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Abstract

Generalizing the notion of matched pair of groups, we define and study matched pairs of locally compact groupoids endowed with Haar systems, in order to give new examples of measured quantum groupoids.

Groupoïdes quantiques mesurés associés aux couples assortis de groupoïdes localement compacts

Résumé

En généralisant la notion de couple assorti de groupes, nous définissons et étudions les paires assorties de groupoides localement compacts munis de systèmes de Haar, afin d'obtenir de nouveaux exemples de groupoïdes quantiques mesurés.

Contents

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Math. classification: 17B37, 22D25, 22A22.

1. **Introduction**

Dealing with locally compact groupoids, we defined in the articles [24] and [23] a notion of pseudo multiplicative unitary and Hopf bimodule in order to generalize, in that framework, classical notions of multiplicative unitary ([2]) and Hopf von Neumann algebras ([9]) which led to locally compact quantum groups ([2], [27], [13]......).

In an other article ([10]), starting with any depth 2 inclusion of von Neumann algebras $M_0 \subset M_1$, with an operator-valued weight T_1 verifying a regularity condition, Michel Enock and the author have given a pseudo-multiplicative unitary generating two Hopf bimodules in duality; one of them acts on M_1 in such a way that M_0 is isomorphic to the fixed point algebra and the von Neumann algebra M_2, given by the basic construction, is isomorphic to the crossed product.

The axiomatic of locally compact quantum groupoids has been developed by Franck Lesieur in [14] and [15] and simplified by M. Enock ([6] Appendice), who has also studied the theory of their actions on von Neumann algebras, generalizing previous results due to S. Vaes ([20]).

The aim of this article is to give a large number of examples of measured quantum groupoids as defined by M. Enock and F. Lesieur. We generalize at the same time the notion of matched pair of finite groupoids ([25]) and of locally compact group ([2], [22], [3], [4]....), in order to obtain such examples coming from a suitable pseudo multiplicative unitary.

In the second paragraph are recalled definitions about locally compact quantum groupoids and their actions on von Neumann algebras.

We precise, in the third chapter, the notion of matched pair of locally compact groupoids, we prove that, for such a pair, there exists a canonical action of each groupoid on the other one, and we give families of examples. Finally we give a canonical pseudo multiplicative unitary which generates their crossed products.

In the fourth chapter we investigate the Hopf bimodule structures of the crossed products given by the pseudo multiplicative unitary, and find suitable Haar operator valued weights for these structures.

We study, in the last chapter, two kinds of examples. The first one is pretty natural and comes from matched pairs of groups actions: a very general example of matched pair of groups is the "ax+b" group ([4] Chap 4), and pentagonal transformations lead also to such actions ([4] Prop 5.1). We prove that, for any locally compact group G, which is a matched pair G_1G_2 in the sense of [3], and which acts on a locally compact space X, then $X \times G_1$, $X \times G_2$ is a matched pair of groupoids in $X \times G$. Moreover, G_1 (resp. G_2) acts, as a group, on the space $X \times G_2$ (resp.$X \times G_1$) in such a way that their usual crossed product is the one obtained using chapter 3. We investigate the quantum groupoid structure given to these crossed products by chapter 4 which is actually different from the one given to any crossed product as the dual of a transformation group. The second example is the farthest possible from groups, it comes from principal groupoids of the form $X \times X$ where $X = X_1 \times X_2$ is the cartesian product of two locally compact spaces, we prove that the structures given by the previous chapters mixes the ones given by the pair groupoids $X_1 \times X_1$ and $X_2 \times X_2$.

Several continuations of this article can be considered. One can weaken the condition $\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{G}^0$, which even with finite groups or groupoids gives substantial examples ([26], [1] 2.8). Also a characterization of these objects in terms of cleft extensions in the spirit of S.Vaes and L.Vainerman [22] should be obtained .

2. Measured quantum groupoids and their actions

2.1. Measured quantum groupoids

Let's recall the definition of a measured quantum groupoid due to Enock which extends Lesieur's works. We use [14], [15] and [6] for general references, in particular we suppose known spatial theory and relative tensor products ([5], [18]) .

Definition 2.1. *A* **measured quantum groupoid** *is a special collection* $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ *such that:*

i) M, N *are two von Neumann algebras,* $\alpha : N \to M$ *and* $\beta : N^o \to M$ *are commuting faithful normal non degenerate representations,*

ii) $\Gamma : M \to M \underset{N}{{}_\beta\star_\alpha} M$ *is a one to one normal morphism such that:*

$$\Gamma(\beta(x)) = 1 \underset{N}{{}_\beta\otimes_\alpha} \beta(x)$$
$$\Gamma(\alpha(x)) = \alpha(x) \underset{N}{{}_\beta\otimes_\alpha} 1$$
$$(\Gamma \underset{N}{{}_\beta\star_\alpha} id)\Gamma = (id \underset{N}{{}_\beta\star_\alpha} \Gamma)\Gamma.$$

iii) T *(resp* T'*) is a faithful semi finite normal operator valued weight from* M *to* $\alpha(N)$ *(resp* $\beta(N)$*) such that:*

$$(id \underset{N}{{}_\beta\star_\alpha} T)\Gamma(x) = T(x) \underset{N}{{}_\beta\otimes_\alpha} 1 \textit{ for any } x \in \mathcal{M}_T^+$$
$$(T' \underset{N}{{}_\beta\star_\alpha} i)\Gamma(x) = 1 \underset{N}{{}_\beta\otimes_\alpha} T'(x) \textit{ for any } x \in \mathcal{M}_{T'}^+$$

iv) ν *is a faithful semi finite normal weight on* N *which is relatively invariant with respect to* T *and* T'*, i.e. for any* $t \in \mathbb{R} : \sigma_t^\Phi \sigma_t^\Psi = \sigma_t^\Psi \sigma_t^\Phi$*, where* $\Phi = \nu \circ \alpha^{-1} \circ T$ *and* $\Psi = \nu \circ \beta^{-1} \circ T'$ *.*

Remark 2.2. The assertion iii) can be replaced by the weights conditions:
iii)' $(id \underset{N}{{}_\beta\star_\alpha} \Phi)\Gamma(x) = T(x)$ for any $x \in \mathcal{M}_T^+$

$(\Psi \underset{N}{{}_\beta\star_\alpha} i)\Gamma(x) = T'(x)$ for any $x \in \mathcal{M}_{T'}^+$

We shall say that the quantum groupoid is commutative (respectively symmetric) if M is abelian (resp. $\varsigma\Gamma = \Gamma$, where $\varsigma : M \underset{N}{{}_\beta\star_\alpha} M \to M \underset{N^o}{{}_\alpha\star_\beta} M$ is the natural flip).

Definition 2.3. ([10] 5.6) *Let N be a von Neumann algebra and ν a faithful normal semifinite weight on N, let α (resp. $\beta, \hat{\beta}$) be a faithful non degenerate representation (resp. anti representations) on a Hilbert space $\mathfrak{H}$ commuting two by two, a* **pseudo multiplicative unitary** *W over the basis $(N, \alpha, \beta, \hat{\beta})$ is a unitary from $\mathfrak{H} \ {}_\beta\!\underset{\nu}{\otimes}_\alpha \mathfrak{H}$ to $\mathfrak{H} \ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} \mathfrak{H}$ such that:*

- *W intertwines $\alpha, \beta, \hat{\beta}$, which means that for any $n \in N$ one has:*
$$W(\alpha(n) \ {}_\beta\!\underset{N}{\otimes}_\alpha 1) = (1 \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} \alpha(n))W$$
$$W(1 \ {}_\beta\!\underset{N}{\otimes}_\alpha \beta(n)) = (1 \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} \beta(n))W$$
$$W(\hat{\beta}(n) \ {}_\beta\!\underset{N}{\otimes}_\alpha 1) = (\hat{\beta}(n) \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} 1)W$$
$$W(1 \ {}_\beta\!\underset{N}{\otimes}_\alpha \hat{\beta}(n)) = (\beta(n) \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} 1))W$$

- *The operator W satisfies the "pentagonal" equation:*

$$(1_{\mathfrak{H}} \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} W)(W \ {}_\beta\!\underset{N}{\otimes}_\alpha 1_{\mathfrak{H}}) =$$
$$= (W \ {}_\beta\!\underset{N^o}{\otimes}_\alpha 1)(\sigma_{\nu^o} \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} 1_{\mathfrak{H}})(1_{\mathfrak{H}} \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} W)\sigma_{2\nu}(1_{\mathfrak{H}} \ {}_\beta\!\underset{N}{\otimes}_\alpha \sigma_{\nu^o})(1_{\mathfrak{H}} \ {}_\beta\!\underset{N}{\otimes}_\alpha W)$$

where σ_{ν^o} is the flip map: $\mathfrak{H} \ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} \mathfrak{H} \to \mathfrak{H} \ {}_{\hat{\beta}}\!\underset{\nu}{\otimes}_\alpha \mathfrak{H}$ and $\sigma_{2\nu}$ is the flip map: $\mathfrak{H} \ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} \mathfrak{H} \ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} \mathfrak{H} \to \mathfrak{H} \ {}_{\hat{\beta}}\!\underset{\nu}{\otimes}_\alpha \mathfrak{H} \ {}_\alpha\!\underset{\nu^o}{\otimes}_{\hat{\beta}} \mathfrak{H}$

