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])}}\right\}. \tag{4.2}$$

We may also assume that

- Letting D_0^+ be the bounded open set such that $\partial D_0^+ = [AB] \cup \widehat{AB}$ we have $\Pi_{\partial\Omega}$, the orthogonal projection on $\partial\Omega$, is well defined and of class C^1 in D_0^+.
- We have
$$1 + 4\|f''\|^2_{L^\infty}\mathrm{diam}(D_0^+) < \frac{16}{9} \tag{4.3}$$
where $\mathrm{diam}(D_0^+) = \sup\{\mathrm{dist}(M,N) \,:\, M,N \in D_0^+\}$. (Here we used (4.1).)

4.2. **Step 2: Iterative construction.** We are now in position to construct the Cantor type set $\mathcal{K}$ as a subset of $\widehat{AB}$. The construction is iterative.

The goal of the construction is to get at step $N \geqslant 0$ a collection of 2^N pairwise disjoint curves included in $\widehat{AB}$ (denoted by $\{K_1^N, ..., K_{2^N}^N\}$) and their chords (denoted by $\{\mathscr{C}_1^N, ..., \mathscr{C}_{2^N}^N\}$).

The idea is standard: at step $N \geqslant 0$ we replace a curve Γ_0 included in $\widehat{AB}$ by two curves included in Γ_0 (see Figure 4.1).

Initialization. We initialize the procedure by letting $K_1^0 := \widehat{AB}$ and $\mathscr{C}_1^0 = \mathscr{C}_0 = [AB]$.

At step $N \geqslant 0$ we have:

- A set of 2^N curves included in $\widehat{AB}$, $\{K_1^N, ..., K_{2^N}^N\}$. The curves K_k^N's are mutually disjoint. We let $\mathcal{K}_N = \cup_{k=1}^{2^N} K_k^N$.
- A set of 2^N chords, $\{\mathscr{C}_1^N, ..., \mathscr{C}_{2^N}^N\}$ such that for $k = 1, ..., 2^N$, $\mathscr{C}_k^N$ is the chord of K_k^N.

Remark 4.1. — (1) Note that since the $\mathscr{C}_k^N$'s are chords of $\widehat{AB}$ and since in the frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$, $\widehat{AB}$ is the graph of a function, none of the chords $\mathscr{C}_k^N$ is vertical, *i.e.*, directed by $\mathbf{e}_2$.

Since the chords $\mathscr{C}_k^N$ are not vertical, for $k \in \{1, ..., 2^N\}$, we may define $\nu_{\mathscr{C}_k^N}$ as the unit vector orthogonal to $\mathscr{C}_k^N$ such that $\nu_{\mathscr{C}_k^N} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$ with $\beta > 0$.

(2) For η satisfying (4.2), if we consider a chord $\mathscr{C}_k^N$ and a straight line D orthogonal to $\mathscr{C}_k^N$ and intersecting $\mathscr{C}_k^N$, then the straight line D intersect K_k^N at exactly one points. This fact is proved in Appendix C.1.

Induction rules. From step $N \geqslant 0$ to step $N+1$ we follow the following rules:

(1) For each $k \in \{1, ..., 2^N\}$, we let η_k^N be the length of $\mathscr{C}_k^N$. Inside the chord $\mathscr{C}_k^N$ we center a segment I_k^N of length $(\eta_k^N)^2$.
(2) With the help of Remark 4.1.2, we may define two distinct points of K_k^N as the intersection of K_k^N with straight lines orthogonal to $\mathscr{C}_k^N$ which pass to the endpoints of I_k^N.
(3) These intersection points are the endpoints of a curve $\tilde{K}_k^N$ included in K_k^N. We let K_{2k-1}^{N+1} and K_{2k}^{N+1} be the connected components of $K_k^N \setminus \overline{\tilde{K}_k^N}$. We let also
   - $\mathscr{C}_{2k-1}^{N+1}$ and $\mathscr{C}_{2k}^{N+1}$ be the corresponding chords;
   - $\mathcal{K}_{N+1} = \cup_{k=1}^{2^{N+1}} K_k^{N+1}$.

Definition 4.2. — A natural terminology consists in defining the father and the sons of a chord or a curve:

- $\mathcal{F}(\mathscr{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathscr{C}_{2k}^{N+1}) = \mathscr{C}_k^N$ is the *father* of the chords $\mathscr{C}_{2k-1}^{N+1}$ and $\mathscr{C}_{2k}^{N+1}$.
  $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$ is the *father* of the curves K_{2k-1}^{N+1} and K_{2k}^{N+1}.
- $\mathcal{S}(\mathscr{C}_k^N) = \{\mathscr{C}_{2k-1}^{N+1}, \mathscr{C}_{2k}^{N+1}\}$ is the set of sons of the chord $\mathscr{C}_k^N$, *i.e.* $\mathcal{F}(\mathscr{C}_{2k-1}^{N+1}) = \mathcal{F}(\mathscr{C}_{2k}^{N+1}) = \mathscr{C}_k^N$.
  $\mathcal{S}(K_k^N) = \{K_{2k-1}^{N+1}, K_{2k}^{N+1}\}$ is the set of sons of the curve K_k^N, *i.e.*, $\mathcal{F}(K_{2k-1}^{N+1}) = \mathcal{F}(K_{2k}^{N+1}) = K_k^N$.

The inductive procedure is represented in Figure 4.1.

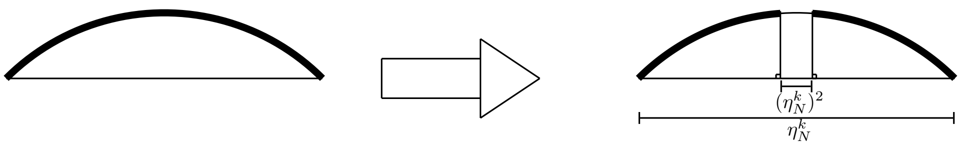

Figure 4.1. Induction step

In Figures 4.2 and 4.3 the two first iterations of the process are represented.

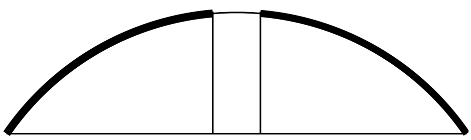

Figure 4.2. First iteration of the process

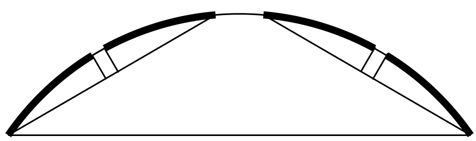

Figure 4.3. Second iteration of the process

We now define the Cantor type set

$$\mathcal{K} = \bigcap_{N \geqslant 0} \overline{\mathcal{K}_N}. \tag{4.4}$$

The Cantor type set $\mathcal{K}$ is fat:

Proposition 4.3. — *We have* $\mathscr{H}^1(\mathcal{K}) > 0$.

This proposition is proved in Appendix C.3.

5. Construction of a sequence of functions

A key argument in the proof of Theorem 1.1 is the use of the coarea formula to calculate a lower bound for (1.2). The coarea formula is applied to a function adapted to the set $\mathcal{K}$.

For $N = 0$ we let

- D_0^+ be the compact set delimited by $K_0 = \widehat{AB}$ and $\mathscr{C}_1^0 := [AB]$ the chord of K_0.
- We recall that we fixed a frame $\mathcal{R}_0 = (A, \mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1 = \overrightarrow{AB}/|\overrightarrow{AB}|$. For $\sigma = (\sigma_1, 0) \in \mathscr{C}_1^0$, we define:

$$I_\sigma \text{ is the connected component of } \{(\sigma_1, t) \in \Omega \,:\, t \leqslant 0\} \text{ which contains } \sigma. \tag{5.1}$$

  (I_σ is a vertical segment included in Ω.)
- $D_0^- = \cup_{\sigma \in \mathscr{C}_1^0} I_\sigma$.
- We now define the maps

$$\begin{array}{rccc} \tilde{\Psi}_0 : & D_0^- & \to & \mathscr{C}_1^0 \\ & x & \mapsto & \Pi_{\mathscr{C}_1^0}(x) \end{array}$$

  and

$$\begin{array}{rccc} \Psi_0 : & D_0^- \cup D_0^+ & \to & \mathscr{C}_1^0 \\ & x & \mapsto & \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_0^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_0(x)] & \text{if } x \in D_0^- \end{cases} \end{array}$$

  where $\Pi_{\partial\Omega}$ is the orthogonal projection on $\partial\Omega$ and $\Pi_{\mathscr{C}_1^0}$ is the orthogonal projection on $\mathscr{C}_1^0$. Note that, in the frame $\mathcal{R}_0$, for $x = (x_1, x_2) \in D_0^-$, we have $\Pi_{\mathscr{C}_1^0}(x) = (x_1, 0)$.

For $N = 1$ and $k \in \{1, 2\}$ we let:

- D_k^1 be the compact set delimited by K_k^1 and $\mathscr{C}_k^1$;
- T_k^1 be the compact right-angled triangle (with its interior) having $\mathscr{C}_k^1$ as side adjacent to the right angle and whose hypothenuse is included in $\mathscr{C}_1^0$;
- H_k^1 be the hypothenuse of T_k^1.

We now define $D_1^- = \tilde{\Psi}_0^{-1}(H_1^1 \cup H_2^1)$, $T_1 = T_1^1 \cup T_2^1$ and $D_1^+ = D_1^1 \cup D_2^1$.

