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On the Calogero–Moser space associated with dihedral groups

Cédric Bonnafé

Abstract

We investigate some geometric properties of the Calogero–Moser spaces associated with a dihedral group. As a consequence, we check in this particular case some conjectures made by the author and Raphaël Rouquier about general Calogero–Moser spaces.

Using the geometry of the associated Calogero–Moser space, R. Rouquier and the author [6] have attached to any finite complex reflection group W several notions (*Calogero–Moser left, right or two-sided cells*, *Calogero–Moser cellular characters*), completing the notion of *Calogero–Moser families* defined by Gordon [13]. If moreover W is a Coxeter group, it is conjectured in [6, Chpt. 15] that these notions coincide with the analogous notions defined using the Hecke algebra by Kazhdan and Lusztig (or Lusztig in the unequal parameters case).

In the present paper, we aim to investigate these conjectures whenever W is a *dihedral group*. Since they are all about the geometry of the Calogero–Moser space, we also study some conjectures in [6, Chpt. 16] about the fixed point subvariety under the action of a group of roots of unity, as well as some other aspects (presentation of the algebra of regular functions; cuspidal points as defined by Bellamy [3] and their associated Lie algebra). We do not prove all the conjectures but we get at least the following results (here, W is a dihedral group of order $2d$, acting on a complex vector space V of dimension 2; we denote by $\mathcal{Z}$ its associated Calogero–Moser space as in [6] and by Z the algebra of regular functions on $\mathcal{Z}$: see Section 3.3 for the definition):

- Calogero–Moser cellular characters and Kazhdan–Lusztig cellular characters coincide (this is [6, Conj. CAR]).
- Calogero–Moser families and Kazhdan–Lusztig families coincide (this is [14, Conj. 1.3]; see also [6, Conj. FAM]). This result is not new: it was already proved by Bellamy [2], but we propose a slightly different proof, based on the computation of cellular characters.
- We give a presentation of $\mathbb{C}[V \times V^*]^W$ (this extends the results of [1] which deal with the case where $d \in \{3,4,6\}$). Using [7], we explain how one could derive from this a presentation of Z: this is done completely only for $d \in \{3,4,6\}$.

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- If d is odd, then Calogero–Moser (left, right or two-sided) cells coincide with the Kazhdan–Lusztig (left, right or two-sided) cells: this is a particular case of [6, Conjs. L and LR]. For proving this fact, we prove that the Galois group defined in [6, Chpt. 5] is equal to the symmetric group $\mathfrak{S}_W$ on the set W.

- If d is odd, then we prove [6, Conj. FIX] about the fixed point subvariety $\mathcal{Z}^{\boldsymbol{\mu}_d}$.

We also investigate special cases using calculations with the software `MAGMA` [8], based on the `MAGMA` package `CHAMP` developed by Thiel [16], and a paper in preparation by Thiel and the author [7]. For instance, we get:

- If $d \in \{3, 4, 6\}$, then we prove [6, Conj. FIX] about the fixed point subvariety $\mathcal{Z}^{\boldsymbol{\mu}_m}$ (for any m).

- If $d = 4$ and the parameters are equal and non-zero (respectively $d = 6$ and the parameters are generic) and if $\mathfrak{m}$ is a Poisson maximal ideal of Z, then we prove that the Lie algebra $\mathfrak{m}/\mathfrak{m}^2$ is isomorphic to $\mathfrak{sl}_3(\mathbb{C})$ (respectively $\mathfrak{sp}_4(\mathbb{C})$). We believe these intriguing examples have their own interest.

Notation. We set $V = \mathbb{C}^2$ and we denote by (x, y) the canonical basis of V and by (X, Y) the dual basis of V^*. We identify $\mathbf{GL}_{\mathbb{C}}(V)$ with $\mathbf{GL}_2(\mathbb{C})$. We also fix a non-zero natural number d, as well as a primitive d-th root of unity $\zeta \in \mathbb{C}^\times$. If $i \in \mathbb{Z}/d\mathbb{Z}$, we denote by ζ^i the element ζ^{i_0}, where i_0 is any representative of i in $\mathbb{Z}$.

We denote by $\mathbb{C}[V]$ the algebra of polynomial functions on V (so that $\mathbb{C}[V] = \mathbb{C}[X, Y]$ is a polynomial ring in two variables) and by $\mathbb{C}(V)$ its fraction field (so that $\mathbb{C}(V) = \mathbb{C}(X, Y)$). We will denote by $\otimes$ the tensor product $\otimes_{\mathbb{C}}$.

1. The dihedral group

1.1. Generators

If $i \in \mathbb{Z}$, we set

$$s_i = \begin{pmatrix} 0 & \zeta^i \\ \zeta^{-i} & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{cases} s = s_0, \\ t = s_1. \end{cases}$$

Note that $s_i = s_{i+d}$ is a reflection of order 2 for all $i \in \mathbb{Z}$ (so that we can write s_i for $i \in \mathbb{Z}/d\mathbb{Z}$). We set

$$W = \langle s, t \rangle.$$

Then W is a dihedral group of order $2d$, and $(W, \{s, t\})$ is a Coxeter system, where

$$s^2 = t^2 = (st)^d = 1.$$

If we need to emphasize the natural number d, we will denote by W_d the group W.

The Coxeter element w_c is given by

$$w_c = ts = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

so that the following equalities are easily checked (for all $i, j \in \mathbb{Z}$)

$$w_c s_i w_c^{-1} = s_{i+2} \qquad \text{and} \qquad s_i s_j = w_c^{i-j}. \tag{1.1}$$

It then follows that

$$s \text{ and } t \text{ are conjugate in } W \text{ if and only if } d \text{ is odd.} \tag{1.2}$$

Note that

$$W = \{w_c^i \mid i \in \mathbb{Z}/d\mathbb{Z}\} \,\dot{\cup}\, \{s_i \mid i \in \mathbb{Z}/d\mathbb{Z}\}. \tag{1.3}$$

The set $\mathrm{Ref}(W)$ of reflections of W is equal to

$$\mathrm{Ref}(W) = \{s_i \mid i \in \mathbb{Z}/d\mathbb{Z}\}. \tag{1.4}$$

Now, let

$$\alpha_i^\vee = \zeta^i x - y \qquad \text{and} \qquad \alpha_i = X - \zeta^i Y,$$

so that

$$s_i(\alpha_i^\vee) = -\alpha_i^\vee \qquad \text{and} \qquad s_i(\alpha_i) = -\alpha_i. \tag{1.5}$$

Finally, we fix a primitive $2d$-th root of unity ξ such that $\xi^2 = \zeta$ and we set

$$\tau = \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix}.$$

Then it is readily seen that

$$\tau s \tau^{-1} = t, \qquad \tau t \tau^{-1} = s \qquad \text{and} \qquad \tau^2 = 1, \tag{1.6}$$

so that $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$.

Remark 1.1. If $d = 2e - 1$ is odd, then $\xi = -\zeta^e$ and so $\tau = -s_e$ induces an inner automorphism of W (the conjugacy by s_e). If d is even, then s and t are not conjugate in W and so τ induces a non-inner automorphism of W.

1.2. **Irreducible characters**

We denote by $\mathbb{1}_W$ the trivial character of W and let $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$. If d is even, then there exist two other linear characters ε_s and ε_t which are characterized by the following properties:

$$\begin{cases} \varepsilon_s(s) = \varepsilon_t(t) = -1, \\ \varepsilon_s(t) = \varepsilon_t(s) = 1. \end{cases}$$

If $k \in \mathbb{Z}$, we set

$$\begin{array}{rccc} \rho_k : & W & \longrightarrow & \mathbf{GL}_2(\mathbb{C}) \\ & s_i & \longmapsto & s_{ki} \\ & w_c^i & \longmapsto & w_c^{ki}. \end{array}$$

It is easily checked from (1.1) that ρ_k is a morphism of groups (that is, a representation of W). If R is any $\mathbb{C}$-algebra, we still denote by $\rho_k : RW \to \mathrm{Mat}_2(R)$ the morphism of algebras induced by ρ_k. The character afforded by ρ_k is denoted by χ_k. The following proposition is well known:

Proposition 1.2. *Let $k \in \mathbb{Z}$. Then:*

(1) $\chi_k = \chi_{-k} = \chi_{k+d}$.

(2) *If d is odd and $k \not\equiv 0 \mod d$, then χ_k is irreducible.*

(3) *If d is even and $k \not\equiv 0$ or $d/2 \mod d$, then χ_k is irreducible.*

(4) $\chi_0 = \mathbb{1}_W + \varepsilon$ *and, if d is even, then* $\chi_{d/2} = \varepsilon_s + \varepsilon_t$.

Corollary 1.3. *Recall that τ is the element of $\mathrm{N}_{\mathbf{GL}_\mathbb{C}(V)}(W)$ defined in Section 1.1.*

(1) *If d is odd, then $|\mathrm{Irr}(W)| = (d+3)/2$ and*

$$\mathrm{Irr}(W) = \{\mathbb{1}_W, \varepsilon\} \,\dot\cup\, \{\chi_k \mid 1 \leqslant k \leqslant (d-1)/2\}.$$

Moreover, τ acts trivially on $\mathrm{Irr}(W)$.

(2) *If d is even, then $|\mathrm{Irr}(W)| = (d+6)/2$ and*

$$\mathrm{Irr}(W) = \{\mathbb{1}_W, \varepsilon, \varepsilon_s, \varepsilon_t\} \,\dot\cup\, \{\chi_k \mid 1 \leqslant k \leqslant (d-2)/2\}.$$

Moreover, ${}^\tau\chi = \chi$ if $\chi \in \mathrm{Irr}(W) \setminus \{\varepsilon_s, \varepsilon_t\}$ while ${}^\tau\varepsilon_s = \varepsilon_t$.

1.3. Some fractions in two variables

We work in the fraction field $\mathbb{C}(V) = \mathbb{C}(X, Y)$. If $1 \leqslant k \leqslant d$, then

$$\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{\zeta^{ki}}{X - \zeta^i} = \frac{dX^{k-1}}{X^d - 1}. \tag{1.7}$$

$$\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{\zeta^{ki}}{X - \zeta^i Y} = \frac{dX^{k-1}Y^{d-k}}{X^d - Y^d}. \tag{1.8}$$

Proof. Let us first prove (1.7). Since $1 \leqslant k \leqslant d$, there exist complex numbers $(\xi_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ such that

$$\frac{dX^{k-1}}{X^d - 1} = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{\xi_i}{X - \zeta^i}.$$

Then

$$\begin{aligned} \xi_i = \lim_{z \to \zeta^i} \frac{dz^{k-1}(z - \zeta^i)}{z^d - 1} &= d\zeta^{(k-1)i} \prod_{\substack{j \in \mathbb{Z}/d\mathbb{Z} \\ j \neq i}} (\zeta^i - \zeta^j)^{-1} \\ &= d\zeta^{(k-1)i-(d-1)i} \prod_{j=1}^{d-1} (1 - \zeta^j)^{-1} = \zeta^{ki}, \end{aligned}$$

and (1.7) is proved.

Now, (1.8) follows easily from (1.7) by replacing X by X/Y. □

If $d = 2e$ is even and $1 \leqslant k \leqslant e$, then

$$\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \frac{\zeta^{2ki}}{X - \zeta^{2i} Y} = \frac{eX^{k-1}Y^{e-k}}{X^e - Y^e}, \tag{1.9}$$

$$\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \frac{\zeta^{k(2i+1)}}{X - \zeta^{2i+1} Y} = -\frac{eX^{k-1}Y^{e-k}}{X^e + Y^e}, \tag{1.10}$$

$$\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \frac{\zeta^{-(k-1)(2i+1)}}{X - \zeta^{2i+1} Y} = \frac{eX^{e-k}Y^{k-1}}{X^e + Y^e}, \tag{1.11}$$

Proof. The equality (1.9) follows from (1.8) by replacing ζ by ζ^2 and d by e. The equality (1.10) follows from (1.9) by replacing Y by ζY (note that $\zeta^e = -1$). Finally, the equality (1.11) follows from (1.10) by replacing k by $e - k + 1$ (note that $\zeta^{-(k-1)(2i+1)} = -(\zeta^e)^{2i+1}\zeta^{-(k-1)(2i+1)} = -\zeta^{(e-k+1)(2i+1)}$). □

2. **Invariants**

The aim of this section is to describe generators and relations for the invariant algebra $\mathbb{C}[V \times V^*]^W$. Note that such results have been obtained if $d \in \{3, 4, 6\}$ in [1]. We set

$$q = xy, \qquad r = x^d + y^d, \qquad Q = XY \qquad \text{and} \qquad R = X^d + Y^d.$$

Then

$$\mathbb{C}[V]^W = \mathbb{C}[Q, R] \qquad \text{and} \qquad \mathbb{C}[V^*]^W = \mathbb{C}[q, r].$$

We set $P_\bullet = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W = \mathbb{C}[q, r, Q, R] \subset \mathbb{C}[V \times V^*]^W$. If $0 \leqslant i \leqslant d$, we set

$$\mathbf{a}_{i,0} = x^{d-i} Y^i + y^{d-i} X^i.$$

Note that

$$\mathbf{a}_{0,0} = r \qquad \text{and} \qquad \mathbf{a}_{d,0} = R.$$

Finally, let

$$\mathbf{eu}_0 = xX + yY.$$

Then $\mathbf{a}_{i,0}, \mathbf{eu}_0 \in \mathbb{C}[V \times V^*]^W$.