Remark 2.4. In fact, measured quantum groupoids and pseudo multiplicative unitaries are closely linked. According to [10] chap. 6, if W is a pseudo multiplicative unitary on $\mathcal{L}(\mathfrak{H})$, it generates two von Neumann algebras M (its right leg) and $\widehat{M}$ (its left leg) and two coproducts Γ and $\widehat{\Gamma}$ on M and $\widehat{M}$ respectively, i.e. two maps verifying definition 2.1 ii). More precisely, for any $m \in M$ and $\hat{m} \in \widehat{M}$, one has: $\Gamma(m) = W^*(1 \ {}_\alpha\!\underset{N^o}{\otimes}_{\hat{\beta}} m)W$ and $\widehat{\Gamma}(\hat{m}) = \sigma_{\nu^o} W(\hat{m} \ {}_\beta\!\underset{N}{\otimes}_\alpha 1)W^* \sigma_\nu$; and conversely, for a given measured quantum groupoid, one can associate a pseudo multiplicative unitary to it, with a manageability condition (implying weak regularity) which leads to a duality theory and the following theorem:

Theorem 2.5. *([14],[15],[6]) Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid, for any n in N let's define $\hat{\beta}(n) = J_\Phi \alpha(n^*) J_\Phi$, then one can associate to $\mathfrak{G}$ a pseudo multiplicative unitary W over the basis $(N, \alpha, \beta, \hat{\beta})$, independent of the choice of T and T', the left leg of which is M and gives Γ. The dual coproduct $\hat{\Gamma}$ on the right leg $\widehat{M}$ leads to a measured quantum groupoid $(N, \hat{M}, \alpha, \hat{\beta}, \widehat{T}, \widehat{T}', \nu)$ denoted $\widehat{\mathfrak{G}}$, the dual of $\mathfrak{G}$.*

To the quantum groupoid $\mathfrak{G}$ one associates a $*$-antiautomorphism of M called the coinverse, which is involutive, and verifies the condition $R \circ \alpha = \beta$ and $\Gamma \circ R = \varsigma_{N^o}(R \underset{N}{{}_\beta\otimes_\alpha} R)\Gamma$.

Let j be the application defined for any $x \in M$ by $j(x) = J_\Phi x^* J_\Phi$, and set $\Gamma^c = (j \underset{N}{{}_\beta\star_\alpha} j)\Gamma \circ j, T^c = j \circ T \circ j, R^c = j \circ R \circ j$.

Then one can consider two other quantum groupoids:

- the commutant $\mathfrak{G}^c = (N^o, M', \hat{\beta}, \hat{\alpha}, \Gamma^c, T^c, R^c T^c R^c, \nu^0)$
- the commutant of the dual:

  $(\widehat{\mathfrak{G}})^c = (N^o, (\widehat{M})', \beta, \hat{\alpha}, (\widehat{\Gamma})^c, (\widehat{T})^c, (\widehat{R})^c(\widehat{T})^c(\widehat{R})^c, \nu^0)$ this last is an important tool for the duality of actions.

2.2. Measured quantum groupoids in action

As quantum groups act on von Neumann algebras, measured quantum groupoids also act on (von Neumann) modules with isomorphic basis. Generalizing in this context what we have done in the finite dimensional situation in [25], M. Enock has given in [6] a nice framework for these actions together with double crossed product theorems. Let's recall some of his definitions.

Definition 2.6. *Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a given measured quantum groupoid, and let A be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A right (resp. left) action of $\mathfrak{G}$ on A is a pair $(b, \mathfrak{a})$ such that:*
i) $b : N \to A$ is an injective $*$-antihomomorphism (resp.morphism),
ii) $\mathfrak{a} : A \to A \underset{N}{{}_b\star_\alpha} M$ (resp. $A \underset{N^o}{{}_b\star_\beta} M$) is an injective $*$-homomorphism,

iii) for all $n \in N$ *one has:* $\mathfrak{a}(b(n)) = 1 \underset{N}{{}_b\otimes_\alpha} \beta(n)$

$$\text{(resp. } \mathfrak{a}(b(n)) = 1 \underset{N^o}{{}_b\otimes_\beta} \alpha(n)\text{)}$$

and one has: $(\mathfrak{a} \underset{N}{{}_b\star_\alpha} id)\mathfrak{a} = (1 \underset{N}{{}_b\star_\alpha} \Gamma)\mathfrak{a}$ *(resp.* $(\mathfrak{a} \underset{N^o}{{}_b\star_\beta} id)\mathfrak{a} = (1 \underset{N^o}{{}_b\star_\beta} \varsigma\Gamma)\mathfrak{a}$*).*

Definition 2.7. *Let* $(b, \mathfrak{a})$ *be a right (resp.left) action of a given measured quantum groupoid* $\mathfrak{G}$ *on a given von Neumann algebra* A*, then:*

i) The crossed product of A *by the action* $(b, \mathfrak{a})$ *is the sub von Neumann algebra of* $A \underset{N}{{}_b\star_\alpha} \mathcal{L}(\mathcal{H}_\phi)$ *(resp.* $A \underset{N^o}{{}_b\star_\beta} \mathcal{L}(\mathcal{H}_\phi)$*) generated by* $\mathfrak{a}(A)$ *and* $1 \underset{N}{{}_b\otimes_\alpha} \widehat{M}'$ *(resp.* $1 \underset{N^o}{{}_b\otimes_\beta} \widehat{M}$*). It will be denoted by* $A \rtimes_{\mathfrak{a}} \mathfrak{G}$*.*

ii) The invariant subalgebra is defined by:

$$A^{\mathfrak{a}} = \{x \in A \cap b(N)'/\mathfrak{a}(x) = x_b \otimes_\alpha 1\}$$

Theorem 2.8. *(*[6] *6.13, 9.3, 9.5, 10.11, 10.12) Let* $(b, \mathfrak{a})$ *be a right action of a given measured quantum groupoid* $\mathfrak{G}$ *on a given von Neumann algebra* A*, and set* $\Phi = \nu \circ \alpha^{-1} \circ T$*, then:*

i) for any $x \in A^+$*, the extended positive element of* A*:*

$$T_{\mathfrak{a}}(x) = (id \underset{\nu}{{}_b\star_\alpha} \Phi)\mathfrak{a}(x)$$

is an extended positive element of $A^{\mathfrak{a}}$ *and* $T_{\mathfrak{a}}$ *is a normal faithful operator valued weight from* A *to* $A^{\mathfrak{a}}$

ii) there exists a unique action $(1_b \otimes_\alpha \hat{\alpha}, \tilde{\mathfrak{a}})$ *of* $\hat{\mathfrak{G}}^c$ *on* $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ *which verifies for any* $x \in A$*,* $y \in \hat{M}'$*:*

$$\tilde{\mathfrak{a}}(\mathfrak{a}(x)) = \mathfrak{a}(x)_{\hat{\alpha}} \otimes_\beta 1$$
$$\tilde{\mathfrak{a}}(1_b \otimes_\alpha y) = 1_b \otimes_\alpha \hat{\Gamma}^c(y)$$

iii) for any $y \in \hat{M}'$*:* $T_{\tilde{\mathfrak{a}}}(1_b \otimes_\alpha y) = 1_b \otimes_\alpha \hat{T}^c(y) = \mathfrak{a}(b(\beta^{-1}(\hat{T}^c(y))))$*,* $T_{\tilde{\mathfrak{a}}}$ *is semi finite, and* $(A \rtimes_\alpha \mathfrak{G})^{\tilde{\mathfrak{a}}} = \mathfrak{a}(A)$*.*

Corollary 2.9. *For any normal semi finite faithful operator valued weight* θ *on* A*, the operator valued weight* $\tilde{\theta} = \mathfrak{a} \circ \theta \circ \mathfrak{a}^{-1} \circ T_{\tilde{\mathfrak{a}}}$ *is normal semi finite faithful on* $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ *and will be called the* **dual operator valued weight** *of* θ*.*

In fact, the examples we shall deal with in this article come from an action of a commutative measured quantum groupoid on a commutative von Neumann algebra, nevertheless, as we shall see, the crossed product will be a substantial non commutative quantum groupoid. So let us see more precisely the commutative situation.

2.3. **The abelian case**

All measured quantum groupoids involved in this article will have a commutative basis, and the Hilbert spaces will be of the form $L^2(X, dx)$ where X is a second countable locally compact space endowed with a Radon measure, hence this will simplify the relative tensor products. Let (Y, dy) be a locally compact space endowed with a Radon measure and let $(L^2(X, dx), \beta)$ (resp. $(L^2(Z, dz), \alpha)$) be a faithful normal representation of $N = L^\infty(Y, ds)$, then the relative tensor product $L^2(X, dx) \underset{N}{{}_\beta\otimes_\alpha} L^2(Z, dz)$ is the completion of the algebraic tensor product $\mathcal{K}(X) \odot \mathcal{K}(Z)$ equipped with the pre-scalar product, defined for any $f_1, f_2 \in \mathcal{K}(X)$ and any $g_1, g_2 \in \mathcal{K}(Z)$ by:

$$\begin{aligned}(f_1 \odot g_1, f_2 \odot g_2) &= (\alpha(\frac{d\omega_{f_1,f_2} \circ \beta}{dy})g_1, g_2) = (\beta(\frac{d\omega_{g_1,g_2} \circ \alpha}{dy})f_1, f_2) \\ &= \int_Z \alpha(\frac{d\omega_{f_1,f_2} \circ \beta}{dy})g_1(z)\overline{g_2(z)}dz \\ &= \int_X \beta(\frac{d\omega_{g_1,g_2} \circ \alpha}{dy})f_1(x)\overline{f_2(x)}dx\end{aligned}$$

In our framework, the relative tensor product $L^2(X, dx) \underset{N}{{}_\beta\otimes_\alpha} L^2(Z, dz)$ will also be viewed as $L^2(X_\beta \times_\alpha Z, dx_\beta\times_\alpha dz)$, where $X_\beta \times_\alpha Z$ is a suitable fibred product of X and Z and $dx_\beta\times_\alpha dz$ is a Radon measure; we shall intensively use this identification throughout this paper.