We first consider the map

$$\begin{array}{rccc} \tilde{\Psi}_1 : & T_1 \cup D_1^- & \to & \mathscr{C}_1^1 \cup \mathscr{C}_2^1 \\ & x & \mapsto & \begin{cases} \Pi_{\mathscr{C}_k^1}(x) & \text{if } x \in T_k^1 \\ \Pi_{\mathscr{C}_k^1}[\tilde{\Psi}_0(x)] & \text{if } x \in D_1^- \end{cases} \end{array}.$$

In Appendix D (Lemma D.1 and Remark D.2), it is proved that the triangles T_1^1 and T_2^1 are disjoint. Thus the map $\tilde{\Psi}_1$ is well defined

By projecting $\mathscr{C}_1^1 \cup \mathscr{C}_2^1$ on $\partial\Omega$ we get

$$\begin{array}{rccc} \Psi_1 : & T_1 \cup D_1^- \cup D_1^+ & \to & \mathcal{K}_1 \\ & x & \mapsto & \begin{cases} \Pi_{\partial\Omega}(x) & \text{if } x \in D_1^+ \\ \Pi_{\partial\Omega}[\tilde{\Psi}_1(x)] & \text{if } x \in T_1 \cup D_1^-. \end{cases} \end{array}$$

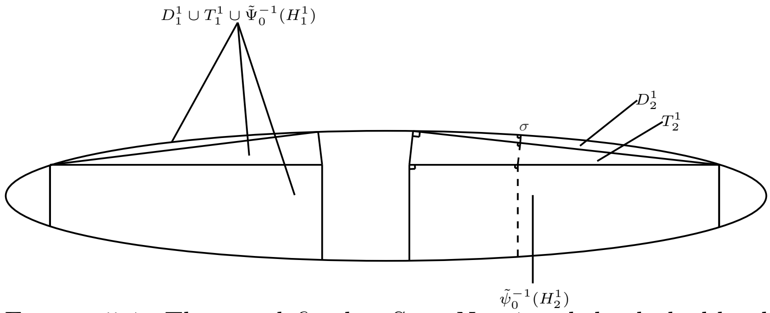

FIGURE 5.1. The sets defined at Step $N = 1$ and the dashed level line of Ψ_1 associated to $\sigma \in \mathcal{K}_1$

For $N \geqslant 1$, we first construct $\tilde{\Psi}_{N+1}$ and then Ψ_{N+1} is obtained from $\tilde{\Psi}_{N+1}$ and $\Pi_{\partial\Omega}$.

For $k \in \{1, ..., 2^{N+1}\}$, we let

- D_k^{N+1} be the compact set delimited by K_k^{N+1} and $\mathscr{C}_k^{N+1}$ (recall that $\mathscr{C}_k^{N+1}$ is the chord associated to K_k^{N+1});
- T_k^{N+1} be the right-angled triangle (with its interior) having $\mathscr{C}_k^{N+1}$ as side adjacent to the right angle and whose hypothenuse is included in $\mathcal{F}(\mathscr{C}_k^{N+1})$. Here $\mathcal{F}(\mathscr{C}_k^{N+1})$ is the father of $\mathscr{C}_k^{N+1}$ (see Definition 4.2);
- $H_k^{N+1} \subset \mathcal{F}(\mathscr{C}_k^{N+1})$ be the hypothenuse of T_k^{N+1}.

We denote

$$T_{N+1} = \bigcup_{k=1}^{2^{N+1}} T_k^{N+1}, \qquad D^{=}_{N+1}\tilde{\Psi}_N^{-1}\Big(\bigcup_{k=1}^{2^{N+1}} H_k^{N+1}\Big) \qquad \text{and} \qquad D^+_{N+1} = \bigcup_{k=1}^{2^{N+1}} D_k^{N+1}.$$

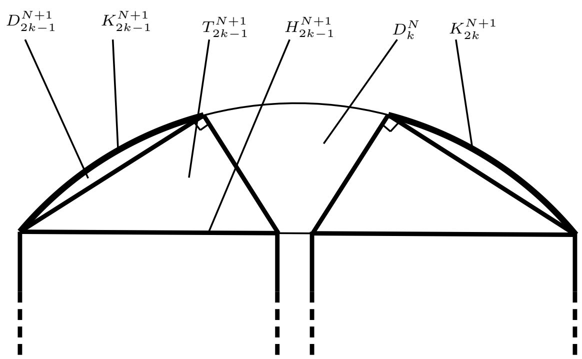

FIGURE 5.2. Induction. The bold lines correspond to the new iteration

Remark 5.1. — It is easy to check that for $N \geqslant 0$:

(1) $T_{N+1} \subset D_N^+$,

(2) if $x \in \overset{\circ}{T}_N$ then $x \notin T_{N'}$ for $N' \geqslant N+1$ (here $T_0 = \emptyset$).

We now define

$$\begin{array}{rccc} \tilde{\Psi}_{N+1}: & T_{N+1} \cup D_{N+1}^- & \to & \cup_{k=1}^{2^{N+1}} \mathscr{C}_k^{N+1} \\ & x & \mapsto & \begin{cases} \Pi_{\mathscr{C}_k^{N+1}}(x) & \text{if } x \in T_k^{N+1} \\ \Pi_{\mathscr{C}_k^{N+1}}[\tilde{\Psi}_N(x)] & \text{if } x \in \tilde{\Psi}_N^{-1}(\cup_{k=1}^{2^{N+1}} H_k^{N+1}). \end{cases} \end{array}$$

In Appendix D (Lemma D.1 and Remark D.2), it is proved that for $N \geqslant 1$, the triangles T_k^N for $k = 1, ..., 2^N$ are mutually disjoint. recursively, we find that all the $\tilde{\Psi}_N$'s are well-defined.

As in the Initialization Step, we get Ψ_{N+1} from $\tilde{\Psi}_{N+1}$ by projecting $\cup_{k=1}^{2^{N+1}} \mathscr{C}_k^{N+1}$ on $\partial\Omega$:

$$\begin{array}{rccc} \Psi_{N+1}: & T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+ & \to & \mathcal{K}_{N+1} \\ & x & \mapsto & \begin{cases} \Pi_{\partial\Omega}[\tilde{\Psi}_{N+1}(x)] & \text{if } x \in T_{N+1} \cup D_{N+1}^- \\ \Pi_{\partial\Omega}(x) & \text{if } x \in D_{N+1}^+. \end{cases} \end{array}$$

It is easy to see that $\Psi_{N+1}(T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+) = \mathcal{K}_{N+1}$.

6. Basic properties of B_∞ and Ψ_N

6.1. **Basic properties of B_∞.** We set $B_N = T_N \cup D_N^+ \cup D_N^-$. It is easy to check that for $N \geqslant 0$ we have $B_{N+1} \subset B_N$ and $\mathcal{K} \subset \partial B_N$. Therefore we may define

$$B_\infty = \cap_{N \geqslant 0} \overline{B_N}$$

which is compact and satisfies $\mathcal{K} \subset \partial B_\infty$.

We are going to prove:

Lemma 6.1. — *The interior of B_∞ is empty.*

Proof of Lemma 6.1. — From Lemma D.1 (and Remark D.2) in Appendix D combined with Hypothesis (4.2), we get two fundamental facts:

(1) The triangles T_1^N, ... ,$T_{2^{N+1}}^N$ are mutually disjoint.

(2) We have:

$$\mathscr{H}^1(H_k^{N+1}) < \frac{\mathscr{H}^1(\mathcal{F}(\mathscr{C}_k^N))}{2}. \tag{6.1}$$

For a non empty set $A \subset \mathbb{R}^2$ we let

$$\mathrm{rad}(A) = \sup\{r \geqslant 0 \,:\, \exists\, x \in A \text{ such that } \overline{B(x,r)} \subset A\}.$$

Note that the topological interior of A is empty if and only if $\mathrm{rad}(A) = 0$.

On the one hand, it is not difficult to check that for sufficiently large N

$$\mathrm{rad}(B_N) = \mathrm{rad}(B_N \cap D_N^-). \tag{6.2}$$

On the other hand, using (6.1) we obtain for $N \geqslant 1$:

$$\mathrm{rad}(B_{N+1} \cap D_{N+1}^-) \leqslant \frac{\mathrm{rad}(B_N \cap D_N^-)}{2}. \tag{6.3}$$

Consequently, by combining (6.2) and (6.3) we get the existence of C_0 such that

$$\mathrm{rad}(B_N) \leqslant \frac{C_0}{2^N}. \tag{6.4}$$

Since $B_\infty = \cap_{N\geqslant 0} B_N$, from (6.4) we get that $\mathrm{rad}(B_\infty) = 0$. □

6.2. **Basic properties of Ψ_N.** We now prove the key estimate for Ψ_N:

Lemma 6.2. — *There exists $b_N = o_N(1)$ such that for $N \geqslant 1$ and U a connected component of B_N, the restriction of Ψ_N to U is $(1+b_N)$-Lipschitz.*

Proof. — Let $N \geqslant 1$ and U be a connected component of B_N. The restriction of $\tilde{\Psi}_N$ to $U \cap (T_N \cup D_N^-)$ is obtained as composition of orthogonal projections on straight lines and thus is 1-Lipschitz.

There exists $b_N = o_N(1)$ such that the projection $P_N := \Pi_{\partial\Omega}$ defined in $\overline{D_N^+}$ is $(1+b_N)$-Lipschitz. The functions Ψ_N are either the composition of $\tilde{\Psi}_N$ with P_N or $\Psi_N = P_N$. Consequently the restriction of Ψ_N to U is $(1+b_N)$-Lipschitz. □

In the following we will not use Ψ_N but "its projection" on $\mathbb{R}$. For $N \geqslant 1$ and $k \in \{1, ..., 2^N\}$, we let $B_k^N := \Psi_N^{-1}(K_k^N)$ and we define

$$\begin{array}{cccc} \Pi_{k,N}: & B_k^N & \to & \mathbb{R} \\ & x & \mapsto & \mathscr{H}^1(\widehat{A\Psi_N(x)}) \end{array}$$

where $\widehat{A\Psi_N(x)} \subset \widehat{AB}$ is defined by (3.1) as the smallest connected component of $\partial\Omega \setminus \{A, \Psi_N(x)\}$ if $\Psi_N(x) \neq A$ and $\widehat{A\Psi_N(x)} = \{A\}$ otherwise.

Lemma 6.3. — *For $N \geqslant 1$ there exists $c_N \in (0,1)$ with $c_N = o_N(1)$ such that for $k \in \{1, ..., 2^N\}$ the function $\Pi_{k,N} : B_k^N \to \mathbb{R}$ is $(1+c_N)$-Lipschitz.*

Proof. — Let $N \geqslant 1$, $k \in \{1, ..., 2^N\}$ and let $x, y \in B_k^N$ be such that $\Psi_N(x) \neq \Psi_N(y)$. It is clear that we have

$$|\Pi_{k,N}(x) - \Pi_{k,N}(y)| = \mathscr{H}^1(\widehat{\Psi_N(y)\Psi_N(x)})$$

where $\widehat{\Psi_N(y)\Psi_N(x)} \subset K_k^N$ is defined by (3.1) as the smallest connected component of $\partial\Omega \setminus \{\Psi_N(y), \Psi_N(x)\}$.