We will now describe some relations between these invariants. For this, let $\mathbf{eu}_0^{(i)} = (xX)^i + (yY)^i$. Then the $\mathbf{eu}_0^{(i)}$'s belong also to $\mathbb{C}[V \times V^*]^W$. As they will appear in relations between generators of $\mathbb{C}[V \times V^*]^W$, we must explain how to express them as polynomials in $\mathbf{eu}_0$. First of all,

$$\begin{aligned}
\mathbf{eu}_0^i &= \sum_{j=0}^{i} \binom{i}{j} (xX)^j (yY)^{i-j} \\
&= \sum_{0 \leqslant j < \frac{i}{2}} \binom{i}{j} (qQ)^j \big((xX)^{i-2j} + (yY)^{i-2j}\big) + \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \binom{i}{\frac{i}{2}} (qQ)^{i/2} & \text{if } i \text{ is even.} \end{cases}
\end{aligned}$$

Therefore,

$$\mathbf{eu}_0^i = \sum_{0 \leqslant j < \frac{i}{2}} \binom{i}{j} (qQ)^j \, \mathbf{eu}_0^{(i-2j)} + \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \binom{i}{\frac{i}{2}} (qQ)^{i/2} & \text{if } i \text{ is even.} \end{cases}$$

So, by triangularity of this formula, an easy induction shows that there exists a family of integers $(n_{i,j})_{0 \leqslant j \leqslant i/2}$ such that

$$\mathbf{eu}_0^{(i)} = \sum_{0 \leqslant j \leqslant \frac{i}{2}} n_{i,j} (qQ)^j \mathbf{eu}_0^{i-2j}, \tag{2.1}$$

with $n_{i,0} = 1$ for all i.

On the other hand, one can check that the following relations hold (for $1 \leqslant i \leqslant j \leqslant d-1$):

$$\begin{cases} (\mathrm{Z}_i^0)\ \mathbf{eu}_0\mathbf{a}_{i,0} = q\mathbf{a}_{i+1,0} + Q\mathbf{a}_{i-1,0} \\ (\mathrm{Z}_{i,j}^0)\ \mathbf{a}_{i,0}\mathbf{a}_{j,0} = q^{d-j}Q^i\mathbf{eu}_0^{(j-i)} + \begin{cases} r\mathbf{a}_{i+j,0} - q^{d-i-j}\,\mathbf{eu}_0^{(i+j)} & \text{if } 2 \leqslant i+j \leqslant d, \\ rR - \mathbf{eu}_0^{(d)} & \text{if } i+j=d, \\ R\mathbf{a}_{i+j-d,0} - Q^{i+j-d}\mathbf{eu}_0^{(2d-i-j)} & \text{if } d \leqslant i+j \leqslant 2d-2. \end{cases} \end{cases}$$

Using (2.1), these last relations can be viewed as relations between q, r, Q, R, $\mathbf{eu}_0$, $\mathbf{a}_{1,0}$, $\mathbf{a}_{2,0}, \ldots, \mathbf{a}_{d-1,0}$.

Theorem 2.1. *The algebra of invariants $\mathbb{C}[V \times V^*]^W$ admits the following presentation:*

$$\begin{cases} \textit{Generators:} & q, r, Q, R, \mathbf{eu}_0, \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \ldots, \mathbf{a}_{d-1,0} \\ \textit{Relations:} & \begin{cases} (\mathrm{Z}_i^0) & \textit{for } 1 \leqslant i \leqslant d-1, \\ (\mathrm{Z}_{i,j}^0) & \textit{for } 1 \leqslant i \leqslant j \leqslant d-1. \end{cases} \end{cases}$$

This presentation is minimal, as well by the number of generators as by the number of relations (there are $d+4$ generators and $(d+2)(d-1)/2$ relations). Moreover, $\mathbb{C}[V \times V^*]^W$ is a free $P_\bullet$-module with basis $(1, \mathbf{eu}_0, \mathbf{eu}_0^2, \mathbf{eu}_0^d, \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \ldots, \mathbf{a}_{d-1,0})$.

Proof. Let H^* denote the subspace of $\mathbb{C}[V]$ defined by

$$H^* = \mathbb{C} \oplus \left(\bigoplus_{i=1}^{d-1} (\mathbb{C}X^i \oplus \mathbb{C}Y^i) \right) \oplus \mathbb{C}(X^d - Y^d).$$

Then H^* is a graded sub-$\mathbb{C}[W]$-module of $\mathbb{C}[V]$. If $0 \leqslant i \leqslant d$, let H_i^* denote the homogeneous component of degree i of H^*. Whenever $1 \leqslant i \leqslant d-1$, it affords χ_i for character whereas H_0^* and H_d^* afford respectively $\mathbb{1}_W$ and ε for characters. Similarly, we define

$$H = \mathbb{C} \oplus \left(\bigoplus_{i=1}^{d-1} (\mathbb{C}x^i \oplus \mathbb{C}y^i) \right) \oplus \mathbb{C}(x^d - y^d).$$

Then H is a graded sub-$\mathbb{C}[W]$-module of $\mathbb{C}[V^*]$. If $0 \leqslant i \leqslant d$, let H_i denote the homogeneous component of degree i of H. Whenever $1 \leqslant i \leqslant d-1$, it affords χ_i for character whereas H_0 and H_d afford respectively $\mathbb{1}_W$ and ε for characters. Moreover, the morphism of $\mathbb{C}[V]^W$-modules $\mathbb{C}[V]^W \otimes H^* \to \mathbb{C}[V]$ induced by the multiplication is a W-equivariant isomorphism and the morphism of $\mathbb{C}[V^*]^W$-module $\mathbb{C}[V^*]^W \otimes H \to \mathbb{C}[V^*]$ induced by the multiplication is a W-equivariant isomorphism. Consequently,

$$\mathbb{C}[V \times V^*]^W = P_\bullet \otimes (H^* \otimes H)^W. \tag{2.2}$$

An easy computation of the subspaces $(H_i^* \otimes H_j)^W$ based on the previous remarks show that

$$(1, \mathbf{eu}_0, \mathbf{eu}_0^{(2)}, \dots, \mathbf{eu}_0^{(d-1)}, (X^d - Y^d)(x^d - y^d), \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \dots, \mathbf{a}_{d-1,0})$$

is a $\mathbb{C}$-basis of $(H \otimes H^*)^W$. By (2.2), it is also a $P_\bullet$-basis of $\mathbb{C}[V \times V^*]^W$. On the other hand,

$$(X^d - Y^d)(x^d - y^d) = 2\mathbf{eu}_0^{(d)} - Rr,$$

so $(1, \mathbf{eu}_0, \mathbf{eu}_0^{(2)}, \dots, \mathbf{eu}_0^{(d)}, \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \dots, \mathbf{a}_{d-1,0})$ is a $P_\bullet$-basis of $\mathbb{C}[V \times V^*]^W$. By (2.1),

$$\mathbb{C}[V \times V^*]^W = P_\bullet \oplus P_\bullet\, \mathbf{eu}_0 \oplus P_\bullet\, \mathbf{eu}_0^2 \oplus \cdots \oplus P_\bullet\, \mathbf{eu}_0^d \oplus P_\bullet\, \mathbf{a}_{1,0} \oplus P_\bullet\, \mathbf{a}_{2,0} \oplus \cdots \oplus P_\bullet\, \mathbf{a}_{d-1,0},$$

which shows the last assertion of the theorem.

It also proves that $\mathbb{C}[V \times V^*]^W = \mathbb{C}[q, r, Q, R, \mathbf{eu}_0, \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \dots, \mathbf{a}_{d-1,0}]$. Let E, A_1, $A_2, \dots, A_{d-1}$ be indeterminates over $\mathbb{C}[q, r, Q, R]$: we have a surjective morphism

$$\mathbb{C}[q, r, Q, R, E, A_1, A_2, \dots, A_{d-1}] \twoheadrightarrow \mathbb{C}[V \times V^*]^W$$

which sends q, r, Q, R, E, A_1, $A_2, \dots, A_{d-1}$ on q, r, Q, R, $\mathbf{eu}_0$, $\mathbf{a}_{1,0}$, $\mathbf{a}_{2,0}, \dots, \mathbf{a}_{d-1,0}$ respectively.

For $1 \leqslant i \leqslant j \leqslant d-1$, let F_i (respectively $F_{i,j}$) denote the element of the polynomial algebra $\mathbb{C}[q, r, Q, R, E, A_1, A_2, \dots, A_{d-1}]$ corresponding to the relation (Z_i^0) (respectively $(Z_{i,j}^0)$). Let A denote the quotient of $\mathbb{C}[q, r, Q, R, E, A_1, A_2, \dots, A_{d-1}]$ by the ideal $\mathfrak{a}$ generated by the F_i's and the $F_{i,j}$'s. We denote by q_0, r_0, Q_0, R_0, E_0, $A_{1,0}$, $A_{2,0}, \dots, A_{d-1,0}$ the respective images of q, r, Q, R, E, A_1, $A_2, \dots, A_{d-1}$ in A. We then have a surjective morphism of bigraded $\mathbb{C}$-algebras $\varphi : A \twoheadrightarrow \mathbb{C}[V \times V^*]^W$. We want to show that φ is an isomorphism. For this, it is sufficient to show that the bi-graded Hilbert series coincide. But,

$$\dim_{\mathbb{C}}^{\mathrm{bigr}}(A) \geqslant \dim_{\mathbb{C}}^{\mathrm{bigr}}(\mathbb{C}[V \times V^*]^W), \tag{2.3}$$

where an inequality between two power series means that we have the corresponding inequality between all the coefficients.

We set $P_0 = \mathbb{C}[q_0, r_0, Q_0, R_0]$. Let

$$A' = P_0 + P_0E_0 + P_0E_0^2 + \cdots + P_0E_0^d + P_0A_{1,0} + P_0A_{2,0} + \cdots + P_0A_{d-1,0}.$$

By construction,

$$\dim_{\mathbb{C}}^{\mathrm{bigr}}(A') \leqslant \dim_{\mathbb{C}}^{\mathrm{bigr}}(\mathbb{C}[V \times V^*]^W). \tag{2.4}$$

We will prove that

$$A' \text{ is a subalgebra of } A. \tag{2.5}$$

For this, taking into account the form of the F_i's and the $F_{i,j}$'s, it is sufficient to show that $E_0^{d+1} \in A'$. But, by (2.1),

$$A_{1,0}A_{d-1,0} = Q_0q_0\left(\sum_{0\leqslant j<(d-2)/2} n_{d-2,j}E_0^{d-2-2j}\right) + R_0r_0 - \sum_{0\leqslant j<d/2} n_{d,j}E_0^{d-2j},$$

and $n_{d,0} = 1$. So

$$E_0^d = -A_{1,0}A_{d-1,0} + Q_0q_0\left(\sum_{0\leqslant j<(d-2)/2} n_{d-2,j}E_0^{d-2-2j}\right) + R_0r_0 - \sum_{1\leqslant j<d/2} n_{d,j}E_0^{d-2j}$$

and so

$$E_0^{d+1} = -E_0A_{1,0}A_{d-1,0} + Q_0q_0\left(\sum_{0\leqslant j<(d-2)/2} n_{d-2,j}E_0^{d-1-2j}\right) + R_0r_0 - \sum_{1\leqslant j<d/2} n_{d,j}E_0^{d+1-2j}.$$

It is then sufficient to show that $E_0A_{1,0}A_{d-1,0} \in A'$. But $E_0A_{1,0}A_{d-1,0} = (q_0A_{2,0} + Q_0r_0)A_{d-1,0}$, which concludes the proof of (2.5).

Since A' contains q_0, r_0, Q_0, R_0, E_0, $A_{1,0}$, $A_{2,0}, \dots, A_{d-1,0}$ and since A is generated by these elements, we have $A = A'$. It follows from (2.3) and (2.4) that $\dim_{\mathbb{C}}^{\mathrm{bigr}}(A) = \dim_{\mathbb{C}}^{\mathrm{bigr}}(\mathbb{C}[V\times V^*]^W)$, which shows that $\varphi : A \to \mathbb{C}[V\times V^*]^W$ is an isomorphism. In other words, this shows that the presentation of $\mathbb{C}[V\times V^*]^W$ given in Theorem 2.1 is correct.