The commutative measured quantum groupoids (i.e M is commutative) are, as expected, coming from measured groupoids in the sense of Jean Renault ([17],[12]). Notwithstanding the fact that a Weyl theorem does not exist in that context, up to some inessential reduction (see [16] theorem 4.1) we can deal with a Hausdorff locally compact groupoid $\mathcal{G}$, we shall suppose it is σ- compact and endowed with a Haar system $\{\lambda^u / u \in \mathcal{G}^0\}$ and a quasi invariant measure ν on $\mathcal{G}^0$, we shall denote $\mu = \int_{\mathcal{G}^0} \lambda^u d\nu(u)$ the integrated measure on $\mathcal{G}$, for any $u \in \mathcal{G}^0$, λ_u will be the image of λ^u by the application $g \mapsto g^{-1}$, and δ will be the Radon Nikodym derivative of μ^{-1} w.r.t. μ . This will allow us to use the $*$-algebra of continuous numerical functions on $\mathcal{G}$ with compact support, which will be noted $\mathcal{K}(\mathcal{G})$.

If $s(g) = g^{-1}g$ (resp. $r(g) = gg^{-1}$) is the source (resp. goal) of any element g of $\mathcal{G}$, then one can define two representations of $N = L^\infty(\mathcal{G}^0, \nu)$ on $M = L^\infty(\mathcal{G}, \mu)$, (which are also antirepresentations) defined for any $f \in L^\infty(\mathcal{G}^0, \nu)$ by:

$$s_{\mathcal{G}}(f) = f \circ s, \quad r_{\mathcal{G}}(f) = f \circ r$$

One easily verifies that for any $f, f' \in \mathcal{K}(\mathcal{G})$ and for ν-almost any $u \in \mathcal{G}^0$, one has:

$$\frac{d\omega_{f,f'} \circ r}{d\nu}(u) = \int_{\mathcal{G}} f(g)\overline{f'(g)}d\lambda^u$$
$$\frac{d\omega_{f,f'} \circ s}{d\nu}(u) = \int_{\mathcal{G}} \delta(v) f(v^{-1})\overline{f'(v^{-1})}d\lambda^u(v)$$

Notation 2.10. One can define for any $i \in \{s, r\}$, $\mathcal{G}^2_{i,j} = \{(g, g') \in \mathcal{G} \times \mathcal{G} / i(g) = j(g')\}$, for instance $\mathcal{G}^2_{s,r} = \mathcal{G}^2$ (the set of pairs of composable elements), let us note $\mu^2_{i,j} = \mu_{i_{\mathcal{G}}} \times_{j_{\mathcal{G}}} \mu$, so $L^2(\mathcal{G}, \mu) \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{i_{\mathcal{G}}}\otimes_{j_{\mathcal{G}}}} L^2(\mathcal{G}, \mu)$ and $L^2(\mathcal{G}^2_{i,j}, \mu^2_{i,j})$ are isomorphic and isometric.

In [24] and [23] we proved that we can associate to $\mathcal{G}$ a pseudo multiplicative unitary $W_{\mathcal{G}} : L^2(\mathcal{G}^2_{s,r}, \mu^2_{s,r}) \to L^2(\mathcal{G}^2_{r,r}, \mu^2_{r,r})$. It is given for any $\xi \in L^2(\mathcal{G}^2_{s,r}, \mu_{s,r})$ and $\mu^2_{r,r}$ almost any $(x, y) \in \mathcal{G}^2_{r,r}$ by:

$$W_{\mathcal{G}}\xi(x, y) = \xi(x, x^{-1}y)$$

The left leg of $W_{\mathcal{G}}$ generates the commutative quantum groupoid:

$$\mathfrak{G}(\mathcal{G}) = (L^\infty(\mathcal{G}^0, \nu), L^\infty(\mathcal{G}, \mu), r_{\mathcal{G}}, s_{\mathcal{G}}, \Gamma_{\mathcal{G}}, T_{\mathcal{G}}, T^{-1}_{\mathcal{G}}, \nu),$$

with the following formulas, for any $f \in \mathcal{K}(\mathcal{G})$:

- **Coproduct**

$\Gamma_{\mathcal{G}}(f)(x, y) = f(xy)$ for any $(x, y) \in G^2_{s_{\mathcal{G}},r_{\mathcal{G}}}$

- **Left and right operator valued weights**

  For μ-almost any element $g \in \mathcal{G}$:

$$T_{\mathcal{G}}(f)(g) = \int_{\mathcal{G}} f(x) d\lambda^{r(g)}(x)$$
$$T_{\mathcal{G}}^{-1}(f)(g) = \int_{\mathcal{G}} f(y) d\lambda_{s(g)}(y)$$

The right leg of $W_{\mathcal{G}}$ generates the symmetric quantum groupoid, which means the coproduct is invariant by a natural flip.

$$\widehat{\mathfrak{G}}(\mathcal{G}) = (L^\infty(\mathcal{G}^0, \nu), \mathcal{L}(\mathcal{G}), r_{\mathcal{G}}, r_{\mathcal{G}}, \widehat{\Gamma}_{\mathcal{G}}, \widehat{T}_{\mathcal{G}}, \widehat{T}_{\mathcal{G}}, \nu),$$

where $\mathcal{L}(\mathcal{G})$ is the left regular algebra of $\mathcal{G}$, which is the sub von Neumann algebra of $\mathcal{L}(L^2(\mathcal{G}, \mu))$ generated by the operators defined for any $f, h \in \mathcal{K}(\mathcal{G})$ by:

$$\lambda(f)h(x) = \int_{\mathcal{G}} f(g) h(g^{-1}x) d\lambda^{r_{\mathcal{G}}(x)}(g)$$

For any $f \in \mathcal{K}(\mathcal{G})$, $\xi \in L^2(\mathcal{G}^2_{r_{\mathcal{G}}, r_{\mathcal{G}}}, \mu_{r_{\mathcal{G}}, r_{\mathcal{G}}})$ and almost any $(x, y) \in \mathcal{G}^2_{r_{\mathcal{G}}, r_{\mathcal{G}}}$, one has:

- **Coproduct**:

  $\hat{\Gamma}_{\mathcal{G}}(\lambda(f))\xi(x, y) = \int_{\mathcal{G}} f(g) \xi(g^{-1}x, g^{-1}y) d\lambda^{r_{\mathcal{G}}(x)}(g)$

- **Left (= right) operator valued weight**

  $\widehat{T}_{\mathcal{G}}(\lambda(f)) = r_{\mathcal{G}}(f \mid \mathcal{G}^0)$

Remark 2.11. Of course one can consider the **right** regular representation of $\mathcal{G}$ which generates in $\mathcal{L}(L^2(\mathcal{G}, \mu))$ the commutant of $\mathcal{L}(\mathcal{G})$ and gives a commutant structure of quantum groupoid:

$$\widehat{\mathfrak{G}}'(\mathcal{G}) = (L^\infty(\mathcal{G}^0, \nu), \mathcal{R}(\mathcal{G}), s_{\mathcal{G}}, s_{\mathcal{G}}, \widehat{\Gamma}'_{\mathcal{G}}, \widehat{T}'_{\mathcal{G}}, \widehat{T}'_{\mathcal{G}}, \nu),$$

2.3.1. The pair groupoid example

We suppose that $\mathcal{G} = X \times X$ where X is an Hausdorff locally compact space together with some Radon measure ν. $\mathcal{G}$ is given its natural locally compact groupoid structure, $\mathcal{G}^0$ is the diagonal of $X \times X$ which will be identified with X, its Haar system is $(\delta_x \otimes \nu)_{x \in X}$ and ν is a (quasi) invariant measure.

The von Neuman algebra $\mathcal{R}(\mathcal{G})$ is isomorphic to $\mathcal{L}(L^2(X,\nu))$ as for any $h \in \mathcal{K}(X^2)$, one has $\rho(h) = 1 \otimes T_h$, where T_h is the integral operator defined for any $\xi \in L^2(X,\nu)$ by:

$$T_h\xi(y) = \int_X h(y,b)\xi(b)d\nu(b)$$

Here the Hilbert space $L^2(\mathcal{G}^2_{s,r},\mu^2_{s,r})$ (resp.$L^2(\mathcal{G}^2_{s,s},\mu^2_{s,s})$) can be identified with $L^2(X^3,\nu^{\otimes 3})$ using map: $((x,y),(y,t)) \to (x,y,t)$ from $\mathcal{G}^2_{s,r}$ to X^3 (resp.$((x,y),(t,y)) \to (x,y,t)$ from $\mathcal{G}^2_{s,s}$ to X^3). With this identification, for any $f \in \mathcal{K}(\mathcal{G})$, and any $(x,y) \in \mathcal{G}$, one has:

$$\Gamma_{\mathcal{G}}(f) = f_{13}, \qquad \widehat{\Gamma}'_{\mathcal{G}}(\rho(f)) = 1 \otimes T_f \otimes 1$$

$$T_{\mathcal{G}}(f))(x) = \int_X f(x,b)d\nu(b) \qquad T^{-1}_{\mathcal{G}}(f)(y) = \int_X f(b,y)d\nu(b)$$

$$\widehat{T}'_{\mathcal{G}}(\rho(f))(x,y) = f(y,y)$$

Let's recall what is a $\mathcal{G}$-space or an action of $\mathcal{G}$ ([12] chap.2),

Notation 2.12. Let X,Y be two Borel spaces. Let us call **fibration** of X by Y any Borel map $\flat : X \to Y$ which is onto. When $Y = \mathcal{G}^0$ and for any $i \in \{s,r\}$, let $X_\flat \times_i \mathcal{G}$ be the fiber product of X and $\mathcal{G}$ which is the set $\{(x,g) \in X \times \mathcal{G} / \flat(x) = i(g)\}$.