Moreover, from Lemma C.3 in Appendix C.2, we have the existence of $C \geqslant 1$ independent of N and k such that for $x, y \in B_k^N$ such that $\Psi_N(x) \neq \Psi_N(y)$ we have (denoting $X := \Psi_N(x), Y := \Psi_N(y)$)

$$\mathrm{dist}\,(X,Y) \leqslant \mathscr{H}^1\left(\widehat{XY}\right) \leqslant \mathrm{dist}\,(X,Y)\,[1 + C\mathrm{dist}\,(X,Y)]$$

and

$$\mathscr{H}^1(K_k^N) \leqslant \mathscr{H}^1(\mathscr{C}_k^N)\left[1 + C\mathscr{H}^1(\mathscr{C}_k^N)\right].$$

From Step 1 in the proof of Proposition 4.4 (Appendix C.3) we have

$$\max_{k=1,\dots,2^N} \mathscr{H}^1(\mathscr{C}_k^N) \leqslant \left(\frac{2}{3}\right)^N.$$

Thus, letting $a_N := \left(\frac{2}{3}\right)^N \left[1 + C\left(\frac{2}{3}\right)^N\right]$, we have $a_N \to 0$, and since $\widehat{XY} \subset K_k^N$ we get:

$$\begin{aligned}\operatorname{dist}(X,Y) &\leqslant \mathscr{H}^1\left(\widehat{XY}\right) \leqslant \mathscr{H}^1(K_k^N)\\ &\leqslant \mathscr{H}^1(\mathscr{C}_k^N)\left[1 + C\mathscr{H}^1(\mathscr{C}_k^N)\right] \leqslant a_N(1 + Ca_N).\end{aligned}$$

Thus, letting $\tilde{a}_N = \max\{a_N(1 + Ca_N), |b_N|\}$ where b_N is defined in Lemma 6.2, we get

$$\begin{aligned}\mathscr{H}^1\left(\widehat{XY}\right) = |\Pi_{k,N}(x) - \Pi_{k,N}(y)| &\leqslant \mathscr{H}^1\left([\Psi_N(y)\Psi_N(x)]\right)(1 + C\tilde{a}_N)\\ &\leqslant (1 + \tilde{a}_N)(1 + C\tilde{a}_N)|x - y|.\end{aligned}$$

Therefore, letting c_N be such that $1+c_N = (1+\tilde{a}_N)(1 + C\tilde{a}_N)$ we have $c_N = o_N(1)$, c_N is independent of $k \in \{1, ..., 2^N\}$ and $\Pi_{k,N}$ is $(1 + c_N)$-Lipschitz. □

7. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1. This is done by contradiction. We assume that there exists a map $u_0 \in BV(\Omega)$ which minimizes (1.2).

7.1. **Upper bound.** The first step in the proof is the estimate

$$\int_\Omega |Du_0| \leqslant \|1\!\!1_\mathcal{K}\|_{L^1(\partial\Omega)} = \mathscr{H}^1(\mathcal{K}). \tag{7.1}$$

This estimate is obtained by proving that for all $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} u_\varepsilon = 1\!\!1_\mathcal{K}$ and

$$\|\nabla u_\varepsilon\|_{L^1(\Omega)} \leqslant (1 + \varepsilon)\|\mathrm{tr}_{\partial\Omega} u_\varepsilon\|_{L^1(\Omega)} = (1 + \varepsilon)\mathscr{H}^1(\mathcal{K}). \tag{7.2}$$

Proposition E.1 in Appendix E gives the existence of such u_ε's.

Clearly (7.2) implies (7.1).

7.2. **Optimality of the upper bound.** In order to have a contradiction we follow the strategy of Spradlin and Tamasan in [3]. We fix a sequence $(u_n)_n \subset C^1(\Omega)$ such that

$$u_n \in W^{1,1}(\Omega) \,:\, u_n \to u \text{ in } L^1(\Omega) \,:\, \int_\Omega |\nabla u_n| \to \int_\Omega |Du_0| \,:\, \mathrm{tr}_{\partial\Omega} u_n = \mathrm{tr}_{\partial\Omega} u_0. \tag{7.3}$$

Note that (7.3) implies

$$\int_F |\nabla u_n| \to \int_F |Du| \text{ for all } F \subset \Omega \text{ relatively closed set.} \tag{7.4}$$

Such a sequence can be obtained *via* partition of unity and smoothing ; see the proof of Theorem 1.17 in [2]. For the convenience of the reader a proof is presented in Appendix A (see Lemma A.1).

For further use, let us note that the sequence $(u_n)_n$ constructed in Appendix A satisfies the following additional property:

$$\left|\begin{array}{l}\text{If } u_0 = 0 \text{ outside a compact set } L \subset \overline{\Omega} \text{ and if } \omega \text{ is an open set}\\ \qquad \text{such that } \operatorname{dist}(\omega, L) > 0 \text{ then, for large } n,\ u_n = 0 \text{ in } \omega\end{array}\right. .$$

For $x \in B_0$ we let

$$V_0(x) = \begin{cases} \nu_{\Pi_{\partial\Omega}(x)} & \text{if } x \in D_0^+ \\ (0,1) & \text{if } x \in D_0^- \end{cases}, \tag{7.5}$$

and for $N \geqslant 0$, $x \in B_{N+1}$ we let

$$V_{N+1}(x) = \begin{cases} V_N(x) & \text{if } x \in B_N \setminus \overset{\circ}{T}{}^{N+1} \\ \nu_{\mathscr{C}_k^{N+1}} & \text{if } x \in \overset{\circ}{T}{}_k^{N+1} \end{cases}, \tag{7.6}$$

where, for $\sigma \in \partial\Omega$, ν_σ is the normal outward of Ω in σ and $\nu_{\mathscr{C}_k^{N+1}}$ is defined in Remark 4.1.1.

We now prove the following lemma.

Lemma 7.1. — *When $N \to \infty$ we may define $V_\infty(x)$ a.e. $x \in B_\infty$ by*

$$\begin{array}{rccc} V_\infty : & B_\infty & \to & \mathbb{R}^2 \\ & x & \mapsto & \lim_{N\to\infty} V_N(x) \end{array} \tag{7.7}$$

Moreover, from dominated convergence, we have:

$$V_N \mathbb{1}_{B_N} \to V_\infty \mathbb{1}_{B_\infty} \text{ in } L^1(\Omega).$$

Proof. — If $x \in B_\infty \setminus \cup_{N\geqslant 1} T_N$, then we have $V_N(x) = V_0(x)$ for all $N \geqslant 1$. Thus $\lim_{N\to\infty} V_N(x) = V_0(x)$.

For a.e. $x \in B_\infty \cap \cup_{N\geqslant 1} T_N$ there exists $N_0 \geqslant 1$ such that $x \in \overset{\circ}{T}_{N_0}$. Therefore for all $N > N_0$ we have $V_N(x) = V_{N_0}(x)$. Consequently $\lim_{N\to\infty} V_N(x) = V_{N_0}(x)$. ☐

This section is devoted to the proof of the following lemma:

Lemma 7.2. — *For all $w \in C^\infty \cap W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} w = \mathbb{1}_{\mathcal{K}}$ we have*

$$\int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| \geqslant \mathscr{H}^1(\mathcal{K})$$

where V_∞ is the vector field defined in (7.7).

Remark 7.3. — Since $|V_\infty(x)| = 1$ for a.e. $x \in B_\infty$, it is clear that Lemma 7.2 implies that for all n we have

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geqslant \mathscr{H}^1(\mathcal{K}).$$

From (7.4) we have:

$$\int_{B_\infty \cap \Omega} |Du_0| \geqslant \mathscr{H}^1(\mathcal{K}).$$

Section 7.3 is devoted to a sharper argument than above to get

$$\int_{B_\infty \cap \Omega} |\nabla u_n| \geqslant \int_{B_\infty \cap \Omega} |\nabla u_n \cdot V_\infty| + \delta$$

with $\delta > 0$ is independent of n. The last estimate will imply $\int_{B_\infty \cap \Omega} |Du_0| \geqslant \mathscr{H}^1(\mathcal{K}) + \delta$ which will be the contradiction we are looking for.

Proof of Lemma 7.2. — We will first prove that for $w \in C^\infty \cap W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} w = 1\!\!\mathrm{I}_{\mathcal{K}}$ we have

$$\int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geqslant \frac{\mathscr{H}^1(\mathcal{K})}{1 + o_N(1)}. \tag{7.8}$$

where V_N is the vector field defined in (7.5) and (7.6).

Granted (7.8), we conclude as follows: if $w \in C^\infty \cap W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} w = 1\!\!\mathrm{I}_{\mathcal{K}}$, then

$$\begin{aligned} \int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| &= \lim_{N\to\infty} \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \\ &\geqslant \lim_{N\to\infty} \frac{\mathscr{H}^1(\mathcal{K})}{1 + o_N(1)} = \mathscr{H}^1(\mathcal{K}), \end{aligned}$$

by dominated convergence.

It remains to prove (7.8). We fix $w \in C^\infty \cap W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} w = 1\!\!\mathrm{I}_{\mathcal{K}}$. Using the Coarea Formula we have for $N \geqslant 1$ and $k \in \{1, ..., 2^N\}$, with the help of Lemma 6.3, we have

$$\begin{aligned} (1 + c_N) \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| &\geqslant \int_{B_N^{(k)} \cap \Omega} |\nabla \Pi_{k,N}| |\nabla w \cdot V_N| \\ &= \int_{\mathbb{R}} \mathrm{d}t \int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N|. \end{aligned}$$

Here, if $\Pi_{k,N}^{-1}(\{t\})$ is non trivial, then $\Pi_{k,N}^{-1}(\{t\})$ is a polygonal line:

$$\Pi_{k,N}^{-1}(\{t\}) = I_{\sigma(t,k,N)} \cup I_{k,N,t}^1 \cup \cdots \cup I_{k,N,t}^{N+1}$$

where

- $\sigma(t,k,N) \in [AB]$ is such that $[AB] \cap \Pi_{k,N}^{-1}(\{t\}) = \{\sigma(t,k,N)\}$,
- $I_{\sigma(t,k,N)}$ is defined in (5.1),
- for $l = 1, ..., N$ we have $I_{k,N,t}^l = \Pi_{k,N}^{-1}(\{t\}) \cap T_{N+1-l}$,
- $I_{k,N,t}^{N+1} = \Pi_{k,N}^{-1}(\{t\}) \cap D_N^+$.