It remains to prove the minimality of this presentation. The minimality of the number of generators follows from [7]. Let us now prove the minimality of the number of relations. For this, let $\mathfrak{p}_0$ denote the bi-graded maximal ideal of $P_\bullet$ and set $B = \mathbb{C}[V\times V^*]^W/\mathfrak{p}_0\mathbb{C}[V\times V^*]^W$. We denote by $e, a_1, a_2, \dots, a_{d-1}$ the respective images of $\mathbf{eu}_0, \mathbf{a}_{1,0}, \mathbf{a}_{2,0}, \dots, \mathbf{a}_{d-1,0}$ in B. Hence,

$$B = \mathbb{C} \oplus \mathbb{C}e \oplus \mathbb{C}e^2 \oplus \cdots \oplus \mathbb{C}e^d \oplus \mathbb{C}a_1 \oplus \mathbb{C}a_2 \oplus \cdots \oplus \mathbb{C}a_{d-1}$$

and B admits the following presentation:

$$\begin{cases} \text{Generators:} & e, a_1, a_2, \dots, a_{d-1} \\ \text{Relations } (1 \leqslant i \leqslant j \leqslant d-1): & \begin{cases} ea_i = 0 \\ a_ia_j = \begin{cases} 0 & \text{if } i+j \neq d, \\ -e^d & \text{if } i+j = d. \end{cases} \end{cases} \end{cases}$$

It is sufficient to prove that the number of relations of B is minimal. By reducing modulo the ideal Be, we get that all the relations of the form $a_ia_j = 0$ or $-e^d$ are necessary. By reducing modulo the ideal $Ba_1 + \cdots + Ba_{i-1} + Ba_{i+1} + \cdots Ba_{d-1}$, we get that the relations $ea_i = 0$ are necessary. □

Remark 2.2. It is easily checked that the element τ defined in Section 1.1 satisfies

$$^{\tau}q = q, \quad ^{\tau}Q = Q, \quad ^{\tau}\mathbf{eu}_0 = \mathbf{eu}_0 \qquad \text{and} \qquad ^{\tau}\mathbf{a}_{i,0} = -\mathbf{a}_{i,0}$$

for all $0 \leqslant i \leqslant d$ (this follows from the fact that $\xi^d = -1$).

Remark 2.3 (Gradings). The algebra $\mathbb{C}[V \times V^*]$ admits a natural $(\mathbb{N} \times \mathbb{N})$-grading, by putting elements of V in bi-degree $(1, 0)$ and elements of V^* in bi-degree $(0, 1)$. This bi-grading is stable under the action of W, so $\mathbb{C}[V \times V^*]^W$ inherits this bi-grading. Note that the generators and the relations given by Theorem 2.1 are bi-homogeneous.

This $(\mathbb{N} \times \mathbb{N})$-grading induces a $\mathbb{Z}$-grading such that any bi-homogeneous element of bi-degree (m, n) is $\mathbb{Z}$-homogeneous of degree $n - m$ (in other words, elements of V have $\mathbb{Z}$-degree -1 while elements of V^* have $\mathbb{Z}$-degree 1).

3. Cherednik algebras

3.1. Definition

We denote by $\boldsymbol{C}$ the $\mathbb{C}$-vector space of maps $\mathrm{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes. If $i \in \mathbb{Z}/d\mathbb{Z}$, we denote by C_i the element of $\boldsymbol{C}^*$ which sends $c \in \boldsymbol{C}$ to c_{s_i}. By (1.1), $C_i = C_{i+2}$. Let $A = C_0$ and $B = C_1$. If d is odd, then $A = B$ and $\mathbb{C}[\boldsymbol{C}] = \mathbb{C}[A]$ whereas, if d is even, then $A \neq B$ (see (1.2)) and $\mathbb{C}[\boldsymbol{C}] = \mathbb{C}[A, B]$.

The *generic rational Cherednik algebra at $t = 0$* is the $\mathbb{C}[\boldsymbol{C}]$-algebra $\mathbf{H}$ defined as the quotient of $\mathbb{C}[\boldsymbol{C}] \otimes \big(\mathrm{T}(V \oplus V^*) \rtimes W\big)$ by the following relations (here, $\mathrm{T}(V \oplus V^*)$ is the tensor algebra of $V \oplus V^*$ over $\mathbb{C}$):

$$\begin{cases} [u, u'] = [U, U'] = 0, \\ [u, U] = -2\sum_{i \in \mathbb{Z}/d\mathbb{Z}} C_i \, \frac{\langle u, \alpha_i \rangle \cdot \langle \alpha_i^\vee, U \rangle}{\langle \alpha_i^\vee, \alpha_i \rangle} \, s_i, \end{cases} \tag{3.1}$$

for $U, U' \in V^*$ and $u, u' \in V$. Note that we have followed the convention of [6].

Given the relations (3.1), the following assertions are clear:

- There is a unique morphism of $\mathbb{C}$-algebras $\mathbb{C}[V] \to \mathbf{H}$ sending $U \in V^* \subset \mathbb{C}[V]$ to the class of $U \in \mathrm{T}(V \oplus V^*) \rtimes W$ in $\mathbf{H}$.

- There is a unique morphism of $\mathbb{C}$-algebras $\mathbb{C}[V^*] \to \mathbf{H}$ sending $u \in V \subset \mathbb{C}[V^*]$ to the class of $u \in \mathrm{T}(V \oplus V^*) \rtimes W$ in $\mathbf{H}$.

- There is a unique morphism of $\mathbb{C}$-algebras $\mathbb{C}W \to \mathbf{H}$ sending $w \in W$ to the class of $w \in \mathrm{T}(V \oplus V^*) \rtimes W$ in $\mathbf{H}$.

- The $\mathbb{C}$-linear map $\mathbb{C}[\boldsymbol{C}] \otimes \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \longrightarrow \mathbf{H}$ induced by the three morphisms defined above and the multiplication map is surjective. Note that it is $\mathbb{C}[\boldsymbol{C}]$-linear.

The last statement is strenghtened by the following fundamental result by Etingof and Ginzburg [11, Thm. 1.3] (see also [6, Thm. 4.1.2]).

Theorem 3.1 (Etingof–Ginzburg)**.** *The multiplication map* $\mathbb{C}[\boldsymbol{C}]\otimes\mathbb{C}[V]\otimes\mathbb{C}W\otimes\mathbb{C}[V^*] \longrightarrow \mathbf{H}$ *is an isomorphism of* $\mathbb{C}[\boldsymbol{C}]$*-modules.*

Remark 3.2. By [6, §3.5.C], the group $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ acts naturally on $\mathbf{H}$. It follows from (1.6) that

$$^{\tau}A = B \qquad \text{and} \qquad ^{\tau}B = A.$$

Here, τ is the element of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ defined in Section 1.1.

Remark 3.3 (Gradings). The algebra $\mathbf{H}$ admits a natural $(\mathbb{N}\times\mathbb{N})$-grading, by putting V in bi-degree $(1,0)$, V^* in bi-degree $(0,1)$, W in degree $(0,0)$ and $\boldsymbol{C}^*$ in degree $(1,1)$ (see for instance [6, §3.2]).

This $(\mathbb{N}\times\mathbb{N})$-grading induces a $\mathbb{Z}$-grading such that any bi-homogeneous element of bi-degree (m,n) is $\mathbb{Z}$-homogeneous of degree $n-m$. In other words, $\deg(V) = -1$, $\deg(V^*) = 1$ and $\deg(W) = 0 = \deg(\boldsymbol{C}^*) = 0$.

3.2. Specialization

Given $c \in \boldsymbol{C}$, we denote by $\mathfrak{C}_c$ the maximal ideal of $\mathbb{C}[\boldsymbol{C}]$ defined by $\mathfrak{C}_c = \{f \in \mathbb{C}[\boldsymbol{C}] \mid f(c) = 0\}$: it is the ideal generated by $(C_i - c_{s_i})_{i\in\mathbb{Z}/d\mathbb{Z}}$. We set

$$\mathbf{H}_c = (\mathbb{C}[\boldsymbol{C}]/\mathfrak{C}_c) \otimes_{\mathbb{C}[\boldsymbol{C}]} \mathbf{H} = \mathbf{H}/\mathfrak{C}_c\mathbf{H}.$$

The $\mathbb{C}$-algebra $\mathbf{H}_c$ is the quotient of the $\mathbb{C}$-algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ by the ideal generated by the following relations:

$$\begin{cases} [u,u'] = [U,U'] = 0, \\ [u,U] = -2\sum_{i\in\mathbb{Z}/d\mathbb{Z}} c_{s_i} \frac{\langle u,\alpha_i\rangle\cdot\langle\alpha_i^\vee,U\rangle}{\langle\alpha_i^\vee,\alpha_i\rangle} s_i, \end{cases} \tag{3.2}$$

for U, $U' \in V^*$ and u, $u' \in V$.

Remark 3.4 (Grading). The ideal $\mathfrak{C}_c$ is not bi-homogeneous (except if $c = 0$) so the algebra $\mathbf{H}_c$ does not inherit from $\mathbf{H}$ an $(\mathbb{N}\times\mathbb{N})$-grading. However, $\mathfrak{C}_c$ is $\mathbb{Z}$-homogeneous, so $\mathbf{H}_c$ still admits a natural $\mathbb{Z}$-grading.

3.3. **Calogero–Moser space**

We denote by Z the centre of $\mathbf{H}$. By [11], it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ so, by Theorem 3.1, it contains the subalgebra

$$P = \mathbb{C}[\boldsymbol{C}] \otimes \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W.$$

Similarly, if $c \in \boldsymbol{C}$, we denote by Z_c the centre of $\mathbf{H}_c$: it turns out [6, Cor. 4.2.7] that Z_c is the image of Z and that the image of P in Z_c is $\boldsymbol{P}_\bullet = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$. Recall also (see for instance [6, Cor. 4.2.7]) that

$$Z \text{ is a free } P\text{-module of rank } |W| \tag{3.3}$$

and that [11]

$$Z \text{ and } Z_c \text{ are integral and integrally closed.} \tag{3.4}$$

Since the $\mathbb{C}$-algebra Z is finitely generated, we can associate to it an irreducible and normal (according to (3.4)) algebraic variety over $\mathbb{C}$, called the *generic Calogero–Moser space*, and which will be denoted by $\mathcal{Z}$. If $c \in \boldsymbol{C}$, we denote by $\mathcal{Z}_c$ the algebraic variety associated with the $\mathbb{C}$-algebra Z_c.

3.4. **About the presentation of** Z

We follow here the method of [7]. If $h \in \mathbf{H}$, it follows from Theorem 3.1 that there exists a unique family of elements $(h_w)_{w\in W}$ of $\mathbb{C}[\boldsymbol{C}] \otimes \mathbb{C}[V] \otimes \mathbb{C}[V^*]$ such that

$$h = \sum_{w\in W} h_w w.$$

We define the $\mathbb{C}[\boldsymbol{C}]$-linear map Trunc : $\mathbf{H} \to \mathbb{C}[\boldsymbol{C}] \otimes \mathbb{C}[V] \otimes \mathbb{C}[V^*]$ by

$$\mathrm{Trunc}(h) = h_1.$$

The next lemma is proved in [7]:

Lemma 3.5. *The restriction of* Trunc *to* Z *yields an isomorphism of bi-graded* $\mathbb{C}[\boldsymbol{C}]$-*modules*

$$\mathrm{Trunc} : Z \xrightarrow{\sim} \mathbb{C}[\boldsymbol{C}] \otimes \mathbb{C}[V \times V^*]^W.$$

We then set $\mathbf{eu} = \mathrm{Trunc}^{-1}(\mathbf{eu}_0)$ and, for $0 \leqslant i \leqslant d$,

$$\mathbf{a}_i = \mathrm{Trunc}^{-1}(\mathbf{a}_{i,0}).$$

An explicit algorithm for computing the inverse map Trunc^{-1} is described in [7]. Note that $\mathrm{Trunc}^{-1}(p) = p$ for $p \in P$, so that $\mathbf{a}_0 = \mathbf{a}_{0,0} = r$ and $\mathbf{a}_d = \mathbf{a}_{d,0} = R$. By [7], the relations $(\mathrm{Z}_i^0)_{1\leqslant i\leqslant d-1}$ and $(\mathrm{Z}_{i,j}^0)_{1\leqslant i\leqslant j\leqslant d-1}$ can be deformed into relations $(\mathrm{Z}_i)_{1\leqslant i\leqslant d-1}$ and $(\mathrm{Z}_{i,j})_{1\leqslant i\leqslant j\leqslant d-1}$ and it follows from Theorem 2.1 and [7] that:

Theorem 3.6. *The centre Z of* $\mathbf{H}$ *admits the following presentation, as a* $\mathbb{C}[\boldsymbol{C}]$*-algebra:*

$$\begin{cases} \textit{Generators:} & q, r, Q, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d-1} \\ \textit{Relations:} & \begin{cases} (\mathrm{Z}_i) & \textit{for } 1 \leqslant i \leqslant d-1, \\ (\mathrm{Z}_{i,j}) & \textit{for } 1 \leqslant i \leqslant j \leqslant d-1. \end{cases} \end{cases}$$

This presentation is minimal, as well by the number of generators as by the number of relations (there are $d+4$ generators and $(d+2)(d-1)/2$ relations). Moreover,

$$Z = P \oplus P\,\mathbf{eu} \oplus P\,\mathbf{eu}^2 \oplus \cdots \oplus P\,\mathbf{eu}^d \oplus P\,\mathbf{a}_1 \oplus P\,\mathbf{a}_2 \oplus \cdots \oplus P\,\mathbf{a}_{d-1}.$$

It must be said that we have no way to determine explicitly the relations $(\mathrm{Z}_{i,j})$ in general: we will describe them precisely only for $d \in \{3,4,6\}$ in Section 8. Note that the information provided by Theorem 3.6 is sufficient enough to be able to prove Theorem 7.1 in Section 7.