Definition 2.13. A (right) $\mathcal{G}$-space is a Borel space X endowed with a fibration $\flat : X \to \mathcal{G}^0$ and a Borel map $(x,g) \mapsto x.g$ from $X_\flat \times_r \mathcal{G}$ to X such that:

i) For all $(x,g) \in X_\flat \times_r \mathcal{G}$, $\flat(x.g) = s(g)$ and $x.\flat(x) = x$

ii) For all $(x,g_1) \in X_\flat \times_r \mathcal{G}$ and all $g_2 \in \mathcal{G}^{s(g_1)}$ then $x.(g_1g_2) = (x.g_1).g_2$

Definition 2.14. A (right) locally compact $\mathcal{G}$-space is a locally compact space X endowed with a structure of (right) $\mathcal{G}$-space such that $\flat$ is open and continuous and $(x,g) \mapsto x.g$ is continuous.

One can also say that $\mathcal{G}$ acts on X. Let us now suppose that X is endowed with a (positive faithful) Radon measure θ such that $\flat_\star\theta$ is absolutely continuous w.r.t. ν, then one easily sees that $b : L^\infty(\mathcal{G}^0,\nu) \to L^\infty(X,\theta)$ defined by $b(f) = f \circ \flat$ is a *(anti)isomorphism, also for any i in

$\{s,r\}$ the von Neumann algebra $L^\infty(X,\theta) \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{\flat}\star_{i_\mathcal{G}}} L^\infty(\mathcal{G},\mu)$ is canonically isomorphic to $L^\infty(X_\flat \times_i \mathcal{G}, \theta_\flat \times_i \mu)$ for a suitable Radon measure $\theta_\flat \times_i \mu$ on $X_\flat \times_i \mathcal{G}$, one easily proves that:

Proposition 2.15. *Let X be a (right) locally compact $\mathcal{G}$-space endowed with a Radon measure θ such that $\flat_\star\theta$ is absolutely continuous w.r.t. ν, let $b : L^\infty(\mathcal{G}^0,\mu) \to L^\infty(X,\theta)$ be defined by $b(f) = f\circ\flat$ for any f in $L^\infty(X,\theta)$, let $\mathfrak{a} : L^\infty(X,\theta) \to L^\infty(X_\flat \times_r \mathcal{G}, \theta_\flat \times_r \mu)$ be defined by $\mathfrak{a}(f)(x,g) = f(x.g)$, then $(b,\mathfrak{a})$ is an action of $\mathfrak{G}(\mathcal{G})$ on $L^\infty(X,\theta)$.*

In the conditions above, if X is second countable, then the crossed product $L^\infty(X,\theta) \rtimes_\mathfrak{a} \mathfrak{G}(\mathcal{G})$ is also $\widehat{\mathfrak{G}}'(X_\flat \times_r \mathcal{G})$, where $X_\flat \times_r \mathcal{G}$ is given its structure of semi direct product. The existence of b, given by Proposition 2.15, leads to integral decompositions $\theta = \int_{\mathcal{G}^0} \theta^u d\nu(u)$, $L^2(X,\theta) = \int_{\mathcal{G}^0} L^2(X^u,\theta^u)d\nu(u)$ and $L^\infty(X,\theta) = \int_{\mathcal{G}^0} L^\infty(X^u)d\nu(u)$, where $X^u = \{x \in X/\flat(x) = u\}$ and the support of θ^u is X^u for any $u \in \mathcal{G}^0$.

For any Radon measure θ' on X, we shall say that θ' is invariant under $\mathfrak{a}$ if and only if, for any $g \in \mathcal{G}$, and any $f \in \mathcal{K}(X^{s(g)})$ one has: $\int f(x)d\theta^{s(g)} = \int f(y.g)d\theta^{r(g)}$.

Lemma 2.16. *If θ and θ' are two Radon measures on X, invariant under $\mathfrak{a}$, then the Radon Nikodym derivative $\frac{d\theta'}{d\theta}$ is an element of $L^\infty(X,\theta)^\mathfrak{a}$.*

Proof: This is easy and a basic consequence of [7] 7.5 to 7.8. ☐

Let us now give a description of the crossed product $L^\infty(X,\theta) \rtimes_\alpha \mathfrak{G}(\mathcal{G})$ using a certain $*$-algebra representation. The vector space $\mathcal{K}(X_\flat \times_r \mathcal{G})$ can be given a $*$-algebra structure denoted by $(\mathcal{K}(X_\flat \times_r \mathcal{G}), \star, \#)$.

For any F, F' in $\mathcal{K}(X_\flat \times_r \mathcal{G})$ and any (x,g) in $X_\flat \times_r \mathcal{G}$, one has:

$$F \star F'(x,g) = \int F(x,h)F'(x.h, h^{-1}g)d\lambda^{r(g)}(h)$$

$$F^\#(x,g) = \overline{F(x.g, g^{-1})}\delta(g^{-1})$$

One can define a representation of $\mathcal{K}(X_b \times_r \mathcal{G})$ in $L^2(X_b \times_r \mathcal{G}, \theta_\flat \times_r \mu)$, let us note it $\mathfrak{R}$, it is defined for any ξ in $L^2(X_b \times_r \mathcal{G}, \theta_\flat \times_r \mu)$, any F in $\mathcal{K}(X_b \times_r \mathcal{G})$ and $\theta_\flat \times_r \mu$-almost any (x,g) in $X_b \times_r \mathcal{G}$ by:

$$\mathfrak{R}(F)\xi(x,g)=\int F(x.g,h)\delta(h)^{-\frac{1}{2}}\xi(x,gh)d\lambda^{s(g)}(h)$$

The crossed product $L^\infty(X,\theta)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G})$ is generated by the image of $\mathfrak{R}$. More precisely, for any $f\in\mathcal{K}(X)$ and any $k\in\mathcal{K}(\mathcal{G})$, one has: $\mathfrak{a}(f)(1_{\mathfrak{b}}\otimes_r\rho(k))=\mathfrak{R}(f\otimes k)$.

3. A generalization of the matched pair procedure

3.1. Measured matched pair of groupoids

Now let's explain a triple extension of the commutative examples (in 2.3), those studied in [23] in finite dimension, and in [22] (or [3]) in the quantum groups case.

All the groupoids involved will be as mentioned in 2.3 till the end; mimicking the case of matched pairs of locally compact groups, let's give the following definition:

Definition 3.1. *Let $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$ be measured groupoid such that $\mathcal{G}_1,\mathcal{G}_2$ are closed subgroupoids of $\mathcal{G}$. We shall say that $\mathcal{G}_1,\mathcal{G}_2$ is a matched pair of measured subgroupoids of $\mathcal{G}$ if and only if:*
i) $\mathcal{G}_1\cap\mathcal{G}_2=\mathcal{G}^0$
ii) $\mathcal{G}_1\mathcal{G}_2:=\{g_1g_2/g_1\in\mathcal{G}_1,g_2\in\mathcal{G}_2^{s(g_1)}\}$ is μ-conegligible in $\mathcal{G}$
iii) the measure ν on $\mathcal{G}^0$ is quasi invariant for the three Haar systems.

Remark 3.2. Condition iii) is absolutely necessary to obtain, as in the case of groups ([3] prop. 3.2), the Haar system of $\mathcal{G}$ from those of $\mathcal{G}_1$ and $\mathcal{G}_2$. for instance if $\mathcal{G}$ is a principal groupoid of the form $X\times X$ where X is any locally compact space, let's choose $\mathcal{G}_1=\mathcal{G}$ and $\mathcal{G}_2=\mathcal{G}^0=X$, then if ν and ν_1 and ν_3 are any Radon measures on X, one can construct the Haar systems $(\delta_x\times\nu)_{x\in X}$ and $(\delta_x\otimes\nu_1)_{x\in X}$ on $\mathcal{G}$ and $\mathcal{G}_1$ and the quasi invariant measures ν and ν_1 respectively, there is no hope to give any formula connecting the Haar systems. We shall now prove such a formula, when the groupoids are given the same quasi invariant measure, using an argument similar to that of [3] prop. 3.2 and the uniqueness condition of Enock ([7] Corollary 7.8) recalled in 2.16.