From the Fundamental Theorem of calculus and from the definition of V_N, denoting

- $I_{\sigma(t,k,N)} = [M_0, M_1]$ (where $M_0 \in \partial\Omega \setminus \widehat{AB}$ and $M_1 = \sigma(t,k,N)$),
- $I_{k,N,t}^l = [M_l, M_{l+1}]$, $l = 1, ..., N+1$ and $M_{N+2} \in K_k^N$,

we have for a.e. $t \in \Pi_{k,N}(K_k^N)$ and using the previous notation,

$$\int_{[M_l, M_{l+1}]} |\nabla w \cdot V_N| \geqslant |w(M_{l+1}) - w(M_l)|.$$

Here we used the convention $w(M_l) = \mathrm{tr}_{\partial\Omega} w(M_l)$ for $l = 0$ and $N + 2$.

Therefore for a.e $t \in \Pi_{k,N}(K_k^N)$ we have

$$\int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N| \geqslant |\mathrm{tr}_{\partial\Omega} w(M_{N+2}) - \mathrm{tr}_{\partial\Omega} w(M_0)| = 1\!\!\mathrm{I}_{\mathcal{K}}(M_{N+2}).$$

Since $\mathcal{K} \subset \mathcal{K}_N = \cup_{k=1}^{2^N} K_k^N$, we may thus deduce that

$$(1 + c_N) \int_{B_N \cap \Omega} |\nabla w \cdot V_N| = (1 + c_N) \sum_{k=1}^{2^N} \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| \geqslant \int_{\widehat{AB}} 1\!\!\mathrm{I}_{\mathcal{K}} = \mathscr{H}^1(\mathcal{K}).$$

The last estimate clearly implies (7.8) and completes the proof of Lemma 7.2. □

7.3. **Transverse argument.** We assumed that there exists a map u_0 which solves Problem (1.2).

We investigate the following dichotomy:

- $u_0 \not\equiv 0$ in $\Omega \setminus B_\infty$;
- $u_0 \equiv 0$ in $\Omega \setminus B_\infty$.

We are going to prove that both cases lead to a contradiction.

7.3.1. *The case $u_0 \not\equiv 0$ in $\Omega\setminus B_\infty$.* We thus have $\displaystyle\int_{\Omega\setminus B_\infty} |u_0| > 0$. In this case, since $(\mathrm{tr}_{\partial\Omega} u_0)_{|\partial\Omega\setminus\partial B_\infty} \equiv 0$, we have

$$\delta := \int_{\Omega\setminus B_\infty} |Du_0| > 0. \tag{7.9}$$

Estimate (7.9) is a direct consequence of the following lemma applied on each connected components of $\Omega \setminus B_\infty$.

Lemma 7.4 (Weak Poincaré lemma). — *Let $\omega \subset \mathbb{R}^2$ be an open connected set. Assume that there exist $x_0 \in \partial\omega$ and $r > 0$ such that $\omega \cap B(x_0, r)$ is Lipschitz.*

If $u \in BV(\omega)$ satisfies $\mathrm{tr}_{\partial\omega\cap B(x_0,r)} = 0$ and $\int_\omega |Du| = 0$ then $u = 0$.

Lemma 7.4 is proved in Appendix B.4.

Recall that we fixed a sequence $(u_n)_n \subset C^1 \cap W^{1,1}(\Omega)$ satisfying (7.3).

In particular, for sufficiently large n, we have

$$\int_{\Omega\setminus B_\infty} |\nabla u_n| > \frac{\delta}{2}.$$

Thus, from Lemma 7.2 and the fact that $|V_\infty(x)| = 1$ for a.e. $x \in B_\infty$,

$$\int_\Omega |\nabla u_n| \geqslant \int_{B_\infty} |\nabla u_n \cdot V_\infty| + \int_{\Omega\setminus B_\infty} |\nabla u_n| \geqslant \mathscr{H}^1(\mathcal{K}) + \frac{\delta}{2}.$$

This implies

$$\int_\Omega |Du_0| = \lim_n \int_\Omega |\nabla u_n| \geqslant \mathscr{H}^1(\mathcal{K}) + \frac{\delta}{2}$$

which is in contradiction with (7.1).

7.3.2. *The case $u_0 \equiv 0$ in $\Omega \setminus B_\infty$.* We first note that, since $\mathrm{tr}_{\partial D_0^+} u_0 \not\equiv 0$, there exists a triangle $T_k^{N_0}$ such that $\int_{T_k^{N_0}} |u_0| > 0$. We fix such a triangle $T_k^{N_0}$ and we let α be the vertex corresponding to the right angle.

We let $\tilde{\mathcal{R}} = (\alpha, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$ be the direct orthonormal frame centered in α where $\tilde{\mathbf{e}}_2 = \nu_{\mathscr{C}_k^{N_0}}$ ($\nu_{\mathscr{C}_k^{N_0}}$ is defined Remark 4.1.1), *i.e.*, the directions of the new frame are given by the side of the right-angle of $T_k^{N_0}$.

It is clear that for $N \geqslant N_0$ we have $V_N \equiv \tilde{\mathbf{e}}_2$ in $\overset{\circ}{T}{}_k^{N_0}$.

By construction of B_∞, $T_k^{N_0} \cap B_\infty$ is a union of segments parallel to $\tilde{e}_2$, *i.e.* $\mathbb{1}_{B_\infty|T_k^{N_0}}(s,t)$ depends only on the first variable "s" in the frame $\tilde{\mathcal{R}}$.

Since $\int_{T_k^{N_0}} |u_0| > 0$, in the frame $\tilde{\mathcal{R}}$, we may find $a, b, c, d \in \mathbb{R}$ such that, considering the rectangle (whose sides are parallel to the direction of $\tilde{\mathcal{R}}$)

$$\mathcal{P} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \, : \, (s,t) \in [a,b] \times [c,d]\} \subset T_k^{N_0}$$

we have

$$\int_{\mathcal{P}} |u_0| > 0.$$

Since from Lemma 6.1 the set B_∞ has an empty interior (and that $\mathbb{I}_{B_\infty|T_k^{N_0}}(s,t)$ depends only on the first variable in the frame $\tilde{\mathcal{R}}$), we may find $a' < b'$ such that

- $[a', b'] \times [c,d] \subset [a,b] \times [c,d]$,
- $\mathcal{S} \cap B_\infty = \emptyset$ with $\mathcal{S} := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \, : \, (s,t) \in \{a', b'\} \times [c,d]\}$
- $\delta := \displaystyle\int_{\mathcal{P}'} |u_0| > 0$ with $\mathcal{P}' := \{\alpha + s\tilde{\mathbf{e}}_1 + t\tilde{\mathbf{e}}_2 \, : \, (s,t) \in [a', b'] \times [c,d]\}$.

Moreover, since $\mathcal{S}$ and B_∞ are compact sets with empty intersection, we may find $\mathcal{V}$, an open neighborhood of $\mathcal{S}$ such that $\operatorname{dist}(\mathcal{V}, B_\infty) > 0$.

Noting that $u_0 \equiv 0$ in $\Omega \setminus B_\infty$, from Lemma A.1 (in Appendix A) it follows that for sufficiently large n we have

- $u_n \equiv 0$ in $\mathcal{S}$,
- $\displaystyle\int_{\mathcal{P}'} |u_n| > \frac{\delta}{2}$.

Consequently, from a standard Poincaré inequality

$$\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_1} u_n| \geqslant \frac{2}{b' - a'} \int_{\mathcal{P}'} |u_n| > \frac{\delta}{b' - a'} =: \delta'.$$

Therefore $\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_1} u_n| > \delta'$, $\int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_2} u_n| \leqslant 2\mathscr{H}^1(\mathcal{K})$ and then by Lemma 3.3 in [3] we obtain:

$$\int_{\mathcal{P}'} |\nabla u_n| \geqslant \int_{\mathcal{P}'} |\partial_{\tilde{\mathbf{e}}_2} u_n| + \frac{\delta'^2}{4\mathscr{H}^1(\mathcal{K}) + \delta'}.$$

Thus, from Lemma 7.2, for sufficiently large n:

$$\int_{\Omega} |\nabla u_n| \geqslant \mathscr{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathscr{H}^1(\mathcal{K}) + \delta'} - o_n(1).$$

From the convergence in BV-norm of u_n to u_0 we have

$$\int_{\Omega} |Du_0| \geqslant \mathscr{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathscr{H}^1(\mathcal{K}) + \delta'}.$$

Clearly this last assertion contradicts (7.1) and ends the proof of Theorem 1.1.

Appendices

Appendix A. A smoothing result

We first state a standard approximation lemma for BV-functions.

Lemma A.1. — *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz open set and let $u \in BV(\Omega)$. There exists a sequence $(u_n)_n \subset C^1(\Omega)$ such that*

(1) $u_n \overset{\text{strictly}}{\longrightarrow} u$ *in the sense that $u_n \to u$ in $L^1(\Omega)$ and $\displaystyle\int_\Omega |\nabla u_n| \to \int_\Omega |Du|$,*

(2) $\operatorname{tr}_{\partial\Omega} u_n = \operatorname{tr}_{\partial\Omega} u$ *for all n,*

(3) *for* $k \in \{1, 2\}$,

$$\int_\Omega |\partial_k u_n| \to \int_\Omega |D_k u| := \sup \left\{ \int_\Omega u \partial_k \xi \,:\, \xi \in C^1_c(\Omega, \mathbb{R}) \text{ and } |\xi| \leqslant 1 \right\},$$

(4) *If* $u = 0$ *outside a compact set* $L \subset \overline{\Omega}$ *and if* ω *is an open set such that* $\operatorname{dist}(\omega, L) > 0$ *then, for large* n, $u_n = 0$ *in* ω.

Proof. — The first assertion is quite standard. It is for example proved in [1][Theorem 1]. We present below the classical example of sequence for such approximation result (we follow the presentation of [2][Theorem 1.17]).

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz open set and let $u \in BV(\Omega)$.

For $n \geqslant 1$, we let $\varepsilon = 1/n$. We may fix $m \in \mathbb{N}^*$ sufficiently large such that letting for $k \in \mathbb{N}$

$$\Omega_k = \left\{ x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\}$$

we have

$$\int_{\Omega \setminus \Omega_0} |Du| < \varepsilon.$$

We fix now $A_1 := \Omega_2$ and for $i \in \mathbb{N} \setminus \{0, 1\}$ we let $A_i = \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$. It is clear that $(A_i)_{i \geqslant 1}$ is a covering of Ω and that each point in Ω belongs to at most three of the sets $(A_i)_{i \geqslant 1}$.