Remark 3.7 (Gradings). The bi-grading and the $\mathbb{Z}$-grading on the algebra $\mathbf{H}$ constructed in Remark 3.3 induce a bi-grading and a $\mathbb{Z}$-grading on Z. Note that the map Trunc is bi-graded, so that the generators given in Theorem 3.6 are bi-homogeneous.

On the other hand, the deformation process for the relations described in [7] respects the bi-grading. So we may, and we will, assume in the rest of this paper that the relations (Z_i) and $(\mathrm{Z}_{i,j})$ given in Theorem 3.6 are bi-homogeneous.

Note. From now on, and until the end of this paper, we fix a parameter $c \in \boldsymbol{C}$ and we set $a = c_s$ and $b = c_t$. Note that, if d is odd, then $a = b$.

3.5. **Poisson bracket**

Recall from [6, §4.4.A] that the algebra Z is endowed with a $\mathbb{C}[\boldsymbol{C}]$-linear Poisson bracket

$$\{\cdot,\cdot\} : Z \times Z \longrightarrow Z,$$

which is a deformation of the Poisson bracket on $Z_0 = \mathbb{C}[V \times V^*]^W$ obtained by restriction of the W-equivariant canonical Poisson bracket on $\mathbb{C}[V \times V^*]$. This Poisson bracket induces a Poisson bracket on Z_c. It satisfies

$$\{q, Q\} = \mathbf{eu} \tag{3.5}$$

(see [9, §4] or [5, §3]).

4. **Calogero–Moser cellular characters**

The aim of this section is to determine, for all values of c, the Calogero–Moser c-cellular characters as defined in [6, §11.1]. It will be given in Table 4.1 at the end of this section.

We will use the alternative definition [6, Thm. 13.4.2], which is more convenient for computational purposes (see also [7]). So, following [6, Chpt. 13], we denote by $\mathbb{C}(V)W$ the group algebra of W over the field $\mathbb{C}(V)$ and we set

$$D_x = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \varepsilon(s_i) c_{s_i} \frac{\langle x, \alpha_i \rangle}{\alpha_i} s_i = - \sum_{i \in \mathbb{Z}/d\mathbb{Z}} c_{s_i} \frac{1}{X - \zeta^i Y} s_i \in \mathbb{C}(V)W$$

$$\text{and} \qquad D_y = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \varepsilon(s_i) c_{s_i} \frac{\langle y, \alpha_i \rangle}{\alpha_i} s_i = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} c_{s_i} \frac{\zeta^i}{X - \zeta^i Y} s_i \in \mathbb{C}(V)W.$$

We denote by $\mathrm{Gau}(W, c)$ the sub-$\mathbb{C}(V)$-algebra of $\mathbb{C}(V)W$ generated by D_x and D_y (it is commutative by [6, §13.4.B]). Note that this algebra is not necessarily split. If L is a simple $\mathrm{Gau}(W, c)$-module, and if $\chi \in \mathrm{Irr}(W)$, we denote by $\mathrm{mult}_{L,\chi}^{\mathrm{CM}}$ the multiplicity of L in a composition series of the module $\mathrm{Res}_{\mathrm{Gau}(W,c)}^{\mathbb{C}(V)W} \mathbb{C}(V)E_\chi$, where E_χ is a $\mathbb{C}W$-module affording the character χ. We then set

$$\gamma_L = \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{mult}_{L,\chi}^{\mathrm{CM}} \ \chi.$$

The set of *Calogero–Moser c-cellular characters* is

$$\mathrm{CellChar}_c^{\mathrm{CM}}(W) = \{\gamma_L \mid L \in \mathrm{Irr}(\mathrm{Gau}(W, c))\}.$$

We will denote by $\mathcal{E}_1^{\mathrm{Gau}}$ (respectively $\mathcal{E}_\varepsilon^{\mathrm{Gau}}$, respectively $\mathcal{L}_k^{\mathrm{Gau}}$) the restriction of $\mathbb{C}(V)E_{\mathbb{1}_W}$ (respectively $\mathbb{C}(V)E_\varepsilon$, respectively $\mathbb{C}(V)E_{\chi_k}$) to $\mathrm{Gau}(W, c)$. If d is even, then the restriction of $\mathbb{C}(V)E_{\varepsilon_s}$ (respectively $\mathbb{C}(V)E_{\varepsilon_t}$) to $\mathrm{Gau}(W, c)$ will be denoted by $\mathcal{E}_s^{\mathrm{Gau}}$ (respectively $\mathcal{E}_t^{\mathrm{Gau}}$).

Remark 4.1. Note that, since $\mathrm{Gau}(W, c) \subset \mathbb{C}(V)W$, every simple $\mathrm{Gau}(W, c)$-module occurs as a composition factor of some $\mathrm{Res}_{\mathrm{Gau}(W,c)}^{\mathbb{C}(V)W} \mathbb{C}(V)E_\chi$, for χ running over $\mathrm{Irr}(W)$.

Notation 4.2. In order to simplify the computation in this section, we set

$$D'_x = \frac{Y^d - X^d}{d} D_x \qquad \text{and} \qquad D'_y = \frac{X^d - Y^d}{d} D_y,$$

so that $\mathrm{Gau}(W, c)$ is the sub-$\mathbb{C}(V)$-algebra of $\mathbb{C}(V)W$ generated by D'_x and D'_y.

4.1. The case where $a = b = 0$

Whenever $a = b = 0$, there is only one Calogero–Moser 0-cellular character [6, Cor. 17.2.3], namely the regular character $\sum_{\chi \in \mathrm{Irr}(W)} \chi(1)\chi$.

4.2. **The case where** $a \neq b = 0$

We will assume here, and only here, that $b = 0 \neq a$: this forces d to be even (and we write $d = 2e$). Let W' be the subgroup of W generated by $s = s_0$ and $s_2 = tst$. Then W' is a dihedral group of order $2e$ and

$$W = \langle t \rangle \ltimes W'.$$

Let c' denote the restriction of c to $\mathrm{Ref}(W')$: then c' is constant (and equal to a) and $\mathrm{Gau}(W,c) = \mathrm{Gau}(W',c')$. It then follows from the definition that the Calogero–Moser c-cellular characters are all the characters of the form $\mathrm{Ind}_{W'}^{W} \gamma'$, where γ' is a Calogero–Moser c'-cellular character of W'. These characters γ' will be determined in the next subsection and so it follows that the list of c-cellular characters of W is

$$1 + \varepsilon_t, \quad \varepsilon_s + \varepsilon, \quad \sum_{k=1}^{(d-2)/2} \chi_k. \tag{4.1}$$

Remark 4.3. If $a = 0$ and $b \neq 0$, then one can use the element $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ of order 2 such that ${}^{\tau}s = t$ and ${}^{\tau}t = s$ constructed in Section 1.1 to be sent back to the previous case. We then deduce from (4.1) that the list of c-cellular characters of W is

$$1 + \varepsilon_s, \quad \varepsilon_t + \varepsilon, \quad \sum_{k=1}^{(d-2)/2} \chi_k. \tag{4.2}$$

Note that we have a semi-direct product decomposition $W = \langle s \rangle \ltimes {}^{n}W'$.

4.3. **The equal parameters case**

We assume here, and only here, that $a = b \neq 0$. Now, if $1 \leqslant k \leqslant d-1$, then

$$\rho_k(D'_x) = -a\frac{Y^d - X^d}{d} \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{1}{X - \zeta^i Y} \begin{pmatrix} 0 & \zeta^{ki} \\ \zeta^{-ki} & 0 \end{pmatrix} = a \begin{pmatrix} 0 & X^{k-1}Y^{d-k} \\ X^{d-k-1}Y^k & 0 \end{pmatrix}$$

by using (1.8). Similarly,

$$\rho_k(D'_y) = a \begin{pmatrix} 0 & X^k Y^{d-k-1} \\ X^{d-k}Y^{k-1} & 0 \end{pmatrix} = \frac{X}{Y}\rho_k(D'_x).$$

If we denote by M the diagonal matrix $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$, then it follows from the previous formulas that

$$\forall\ 1 \leqslant k \leqslant d-2,\ \forall\ D \in \mathrm{Gau}(W,c),\ M\rho_k(D)M^{-1} = \rho_{k+1}(D). \tag{4.3}$$

This implies that

$$\forall\ 1 \leqslant k \leqslant d-2,\ \mathcal{L}_k^{\mathrm{Gau}} \simeq \mathcal{L}_{k+1}^{\mathrm{Gau}}. \tag{4.4}$$

Since $\mathrm{Tr}(\rho_k(D'_x)) = \mathrm{Tr}(\rho_k(D'_y)) = 0$, the nature of the restriction of ρ_k to $\mathrm{Gau}(W, c)$ depends on whether $-\det(\rho_k(D'_x)) = a^2 X^d Y^{d-2}$ is a square in $\mathbb{C}(V)$. Two cases may occur:

First case: assume that d is odd

Then $-\det(\rho_k(D'_x))$ is not a square in $\mathbb{C}(V)$ (for $1 \leqslant k \leqslant d-1$), so it follows that $\mathcal{L}_k^{\mathrm{Gau}}$ is simple (but not absolutely simple) and it follows from (4.4) that

$$\mathrm{Irr}(\mathrm{Gau}(W, c)) = \{\mathcal{E}_1^{\mathrm{Gau}}, \mathcal{E}_\varepsilon^{\mathrm{Gau}}, \mathcal{L}_1^{\mathrm{Gau}}\}. \tag{4.5}$$

Moreover, the list of c-cellular characters is given in this case by

$$\mathbb{1}_W, \quad \varepsilon \quad \text{and} \quad \sum_{k=1}^{(d-1)/2} \chi_k. \tag{4.6}$$

Second case: assume that d is even

In this case, it is easily checked that $\mathcal{E}_1^{\mathrm{Gau}}$, $\mathcal{E}_\varepsilon^{\mathrm{Gau}}$, $\mathcal{E}_s^{\mathrm{Gau}}$ and $\mathcal{E}_t^{\mathrm{Gau}}$ are four non-isomorphic simple $\mathrm{Gau}(W, c)$-modules. Also, if $1 \leqslant k \leqslant d-1$, then

$$\mathcal{L}_k^{\mathrm{Gau}} \simeq \mathcal{E}_s^{\mathrm{Gau}} \oplus \mathcal{E}_t^{\mathrm{Gau}},$$

by (4.4). Therefore,

$$\mathrm{Irr}(\mathrm{Gau}(W, c)) = \{\mathcal{E}_1^{\mathrm{Gau}}, \mathcal{E}_\varepsilon^{\mathrm{Gau}}, \mathcal{E}_s^{\mathrm{Gau}}, \mathcal{E}_t^{\mathrm{Gau}}\}. \tag{4.7}$$

and the list of c-cellular characters is given in this case by

$$\mathbb{1}_W, \quad \varepsilon, \quad \varepsilon_s + \sum_{k=1}^{(d-2)/2} \chi_k \quad \text{and} \quad \varepsilon_t + \sum_{k=1}^{(d-2)/2} \chi_k. \tag{4.8}$$

4.4. The opposite parameters case

We assume here, and only here, that $b = -a \neq 0$. This forces d to be even. Then, using the automorphism of $\mathbf{H}$ induced by the linear character ε_s (see [6, §3.5.B]), one can pass from the equal parameter case to the opposite parameter case by tensorizing by ε_s. Therefore, the list of c-cellular characters is given in this case by

$$\varepsilon_s, \quad \varepsilon_t, \quad \mathbb{1}_W + \sum_{k=1}^{(d-2)/2} \chi_k \quad \text{and} \quad \varepsilon + \sum_{k=1}^{(d-2)/2} \chi_k. \tag{4.9}$$

4.5. **The generic case**

We assume here, and only here, that $ab(a^2 - b^2) \neq 0$ (so that we are not in the cases covered by the previous subsections). Note that this forces d to be even. We will prove that the list of c-cellular characters is given in this case by

$$\mathbb{1}_W, \quad \varepsilon, \quad \varepsilon_s, \quad \varepsilon_t \quad \text{and} \quad \sum_{k=1}^{(d-2)/2} \chi_k. \tag{4.10}$$

Proof. We have, for $1 \leqslant k \leqslant e-1$,

$$\rho_k(D'_x) = \frac{X^d - Y^d}{d}\left(a \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \frac{1}{X - \zeta^{2i}Y} \begin{pmatrix} 0 & \zeta^{2ki} \\ \zeta^{-2ki} & 0 \end{pmatrix} + b \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \frac{1}{X - \zeta^{2i+1}Y} \begin{pmatrix} 0 & \zeta^{k(2i+1)} \\ \zeta^{-k(2i+1)} & 0 \end{pmatrix}\right).$$

So it follows from (1.9), (1.10) and (1.11) that

$$\rho_k(D'_x) = \frac{1}{2}\begin{pmatrix} 0 & X^{k-1}Y^{e-k}\big((a-b)X^e + (a+b)Y^e\big) \\ X^{e-k-1}Y^k\big((a+b)X^e + (a-b)Y^e\big) & 0 \end{pmatrix}.$$

The matrix $\rho_k(D'_y)$ can be computed similarly and we can deduce that,

$$\forall\ 1 \leqslant k \leqslant e-2,\ \forall\ D \in \mathrm{Gau}(W,c),\ M\rho_k(D)M^{-1} = \rho_{k+1}(D).$$

Therefore,

$$\forall\ 1 \leqslant k \leqslant e-2,\ \mathcal{L}_k^{\mathrm{Gau}} \simeq \mathcal{L}_{k+1}^{\mathrm{Gau}}. \tag{4.11}$$

Moreover,

$$-\det(\rho_k(D'_x)) = \frac{1}{4}X^{e-2}Y^e\big((a-b)X^e + (a+b)Y^e\big)\big((a+b)X^e + (a-b)Y^e\big).$$

Since $ab(a^2 - b^2) \neq 0$, $-\det(\rho_k(D'_x))$ is not a square in $\mathbb{C}(V)$, and so $\mathcal{L}_k^{\mathrm{Gau}}$ is simple (but not absolutely simple) for $1 \leqslant k \leqslant e-1$.