Lemma 3.3. *1) The von Neumann fiber product* $L^\infty(\mathcal{G}_1,\mu_1) \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{s_1}\star_{s_2}} L^\infty(\mathcal{G}_2,\mu_2)$ *is isomorphic to* $L^\infty(\mathcal{G}_{1s}\times_s\mathcal{G}_2, \mu_{1s}\times_s\mu_2)$*, where the measure* $\mu_{1s}\times_s\mu_2$ *is given for any* $f \in \mathcal{K}(\mathcal{G}_{1s}\times_s\mathcal{G}_2)$ *by:*

$$(\mu_{1s}\times_s\mu_2)(f) = \\ = \int_{\mathcal{G}^0}\int_{\mathcal{G}_{1r}\times_r\mathcal{G}_2} f(g_1^{-1}, g_2^{-1})\delta_1(g_1^{-1})\delta_2(g_2^{-1})d\lambda_1^u(g_1)d\lambda_2^u(g_2)d\nu(u)$$

2) The restriction to $L^\infty(\mathcal{G}_{1s}\times_s\mathcal{G}_2, \mu_{1s}\times_s\mu_2)$ *of the natural action by left multiplication of* $\mathfrak{G}(\mathcal{G}_1\times\mathcal{G}_2)$ *on* $L^\infty(\mathcal{G}_1\times\mathcal{G}_2, \mu_1\times\mu_2)$ *is well defined; if* $(\mathfrak{a}, r_{1s}\otimes_s r_2)$ *is this action, then for any* $f \in L^\infty(\mathcal{G}_{1s}\times_s\mathcal{G}_2, \mu_{1s}\times_s\mu_2)$*,* $\mu_{1s}\times_s\mu_2$*-any element* $(g_1,g_2) \in G_{1s}\times_s\mathcal{G}_2$*,* μ_1*-any* $h_1 \in \mathcal{G}_1$*,* μ_2*-any* $h_2 \in \mathcal{G}_2$ *,one has:*

$$\mathfrak{a}(f)(g_1,g_2,h_1,h_2) = f(h_1^{-1}g_1, h_2^{-1}g_2)$$

and $\mu_{1s}\times_s\mu_2$ *is invariant under* $\mathfrak{a}$ *in the sense of the end of Paragraph 2.3.*

Proof: The assertion 1) is a simple calculation (see for instance [23] 3.1) and the second one, an obvious consequence of 1), as one deals with the left multiplication of $\mathcal{G}_1\times\mathcal{G}_2$ and its canonical Haar system. □

Lemma 3.4. *Let* $\tilde{\mu}$ *be the measure on* $\mathcal{G}_{1s}\times_s\mathcal{G}_2$ *defined for any* $f \in \mathcal{K}(\mathcal{G}_{1s}\times_s\mathcal{G}_2)$ *by:*

$$\tilde{\mu}(f) = \int_{u\in\mathcal{G}^0}\int_{\mathcal{G}_1\mathcal{G}_2} \tilde{f}(\theta^{-1}(g))d\lambda^u(g)d\nu(u)$$

where the map $\theta : \mathcal{G}_{1s}\times_s\mathcal{G}_2 \to \mathcal{G}_1\mathcal{G}_2$ *is given by* $\theta(g_1,g_2) = g_1g_2^{-1}$ *and* $\tilde{f}(g_1,g_2) = f(g_1,g_2)\delta(g_2)$*. Then* $\tilde{\mu}$ *is invariant under the action* $\mathfrak{a}$*.*

Proof: This is the same argument as in Lemma 4.10 of [22]. □

Proposition 3.5. *Let* $\mathcal{G}_1, \mathcal{G}_2$ *be a matched pair of measured subgroupoids of* $\mathcal{G}$*, and let* δ_i *be the modular function of* ν *(relatively to* $\mathcal{G}_i$*) for* $i = 1,2$*, then up to normalization of the Haar systems, for any* $u \in \mathcal{G}^0$ *and any* $f \in \mathcal{K}(\mathcal{G})$*, one has:*

$$\int f d\lambda^u = \iint_{\mathcal{G}_1\times\mathcal{G}_2} f(g_1g_2)\delta(g_2))\delta_2(g_2^{-1})d\lambda_2^{s(g_1)}(g_2)d\lambda_1^u(g_1)$$

$$= \iint_{\mathcal{G}_2\times\mathcal{G}_1} f(g_2g_1)\delta(g_1)\delta_1(g_1^{-1})d\lambda_1^{s(g_2)}(g_1)d\lambda_2^u(g_2)$$

Proof: Due to the Lemmas 3.3 and 3.4, one can apply Lemma 2.16 to $\mu_{1s}\times_s\mu_2$ and $\tilde{\mu}$, so there exists a function $h \in L^\infty(\mathcal{G}_{1s}\times_s\mathcal{G}_2, \mu_{1s}\times_s\mu_2)^{\mathfrak{a}}$ such that $\tilde{\mu} = h(\mu_{1s}\times_s\mu_2)$, hence for $\mu_{1s}\times_s\mu_2$-any $(g_1, g_2) \in \mathcal{G}_{1s}\times_s\mathcal{G}_2$: $h(g_1, g_2) = h(s(g_1), s(g_2))$. For any $F \in \mathcal{K}(\mathcal{G})$, let us define the function $f \in L^\infty(\mathcal{G}_{1s}\times_s\mathcal{G}_2, \mu_{1s}\times_s\mu_2)$ by $f(g_1, g_2) = F(g_1 g_2^{-1})\delta(g_2^{-1})$ for any $(g_1, g_2) \in \mathcal{G}_{1s}\times_s\mathcal{G}_2$; then $\tilde{f} \circ \theta = F$, so one has:

$$
\begin{aligned}
&\int_{u\in\mathcal{G}^0}\int_{\mathcal{G}_1\mathcal{G}_2} F(g))d\lambda^u(g)d\nu(u) = \\
&= \int_{u\in\mathcal{G}^0}\int_{\mathcal{G}_1\mathcal{G}_2} \tilde{f}(\theta^{-1}(g))d\lambda^u(g)d\nu(u) = \tilde{\mu}(f) = (\mu_{1s}\times_s\mu_2)(hf) = \\
&= \int_{\mathcal{G}^0}\int_{\mathcal{G}_{1r}\times_r\mathcal{G}_2} h(g_1^{-1}, g_2^{-1})f(g_1^{-1}, g_2^{-1})\delta_1(g_1^{-1})\delta_2(g_2^{-1})d\lambda_1^u(g_1)\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times d\lambda_2^u(g_2)d\nu(u) \\
&= \int_{\mathcal{G}^0}\int_{\mathcal{G}_{1r}\times_r\mathcal{G}_2} h(r(g_1), r(g_1))F(g_1^{-1}g_2)\delta_1(g_1^{-1})\delta(g_2)\delta_2(g_2^{-1})d\lambda_2^u(g_2)\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times d\lambda_1^u(g_1)d\nu(u) \\
&= \int_{\mathcal{G}_1}(\int_{\mathcal{G}_2} h(r(g_1), r(g_1))F(g_1^{-1}g_2)\delta_1(g_1^{-1})\delta(g_2)\delta_2(g_2^{-1})d\lambda_2^{r(g_1)}(g_2))\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times d\mu_1(g_1) \\
&= \int_{\mathcal{G}_1}(\int_{\mathcal{G}_2} h(s(g_1), s(g_1))F(g_1 g_2)\delta(g_2)\delta_2(g_2^{-1})d\lambda_2^{s(g_1)}(g_2))d\mu_1(g_1) \\
&= \int_{\mathcal{G}^0}\int_{\mathcal{G}_{1r}\times_r\mathcal{G}_2} F(g_1 g_2)\delta(g_2)\delta_2(g_2^{-1})\lambda_2^{s(g_1)}(g_2)h(s(g_1), s(g_1))d\lambda_1^u(g_1)d\nu(u)
\end{aligned}
$$

This gives the first equality of the Proposition, if one replaces the Haar system λ_1^u by $k\lambda_1^u$, where k is defined by $k(g_1) = h(s(g_1), s(g_1))$ which is still a Haar system (as $k(g_1)$ depends only on $s(g_1)$). The second equality is proven a similar way. □

Remark 3.6. If $\mathcal{G}_1, \mathcal{G}_2$ is a measured matched pair, as $\mathcal{G}_2\mathcal{G}_1 = (\mathcal{G}_1\mathcal{G}_2)^{-1}$ and $\mathcal{G}_1\mathcal{G}_2 \cap \mathcal{G}_2\mathcal{G}_1$ is conegligible then $\mathcal{G}_2, \mathcal{G}_1$ is also a measured matched pair.

3.2. Families of examples

3.2.1. Simplest examples

- If $\mathcal{G}$ is a group then a matched pair of groupoids is a matched pair of groups as in [3].

- If $\mathcal{G}$ is finite then a matched pair is exactly a matched pair as defined in [25].

- If $\mathcal{G}$ is a group bundle (i.e. $s = r$), as for any $u \in \mathcal{G}^0$, $\mathcal{G}^u = \mathcal{G}^u_u$ is a group, the matched pairs are exactly group bundles over G^0 such that for any $u \in \mathcal{G}^0$, $\mathcal{G}^u_1, \mathcal{G}^u_2$ is a matched pair of groups in $\mathcal{G}^u$. The Haar systems being continuous families of Haar measures, in that case, proposition 3.5 is a completely direct consequence of [3] prop. 3.2.

3.2.2. Action of a matched pair of groups

If $\mathcal{G}$ is a locally compact transformation groupoid of the form $X \times G$, where G is a group matched pair G_1G_2 in the sense of [3], acting on the right on X. Let $\mathcal{G}_1$ be equal to $X \times G_1$, $\mathcal{G}_2$ be equal to $X \times G_2$. One can consider the canonical Haar systems $(\delta_x \times ds)_{x\in X}$, $(\delta_x \times ds_1)_{x\in X}$ $(\delta_x \times ds_2)_{x\in X}$ associated with the Haar measures of the groups, hence $\mathcal{G}_1$, $\mathcal{G}_2$ is clearly a matched pair.