We let $(\varphi_i)_{i \geqslant 1}$ be a partition of unity subordinate to the covering $(A_i)_{i \geqslant 1}$, *i.e.*, $\varphi_i \in C^\infty_c(A_i)$, $0 \leqslant \varphi_i \leqslant 1$ and $\sum_{i \geqslant 1} \varphi_i = 1$ in Ω.

We let $\eta \in C^\infty_c(\mathbb{R}^2)$ be such that $\operatorname{supp}(\eta) \subset B(0, 1)$, $\eta \geqslant 0$, $\int \eta = 1$ and for $x \in \mathbb{R}^2$ $\eta(x) = \eta(|x|)$. For $t > 0$ we let $\eta_t = t^{-2} \eta(\cdot / t)$.

As explained in [2], for $i \geqslant 1$, we may choose $\varepsilon_i \in (0, \varepsilon)$ sufficiently small such that

$$\begin{cases} \operatorname{supp}(\eta_{\varepsilon_i} * (u\varphi_i)) \subset A_i \\ \displaystyle\int_\Omega |\eta_{\varepsilon_i} * (u\varphi_i) - u\varphi_i| < \frac{\varepsilon}{2^i} \\ \displaystyle\int_\Omega |\eta_{\varepsilon_i} * (u\nabla\varphi_i) - u\nabla\varphi_i| < \frac{\varepsilon}{2^i}. \end{cases}.$$

Here $*$ is the convolution operator.

Define

$$u_n := \sum_{i \geqslant 1} \eta_{\varepsilon_i} * (u\varphi_i).$$

In some neighborhood of each point $x \in \Omega$ there are only finitely many nonzero terms in the sum defining u_n. Thus u_n is well defined and smooth in Ω.

Moreover, we may easily check that

$$\|u_n - u\|_{L^1(\Omega)} + \left| \int_\Omega |Du| - \int_\Omega |\nabla u_n| \right| < \varepsilon \text{ (here } \varepsilon = 1/n).$$

The previous estimate proves that (u_n) satisfies the first assertion, *i.e.* $u_n \overset{\text{strictly}}{\longrightarrow} u$.

As claimed in [2][Remark 2.12] we have $\operatorname{tr}_{\partial\Omega} u_n = \operatorname{tr}_{\partial\Omega} u$ for all n. Thus the second assertion is satisfied.

We now prove the third assertion. Since $u_n \to u$ in $L^1(\Omega)$, by inferior semi continuity we easily get for $k \in \{1,2\}$

$$\int_\Omega |D_k u| \leqslant \liminf_{n\to\infty} \int_\Omega |\partial_k u_n|.$$

We now prove $\displaystyle\int_\Omega |D_k u| \geqslant \limsup_{n\to\infty} \int_\Omega |\partial_k u_n|$.

Let $\xi \in C^1_c(\Omega, \mathbb{R})$ with $|\xi| \leqslant 1$. Since η is a symmetric mollifier and $\sum \varphi_i = 1$ we have

$$\begin{aligned}
\int_\Omega u_n \partial_k \xi &= \sum_{i\geqslant 1} \int_\Omega \eta_{\varepsilon_i} * (u\varphi_i)\partial_k \xi \\
&= \sum_{i\geqslant 1} \int_\Omega u\varphi_i \partial_k(\eta_{\varepsilon_i} * \xi) \\
&= \sum_{i\geqslant 1} \int_\Omega u\partial_k[\varphi_i(\eta_{\varepsilon_i} * \xi)] - \sum_{i\geqslant 1} \int_\Omega u\partial_k\varphi_i(\eta_{\varepsilon_i} * \xi) \\
&= \sum_{i\geqslant 1} \int_\Omega u\partial_k[\varphi_i(\eta_{\varepsilon_i} * \xi)] - \sum_{i\geqslant 1} \int_\Omega \xi\left[\eta_{\varepsilon_i} * (u\partial_k\varphi_i) - u\partial_k\varphi_i\right].
\end{aligned}$$

On the one hand we have (note that $\varphi_i(\eta_{\varepsilon_i} * \xi) \in C^1_c(A_i)$ and $|\varphi_i(\eta_{\varepsilon_i} * \xi)| \leqslant 1$)

$$\begin{aligned}
\left|\sum_{i\geqslant 1} \int_\Omega u\partial_k[\varphi_i(\eta_{\varepsilon_i} * \xi)]\right| &= \left|\int_{A_1} u\partial_k[\varphi_i(\eta_{\varepsilon_i} * \xi)] + \sum_{i\geqslant 2} \int_{A_i} u\partial_k[\varphi_i(\eta_{\varepsilon_i} * \xi)]\right| \\
&\leqslant \int_\Omega |D_k u| + \sum_{i\geqslant 2} \int_{A_i} |D_k u| \\
&\leqslant \int_\Omega |D_k u| + 3\int_{\Omega\setminus\Omega_0} |D_k u| \\
&\leqslant \int_\Omega |D_k u| + 3\varepsilon.
\end{aligned}$$

Here we used that each point in Ω belongs to at most three of the sets $(A_i)_{i\geqslant 1}$, for $i \geqslant 2$ we have $A_i \subset \Omega \setminus \Omega_0$ and

$$\int_{\Omega\setminus\Omega_0} |D_k u| \leqslant \int_{\Omega\setminus\Omega_0} |Du| < \varepsilon.$$

On the other hand, since for $i \geqslant 1$ $\displaystyle\int_\Omega |\eta_{\varepsilon_i} * (u\nabla\varphi_i) - u\nabla\varphi_i| < \frac{\varepsilon}{2^i}$, we get

$$\left|\sum_{i\geqslant 1} \int_\Omega \xi\left[\eta_{\varepsilon_i} * (u\partial_k\varphi_i) - u\partial_k\varphi_i\right]\right| \leqslant \sum_{i\geqslant 1} \int_\Omega |\eta_{\varepsilon_i} * (u\partial_k\varphi_i) - u\partial_k\varphi_i| < \varepsilon.$$

Consequently

$$\sup\left\{\int_\Omega u_n\partial_k\xi \,:\, \xi \in C^1_c(\Omega,\mathbb{R}) \text{ and } |\xi| \leqslant 1\right\} = \int_\Omega |\partial_k u_n| \leqslant \int_\Omega |D_k u| + 4\varepsilon$$

and thus

$$\limsup_n \int_\Omega |\partial_k u_n| \leqslant \int_\Omega |D_k u|.$$

This inequality in conjunction with

$$\liminf_n \int_\Omega |\partial_k u_n| \geqslant \int_\Omega |D_k u|$$

proves the third assertion of Lemma A.1.

The last assertion of Lemma A.1 is a direct consequence of the definition of the u_n's. □

Appendix B. Proofs of Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 7.4

B.1. **Proof of Lemma 2.2.** Let $u \in BV(\mathcal{Q})$. We prove that

$$\int_\mathcal{Q} |D_2 u| \geqslant \int_0^1 |\mathrm{tr}_{\partial\mathcal{Q}} u(\cdot, 0) - \mathrm{tr}_{\partial\mathcal{Q}} u(\cdot, 1)|.$$

From Lemma A.1, there exists $(u_n)_n \subset C^1(\mathcal{Q})$ with $\mathrm{tr}_{\partial\mathcal{Q}} u_n = \mathrm{tr}_{\partial\mathcal{Q}} u$ and such that $u_n \overset{\text{strictly}}{\longrightarrow} u$ and

$$\int_\mathcal{Q} |\partial_2 u_n| \to \int_\mathcal{Q} |D_2 u|.$$

From Fubini's theorem and the Fundamental theorem of calculus we have

$$\begin{aligned}
\int_\mathcal{Q} |\partial_2 u_n| &= \int_0^1 \mathrm{d}x_1 \int_0^1 |\partial_2 u_n(x_1, x_2)| \mathrm{d}x_2 \\
&\geqslant \int_0^1 \mathrm{d}x_1 \left| \int_0^1 \partial_2 u_n(x_1, x_2) \mathrm{d}x_2 \right| \\
&= \int_0^1 \mathrm{d}x_1 \, |\mathrm{tr}_{\partial\mathcal{Q}} u_n(x_1, 1) - \mathrm{tr}_{\partial\mathcal{Q}} u_n(x_1, 0)| \\
&= \int_0^1 |\mathrm{tr}_{\partial\mathcal{Q}} u(\cdot, 1) - \mathrm{tr}_{\partial\mathcal{Q}} u(\cdot, 0)| \,.
\end{aligned}$$

Since $\int_\mathcal{Q} |\partial_2 u_n| \to \int_\mathcal{Q} |D_2 u|$, Lemma 2.2 is proved.

B.2. **Proof of Lemma 2.3.** Let Ω be a planar open set. Let $u \in BV(\Omega)$ be such that

$$\int_\Omega |Du| = \int_\Omega |D_2 u|.$$

We prove that $\displaystyle\int_\Omega |D_1 u| = 0$. We argue by contradiction and we assume that $\displaystyle\int_\Omega |D_1 u| > 0$, *i.e.*, there exists $\xi \in C^1_c(\Omega)$ such that $|\xi| \leqslant 1$ and

$$\eta := \int_\Omega u \partial_1 \xi > 0.$$

Let $(\xi_n)_n \subset C^1_c(\Omega)$ be such that $|\xi_n| \leqslant 1$ and

$$\eta_n := \int_\Omega u \partial_2 \xi_n \to \int_\Omega |D_2 u|.$$

For $(\alpha,\beta)\in\{x\in\mathbb{R}^2\ :\ |x|\leqslant 1\}$ we let $\xi^{(n)}_{\alpha,\beta}=(\alpha\xi,\beta\xi_n)\in C^1_c(\Omega,\mathbb{R}^2)$. Clearly, $|\xi^{(n)}_{\alpha,\beta}|\leqslant 1$ and

$$\int_\Omega |Du|\geqslant\int_\Omega u\mathrm{div}(\xi^{(n)}_{\alpha,\beta})=\alpha\eta+\beta\eta_n. \tag{B.1}$$

If we maximize the right hand side of (B.1) w.r.t. $(\alpha,\beta)\in\{x\in\mathbb{R}^2\ :\ |x|\leqslant 1\}$, then we find with $(\alpha,\beta)=\left(\frac{\eta}{\sqrt{\eta^2+\eta_n^2}},\frac{\eta_n}{\sqrt{\eta^2+\eta_n^2}}\right)$ that

$$\int_\Omega |Du|\geqslant\sqrt{\eta^2+\eta_n^2}\underset{n\to\infty}{\to}\sqrt{\eta^2+\left(\int_\Omega |Du|\right)^2}>\int_\Omega |Du|.$$

This is a contradiction.