Moreover, an easy computation shows that $\mathcal{E}_1^{\mathrm{Gau}}$, $\mathcal{E}_\varepsilon^{\mathrm{Gau}}$, $\mathcal{E}_s^{\mathrm{Gau}}$ and $\mathcal{E}_t^{\mathrm{Gau}}$ are pairwise non-isomorphic simple $\mathrm{Gau}(W,c)$-modules. So it follows from (4.11) that

$$\mathrm{Irr}(\mathrm{Gau}(W,c)) = \{\mathcal{E}_1^{\mathrm{Gau}}, \mathcal{E}_\varepsilon^{\mathrm{Gau}}, \mathcal{E}_s^{\mathrm{Gau}}, \mathcal{E}_t^{\mathrm{Gau}}, \mathcal{L}_1^{\mathrm{Gau}}\} \tag{4.12}$$

and that (4.10) holds. □

Parameters	$d = 2e$ (even)	$d = 2e - 1$ (odd)
$a = b = 0$	$\sum_{\chi \in \mathrm{Irr}(W)} \chi(1)\chi$	$\sum_{\chi \in \mathrm{Irr}(W)} \chi(1)\chi$
$a \neq b = 0$	$\mathbb{1}_W + \varepsilon_t,\ \ \varepsilon_s + \varepsilon,\ \ \sum_{k=1}^{e-1} \chi_k$	
$a = 0 \neq b$	$\mathbb{1}_W + \varepsilon_s,\ \ \varepsilon_t + \varepsilon,\ \ \sum_{k=1}^{e-1} \chi_k$	
$a = b \neq 0$	$\mathbb{1}_W,\ \ \varepsilon,\ \ \varepsilon_s + \sum_{k=1}^{e-1} \chi_k,\ \ \varepsilon_t + \sum_{k=1}^{e-1} \chi_k$	$\mathbb{1}_W,\ \ \varepsilon,\ \ \sum_{k=1}^{e-1} \chi_k$
$a = -b \neq 0$	$\varepsilon_s,\ \ \varepsilon_t,\ \ \mathbb{1}_W + \sum_{k=1}^{e-1} \chi_k,\ \ \varepsilon + \sum_{k=1}^{e-1} \chi_k$	
$ab(a^2 - b^2) \neq 0$	$\mathbb{1}_W,\ \ \varepsilon_s,\ \ \varepsilon_t,\ \ \varepsilon,\ \ \sum_{k=1}^{e-1} \chi_k$	

Table 4.1. Calogero–Moser cellular characters of W

4.6. **Conclusion**

The following Table 4.1 gathers all the possible list of cellular characters of W, according to the values of the parameters a and b.

Remark 4.4. Whenever $a, b \in \mathbb{R}$, the Kazhdan–Lusztig cellular characters for the dihedral groups are easily computable (see for instance [15]) and a comparison with Table 4.1 shows that they coincide with Calogero–Moser cellular characters: this is [6, Conj. CAR] for dihedral groups.

5. **Calogero–Moser families**

The aim of this section is to compute the *Calogero–Moser c-families* of W (as defined in [6, §9.2]) for all values of c. The result is given in Table 5.1. Note that this result is not new: the Calogero–Moser families have been computed by Bellamy in his thesis [2]. We provide here a different proof, which uses the computation of Calogero–Moser cellular characters.

5.1. **Families**

To any irreducible character χ, Gordon [13] associates a simple $\mathbf{H}_c$-module $\mathcal{L}_c(\chi)$ (we follow the convention of [6, Prop. 9.1.3]). We denote by $\Omega_\chi^c : Z \to \mathbb{C}$ the morphism defined by the following property: if $z \in Z$, then $\Omega_\chi^c(z)$ is the scalar by which z acts on

$\mathcal{L}_c(\chi)$ (by Schur's Lemma). We say that two irreducible characters χ and χ' *belong to the same Calogero–Moser c-family* if $\Omega^c_\chi = \Omega^c_{\chi'}$ (see [13] or [6, Lem. 9.2.3]: note that Calogero–Moser families are called *Calogero–Moser blocks* in [13]). We give here a different proof of a theorem of Bellamy [2]:

Theorem 5.1 (Bellamy)**.** *Let $c \in \boldsymbol{C}$ and let χ and χ' be two irreducible characters of W. Then χ and χ' lie in the same Calogero–Moser family if and only if $\Omega^c_\chi(\mathbf{eu}) = \Omega^c_{\chi'}(\mathbf{eu})$. Consequently, the Calogero–Moser families are given by Table 5.1.*

Proof. By [6, Prop. 7.3.2], the values of $\Omega^c_\chi(\mathbf{eu})$ are given as follows:

(a) If $d = 2e - 1$ is odd, then $\begin{cases} \Omega^c_{\mathbb{1}_W}(\mathbf{eu}) = da, \\ \Omega^c_{\varepsilon}(\mathbf{eu}) = -da, \\ \Omega^c_{\chi_k}(\mathbf{eu}) = 0 \qquad \text{if } 1 \leqslant k \leqslant e-1. \end{cases}$

(b) If $d = 2e$ is even, then $\begin{cases} \Omega^c_{\mathbb{1}_W}(\mathbf{eu}) = e(a+b), \\ \Omega^c_{\varepsilon}(\mathbf{eu}) = -e(a+b), \\ \Omega^c_{\varepsilon_s}(\mathbf{eu}) = e(a-b), \\ \Omega^c_{\varepsilon_t}(\mathbf{eu}) = e(b-a), \\ \Omega^c_{\chi_k}(\mathbf{eu}) = 0 \qquad \text{if } 1 \leqslant k \leqslant e-1. \end{cases}$

But two irreducible characters occuring in the same Calogero–Moser c-cellular character necessarily belong to the same Calogero–Moser c-family [6, Prop. 11.4.2]. So the Theorem follows from (a), (b) and Table 4.1. ◻

Remark 5.2. Whenever $a, b \in \mathbb{R}$, the Kazhdan–Lusztig families for the dihedral groups are easily computable (see for instance [15]) and a comparison with Table 5.1 shows that they coincide with Calogero–Moser families: this is [6, Conj. FAM] for dihedral groups. Note that this was already proved by Bellamy [2].

5.2. **Cuspidal families**

Recall that the algebras Z and Z_c are endowed with a Poisson bracket $\{,\}$. This Poisson structure has been used by Bellamy [3] to define the notion of *cuspidal* Calogero–Moser families. If $\mathcal{F}$ is a Calogero–Moser c-family, we set $\mathfrak{m}^c_{\mathcal{F}} = \operatorname{Ker}(\Omega^c_\chi) \subset Z_c$, where χ is some (or any) element of $\mathcal{F}$ (note that Ω^c_χ factorizes through the projection $Z \twoheadrightarrow Z_c$). The Calogero–Moser c-family $\mathcal{F}$ is called *cuspidal* if $\{\mathfrak{m}^c_{\mathcal{F}}, \mathfrak{m}^c_{\mathcal{F}}\} \subset \mathfrak{m}^c_{\mathcal{F}}$. They have been determined for most of the Coxeter groups by Bellamy and Thiel [4]. In our case, we recall here their result, as well as a proof for the sake of completeness.

Parameters	$d = 2e$ (even)	$d = 2e - 1$ (odd)
$a = b = 0$	$\mathrm{Irr}(W)$	$\mathrm{Irr}(W)$
$a \neq b = 0$	$\{\mathbb{1}_W, \varepsilon_t\}$, $\{\varepsilon_s, \varepsilon\}$, $\{\chi_1, \dots, \chi_{e-1}\}$	
$a = 0 \neq b$	$\{\mathbb{1}_W, \varepsilon_s\}$, $\{\varepsilon_t, \varepsilon\}$, $\{\chi_1, \dots, \chi_{e-1}\}$	
$a = b \neq 0$	$\{\mathbb{1}_W\}$, $\{\varepsilon\}$, $\{\varepsilon_s, \varepsilon_t, \chi_1, \dots, \chi_{e-1}\}$	$\{\mathbb{1}_W\}$, $\{\varepsilon\}$, $\{\chi_1, \dots, \chi_{e-1}\}$
$a = -b \neq 0$	$\{\varepsilon_s\}$, $\{\varepsilon_t\}$, $\{\mathbb{1}_W, \varepsilon, \chi_1, \dots, \chi_{e-1}\}$	
$ab(a^2 - b^2) \neq 0$	$\{\mathbb{1}_W\}$, $\{\varepsilon_s\}$, $\{\varepsilon_t\}$, $\{\varepsilon\}$, $\{\chi_1, \dots, \chi_{e-1}\}$	

TABLE 5.1. Calogero–Moser families of W

Proposition 5.3. *The list of cuspidal Calogero–Moser families is given by Table 5.2. The following properties hold:*

(1) *A Calogero–Moser family $\mathcal{F}$ is cuspidal if and only if $|\mathcal{F}| \geqslant 2$ and $\chi_1 \in \mathcal{F}$ (and then $\chi_k \in \mathcal{F}$ for all $1 \leqslant k < d/2$).*

(2) *There is at most one cuspidal family. If $d \geqslant 5$, there is always exactly one cuspidal family.*

Proof. The main observation is that $\{q, Q\} = \mathbf{eu}$ (see (3.5)). This implies that, if χ belongs to a cuspidal family, then $\Omega^c_\chi(\mathbf{eu}) = 0$. Since it follows from Table 5.1 that the Calogero–Moser c-families are determined by the values of $\Omega^c_\chi(\mathbf{eu})$, this implies that there is at most one cuspidal Calogero–Moser c-family, and that it must contain χ_1 (and χ_k, for $1 \leqslant k < d/2$).

Also, since a Calogero–Moser c-family of cardinality 1 cannot be cuspidal [13, §5.2], this shows the "only if" part of (1). It remains to prove the "if" part of (1). So assume that $\chi_1 \in \mathcal{F}$ and that $|\mathcal{F}| \geqslant 2$. By Bellamy theory [3, Introduction], there exists a non-trivial

Parameters	$d = 2e$ (even)	$d = 2e - 1$ (odd)
$a = b = 0$	$\mathrm{Irr}(W)$	$\mathrm{Irr}(W)$, $e \geqslant 2$
$a \neq b = 0$	$\{\chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 3$	
$a = 0 \neq b$	$\{\chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 3$	
$a = b \neq 0$	$\{\varepsilon_s, \varepsilon_t, \chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 2$	$\{\chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 3$
$a = -b \neq 0$	$\{\mathbb{1}_W, \varepsilon, \chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 2$	
$ab(a^2 - b^2) \neq 0$	$\{\chi_1, \ldots, \chi_{e-1}\}$, $e \geqslant 3$	

TABLE 5.2. Calogero–Moser cuspidal families of W

parabolic subgroup W' of W and a cuspidal Calogero–Moser c'-family $\mathcal{F}'$ of W' (here, c' denotes the restriction of c to $\mathrm{Ref}(W')$) which are associated with $\mathcal{F}$. Again, by [3, Main Theorem, Introduction], $|\mathcal{F}| = |\mathcal{F}'|$. We must show that $W = W'$. So assume that $W' \neq W$. Then $|W'| = 2$ and so $|\mathcal{F}'| \leqslant 2$ and c' must be equal to 0. This forces $|\mathcal{F}| = 2$ and $ab = 0$ (and $c \neq 0$). This can only occur in type G_2: but the explicit computation of the Poisson bracket in type G_2 shows that $\mathcal{F}$ is necessarily cuspidal. □

If $\mathcal{F}$ is a cuspidal Calogero–Moser c-family, then the Poisson bracket $\{\cdot,\cdot\}$ stabilizes the maximal ideal $\mathfrak{m}_{\mathcal{F}}^c$ and so it induces a Lie bracket $[\cdot,\cdot]$ on the cotangent space $\mathfrak{Lie}_c(\mathcal{F}) = \mathfrak{m}_{\mathcal{F}}^c/(\mathfrak{m}_{\mathcal{F}}^c)^2$. It is a question to determine in general the structure of this Lie algebra. We would like to emphasize here the following two particular intriguing examples (a proof will be given in Section 8, using explicit computations).