Let ν be a quasi invariant measure on $\mathcal{G}^0 = X$ w.r.t. the action of G, let ρ be the Radon Nikodym cocycle for ν and the action, this means that for any $g \in G$ and $h \in \mathcal{K}(X)$ one has $\int h(x.g)d\nu(x) = \int \rho(x,g)h(x)d\nu(x)$. Let Δ (resp. Δ_1, resp. Δ_2) be the modular function of the group G (resp. G_1, resp. G_2), hence by [17] 3.21 one has: $\delta(x,g) = \frac{\Delta(g)}{\rho(x,g)}$ (resp $\delta_i(x,g) = \frac{\Delta_i(g)}{\rho(x,g)}$ for $i = 1,2$). Therefore due to [3] prop. 3.2, for any $f \in \mathcal{K}(X \times \mathcal{G})$ and any $x \in X$, one has:

$$
\begin{aligned}
&\delta_x \times ds(f) = \int f(x,g)ds(g) \\
&= \int_{G_1}\int_{G_2} f(x,g_1g_2)\Delta(g_2)\Delta_2(g_2)^{-1}ds_2(g_2)ds_1(g_1) \\
&= \int_{G_1}\int_{G_2} f((x,g_1)(xg_1,g_2))\frac{\Delta(g_2)}{\rho(xg_1,g_2)}(\frac{\Delta_2(g_2)}{\rho(xg_1,g_2)})^{-1}ds_2(g_2)ds_1(g_1) \\
&= \int_{G_1}\int_{G_2} f((x,g_1)(xg_1,g_2))\delta(xg_1,g_2)\delta_2(xg_1,g_2)^{-1}ds_2(g_2)ds_1(g_1) \\
&= \int_{\mathcal{G}_1}\int_{\mathcal{G}_2} f((y,g_1)(z,g_2))\delta(z,g_2)\delta_2(z,g_2)^{-1}(\delta_{s(x,g_1)} \times ds_2)(z,g_2)\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (\delta_x \times ds_1)(y,g_1)
\end{aligned}
$$

which gives Proposition 3.5 in that case.

3.3. The case of principal and transitive groupoids

Let's describe what are matched pairs in the case where $\mathcal{G}$ is principal proper and transitive.

So we shall suppose that $\mathcal{G} = X \times X$ where X is an Hausdorff locally compact space together with some Radon measure ν, and $\mathcal{G}$ is given its natural locally compact groupoid structure, Haar system and quasi invariant measure ν, as it was explained in remark 3.2.

Let us describe a family of examples and let us show that all matched pairs of a principal proper and transitive groupoid are of this type when X is compact.

We shall suppose that X is equal to $X_1 \times X_2$ where X_1 and X_2 are two Hausdorff locally compact spaces and $\nu = \nu_1 \times \nu_2$, where ν_i is a Radon measure on X_i for $i = 1, 2$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the equivalence relations associated with the natural projections, so one has:

$$\forall (a,b),(c,d) \in X : \quad (a,b)\mathcal{R}_1(c,d) \;\text{ iff }\; a = c$$

$$\forall (a,b),(c,d) \in X : \quad (a,b)\mathcal{R}_2(c,d) \;\text{ iff }\; b = d$$

Let $\mathcal{G}_i$ be the sub groupoids of $\mathcal{G} = X \times X$ associated with $\mathcal{R}_i$ for $i = 1, 2$. One has:

$$\mathcal{G}_1 = \bigsqcup_{x_1 \in X_1} \{x_1\} \times X_2 \times \{x_1\} \times X_2$$

$$\mathcal{G}_2 = \bigsqcup_{x_2 \in X_2} X_1 \times \{x_2\} \times X_1 \times \{x_2\}$$

As X is Hausdorff, clearly $\mathcal{G}_1$ and $\mathcal{G}_2$ are closed in $\mathcal{G}$. For any $(x_1, x_2) \in X (= \mathcal{G}^0)$, let $\lambda^{(x1,x2)}$ be equal to $\delta_{(x_1,x_2)} \times \nu$ which gives the canonical Haar system of $\mathcal{G}$ and let's define two other Radon measures on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively by the formulas:

$$\lambda_1^{(x1,x2)} = \delta_{(x_1,x_2)} \times \delta_{x_1} \times \nu_2$$

$$\lambda_2^{(x1,x2)} = \delta_{(x_1,x_2)} \times \nu_1 \times \delta_{x_2}$$

Lemma 3.7. *The pair* $(\mathcal{G}_1, (\lambda_1^u)_{u\in X}, \nu)$, $(\mathcal{G}_2, (\lambda_2^u)_{u\in X}, \nu)$ *is a matched pair in the measured groupoid* $(\mathcal{G}, (\lambda^u)_{u\in X}, \nu)$.

Proof: For any $(a, b, c, d) \in \mathcal{G}$, one has:

$$(a, b, c, d) = (a, b, a, d).(a, d, c, d)$$

so $\mathcal{G} = \mathcal{G}_1\mathcal{G}_2$, and $\mathcal{G}_1 \cap \mathcal{G}_2 = \bigsqcup_{(x_1,x_2)\in X} \{x_1\} \times \{x_2\} \times \{x_1\} \times \{x_2\} = \mathcal{G}^0$.

For any $(x_1, x_2) \in X$, the support of $\lambda_1^{(x_1,x_2)}$ is clearly $\mathcal{G}_1^{(x_1,x_2)}$. For any (x_1, x_2, x_1, z_2) in $\mathcal{G}_1$ and for any $f \in \mathcal{K}(\mathcal{G}_1)$, one has:

$$\begin{aligned}\int_{\mathcal{G}_1} f((x_1, x_2, x_1, z_2)t) d\lambda_1^{(x_1,z_2)}(t) &= \int_{X_1\times X_2} f(x_1, x_2, t_1, t_2)\delta_{x_1}(t_1) d\nu_2(t_2)\\ &= \int_{X_2} f(x_1, x_2, x_1, t_2) d\nu_2(t_2)\\ &= \int_{\mathcal{G}_1} f(t) d\lambda_1^{(x_1,x_2)}(t)\end{aligned}$$

One easily deduces that $(\lambda_1^u)_{u\in X}$ is a continuous Haar system for $\mathcal{G}_1$. Let μ_1 be equal to $\int_X \lambda_1^u d\nu(u)$, for any f in $\mathcal{K}(\mathcal{G}_1)$ one has:

$$\begin{aligned}&\int_{\mathcal{G}_1} f(z^{-1}) d\mu_1(z) =\\ &= \int_X \int_{\mathcal{G}_1} f((y_1, y_2, z_1, z_2)^{-1})\delta_{(x_1,x_2)} \times \delta_{x_1} \times d\nu_2(y_1, y_2, z_1, z_2) d\nu(x_1, x_2)\\ &= \int_X \int_{X_2} f((x_1, x_2, x_1, z_2)^{-1}) d\nu_2(z_2) d\nu(x_1, x_2)\\ &= \int_{X_1} \int_{X_2} \int_{X_2} f(x_1, z_2, x_1, x_2) d\nu_2(z_2) d\nu_1(x_1) d\nu_2(x_2)\end{aligned}$$

Hence using Fubini's theorem, one has : $\int_{\mathcal{G}_1} f(z^{-1})d\mu_1(z) = \int_{\mathcal{G}_1} f(z)d\mu_1(z)$, so ν is quasi invariant, even invariant, relatively to Haar system $(\lambda_1^u)_{u\in X}$ and $(\mathcal{G}_1, (\lambda_1^u)_{u\in X}, \nu)$ is a measured groupoid. For similar reasons, groupoid $(\mathcal{G}_2, (\lambda_2^u)_{u\in X}, \nu)$ is also a measured groupoid and ν is invariant for the three measured groupoids, so $(\mathcal{G}_1, (\lambda_1^u)_{u\in X}, \nu)$, $(\mathcal{G}_2, (\lambda_2^u)_{u\in X}, \nu)$ is a matched pair. □

Let us verify proposition 3.5 in that case, for any $h \in \mathcal{K}(\mathcal{G})$ and any $(x_1, x_2) \in X$, one has:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} h((a_1,a_2,b_1,b_2)(c_1,c_2,d_1,d_2))d\lambda_2^{s(a_1,a_2,b_1,b_2)}(c_1,c_2,d_1,d_2)\times \\ \times\, d\lambda_1^{(x_1,x_2)}(a_1,a_2,b_1,b_2) =$$

$$\begin{aligned}
&= \int_{X_2}\int_{\mathcal{G}_1} h((x_1,x_2,x_1,b_2)(c_1,c_2,d_1,d_2))d\lambda_2^{(x_1,b_2)}(c_1,c_2,d_1,d_2)d\nu_2(b_2)\\
&= \int_{X_2}\int_{X_1} h(x_1,x_2,d_1,b_2)d\nu_1(d_1)d\nu_2(b_2)\\
&= \int_{X_1}\int_{X_2} h(x_1,x_2,d_1,b_2)d\nu_1(d_1)d\nu_2(b_2)\\
&= \int_{\mathcal{G}} h(z)d\lambda^{(x_1,x_2)}(z)
\end{aligned}$$

This proves that proposition 3.5 is here a reformulation of Fubini's theorem.

Finally, let's prove that we have described all possible examples when X is Hausdorff and compact. So we suppose given a matched pair $\mathcal{G}_1$, $\mathcal{G}_2$ in $X\times X$, there exist two equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ associated to two partitions $(X^1_\alpha), (X^2_\beta)$ of X, using classical arguments (see [11] chap 1 par.4), $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed and Hausdorff, so each element of the partitions is closed as a subset of X, hence compact in X. As well-known, due to the fact that ν is quasi invariant for $\mathcal{G}_i$ $(i = 1, 2)$, there exist Radon measures Λ_i on $X/\mathcal{R}_i$ and Borel functions $h_i : X \to \mathbb{R}^+_\star$ such that if one denotes $\alpha_i : X \to X/\mathcal{R}_i$ the usual projection, one has $\mu = h_i(\Lambda_i \circ \alpha_i)$.