B.3. **Proof of Lemma 2.4.** Let $u\in BV(\mathcal{Q})$ satisfying $\mathrm{tr}_{\partial\mathcal{Q}}u=0$ in $\{0\}\times[0,1]$. We are going to prove that

$$\int_{\mathcal{Q}}|u|\leqslant\int_{\mathcal{Q}}|D_1u|.$$

Let $(u_n)_n\subset C^1(\Omega)$ be given by Lemma A.1. Using the Fundamental theorem of calculus we have for $(x_1,x_2)\in\mathcal{Q}$ that

$$|u_n(x_1,x_2)|\leqslant\int_0^{x_1}|\partial_1u_n(t,x_2)|\mathrm{d}t\leqslant\int_0^1|\partial_1u_n(t,x_2)|\mathrm{d}t.$$

Therefore, from Fubini's theorem, we get

$$\int_{\mathcal{Q}}|u_n|\leqslant\int_{\mathcal{Q}}\mathrm{d}x_1\mathrm{d}x_2\int_0^1|\partial_1u_n(t,x_2)|\mathrm{d}t=\int_0^1\mathrm{d}x_2\int_0^1|\partial_1u_n(t,x_2)|\mathrm{d}t=\int_Q|\partial_1u_n|.$$

It suffices to see that $\int_{\mathcal{Q}}|u_n|\to\int_{\mathcal{Q}}|u|$ and $\int_Q|\partial_1u_n|\to\int_Q|D_1u|$ to get the result.

B.4. **Proof of Lemma 7.4.** Let $\omega\subset\mathbb{R}^2$ be an open connected set. Assume there exist $x_0\in\partial\omega$ and $r>0$ such that $\omega\cap B(x_0,r)$ is Lipschitz.

Let $u\in BV(\omega)$ satisfying $\mathrm{tr}_{\partial\omega\cap B(x_0,r)}u=0$ and $\int_\omega|Du|=0$. We are going to prove that $u=0$. On the one hand, since $\int_\omega|Du|=0$, we get $u=C$ with $C\in\mathbb{R}$ a constant. We thus have $\mathrm{tr}_{\partial\omega\cap B(x_0,r)}u=C$. Consequently $C=0$ and $u\equiv 0$.

Appendix C. Results related to the Cantor set $\mathcal{K}$

C.1. **Justification of Remark 4.1.**(1)**.** We prove the following lemma:

Lemma C.1. — *Let $\eta>0$ and let $f\in C^2([0,\eta],\mathbb{R})$ be such that*

$$\eta<\frac{1}{2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])}}.$$

We denote C_f the graph of f in an orthonormal frame $\mathcal{R}_0$.

For $0\leqslant a<b\leqslant\eta$, denoting $\mathscr{C}$ the chord $[(a,f(a)),(b,f(b))]$, for any straight line D orthogonal to $\mathscr{C}$ such that $D\cap\mathscr{C}\neq\emptyset$, the straight line D intersects $C_{f,a,b}$ at exactly one point, where $C_{f,a,b}$ is the part of C_f delimited by $(a,f(a))$ and $(b,f(b))$.

Remark C.2. — We may state an analog result with $f\in C^1$ where we use the modulus of continuity of f' instead of $\|f''\|_\infty$ in the hypothesis.

Proof. — The key point here is uniqueness. Indeed, for $0 \leqslant a < b \leqslant \eta$ and $\mathscr{C}, D$ as in the lemma, we may easily prove that $C_{f,a,b} \cap D \neq \emptyset$ by solving an equation. (We do not use $\eta < (2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])})^{-1}$ for the existence)

In contrast with the existence of an intersection point, its uniqueness is valid only for η not too large. To prove uniqueness we argue by contradiction and we consider f and η as in lemma and we assume that there exist two points $0 \leqslant a < b \leqslant \eta$ such that there exist $a \leqslant x < y \leqslant b$ such that the segments $[(x, f(x)), (y, f(y))]$ and $[(a, f(a)), (b, f(b))]$ are orthogonal. Note that with this hypothesis the straight line $D := ((x, f(x)), (y, f(y)))$ is orthogonal to the chord $\mathscr{C} := [(a, f(a)), (b, f(b))]$.

So we get

$$\frac{f(y) - f(x)}{y - x} = -\frac{b - a}{f(b) - f(a)}.$$

From the Mean Value Theorem, there exist $c \in (x, y)$ and $\tilde{c} \in (a, b)$ such that $f'(c) = -\dfrac{1}{f'(\tilde{c})}$. Consequently

$$f'(c) \times [f'(\tilde{c}) - f'(c)] = -1 - [f'(c)]^2. \tag{C.1}$$

From the hypothesis $\eta < (2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])})^{-1}$, we have

$$|f'(\tilde{c}) - f'(c)| \leqslant \eta\|f''\|_{L^\infty([0,\eta])} < \frac{1}{2\|f'\|_{L^\infty([0,\eta])}}.$$

Therefore, we get

$$|f'(c) \times [f'(\tilde{c}) - f'(c)]| < \frac{1}{2}$$

which is in contradiction with (C.1). □

C.2. **Two preliminary results.** We first prove a standard result which states that the length of a *small* chord is a *good* approximation for the length of a curve.

Lemma C.3. — *Let $0 < \eta < 1$ and let $f \in C^2([0, \eta], \mathbb{R}^+)$. We fix an orthonormal frame and we denote C_f the graph of f in the orthonormal frame. Let $A = (a, f(a)), B = (b, f(b)) \in C_f$ (with $0 \leqslant a < b \leqslant \eta$) and let $\mathscr{C} = [AB]$ be the chord of C_f joining A and B. We denote $\widehat{AB}$ the arc of C_f with endpoints A and B.*

We have

$$\mathscr{H}^1(\mathscr{C}) \leqslant \mathscr{H}^1(\widehat{AB}) \leqslant \mathscr{H}^1(\mathscr{C})\left\{1 + (b - a)\|f''\|_{L^\infty}[2\|f'\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\right\}.$$

Proof. — The estimate $\mathscr{H}^1(\mathscr{C}) \leqslant \mathscr{H}^1(\widehat{AB})$ is standard, we thus prove the second inequality.

On the one hand

$$\mathscr{H}^1(\mathscr{C}) = \sqrt{(a - b)^2 + [f(a) - f(b)]^2} = (b - a)\sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2}.$$

On the other hand

$$\mathscr{H}^1(\widehat{AB}) = \int_a^b \sqrt{1 + f'^2}.$$

With the help of the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$

Applying once again the Mean Value Theorem (to f'), for $x \in [a,b]$ there exists c_x between c and x such that

$$f'(x) = f'(c) + f''(c_x)(x-c).$$

Consequently for $x \in [a,b]$ we have:

$$\begin{aligned}\sqrt{1+f'(x)^2} &= \sqrt{1+[f'(c)+f''(c_x)(x-c)]^2}\\ &= \sqrt{1+f'(c)^2}\sqrt{1+\frac{2f'(c)f''(c_x)(x-c)+f''(c_x)^2(x-c)^2}{1+f'(c)^2}}\\ &\leqslant \sqrt{1+\left(\frac{f(a)-f(b)}{a-b}\right)^2}\left[1+2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b-a)+\|f''\|^2_{L^\infty}(b-a)^2\right].\end{aligned}$$

Thus we have

$$\begin{aligned}\mathscr{H}^1(\widehat{AB}) &= \int_a^b \sqrt{1+f'(x)^2}\,\mathrm{d}x\\ &\leqslant (b-a)\sqrt{1+\left(\frac{f(a)-f(b)}{a-b}\right)^2}\\ &\qquad\qquad \left[1+2\|f'\|_{L^\infty}\|f''\|_{L^\infty}(b-a)+\|f''\|^2_{L^\infty}(b-a)^2\right]\\ &= \mathscr{H}^1(\mathscr{C})\left\{1+(b-a)\|f''\|_{L^\infty}[2\|f'\|_{L^\infty}+\|f''\|_{L^\infty}(b-a)]\right\}. \quad \square\end{aligned}$$

We now state another technical lemma which gives an upper bound for the height of the curve w.r.t. its chord.

Lemma C.4. — *Let $0 \leqslant a < b \leqslant \eta$, $f \in C^2([0,\eta],\mathbb{R}^+)$ be a strictly concave function and let C_f be the graph of f in an orthonormal frame. Let $A=(a,f(a))$ and $B=(b,f(b))$ be two points of C_f.*

Assume that we have

$$\eta < \frac{1}{2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])}}$$

in order to define for $C \in [AB]$ (with the help of Lemma C.1) $\tilde{C}$ as the unique intersection point of C_f with the line orthogonal to $[AB]$ passing by C.

We have

$$\mathscr{H}^1([C\tilde{C}]) \leqslant \frac{(b-a)^2\|f''\|_{L^\infty}}{8}.$$

Proof. — Let $0 \leqslant a < b \leqslant \eta$, $f \in C^2([0,\eta],\mathbb{R}^+)$ be as in Lemma C.4.

We consider the function

$$\begin{array}{cccc} g: & [0,\eta] & \to & \mathbb{R}\\ & x & \mapsto & f(x)-\left[\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)\right].\end{array}$$

It is clear that g is non negative since f is strictly concave.

For $C \in [AB]$, we let $\tilde{C}$ be as in Lemma C.4. Then we have

$$\sup_{C\in[AB]} \mathscr{H}^1([C\tilde{C}]) = \max_{[0,\eta]} g.$$

Thus, it suffices to prove $\max_{[0,\eta]} g \leqslant \dfrac{(b-a)^2\|f''\|_{L^\infty}}{8}$.

Since g is C^1 and $g(a) = g(b) = 0$, there exists $c \in (a, b)$ such that

$$g(c) = \max_{[0,\eta]} g \text{ and } g'(c) = 0.$$

Let $t \in \{a, b\}$ be such that $|t - c| \leqslant \dfrac{b-a}{2}$. Using a Taylor expansion, there exists $\tilde{c}$ between c and t such that

$$0 = g(t) = g(c) + (t - c)g'(c) + \frac{(t-c)^2}{2} g''(\tilde{c}).$$

Thus

$$0 \leqslant \max_{[0,\eta]} g = g(c) = -\frac{(t-c)^2}{2} g''(\tilde{c}) \leqslant \frac{(b-a)^2 \|f''\|_{L^\infty}}{8}.$$

The last inequality completes the proof. □

C.3. **Proof of Proposition 4.4.** We prove that

$$\liminf_{N\to\infty} \mathscr{H}^1(\mathcal{K}_N) > 0. \tag{C.2}$$

Step 1. We prove that $\displaystyle\max_{k=1,\dots,2^N} \mathscr{H}^1(\mathscr{C}_k^N) \leqslant \left(\frac{2}{3}\right)^N$.