Theorem 5.4. *Let $\mathcal{F}$ be a cuspidal Calogero–Moser c-family of W. Then:*

(1) *If $d = 4$ (i.e. if W is of type B_2) and $a = b \neq 0$, then $\mathfrak{Lie}_c(\mathcal{F}) \simeq \mathfrak{sl}_3(\mathbb{C})$ is a simple Lie algebra of type A_2.*

(2) *If $d = 6$ (i.e. if W is of type G_2) and $ab(a^2 - b^2) \neq 0$, then $\mathfrak{Lie}_c(\mathcal{F}) \simeq \mathfrak{sp}_4(\mathbb{C})$ is a simple Lie algebra of type B_2.*

6. Calogero–Moser cells

Let $c \in \boldsymbol{C}$. The main theme of [6] is a construction of partitions of W into Calogero–Moser left, right and two-sided c-cells, using a Galois closure M of the field extension

Frac(Z)/Frac(P). Let G denote the Galois group of the field extension M/Frac(P): the Calogero–Moser cells are defined [6, Definition 6.1.1] as orbits of particular subgroups of G. Our aim in this section is to prove [6, Conjs. L and LR] whenever d is odd. We first start by trying to determine the Galois group G.

It is proved in [6, §5.1.C] that there is an embedding

$$G \hookrightarrow \mathfrak{S}_W$$

(here, $\mathfrak{S}_W$ denotes the symmetric group on the set W, and we identify G with its image) such that

$$\iota(W \times W) \subset G,$$

where $\iota : W \times W \longrightarrow \mathfrak{S}_W$ denotes the morphism obtained by letting $W \times W$ act by left and right translations $((x, y) \cdot z = xzy^{-1})$. Let $\mathfrak{A}_W$ denote the alternating group on W.

Theorem 6.1. *If d is odd, then $G = \mathfrak{S}_W$.*

Proof. We first prove an easy lemma about finite permutation groups. Recall that a subgroup Γ of the symmetric group $\mathfrak{S}_n$ is called *primitive* if it is transitive and if the stabilizer of an element of $\{1, 2, \ldots, n\}$ is a maximal subgroup of Γ.

Lemma 6.2. *Let Γ be a subgroup of $\mathfrak{S}_W$. We assume that:*

(1) *d is odd;*

(2) *Γ contains $\iota(W \times W)$;*

(3) *Γ is primitive.*

Then $\Gamma = \mathfrak{S}_W$.

Proof of Lemma 6.2. Since d is odd, the action of $\sigma = \iota(w_c, w_c)$ on W is a cycle of length d (it fixes $\langle w_c \rangle$ and acts by a cycle on $W \setminus \langle w_c \rangle$ by (1.1)). Moreover, $\iota(w_c, 1)$ and $\iota(1, w_c)$ belong to the centralizer of $\iota(w_c, w_c)$ in Γ, so $C_\Gamma(\sigma) \neq \langle \sigma \rangle$. Since Γ is primitive, it follows from [10, Ex. 7.4.12] that $\Gamma = \mathfrak{S}_W$ or $\mathfrak{A}_W$.

But note that the action of $\iota(s, 1)$ is a product of d transpositions, so it is an odd permutation (because d is odd). Therefore, $\Gamma \neq \mathfrak{A}_W$ and the Lemma is proved. □

Assume that d is odd. Let us recall some results from [6, Chpt. 10] about Calogero–Moser two-sided cells. First, Calogero–Moser two-sided c-cells are defined as the orbits of some subgroup of $\mathfrak{S}_W$. Also [6, Thm. 10.2.7], there is a bijection between Calogero–Moser c-families and Calogero–Moser two-sided c-cells and, if C is the two-sided c-cell corresponding to the c-family $\mathcal{F}$, then

$$|C| = \sum_{\chi \in \mathcal{F}} \chi(1)^2.$$

Applied to the case where $a = b \neq 0$, Table 5.1 shows that there is a subgroup $\bar{I}$ of G which has three orbits for its action on W, of respective lengths 1, 1 and $2d - 2$. Since $\iota(W \times W)$ is transitive on W, G is also transitive and we may assume that one of the two orbits of length 1 is the singleton $\{1\}$. Let ΔW denote the diagonal in $W \times W$. Its action on W is by conjugacy: it has only one fixed point (because the center of W is trivial). This proves that the subgroup $\langle I, \iota(\Delta W)\rangle$ acts transitively on $W \setminus \{1\}$. So G is 2-transitive and, in particular, primitive. The Theorem now follows from Lemma 6.2 above. □

Corollary 6.3. *If d is odd, then the Conjectures* [6, Conjs. LR and L] *hold.*

Proof. Assume that $c \in \boldsymbol{C}$ takes real values. The computation of Calogero–Moser c-families and c-cellular characters shows that, if we choose randomly two prime ideals as in [6, Chpt. 15], then the associated Calogero–Moser two-sided or left c-cells have the same sizes as the Kazhdan–Lusztig two-sided or left c-cells respectively (see [6, Chpts. 10 and 11]). Since the Galois group G coincides with $\mathfrak{S}_W$, we can manage to change the prime ideals so that Calogero–Moser and Kazhdan–Lusztig c-cells coincide. □

Remark 6.4. Let

$$\mathfrak{S}_W^B = \{\sigma \in \mathfrak{S}_W \mid \forall\, w \in W,\ \sigma(w_0 w) = w_0 \sigma(w)\} \quad \text{and} \quad \mathfrak{S}_W^D = \mathfrak{S}_W^B \cap \mathfrak{A}_W.$$

Note that, in our case, $\mathfrak{S}_W^B$ (respectively $\mathfrak{S}_W^D$) is a Weyl group of type B_d (respectively D_d) and that $\mathfrak{S}_W^D$ is a normal subgroup of $\mathfrak{S}_W^B$ of index 2.

Assume here, and only here, that d is *even*. It then follows from [6, Prop. 5.5.2] that $G \subset \mathfrak{S}_W^B$. We would bet a few euros (but not more) that $G = \mathfrak{S}_W^D$. This has been checked for $d = 4$ in [6, Thm. 19.6.1] and it will be checked in (8.4) whenever $d = 6$. Let us just prove a few general facts.

First, let $\bar{W} = W/\mathrm{Z}(W)$ (it is a dihedral group of order d) and let $\bar{\iota} : \bar{W} \times \bar{W} \to \mathfrak{S}_{\bar{W}}$ denote the morphism induced by the action by left and right translations. Let $\bar{G}$ denote the image of G in $\mathfrak{S}_{\bar{W}}$ (indeed $G \subset \mathfrak{S}_W^B$ and there is a natural morphism $\mathfrak{S}_W^B \to \mathfrak{S}_{\bar{W}}$). Then, if $a = b \neq 0$, the Calogero–Moser two-sided c-cells have cardinalities 1, 1 and $2d - 2$ (by [6, Thm. 10.2.7]) so it follows from the definition of Calogero–Moser cells that there exists a subgroup I_1 of G whose orbits have cardinalities 1, 1 and $2d - 2$. Therefore, the image $\bar{I}_1$ of I_1 in $\bar{G}$ has orbits of cardinalities 1 and $d - 1$. Consequently, $\bar{G}$ is 2-transitive. Similarly, taking c such that $ab(a^2 - b^2) \neq 0$, we get that there is a subgroup $\bar{I}_2$ of $\bar{G}$ whose orbits have cardinalities 1, 1 and $d - 2$. Therefore,

$$\bar{G} \text{ is 3-transitive.} \tag{6.1}$$

On the other hand,

$$\bar{G} \text{ contains } \bar{\iota}(\bar{W} \times \bar{W}). \tag{6.2}$$

As a consequence, we get

$$\text{If } d/2 \text{ is odd, then } \bar{G} = \mathfrak{S}_{\bar{W}}. \tag{6.3}$$

Indeed, this follows from Lemma 6.2.

7. Fixed points

The $\mathbb{Z}$-grading on the $\mathbb{C}$-algebra $\mathbf{H}$ (defined in Remark 3.3) induces an action of the group $\mathbb{C}^\times$ on $\mathbf{H}$ as follows [6, §3.5.A]. If $\xi \in \mathbb{C}^\times$ then:

- If $y \in V$, then ${}^\xi y = \xi^{-1} y$.
- If $x \in V^*$, then ${}^\xi x = \xi x$.
- If $w \in W$, then ${}^\xi w = w$.
- If $s \in \mathrm{Ref}(W)$, then ${}^\xi C_s = C_s$.

So the center Z inherits an action of $\mathbb{C}^\times$, which may be viewed as a $\mathbb{C}^\times$-action on the Calogero–Moser space $\mathcal{Z}$, which stabilizes all the fibers $\mathcal{Z}_c$ (for $c \in \boldsymbol{C}$).

Now, if $m \in \mathbb{N}^*$, we denote by $\boldsymbol{\mu}_m$ the group of m-th root of unity in $\mathbb{C}^\times$. In [6, Conj. FIX], R. Rouquier and the author conjecture that all the irreducible components of the fixed point variety $\mathcal{Z}^{\boldsymbol{\mu}_m}$ are isomorphic to the Calogero–Moser space of some other complex reflection groups (here, $\mathcal{Z}^{\boldsymbol{\mu}_m}$ is endowed with its reduced structure). This conjecture will be checked for $d \in \{3, 4, 6\}$ and any m in Section 8.

Theorem 7.1. *Assume that d is odd. Then*

$$\mathcal{Z}^{\boldsymbol{\mu}_d} \simeq \{(a, u, v, e) \in \mathbb{C}^4 \mid (e - da)(e + da)e^{d-2} = uv\}.$$

Remark 7.2. By [6, Thm. 18.2.4], the above Theorem shows that $\mathcal{Z}_c^{\boldsymbol{\mu}_d}$ is isomorphic to the Calogero–Moser space associated with the cyclic group of order d and some parameters, so it proves [6, Conj. FIX] in this case.

Proof. The case where $d = 1$ is not interesting, so we assume that $d \geqslant 3$. Let $\mathfrak{I}$ denote the ideal of Z generated by $\{{}^\zeta z - z \mid z \in Z\}$. Then the algebra of regular functions on $\mathcal{Z}^{\boldsymbol{\mu}_d}$ is $Z/\sqrt{\mathfrak{I}}$. We will describe $Z/\mathfrak{I}$, and this will prove that $\mathfrak{I} = \sqrt{\mathfrak{I}}$ in this case. Therefore,

$$\mathfrak{I} = \langle q, Q, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{d-1} \rangle,$$

and so $Z/\mathfrak{I}$ is generated by the images of A, r, R and $\mathbf{eu}$. In the quotient $Z/\mathfrak{I}$, all the equations of $\mathbb{Z}$-degree which is not divisible by d are automatically fulfiled, so it only remains the equations $(\mathrm{Z}_{i,d-i})$ (which is bi-homogeneous of bi-degree (d,d)). Also, note that $(\mathrm{Z}^0_{i,d-i})$ implies that

$$\mathbf{eu}^d = rR \mod \langle \mathfrak{I}, A\rangle.$$

The only bi-homogeneous monomials in A, r, R and $\mathbf{eu}$ of bi-degree (d,d) are rR and the $\mathbf{eu}^k A^{d-k}$ (for $0 \leqslant k \leqslant d$). Therefore, the above equation implies that there exist complex numbers $\lambda_0, \dots, \lambda_{d-1}$ such that

$$\mathbf{eu}^d + \lambda_{d-1}A\mathbf{eu}^{d-1} + \cdots + \lambda_1 A^{d-1}\mathbf{eu} + \lambda_0 A^d \equiv rR \mod \mathfrak{I}. \tag{7.1}$$

On the other hand, it follows from [6, Cor. 9.4.4] that

$$(\mathbf{eu} - dA)(\mathbf{eu} + dA)\mathbf{eu}^{2d-2} = \prod_{\chi \in \mathrm{Irr}(W)} (\mathbf{eu} - \Omega_\chi(\mathbf{eu}))^{\chi(1)^2} \equiv 0 \mod \langle q, Q, r, R\rangle.$$

So (7.1) implies that the polynomial $\mathbf{t}^d + \lambda_{d-1}A\mathbf{t}^{d-1} + \cdots + \lambda_1 A^{d-1}\mathbf{t} + \lambda_0 A^d$ divides $(\mathbf{t} - dA)(\mathbf{t} + dA)\mathbf{t}^{2d-2}$ in $\mathbb{C}[A][\mathbf{t}]$.

Since all the $\mathbb{C}^\times$-fixed points belong to $\mathcal{Z}^{\boldsymbol{\mu}_d}$, this implies that $\mathbf{t} - dA$ and $\mathbf{t} + dA$ both divide $\mathbf{t}^d + \lambda_{d-1}A\mathbf{t}^{d-1} + \cdots + \lambda_1 A^{d-1}\mathbf{t} + \lambda_0 A^d$. Therefore,

$$\mathbf{t}^d + \lambda_{d-1}A\mathbf{t}^{d-1} + \cdots + \lambda_1 A^{d-1}\mathbf{t} + \lambda_0 A^d = (\mathbf{t} - dA)(\mathbf{t} + dA)\mathbf{t}^{d-2},$$

and so there remains only one relations in the quotient $Z/\mathfrak{I}$, namely

$$(\mathbf{eu} - dA)(\mathbf{eu} + dA)\mathbf{eu}^{d-2} \equiv rR \mod \mathfrak{I},$$

as desired. □

8. Examples

We are interested here in the cases where $d \in \{3, 4, 6\}$. These are the Weyl groups of rank 2 (of type A_2, B_2 or G_2). For each of these cases, we give a complete presentation of the centre Z of $\mathbf{H}$ (using the algorithms developed in [7]). For the sake of completeness, we give the minimal polynomial of $\mathbf{eu}$ over P, as the Galois group G is defined as the Galois group of this polynomial over the fraction field of P (see [6, §5.1.D]): it will only be used in type G_2 for proving that $G = \mathfrak{S}_W^D$ in this case. We use these explicit computations to check some of the facts that have been stated earlier in this paper. Most of the computations are done using `MAGMA` [8].