Proposition 3.8. *The map* $\alpha_1 \times \alpha_2 : \quad X \to X/\mathcal{R}_1 \times X/\mathcal{R}_2$ *defined for any* $x \in X$ *by:* $(\alpha_1 \times \alpha_2)(x) = (\alpha_1(x), \alpha_2(x))$*, realizes a homeomorphism of compact spaces, and up to normalization* $(\alpha_1 \times \alpha_2)(\mu) = \Lambda_1 \otimes \Lambda_2$*.*

Proof: The application $\alpha_1 \times \alpha_2$ is clearly continuous, let $x, y \in X$ be such that $(\alpha_1 \times \alpha_2)(x) = (\alpha_1 \times \alpha_2)(y)$ then $(x, y) \in \mathcal{G}_1 \cap \mathcal{G}_2$ so $x = y$, hence $\alpha_1 \times \alpha_2$ is injective. As $\mathcal{G}_1\mathcal{G}_2$ is μ conegligible in $\mathcal{G}$ and compact its complementary is a negligible open set so it is empty and $\mathcal{G}_1\mathcal{G}_2 = \mathcal{G}$, this implies that for any $x, y \in X$, there exists $t \in X$ such that $(x, t) \in \mathcal{G}_1$ and $(t, y) \in \mathcal{G}_2$, hence $\alpha_1(x) = \alpha_1(t)$ and $\alpha_2(y) = \alpha_2(t)$ which means that $(\alpha_1 \times \alpha_2)(t) = (\alpha_1(x), \alpha_2(y))$, so $\alpha_1 \times \alpha_2$ is onto, the lemma follows easily. □

3.4. **The mutual actions of a matched pair of groupoids**

Let's give a generalization to matched pairs of groupoids of the well-known fact that matched pairs of groups act one on the other.

Remark 3.9. As in [4] Chap 2, if $(\mathcal{G}_1, (\lambda_1^u), \nu), (\mathcal{G}_2, (\lambda_2^u), \nu)$ is a given measured matched pair, then for $i = 1$ and 2, there exist Borel functions $p_i : \mathcal{G} \to \mathcal{G}_i$ such that, for any $g \in \mathcal{G}_1\mathcal{G}_2$, $g = p_1(g)p_2(g)$. Of course, there also exists two borel almost everywhere defined maps $p'_i : \mathcal{G} \to \mathcal{G}_i$ such that $g = p'_2(g)p'_1(g)$, for any $g \in \mathcal{G}_2\mathcal{G}_1$; so on the μ- conegligible set $\mathcal{G}_1\mathcal{G}_2 \cap \mathcal{G}_2\mathcal{G}_1$, one has: $g = p_1(g)p_2(g) = p'_2(g)p'_1(g)$. In this framework, new representations appear, the **middle** ones:

Lemma 3.10. *For* μ*-almost any* g *in* $\mathcal{G}$*, one has:* $s \circ p_1(g) = r \circ p_2(g), r \circ p'_1(g) = s \circ p'_2(g)$*, so there exist two* μ*-almost everywhere defined maps such that:*

$$m = s \circ p_1 = r \circ p_2 \quad , \quad \hat{m} = s \circ p'_2 = r \circ p'_1$$

let us note $m_{\mathcal{G}} : f \mapsto f \circ m$ *(resp.*$\hat{m}_{\mathcal{G}} : f \mapsto f \circ \hat{m}$*) the associated representation of* $L^\infty(\mathcal{G}^0, \nu)$*.*

Proof: As for any $g \in \mathcal{G}_1\mathcal{G}_2$, $p_1(g)$ and $p_2(g)$ are composable, the lemma is obvious. □

Lemma 3.11. *For $i = 1$ or 2, let us note $s_i = s_{\mathcal{G}_i}$ and $r_i = r_{\mathcal{G}_i}$. There exists an isomorphism $U : L^2(\mathcal{G},\mu) \to L^2(\mathcal{G}_2,\mu_2)\,{}_{s_2}\!\underset{\nu}{\otimes}_{r_1} L^2(\mathcal{G}_1,\mu_1) (= L^2(\mathcal{G}_{2\,s_2}\!\times_{r_1}\mathcal{G}_1, \mu_{2\,s_2}\!\times_{r_1}\mu_1))$ such that for any $\xi \in \mathcal{K}(\mathcal{G})$ and $\mu_{2\,s_2}\!\times_{r_1}\mu_1$-almost any $(g_2,g_1) \in \mathcal{G}_{2\,s_2}\!\times_{r_1}\mathcal{G}_1$, one has:*

$$U\xi(g_2,g_1) = (\frac{\delta}{\delta_1})^{\frac{1}{2}}(g_1)\xi(g_2g_1)$$

it gives an isomorphism between $L^\infty(\mathcal{G},\mu)$ and $L^\infty(\mathcal{G}_2,\mu_2)_{s_2}\!\underset{\nu}{\star}_{r_1} L^\infty(\mathcal{G}_1,\mu_1)$; these two are also isomorphic to $L^\infty(\mathcal{G}_{2\,s_2}\!\times_{r_1}\mathcal{G}_1, \mu_{2\,s_2}\!\times_{r_1}\mu_1)$.

Proof: This is an obvious consequence of Proposition 3.5 . □

Proposition 3.12. *Let $\mathfrak{a}$ be the map: $L^\infty(\mathcal{G}_2,\mu_2) \to L^\infty(\mathcal{G}_2,\mu_2)\ {}_{s_2}\!\underset{\nu}{\star}_{r_1} L^\infty(\mathcal{G}_1,\mu_1)$ (resp. $\hat{\mathfrak{a}} : L^\infty(\mathcal{G}_1,\mu_1) \to L^\infty(\mathcal{G}_1,\mu_1)\ {}_{r_1}\!\underset{\nu}{\star}_{s_2} L^\infty(\mathcal{G}_2,\mu_2)$) be the map defined for any $f \in L^\infty(\mathcal{G}_2,\mu_2)$ (resp.any $h \in L^\infty(\mathcal{G}_1,\mu_1)$) and almost any $(g_2,g_1) \in \mathcal{G}_{2\,s_2}\!\times_{r_1}\mathcal{G}_1$ (resp. $(g_1,g_2) \in \mathcal{G}_{1\,r_1}\!\times_{s_2}\mathcal{G}_2$) by:*

$$\mathfrak{a}(f)(g_2,g_1) = f(p_2(g_2g_1)) \quad (\text{resp.} \quad \hat{\mathfrak{a}}(h)(g_1,g_2) = h(p_1(g_2g_1)))$$

Then the pair $(s_2,\mathfrak{a})$ (resp. $(r_1,\hat{\mathfrak{a}})$) is a right (resp.left) action of $\mathfrak{G}(\mathcal{G}_1)$ (resp.$\mathfrak{G}(\mathcal{G}_2)$) on $L^\infty(\mathcal{G}_2,\mu_2)$ (resp.$L^\infty(\mathcal{G}_1,\mu_1)$). Moreover one has: $\mathfrak{a}\circ r_2 = \hat{\mathfrak{a}}\circ s_1 = m$ and $\mathfrak{a}\circ s_2 = s$.

Proof: For all $\phi \in L^\infty(\mathcal{G}^0,\nu)$, and $\mu_{2\,s_2}\!\times_{r_1}\mu_1$-almost any $(g_2,g_1) \in \mathcal{G}_2 \times \mathcal{G}_1$ one has:

$$\begin{aligned}\mathfrak{a}(s_2(\phi))(g_2,g_1) &= s_2(\phi))(p_2(g_2g_1)) = \phi(s(p_2(g_2g_1))) = \phi(s(g_1)) \\ &= (1_{s_2}\!\otimes_{r_1} s_1(\phi))(g_2,g_1)\end{aligned}$$

so $\mathfrak{a}(s_2(\phi)) = 1_{s_2}\!\otimes_{r_1} s_1(\phi)$, up to the identification of 3.11, the relation $\mathfrak{a}\circ r_2 = m$ is obtained in the same way.

Let f be any element of $L^\infty(\mathcal{G}_2,\mu_2)$ and let (g_2,g_1,h_1) be any element of $\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_1$ such that such that g_2g_1 and $g_2g_1h_1$ exist and are in $\mathcal{G}_1\mathcal{G}_2$. On the one hand, we have:

$$(\mathfrak{a}\ \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{s_2}\!\star_{r_1}}\ i)\mathfrak{a}(f)(g_2,g_1,h_1) = \mathfrak{a}(f)(p_2(g_2g_1),h_1)) = f(p_2(p_2(g_2g_1)h_1))$$

on the other hand:

$$(i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{s_2}\star_{r_1}} \Gamma_1)\mathfrak{a}(f)(g_2,g_1,h_1) = \mathfrak{a}(f)(g_2,g_1h_1)) = f(p_2(g_2g_1h_1))$$

But one has $p_2(p_2(g_2g_1)h_1) = p_2(p_1(g_2g_1)p_2(g_2g_1)h_1) = p_2(g_2g_1h_1)$, hence:

$$(\mathfrak{a} \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{s_2}\star_{r_1}} i)\mathfrak{a} = (i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_{s_2}\star_{r_1}} \Gamma_1)\mathfrak{a}$$

The demonstration for $\hat{\mathfrak{a}}$ is quite similar. □

Remark 3.13. The crossed product $L^\infty(\mathcal{G}_2,\mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ is, up to an isomorphism, the image in $\mathcal{L}(L^2(\mathcal{G},\mu))$, of the map $\mathcal{R}$, defined for any $F \in \mathcal{K}(\mathcal{G}_{2\,s_2}\!\times_{r_1}\mathcal{G}_1)$ and μ-almost any $g \in \mathcal{G}$ by:

$$\mathcal{R}(F)\xi(g) = \int_{\mathcal{G}_1} F(p_2(g),g_1')\delta(g_1')^{-\frac{1}{2}}\xi(gg_1')d\lambda_1^{s(g)}(g_1')$$

Remark 3.14. Proposition 3.12 generalizes the fact, proven in [22] chap.4, that if G_1, G_2 is a matched pair of groups, then there exists a canonical action of G_1 on $L^\infty(G_2)$ (resp. G_2 on $L^\infty(G_1)$) coming from a map β (resp. α) such that, up to negligible sets, for any $g_i \in G_i$ $(i = 1, 2)$, one has $g_1g_2^{-1} = \beta_{g_1}(g_2)^{-1}\alpha_{g_2}(g_1)$.