For $N \geqslant 1$ we let $\{K_k^N : k = 1, ..., 2^N\}$ be the set of the connected components of $\mathcal{K}_N$. We let $\mathscr{C}_k^N$ be the chord of K_k^N and we define $\mu_N = \max_{k=1,\dots,2^N} \mathscr{H}^1(\mathscr{C}_k^N)$. Note that by (4.2) we have $\mu_0 < 1$.

We first prove that for $N \geqslant 0$ we have

$$\mu_{N+1} \leqslant \frac{2}{3}\mu_N. \tag{C.3}$$

By induction (C.3) implies (since $\mu_0 < 1$)

$$\mu_N \leqslant \left(\frac{2}{3}\right)^N. \tag{C.4}$$

In order to get (C.3), we prove that for $N \geqslant 1$ and K_k^N a connected component of $\mathcal{K}_N$ and $\mathscr{C}_k^N$ its chord, we have

$$\mathscr{H}^1(\mathscr{C}) \leqslant \frac{2\mathscr{H}^1(\mathscr{C}_k^N)}{3} \text{ for } \mathscr{C} \in \mathcal{S}(\mathscr{C}_k^N) \tag{C.5}$$

(see Definition 4.2 for $\mathcal{S}(\cdot)$, the set of sons of a chord).

Let $N \geqslant 1$. For $k \in \{1, ..., 2^N\}$, we let K_k^N be a connected component of $\mathcal{K}_N$. We let $K_{2k-1}^{N+1}, K_{2k}^{N+1} \in \mathcal{S}(K_k^N)$ be the curve obtained from K_k^N in the induction step.

For $\tilde{k} \in \{2k-1, 2k\}$, we let $\mathscr{C}_{\tilde{k}}^{N+1}$ be the chords of $K_{\tilde{k}}^{N+1}$.

In the frame $\mathcal{R}_0$, we may define four points of Γ,

$$(a_1, f(a_1)), (b_1, f(b_1)), (a_2, f(a_2)), (b_2, f(b_2)),$$

with $0 < a_1 < b_1 < a_2 < b_2 < \eta$, such that:

- the endpoints of K_{2k-1}^{N+1} are $(a_1, f(a_1))$ and $(b_1, f(b_1))$;
- the endpoints of K_{2k}^{N+1} are $(a_2, f(a_2))$ and $(b_2, f(b_2))$;
- the endpoints of K_k^N are $(a_1, f(a_1))$ and $(b_2, f(b_2))$.

In the frame $\mathcal{R}_0$ we let $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ be the coordinates of the points of $\mathscr{C}_k^N$ such that for $l \in \{1, 2\}$, the triangles whose vertices are $\{(a_l, f(a_l)); (b_l, f(b_l)); (\alpha_l, \beta_l)\}$ are right angled in (α_l, β_l).

We denote

- $\mathcal{I}_1$ the segment $[(b_1, f(b_1)); (\alpha_1, \beta_1)]$;
- $\mathcal{I}_2$ the segment $[(a_2, f(a_2)); (\alpha_2, \beta_2)]$.

From the construction of K_{2k-1}^{N+1} and K_{2k}^{N+1} and from the Pythagorean theorem we have for $l = 1, 2$

$$\mathscr{H}^1(\mathscr{C}_{2k-2+l}^{N+1})^2 = \mathscr{H}^1(\mathcal{I}_l)^2 + \left(\frac{\mathscr{H}^1(\mathscr{C}_k^N) - \mathscr{H}^1(\mathscr{C}_k^N)^2}{2}\right)^2.$$

Using Lemma C.4 we get that

$$\mathscr{H}^1(\mathcal{I}_l) \leqslant (b_2 - a_1)^2 \|f''\|_{L^\infty}.$$

On the other hand we have obviously $b_2 - a_1 \leqslant \mathscr{H}^1(\mathscr{C}_k^N)$. Consequently we get

$$\begin{aligned}
\mathscr{H}^1(\mathscr{C}_{2k-2+l}^{N+1})^2 &\leqslant \mathscr{H}^1(\mathscr{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \left(\frac{\mathscr{H}^1(\mathscr{C}_k^N) - \mathscr{H}^1(\mathscr{C}_k^N)^2}{2}\right)^2 \\
&\leqslant \mathscr{H}^1(\mathscr{C}_k^N)^4 \|f''\|_{L^\infty}^2 + \frac{\mathscr{H}^1(\mathscr{C}_k^N)^2}{4}.
\end{aligned}$$

Therefore

$$\mathscr{H}^1(\mathscr{C}_{2k-2+l}^{N+1}) \leqslant \frac{\mathscr{H}^1(\mathscr{C}_k^N)}{2}\sqrt{1 + 4\|f''\|_{L^\infty}^2 \mathscr{H}^1(\mathscr{C}_k^N)^2},$$

thus using (4.3) we get

$$\mathscr{H}^1(\mathscr{C}_{2k-2+l}^{N+1}) \leqslant \frac{2\mathscr{H}^1(\mathscr{C}_k^N)}{3}.$$

The last estimate gives (C.5) and thus (C.4) holds.

Step 2. We prove that $\liminf_{N\to\infty} \sum_{k=1}^{2^N} \mathscr{H}^1(\mathscr{C}_k^N) > 0$.

For $N \geqslant 1$, we let

$$c_N = \sum_{k=1}^{2^N} \mathscr{H}^1(\mathscr{C}_k^N).$$

The main ingredient in this step consists in noting that a son of $\mathscr{C}_k^N$ is a hypothenuse of a right angled triangle which admits a cathetus of length

$$\frac{\mathscr{H}^1(\mathscr{C}_k^N) - \mathscr{H}^1(\mathscr{C}_k^N)^2}{2}.$$

Consequently we have

$$\mathscr{H}^1(\mathscr{C}_{2k-1}^{N+1}) + \mathscr{H}^1(\mathscr{C}_{2k}^{N+1}) \geqslant \mathscr{H}^1(\mathscr{C}_k^N) - \mathscr{H}^1(\mathscr{C}_k^N)^2.$$

Thus, summing the previous inequality for $k = 1, ..., 2^N$ we get

$$\begin{aligned} c_{N+1} &= \sum_{k=1}^{2^N} \mathscr{H}^1(\mathscr{C}_{2k-1}^{N+1}) + \mathscr{H}^1(\mathscr{C}_{2k}^{N+1}) \geqslant \sum_{k=1}^{2^N} \mathscr{H}^1(\mathscr{C}_k^N)[1 - \mathscr{H}^1(\mathscr{C}_k^N)] \\ &\geqslant c_N(1-\mu_N) \geqslant c_N \left[1 - \left(\frac{2}{3}\right)^N\right]. \end{aligned}$$

By induction for $N \geqslant 2$

$$c_N \geqslant c_1 \prod_{k=1}^{N-1} \left[1 - \left(\frac{2}{3}\right)^k\right] = c_1 \times \exp\left[\sum_{k=1}^{N-1} \ln \left[1 - \left(\frac{2}{3}\right)^k\right]\right].$$

It is clear that $\liminf_N \sum_{l=1}^{N-1} \ln \left[1 - \left(\frac{2}{3}\right)^k\right] > -\infty$, thus $\liminf_N c_N > 0$.

Step 3. We prove (C.2).

Since for a connected component K_k^N of $\mathcal{K}_N$ and its chord $\mathscr{C}_k^N$ we have

$$\mathscr{H}^1(K_k^N) \geqslant \mathscr{H}^1(\mathscr{C}_k^N),$$

we obtain (C.2) from Step 2.

Appendix D. A fundamental ingredient in the construction of the $\tilde{\Psi}_N$'s

In this section we use the notation of Sections 4 and 5.

Lemma D.1. — *Let $\gamma \subset \widehat{AB}$ be a curve and let $\mathscr{C}$ be its chord. We let γ_1, γ_2 be the curves included in γ obtained by the induction construction represented Figure 4.1 (section 4.2). For $l = 1, 2$, we denote also by $\mathscr{C}_l$ the chord of γ_l and by T_l the right-angled triangle having $\mathscr{C}_l$ as side of the right-angle and having its hypothenuse included in $\mathscr{C}$.*

If $\mathscr{H}^1(\mathscr{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$, then the hypothenuses of the triangles T_1 and T_2 have their length strictly lower than $\dfrac{\mathscr{H}^1(\mathscr{C})}{2}$. In particular the triangles T_1 and T_2 are disjoint.

Remark D.2. — From (4.2), we know that $\mathscr{C}_0 = \mathscr{C}_1^0$ is such that $\mathscr{H}^1(\mathscr{C}_1^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$. From (C.3) we have that for $N \geqslant 1$ and $k \in \{1, ..., 2^N\}$ we have $\mathscr{H}^1(\mathscr{C}_k^N) < \mathscr{H}^1(\mathscr{C}_1^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$.

Therefore with the help of Lemma D.1, for $N \geqslant 1$, the triangles T_k^N's are pairwise disjoint.

Proof. — We model the statement by denoting $\{M, Q\}$ the set of endpoints of γ and N and P are points such that:

- M, N are the endpoints of γ_1,
- P, Q are the endpoints of γ_2.

We denote $\delta := \mathscr{H}^1([MQ]) = \mathscr{H}^1(\mathscr{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty}^2)^{-2}\}$.

We fix an orthonormal frame $\tilde{\mathcal{R}}$ with the origin in M, with the x-axis (MQ) and such that N, P, Q have respectively for coordinates (x_1, y_1), (x_2, y_2) and $(x_3, 0)$, where $0 < x_1 < x_2 < x_3$ and $y_1, y_2 > 0$.