In this section, we denote by z_χ the point of $\mathcal{Z}_c$ corresponding to the maximal ideal $\operatorname{Ker}\Omega_\chi^c$ of Z_c.

8.1. **Type** A_2

We work here under the following hypothesis:

Note. We assume in this subsection, and only in this subsection, that $d = 3$. In other words, W is a Weyl group of type A_2.

Using `MAGMA` and [7], we can compute effectively the generators of the $\mathbb{C}[\boldsymbol{C}]$-algebra Z and we obtain the following presentation for Z (note that $\mathbb{C}[\boldsymbol{C}] = \mathbb{C}[A]$ because $A = B$):

Proposition 8.1. *The $\mathbb{C}[A]$-algebra Z admits the following presentation:*

$$\begin{cases} \textit{Generators:} & q, r, Q, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2 \\ \textit{Relations:} & \begin{cases} \mathbf{eu}\,\mathbf{a}_1 = q\,\mathbf{a}_2 + rQ \\ \mathbf{eu}\,\mathbf{a}_2 = Q\,\mathbf{a}_1 + qR \\ \mathbf{a}_1^2 = 4q^2Q + r\,\mathbf{a}_2 - q\,\mathbf{eu}^2 + 9A^2\,q \\ \mathbf{a}_1\,\mathbf{a}_2 = 4qQ\,\mathbf{eu} + rR - \mathbf{eu}^3 + 9A^2\,\mathbf{eu} \\ \mathbf{a}_2^2 = 4qQ^2 + R\,\mathbf{a}_1 - Q\,\mathbf{eu}^2 + 9A^2\,Q \end{cases} \end{cases}$$

The minimal polynomial of **eu** is given by

$$\mathbf{t}^6 - (6qQ + 9A^2)\,\mathbf{t}^4 - rR\,\mathbf{t}^3 + 9(q^2Q^2 + 2A^2qQ)\,\mathbf{t}^2 + 3qrQR\,\mathbf{t} + q^3R^2 + r^2Q^3 - 4q^3Q^3 - 9A^2q^2Q^2. \quad (8.1)$$

We conclude by proving [6, Conj. FIX] in this case (about the variety $\mathcal{Z}^{\boldsymbol{\mu}_m}$). Note that the only interesting case is where m divides the order of an element of W. So $m \in \{1, 2, 3\}$. The case $m = 1$ is stupid while the case $m = 3$ is treated in Theorem 7.1:

Proposition 8.2. *The $\mathbb{C}[A]$-algebra $\mathbb{C}[\mathcal{Z}^{\boldsymbol{\mu}_2}]$ admits the following presentation:*

$$\begin{cases} \textit{Generators:} & q, Q, \mathrm{eu} \\ \textit{Relations:} & \begin{cases} q(\mathrm{eu}^2 - 9A^2 - 4qQ) = 0, \\ \mathrm{eu}(\mathrm{eu}^2 - 9A^2 - 4qQ) = 0, \\ Q(\mathrm{eu}^2 - 9A^2 - 4qQ) = 0, \end{cases} \end{cases}$$

In particular, if $a \neq 0$, then the variety $\mathcal{Z}_c^{\boldsymbol{\mu}_2}$ has two irreducible components:

(1) *A component of dimension* 2 *defined by the equation* $\mathrm{eu}^2 - 9a^2 - 4qQ = 0$ *(which contains the points z_1 and z_ε).*

(2) *An isolated point, which is equal to z_{χ_1}.*

8.2. Type B_2

We work here under the following hypothesis:

Note. We assume in this subsection, and only in this subsection, that $d = 4$. In other words, W is a Weyl group of type B_2.

Using `MAGMA` and [7], we can compute effectively the generators of the $\mathbb{C}[\mathcal{C}]$-algebra Z and we obtain the following presentation for Z (note that $A \neq B$):

Proposition 8.3. *The $\mathbb{C}[A, B]$-algebra Z admits the following presentation:*

$$\begin{cases} \textit{Generators: } q, r, Q, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \\ \textit{Relations: } \begin{cases} \mathbf{eu}\,\mathbf{a}_1 = q\,\mathbf{a}_2 + rQ - 2(A^2 - B^2)\,q \\ \mathbf{eu}\,\mathbf{a}_2 = q\,\mathbf{a}_3 + Q\,\mathbf{a}_1 - 2(A^2 - B^2)\,\mathbf{eu} \\ \mathbf{eu}\,\mathbf{a}_3 = qR + Q\,\mathbf{a}_2 - 2(A^2 - B^2)\,Q \\ \mathbf{a}_1^2 = r\,\mathbf{a}_2 - q^2\,\mathbf{eu}^2 + 4q^3Q + 2(A^2 - B^2)\,r + 8(A^2 + B^2)\,q^2 \\ \mathbf{a}_1\,\mathbf{a}_2 = r\,\mathbf{a}_3 - q\,\mathbf{eu}^3 + 4q^2Q\,\mathbf{eu} + 2(A^2 - B^2)\,\mathbf{a}_1 + 8(A^2 + B^2)\,q\,\mathbf{eu} \\ \mathbf{a}_1\,\mathbf{a}_3 = rR - \mathbf{eu}^4 + 5qQ\mathbf{eu}^2 - 4q^2Q^2 + 4(A^2 - B^2)\,\mathbf{a}_2 \\ \qquad\quad +8(A^2 + B^2)\,\mathbf{eu}^2 - 8(A^2 + B^2)\,qQ - 8(A^2 - B^2)^2 \\ \mathbf{a}_2^2 = rR - \mathbf{eu}^4 + 4qQ\,\mathbf{eu}^2 + 4(A^2 - B^2)\,\mathbf{a}_2 \\ \qquad\quad +8(A^2 + B^2)\,\mathbf{eu}^2 - 4(A^2 - B^2)^2 \\ \mathbf{a}_2\,\mathbf{a}_3 = R\,\mathbf{a}_1 - Q\,\mathbf{eu}^3 + 4qQ^2\,\mathbf{eu} + 2(A^2 - B^2)\,\mathbf{a}_3 + 8(A^2 + B^2)\,Q\,\mathbf{eu} \\ \mathbf{a}_3^2 = R\,\mathbf{a}_2 - Q^2\,\mathbf{eu}^2 + 4qQ^3 + 2(A^2 - B^2)\,R + 8(A^2 + B^2)\,Q^2 \end{cases} \end{cases}$$

Now, let

$$\begin{aligned} f_4(\mathbf{t}) = \mathbf{t}^4 - 8(qQ + A^2 + B^2)\,\mathbf{t}^3 + \Big(20q^2Q^2 - rR + 32(A^2 + B^2)qQ + 16(A^2 - B^2)^2\Big)\,\mathbf{t}^2 \\ - 4\Big(4q^3Q^3 - qrQR + 2(A^2 - B^2)(q^2R + rQ^2) + 8(A^2 + B^2)q^2Q^2\Big)\,\mathbf{t} + (q^2R - rQ^2)^2. \end{aligned}$$

Then

$$\text{The minimal polynomial of } \mathbf{eu} \text{ is } f_4(\mathbf{t}^2). \tag{8.2}$$

8.2.1. Cuspidal point

We aim to prove here Theorem 5.4(1). By Table 5.2, there is a cuspidal point in $\mathcal{Z}_c$ if and only if $a^2 = b^2$. Using the automorphism of $\mathbf{H}$ (and so, of Z) induced by the linear character ε_s (see [6, §3.5.B]), we may reduce to the case where

$$a = b.$$

$$\begin{array}{llll}
\mathfrak{q} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & & \mathfrak{eu} \longmapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\
\mathfrak{r} \longmapsto 8a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & & \mathfrak{a}_2 \longmapsto \frac{8a}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathfrak{Q} \longmapsto \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} & & \mathfrak{a}_1 \longmapsto 4a \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
\mathfrak{R} \longmapsto 32a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \mathfrak{a}_3 \longmapsto 8a \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{array}$$

Table 8.1. Definition of $\aleph_c$ for $d = 4$

Then there is only one cuspidal family $\mathcal{F}$ in $\mathcal{Z}_c$ (the one containing χ_1). It is easily checked that $\mathfrak{m}_{\mathcal{F}}^c = \langle q, r, Q, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle_{Z_c}$ and it is readily checked from the presentation given in Proposition 8.3 that the cotangent space $\mathfrak{Lie}_c(\mathcal{F}) = \mathfrak{m}_{\mathcal{F}}^c/(\mathfrak{m}_{\mathcal{F}}^c)^2$ has dimension 8: so a basis is given by the images $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{eu}$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ of q, r, Q, R, $\mathbf{eu}$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ respectively.

The computation of the Poisson bracket can be done using the `MAGMA` package `CHAMP`: writing the result modulo $(\mathfrak{m}_{\mathcal{F}}^c)^2$ gives the Lie bracket on $\mathfrak{Lie}_c(\mathcal{F})$. We can then deduce that:

Proposition 8.4. *Assume here that $a = b$. Then the linear map $\aleph_c : \mathfrak{Lie}_c(\mathcal{F}) \longrightarrow \mathfrak{sl}_3(\mathbb{C})$ defined in Table 8.1 is a morphism of Lie algebras. It is an isomorphism if $a \neq 0$.*

8.2.2. **Fixed points**

The next result follows immediately from the presentation given in Proposition 8.3:

Proposition 8.5. *The $\mathbb{C}[A, B]$-algebra $\mathbb{C}[\mathcal{Z}^{\boldsymbol{\mu}_4}]$ admits the following presentation:*

$$\begin{cases} \textit{Generators:} & r, R, \mathbf{eu}, \mathbf{a}_2 \\ \textit{Relations:} & \begin{cases} \mathbf{eu}\,(\mathbf{a}_2 + 2(A^2 - B^2) = 0 \\ r\,(\mathbf{a}_2 + 2(A^2 - B^2) = 0 \\ R\,(\mathbf{a}_2 + 2(A^2 - B^2) = 0 \\ (\mathbf{a}_2 - 2(A^2 - B^2))(\mathbf{a}_2 + 2(A^2 - B^2)) = 0 \\ (\mathbf{eu} - 2(A+B))(\mathbf{eu} + 2(A+B))(\mathbf{eu} - 2(A-B))(\mathbf{eu} + 2(A-B)) \\ \qquad = rR + 4(A^2 - B^2)(\mathbf{a}_2 + 2(A^2 - B^2)) \end{cases} \end{cases}$$

In particular:

(1) *If $a^2 = b^2$, then $\mathcal{Z}_c^{\boldsymbol{\mu}_4}$ is irreducible and is equal to*

$$\{(e, u, v) \in \mathbb{C}^3 \mid (e - 4a)(e + 4a)e^2 = uv\}.$$

(2) *If $a^2 \neq b^2$, then $\mathcal{Z}_c^{\boldsymbol{\mu}_4}$ has two irreducible components:*

(a) *The one of maximal dimension which is equal to*

$$\{(e, u, v) \in \mathbb{C}^3 \mid (e - 2(a+b))(e + 2(a+b))(e - 2(a-b))(e + 2(a-b)) = uv\},$$

and which contains $z_{\mathbb{1}}$, z_{ε}, z_{ε_s} and z_{ε_t}.

(b) *An isolated point (corresponding to the maximal ideal $\langle r, R, \mathbf{eu}, \mathbf{a}_2 - 2(a^2 - b^2)\rangle$), which is equal to z_{χ_1}.*

8.3. **Type** G_2

We work here under the following hypothesis:

Note. We assume in this subsection, and only in this subsection, that $d = 6$. In other words, W is a Weyl group of type G_2.