By construction we have

$$x_1 = \frac{\delta - \delta^2}{2},\ x_2 = \frac{\delta + \delta^2}{2} \text{ and } x_3 = \delta.$$

Moreover, arguing as in the proof of Lemma C.4 we have (recall that $\widehat{AB}$ is the graph of a function f in an other orthonormal frame):

$$0 < y_1, y_2 \leqslant \delta^2 \|f''\|_{L^\infty}.$$

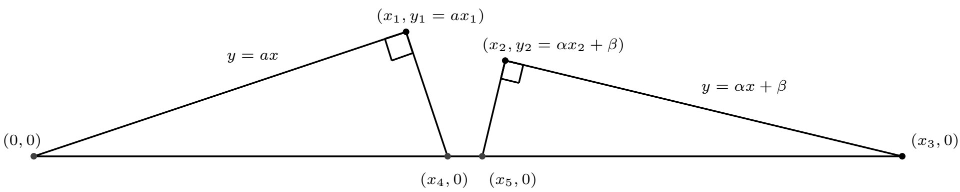

Figure D.1. Model problem

From these points, in Section 4.2, we defined two right-angled triangles having their hypothenuses contained in the x-axis.

The first triangle admits for vertices the origin $(0, 0)$, (x_1, y_1) and a point of the x-axis $(x_4, 0)$. This triangle is right angled in (x_1, y_1). In the frame $\tilde{\mathcal{R}}$, one of the side of the right-angle is included in the line parametrized by the cartesian equation $y = ax$. Since $\delta \leqslant 1/2$, we have

$$|a| = \left|\frac{y_1}{x_1}\right| \leqslant \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leqslant 4\|f''\|_{L^\infty}\delta.$$

The second triangle admits for vertices (x_2, y_2), $(x_3, 0)$ and a point of the x-axis $(x_5, 0)$. This triangle is right-angled in (x_2, y_2). In the frame $\tilde{\mathcal{R}}$, one of the side of the right-angle is included in the line parametrized by the cartesian equation $y = \alpha x + \beta$, where

$$|\alpha| = \left|\frac{y_2}{x_2 - x_3}\right| \leqslant \frac{2\delta^2 \|f''\|_{L^\infty}}{\delta - \delta^2} \leqslant 4\|f''\|_{L^\infty}\delta.$$

The proof of the proposition consists in obtaining

$$x_4 < \frac{x_3}{2} \text{ and } x_3 - x_5 < \frac{x_3}{2}.$$

We get the first estimate. With the help of Pythagorean theorem we have

$$x_1^2 + y_1^2 + (x_1 - x_4)^2 + y_1^2 = x_4^2.$$

By noting that $y_1 = ax_1$ we have

$$x_4 = (1 + a^2)x_1.$$

Thus:

$$\begin{aligned} x_4 < \frac{x_3}{2} \quad &\iff \quad (1+a^2)\frac{\delta-\delta^2}{2} < \frac{\delta}{2} \\ &\Longleftarrow \quad (1+16\|f''\|^2_{L^\infty}\delta^2)(1-\delta) < 1 \\ &\iff \quad \delta-\delta^2 < \frac{1}{16\|f''\|^2_{L^\infty}} \\ &\Longleftarrow \quad \delta < \frac{1}{16\|f''\|^2_{L^\infty}}. \end{aligned}$$

Following the same strategy we get that if $\delta < \dfrac{1}{16\|f''\|^2_{L^\infty}}$ then $x_3 - x_5 < \dfrac{x_3}{2}$. □

Appendix E. Adaptation of a result of Giusti in [2]

In this appendix we present briefly the proof of Theorem 2.16 and Remark 2.17 in [2]. The argument we present below follows the proof of Theorem 2.15 in [2].

Proposition E.1. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^2 and let $h \in L^1(\partial\Omega)$. For all $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,1}(\Omega)$ such that $\mathrm{tr}_{\partial\Omega} u_\varepsilon = h$ and*

$$\|u_\varepsilon\|_{W^{1,1}(\Omega)} := \|u_\varepsilon\|_{L^1(\Omega)} + \|\nabla u_\varepsilon\|_{L^1(\Omega)} \leqslant (1+\varepsilon)\|h\|_{L^1(\Omega)}.$$

Proof. — We sketch the proof of Proposition E.1. Let $h \in L^1(\partial\Omega)$ and let $\varepsilon > 0$ be sufficiently small such that

$$(1+\varepsilon^2)^2 + \varepsilon^2 + \varepsilon^4 < 1 + \frac{\varepsilon}{2} \text{ and } (1+\varepsilon^2)\varepsilon^2 < \frac{\varepsilon}{2}.$$

.

Step 1. We may consider $\eta > 0$ sufficiently small such that in $\Omega_\eta := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \eta\}$ we have:

(1) The function
$$\begin{array}{rccc} d: & \Omega_\eta & \to & (0,\eta) \\ & x & \mapsto & \mathrm{dist}(x,\partial\Omega) \end{array}$$
is of class C^1 and satisfies $|\nabla d| \geqslant 1/2$,

(2) The orthogonal projection on $\partial\Omega$, $\Pi_{\partial\Omega}$ is Lipschitz.

We now fix a sequence $(h_k)_k \subset C^\infty(\partial\Omega)$ such that $h_k \xrightarrow{L^1} h$. We may assume that (up to replace the first term and to consider an extraction):

(1) $h_0 \equiv 0$,
(2) $\sum_{k\geqslant 0} \|h_{k+1} - h_k\|_{L^1} \leqslant (1+\varepsilon^2)\|h\|_{L^1}$.

And finally we fix a decreasing sequence $(t_k)_k \subset \mathbb{R}^*_+$ such that

(1) $t_0 < \min(\eta, \varepsilon^2)$ is sufficiently small such that
   - $4t_0 \max(1; \|\nabla\Pi_{\partial\Omega}\|_{L^\infty}) \times \max(1, \sup_k \|h_k\|_{L^1}) < \min(\varepsilon^2, \varepsilon^2\|h\|_{L^1})$,
   - for $\varphi \in L^1(\partial\Omega)$ we have for $s \in (0, t_0)$
   $$\int_{d^{-1}(\{s\})} |\varphi \circ \Pi_{\partial\Omega}(x)| \leqslant (1+\varepsilon^2)\int_{\partial\Omega} |\varphi(x)|.$$

(2) For $k \geqslant 1$ we have $t_k \leqslant \dfrac{t_0\|h\|_{L^1}}{2^k(1 + \|\nabla h_k\|_{L^\infty} + \|\nabla h_{k+1}\|_{L^\infty})}$.

Step 2. We define a map $u_\varepsilon : \Omega \to \mathbb{R}$ by

$$x \mapsto \begin{cases} \dfrac{d(x) - t_{k+1}}{t_k - t_{k+1}} h_k \circ \Pi_{\partial\Omega}(x) + \dfrac{t_k - d(x)}{t_k - t_{k+1}} h_{k+1} \circ \Pi_{\partial\Omega}(x) & \text{if } d(x) \in [t_{k+1}, t_k) \\ 0 & \text{otherwise.} \end{cases}$$

We may easily check that u_ε is locally Lipschitz and thus weakly differentiable.

From the coarea formula and a standard change of variable we have

$$\begin{aligned}
\|u_\varepsilon\|_{L^1} &\leqslant 2 \int_{\{d \leqslant t_0\}} |u_\varepsilon||\nabla d| \\
&\leqslant 2 \int_0^{t_0} \mathrm{d}s \int_{d^{-1}(\{s\})} |u_\varepsilon| \mathrm{d}x \\
&\leqslant 2 \sum_{k \geqslant 0} \int_{t_{k+1}}^{t_k} \mathrm{d}s \int_{d^{-1}(\{s\})} |u_\varepsilon| \mathrm{d}x \\
&\leqslant 2 \sum_{k \geqslant 0} \int_{t_{k+1}}^{t_k} \mathrm{d}s \int_{d^{-1}(\{s\})} [|h_k \circ \Pi_{\partial\Omega}(x)| + |h_{k+1} \circ \Pi_{\partial\Omega}(x)|] \mathrm{d}x \\
&\leqslant 2(1+\varepsilon^2) \sum_{k \geqslant 0} \int_{t_{k+1}}^{t_k} \mathrm{d}s \int_{\partial\Omega} [|h_k(x)| + |h_{k+1}(x)|] \mathrm{d}x \\
&\leqslant 2(1+\varepsilon^2) \sum_{k \geqslant 0} (t_k - t_{k+1})(\|h_k\|_{L^1} + \|h_{k+1}\|_{L^1}) \\
&\leqslant 4(1+\varepsilon^2) t_0 \sup_k \|h_k\|_{L^1} \\
&\leqslant (1+\varepsilon^2)\varepsilon^2 \|h\|_{L^1} \\
&\leqslant \frac{\varepsilon}{2} \|h\|_{L^1}.
\end{aligned}$$

We now estimate $\|\nabla u_\varepsilon\|_{L^1}$. It is easy to check that if $d(x) \in (t_{k+1}, t_k)$ then we have

$$\begin{aligned}
|\nabla u_\varepsilon(x)| \leqslant |\nabla d(x)| \Big[& \frac{|h_k \circ \Pi_{\partial\Omega}(x) - h_{k+1} \circ \Pi_{\partial\Omega}(x)|}{t_k - t_{k+1}} \\
& + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} [|\nabla h_k| \circ \Pi_{\partial\Omega}(x) + |\nabla h_{k+1}| \circ \Pi_{\partial\Omega}(x)] \Big].
\end{aligned}$$

Consequently we get

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^1} \leqslant (1+\varepsilon^2) \sum_{k \geqslant 0} \Big\{ & \int_{t_{k+1}}^{t_k} \frac{\|h_{k+1} - h_k\|_{L^1}}{t_k - t_{k+1}} \\
& + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} (t_k - t_{k+1})(\|\nabla h_{k+1}\|_{L^1} + \|\nabla h_k\|_{L^1}) \Big\} \\
\leqslant (1+\varepsilon^2)&[(1+\varepsilon^2)\|h\|_{L^1} + 2\|\nabla \Pi_{\partial\Omega}\|_{L^\infty} t_0 \|h\|_{L^1}] \\
\leqslant (1+\varepsilon^2)&[(1+\varepsilon^2) + \varepsilon^2]\|h\|_{L^1} \\
\leqslant (1+\varepsilon/2)&\|h\|_{L^1}.
\end{aligned}$$

Consequently $u_\varepsilon \in W^{1,1}(\Omega)$ and $\|u_\varepsilon\|_{W^{1,1}} \leqslant (1+\varepsilon)\|h\|_{L^1}$.

In order to end the proof it suffices to check that $\mathrm{tr}_{\partial\Omega}(u_\varepsilon) = h$. The justification of this property follows the argument of Lemma 2.4 in [2]. $\square$

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