Using `MAGMA` and [7], we can compute effectively the generators of the $\mathbb{C}[\mathcal{C}]$-algebra Z and we obtain the following presentation for Z (note that $A \neq B$):

Proposition 8.6. *The $\mathbb{C}[A, B]$-algebra Z admits the following presentation:*

$$\begin{cases} \textit{Generators:} & r, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5 \\ \textit{Relations:} & \textit{see Table 8.2} \end{cases}$$

$$\begin{cases}
\mathbf{eu}\mathbf{a}_1 = q\mathbf{a}_2 + rQ - 3(A^2-B^2)q^2 \\
\mathbf{eu}\mathbf{a}_2 = q\mathbf{a}_3 + Q\mathbf{a}_1 - 3(A^2-B^2)q\mathbf{eu} \\
\mathbf{eu}\mathbf{a}_3 = q\mathbf{a}_4 + Q\mathbf{a}_2 - 3(A^2-B^2)\mathbf{eu}^2 + 3(A^2-B^2)qQ \\
\mathbf{eu}\mathbf{a}_4 = q\mathbf{a}_5 + Q\mathbf{a}_3 - 3(A^2-B^2)Q\mathbf{eu} \\
\mathbf{eu}\mathbf{a}_5 = qR + Q\mathbf{a}_4 - 3(A^2-B^2)Q^2 \\
\mathbf{a}_1^2 = r\mathbf{a}_2 - q^4\mathbf{eu}^2 + 4q^5Q + 6(A^2-B^2)qr + 18(A^2+B^2)q^4 \\
\mathbf{a}_1\mathbf{a}_2 = r\mathbf{a}_3 - q^3\mathbf{eu}^3 + 4q^4Q\mathbf{eu} + 3(A^2-B^2)r\mathbf{eu} + 18(A^2+B^2)q^3\mathbf{eu} + 3(A^2-B^2)q\mathbf{a}_1 \\
\mathbf{a}_1\mathbf{a}_3 = r\mathbf{a}_4 - q^2\mathbf{eu}^4 + 5q^3Q\mathbf{eu}^2 - 4q^4Q^2 + 18(A^2+B^2)q^2\mathbf{eu}^2 \\
\qquad +6(A^2-B^2)q\mathbf{a}_2 + 3(A^2-B^2)Qr - 18(A^2+B^2)q^3Q - 18(A^2-B^2)^2q^2 \\
\mathbf{a}_1\mathbf{a}_4 = r\mathbf{a}_5 - q\mathbf{eu}^5 + 6q^2Q\mathbf{eu}^3 - 8q^3Q^2\mathbf{eu} + 18(A^2+B^2)q\mathbf{eu}^3 \\
\qquad +9(A^2-B^2)q\mathbf{a}_3 - 36(A^2+B^2)q^2Q\mathbf{eu} + 3(A^2-B^2)Q\mathbf{a}_1 - 54(A^2-B^2)^2q\mathbf{eu} \\
\mathbf{a}_1\mathbf{a}_5 = rR - \mathbf{eu}^6 + 7qQ\mathbf{eu}^4 - 13q^2Q^2\mathbf{eu}^2 + 4q^3Q^3 + 18(A^2+B^2)\mathbf{eu}^4 + 9(A^2-B^2)q\mathbf{a}_4 \\
\qquad -54(A^2+B^2)qQ\mathbf{eu}^2 + 9(A^2-B^2)Q\mathbf{a}_2 + 18(A^2+B^2)q^2Q^2 \\
\qquad -81(A^2-B^2)^2\mathbf{eu}^2 + 27(A^2-B^2)^2qQ \\
\mathbf{a}_2^2 = r\mathbf{a}_4 - q^2\mathbf{eu}^4 + 4q^3Q\mathbf{eu}^2 + 18(A^2+B^2)q^2\mathbf{eu}^2 + 6(A^2-B^2)q\mathbf{a}_2 \\
\qquad +6(A^2-B^2)Qr - 9(A^2-B^2)^2q^2 \\
\mathbf{a}_2\mathbf{a}_3 = r\mathbf{a}_5 - q\mathbf{eu}^5 + 5q^2Q\mathbf{eu}^3 - 4q^3Q^2\mathbf{eu} + 18(A^2+B^2)q\mathbf{eu}^3 + 9(A^2-B^2)q\mathbf{a}_3 \\
\qquad -18(A^2+B^2)q^2Q\mathbf{eu} + 6(A^2-B^2)Q\mathbf{a}_1 - 36(A^2-B^2)^2q\mathbf{eu} \\
\mathbf{a}_2\mathbf{a}_4 = rR - \mathbf{eu}^6 + 6qQ\mathbf{eu}^4 - 8q^2Q^2\mathbf{eu}^2 + 18(A^2+B^2)\mathbf{eu}^4 + 12(A^2-B^2)q\mathbf{a}_4 \\
\qquad -36(A^2+B^2)qQ\mathbf{eu}^2 + 12(A^2-B^2)Q\mathbf{a}_2 - 81(A^2-B^2)^2\mathbf{eu}^2 + 18(A^2-B^2)^2qQ \\
\mathbf{a}_2\mathbf{a}_5 = R\mathbf{a}_1 - Q\mathbf{eu}^5 + 6qQ^2\mathbf{eu}^3 - 8q^2Q^3\mathbf{eu} + 3(A^2-B^2)q\mathbf{a}_5 + 18(A^2+B^2)Q\mathbf{eu}^3 \\
\qquad +9(A^2-B^2)Q\mathbf{a}_3 - 36(A^2+B^2)qQ^2\mathbf{eu} - 54(A^2-B^2)^2Q\mathbf{eu} \\
\mathbf{a}_3^2 = rR - \mathbf{eu}^6 + 6qQ\mathbf{eu}^4 - 9q^2Q^2\mathbf{eu}^2 + 4q^3Q^3 + 18(A^2+B^2)\mathbf{eu}^4 - 36(A^2+B^2)qQ\mathbf{eu}^2 \\
\qquad +12(A^2-B^2)q\mathbf{a}_4 + 12(A^2-B^2)Q\mathbf{a}_2 + 18(A^2+B^2)q^2Q^2 - 72(A^2-B^2)^2\mathbf{eu}^2 + 36(A^2-B^2)^2qQ \\
\mathbf{a}_3\mathbf{a}_4 = R\mathbf{a}_1 - Q\mathbf{eu}^5 + 5qQ^2\mathbf{eu}^3 - 4q^2Q^3\mathbf{eu} + 6(A^2-B^2)q\mathbf{a}_5 + 18(A^2+B^2)Q\mathbf{eu}^3 \\
\qquad +9(A^2-B^2)Q\mathbf{a}_3 - 18(A^2+B^2)qQ^2\mathbf{eu} - 36(A^2-B^2)^2Q\mathbf{eu} \\
\mathbf{a}_3\mathbf{a}_5 = R\mathbf{a}_2 - Q^2\mathbf{eu}^4 + 5qQ^3\mathbf{eu}^2 - 4q^2Q^4 + 18(A^2+B^2)Q^2\mathbf{eu}^2 + 6(A^2-B^2)Q\mathbf{a}_4 \\
\qquad +3(A^2-B^2)qR - 18(A^2+B^2)qQ^3 - 18(A^2-B^2)^2Q^2 \\
\mathbf{a}_4^2 = R\mathbf{a}_2 - Q^2\mathbf{eu}^4 + 4Q^3q\mathbf{eu}^2 + 18(A^2+B^2)Q^2\mathbf{eu}^2 + 6(A^2-B^2)Q\mathbf{a}_4 \\
\qquad +6(A^2-B^2)qR - 9(A^2-B^2)^2Q^2 \\
\mathbf{a}_4\mathbf{a}_5 = R\mathbf{a}_3 - Q^3\mathbf{eu}^3 + 4qQ^4\mathbf{eu} + 3(A^2-B^2)R\mathbf{eu} + 18(A^2+B^2)Q^3\mathbf{eu} + 3(A^2-B^2)Q\mathbf{a}_5 \\
\mathbf{a}_5^2 = R\mathbf{a}_4 - Q^4\mathbf{eu}^2 + 4qQ^5 + 6(A^2-B^2)QR + 18(A^2+B^2)Q^4
\end{cases}$$

TABLE 8.2. Presentation of Z whenever $d = 6$

Now, let

$$f_6(\mathbf{t}) = \mathbf{t}^6 - 6\big(2qQ + 3(A^2+B^2)\big)\,\mathbf{t}^5 + 9\Big(6q^2Q^2 + 16(A^2+B^2)qQ + 9(A^2-B^2)^2\Big)\,\mathbf{t}^4$$
$$- \Big(rR + 112q^3Q^3 + 396(A^2+B^2)q^2Q^2 + 324(A^2-B^2)^2qQ\Big)\,\mathbf{t}^3$$

$$\begin{aligned}
&+ 3\Big(2rqRQ + 35q^4Q^4 - 6(A^2 - B^2)(rQ^3 + q^3R) + 144(A^2 + B^2)q^3Q^3 \\
&\qquad\qquad + 162(A^2 - B^2)^2q^2Q^2\Big)\,\mathbf{t}^2 \\
&- 9\Big(rq^2RQ^2 + 4q^5Q^5 - 4(A^2 - B^2)(rqQ^4 + q^4RQ) + 18(A^2 + B^2)q^4Q^4 \\
&\qquad\qquad + 36(A^2 - B^2)^2q^3Q^3\Big)\,\mathbf{t} \\
&+ \Big(rQ^3 + q^3R - 9(A^2 - B^2)q^2Q^2\Big)^2
\end{aligned}$$

Then

$$\text{the minimal polynomial of } \mathbf{eu} \text{ is } f_6(\mathbf{t}^2). \tag{8.3}$$

8.3.1. Galois group

Since $f_6(0)$ is a square in P, it follows from [6, (B.6.1)] that the discriminant of $f_6(\mathbf{t}^2)$ is a square in P. Therefore, the Galois group G of the polynomial $f_6(\mathbf{t})$ is contained in $\mathfrak{A}_W$. Moreover, it follows from Remark 6.4 that G is contained in $\mathfrak{S}_W^B$. Therefore, $G \subset \mathfrak{S}_W^D$. In fact,

$$G = \mathfrak{S}_W^D. \tag{8.4}$$

Proof. Let G_1 denote the stabilizer of $1 \in W$ in $G \subset \mathfrak{S}_W$. By the computation of Calogero–Moser families, G_1 contains a subgroup admitting an orbit of cardinality 10 (case $a = b \neq 0$). Consequently:

(1) G_1 contains a subgroup admitting an orbit of cardinality 10.

(2) G contains $\iota(W \times W)$.

An easy computation with the software `GAP4` [12] shows that the only subgroup G of $\mathfrak{S}_W^D$ satisfying (1) and (2) is $\mathfrak{S}_W^D$. □

8.3.2. Fixed points

The next result follows immediately from the presentation given in Proposition 8.6:

Proposition 8.7. *The* $\mathbb{C}[A, B]$*-algebra* $\mathbb{C}[\mathcal{Z}^{\mu_6}]$ *admits the following presentation:*

$$\begin{cases} \textit{Generators:} & r, R, \mathbf{eu} \\ \textit{Relation:} & (\mathbf{eu} - 3(A + B))(\mathbf{eu} + 3(A + B))(\mathbf{eu} - 3(A - B))(\mathbf{eu} + 3(A - B))\mathbf{eu}^2 = rR \end{cases}$$

This proves [6, Conj.FIX] *in this case.*

$$\mathfrak{r} \longmapsto -324(a^2-b^2)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} \qquad \mathbf{a}_1 \longmapsto -54(a^2-b^2)\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

$$\mathfrak{q} \longmapsto \begin{pmatrix}0&0&0&0\\-3&0&0&0\\0&-4&0&0\\0&0&3&0\end{pmatrix} \qquad \mathbf{a}_2 \longmapsto 6(a^2-b^2)\begin{pmatrix}0&0&0&0\\3&0&0&0\\0&-2&0&0\\0&0&-3&0\end{pmatrix}$$

$$\mathfrak{R} \longmapsto 9(a^2-b^2)\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \qquad \mathbf{a}_3 \longmapsto 3(a^2-b^2)\begin{pmatrix}-3&0&0&0\\0&2&0&0\\0&0&-2&0\\0&0&0&3\end{pmatrix}$$

$$\mathfrak{Q} \longmapsto \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&-1\\0&0&0&0\end{pmatrix} \qquad \mathbf{a}_4 \longmapsto 3(a^2-b^2)\begin{pmatrix}0&-2&0&0\\0&0&1&0\\0&0&0&2\\0&0&0&0\end{pmatrix}$$

$$\mathfrak{eu} \longmapsto \begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix} \qquad \mathbf{a}_5 \longmapsto (-9/2)(a^2-b^2)\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

TABLE 8.3. Definition of $\aleph_c$ for $d = 6$

8.3.3. **Lie algebra at cuspidal point**

Recall from Proposition 5.3 that there is a unique cuspidal Calogero–Moser c-family $\mathcal{F}$: it is the one which contains χ_1 (this fact does not depend on the parameter c; however, the cardinality of $\mathcal{F}$ depends on the parameter). It corresponds to the maximal ideal $\mathfrak{m} = \langle q, r, Q, R, \mathbf{eu}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\rangle$ of Z_c. It follows from the presentation of Z given by Proposition 8.6 that the cotangent space $\mathfrak{Lie}_c(\mathcal{F}) = \mathfrak{m}_{\mathcal{F}}^c/(\mathfrak{m}_{\mathcal{F}}^c)^2$ has dimension10 (a basis is given by the images $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{eu}$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, $\mathfrak{a}_4$ and $\mathfrak{a}_5$ of q, r, Q, R, $\mathbf{eu}$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$ in $\mathfrak{m}$ respectively). The Poisson bracket (and so the Lie bracket in $\mathfrak{Lie}_c(\mathcal{F})$) can then be computed explicitly using the MAGMA package CHAMP. We can then deduce the following result:

Proposition 8.8. *Let $\aleph_c : \mathfrak{Lie}_c(\mathcal{F}) \longrightarrow \mathfrak{sp}_4(\mathbb{C})$ be the linear map defined by Table 8.3. It is a morphism of Lie algebras. Moreover:*

(1) *If $a^2 \neq b^2$, then $\aleph_c$ is an isomorphism of Lie algebras.*

(2) *If $a^2 = b^2$, then its image is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (with basis $\aleph_c(\mathfrak{q})$, $\aleph_c(\mathfrak{Q})$ et $\aleph_c(\mathfrak{eu})$) and its kernel is commutative, of dimension7 (as a module for $\mathfrak{sl}_2(\mathbb{C})$, it is irreducible).*

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