3.5. A pseudo multiplicative unitary associated with a matched pair

Lemma 3.15. *For any $f, f' \in \mathcal{K}(\mathcal{G})$, ν-almost any $u \in \mathcal{G}^0$, one has:*

$$\frac{d\omega_{f,f'}\circ m_{\mathcal{G}}}{d\nu}(u) = \int_{\mathcal{G}_1\times\mathcal{G}_2} f\overline{f'}(g_1^{-1}g_2)\delta_1(g_1^{-1})\delta(g_2)\delta_2(g_2^{-1})d\lambda_1^u(g_1)d\lambda_2^u(g_2)$$

Proof: For all $f, f' \in \mathcal{K}(\mathcal{G})$, $h \in \mathcal{K}(\mathcal{G}^0)$, and ν-almost any element $u \in \mathcal{G}^0$, one has:

$$(\omega_{f,f'}\circ m_{\mathcal{G}})(h) =$$
$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}} h(m(g))(f\overline{f'})(g)d\lambda^u(g)d\mu(u) =$$
$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_1\times\mathcal{G}_2} h(m(g_1g_2))(f\overline{f'})(g_1g_2)(\delta\delta_2^{-1})(g_2)d\lambda_2^{s(g_1)}(g_2)d\lambda_1^u(g_1)d\mu(u) =$$

by 3.5

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_1\times\mathcal{G}_2} h(s(g_1))(f\overline{f'})(g_1g_2)(\delta\delta_2^{-1})(g_2)d\lambda_2^{s(g_1)}(g_2)d\lambda_1^u(g_1)d\mu(u)$$
$$= \int_{\mathcal{G}_1} h(s(g_1))\int_{\mathcal{G}_2}(f\overline{f'})(g_1g_2)(\delta\delta_2^{-1})(g_2)d\lambda_2^{s(g_1)}(g_2)d\nu_1(g_1)$$
$$= \int_{\mathcal{G}_1} h(r(g_1))\delta_1(g_1^{-1})\int_{\mathcal{G}_2}(f\overline{f'})(g_1^{-1}g_2)(\delta\delta_2^{-1})(g_2)d\lambda_2^{r(g_1)}(g_2)d\nu_1(g_1)$$
$$\text{by making the change of variable}: \mathrm{g}_1\mapsto \mathrm{g}_1^{-1}$$
$$= \int_{\mathcal{G}^0} h(u)\int_{\mathcal{G}_1}\int_{\mathcal{G}_2}(f\overline{f'})(g_1^{-1}g_2)\delta_1(g_1^{-1})(\delta\delta_2^{-1})(g_2)d\lambda_2^{r(g_1)}(g_2)d\lambda_1^u(g_1)d\nu(u)$$
$$= \int_{\mathcal{G}^0} h(u)\int_{\mathcal{G}_1\times\mathcal{G}_2}(f\overline{f'})(g_1^{-1}g_2)\delta_1(g_1^{-1})(\delta\delta_2^{-1})(g_2)d\lambda_2^{u}(g_2)d\lambda_1^u(g_1)d\nu(u)$$

the lemma follows. □

Notation 3.16. For any $i,j \in \{s,r,m,\hat{m}\}$, let us note $\mu^2_{i,j} = \mu_{i_\mathcal{G}} \times_{j_\mathcal{G}} \mu$ and $\mathcal{G}_{i,j} = \{(g,g') \in \mathcal{G}/i(g) = j(g')\}$, therefore $L^2(\mathcal{G},\mu)\ {}_{i_\mathcal{G}}\underset{\nu}{\otimes}{}_{j_\mathcal{G}}\ L^2(\mathcal{G},\mu)$ is isomorphic to $L^2(\mathcal{G}^2_{i,j},\mu^2_{i,j})$.

Lemma 3.17. *For any $f \in \mathcal{K}(\mathcal{G}\times\mathcal{G})$ one has:*

$$\mu^2_{m,r}(f) = \int_{\mathcal{G}^0}\int_{\mathcal{G}^2_{m,r}} f(g,g')d\lambda^{m(g)}(g')d\lambda^u(g)d\nu(u)$$

$$\mu^2_{s,m}(f) = \int_{\mathcal{G}^0}\int_{\mathcal{G}^2_{s,m}} \delta(g^{-1})f(g^{-1},g')d\lambda^{m(g')}(g)d\lambda^u(g')d\nu(u),$$

Proof: This is easy computations. □

Let's define an important pseudo multiplicative unitary, which generalizes at the same time, the multiplicative unitary of [4] 3.2 and the multiplicative partial isometry $I_{\mathcal{H},\mathcal{K}}$ of [25] Definition 4.1.5.

Proposition 3.18. *Let $W_{\mathcal{G}_1,\mathcal{G}_2} : L^2(\mathcal{G}^2_{s,m},\mu^2_{s,m}) \to L^2(\mathcal{G}^2_{m,r},\mu^2_{m,r})$ be the operator defined for any $\xi \in L^2(\mathcal{G}^2_{s,m},\mu^2_{s,m})$ and $\mu^2_{m,r}$-almost any (x,y) in $\mathcal{G}^2_{m,r}$ by:*

$$W_{\mathcal{G}_1,\mathcal{G}_2}\xi(x,y) = D(x,y)^{\frac{1}{2}}\xi(\theta(x,y))$$

where $\theta(x,y) = (xp_1(p_2(x)^{-1}y),\ \ p_2(x)^{-1}y)$, and $D(x,y)$ is the Radon-Nikodym derivative $\frac{d\mu^2_{s,m}\circ\theta}{d\mu^2_{m,r}}$. Up to the identification of 3.16, $W_{\mathcal{G}_1,\mathcal{G}_2}$ is a pseudo multiplicative unitary over the basis $(L^\infty(\mathcal{G}^0,\nu), m_{\mathcal{G}}, s_{\mathcal{G}}, r_{\mathcal{G}})$.

Proof: Mimicking [4] 3.2, let us consider two maps, ω_1 and ω_2, defined on $\mathcal{G}^2_{m,r}\cap\mathcal{G}_2\mathcal{G}_1\times\mathcal{G}$ and on $\mathcal{G}^2_{s,r}\cap\mathcal{G}\ \times\mathcal{G}_2\mathcal{G}_1$ respectively by: $\omega_1(x,y) = (x, p_2(x)^{-1}y)$ and $\omega_2(x,y) = (xp_1(y), y))$. Obviously, one has: $Im\omega_1\subset\mathcal{G}^2_{s,r}$ (resp. $Im\omega_2\subset\mathcal{G}^2_{s,m}$). Due to Lemma 3.17, for any $f\in\mathcal{K}(\mathcal{G}\times\mathcal{G})$, one has:

$$\mu^2_{m,r}(f\circ\omega_1) = \mu^2_{s,r}(f), \qquad \mu^2_{s,r}(f\circ\omega_2) = \mu^2_{s,m}(\Delta f)$$

where $\Delta(x,y) = \delta(p_1(y))$. So one can define two unitaries, first $W_1 : L^2(\mathcal{G}^2_{s,r},\mu^2_{s,r}) \to L^2(\mathcal{G}^2_{m,r},\mu^2_{m,r})$ and $W_2 : L^2(\mathcal{G}^2_{s,m},\mu^2_{s,m}) \to L^2(\mathcal{G}^2_{s,r},\mu^2_{s,r})$ such that $W_1\xi = \xi\circ\omega_1$ and $W_2\eta = \Delta^{-1/2}\eta\circ\omega_2$. Their composition $W_{\mathcal{G}_1,\mathcal{G}_2} = W_1W_2$ is defined for any $\xi\in L^2(\mathcal{G}^2_{s,m},\mu^2_{s,m})$ and $\mu^2_{m,r}$-almost any (x,y) in $\mathcal{G}^2_{m,r}$ by:

$$W_{\mathcal{G}_1,\mathcal{G}_2}\xi(x,y) = D(x,y)^{\frac{1}{2}}\xi(\theta(x,y))$$

where D is the Radon Nikodym derivative $\frac{d\mu^2_{s,m}\circ\theta}{d\mu^2_{m,r}}$ and $\theta(x,y) = \omega_2\circ\omega_1(x,y) = (xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)$. $W_{\mathcal{G}_1,\mathcal{G}_2}$ is obviously a unitary and the fact that this is a pseudo multiplicative unitary is essentially Proposition 4.1.6 in [25]. □

Remark 3.19. Using the identification of 3.11, $W_{\mathcal{G}_1,\mathcal{G}_2}$ is also a unitary:

$$[L^2(\mathcal{G}_2,\mu_2)\ {}_{s_2}\underset{\nu}{\otimes}{}_{r_1}\ L^2(\mathcal{G}_1,\mu_1)]\ {}_{s_1}\underset{\nu}{\otimes}{}_{m}\ [L^2(\mathcal{G}_2,\mu_2)\ {}_{s_2}\underset{\nu}{\otimes}{}_{r_1}\ L^2(\mathcal{G}_1,\mu_1)]$$
$$\to [L^2(\mathcal{G}_2,\mu_2)\ {}_{s_2}\underset{\nu}{\otimes}{}_{r_1}\ L^2(\mathcal{G}_1,\mu_1)]\ {}_{m}\underset{\nu}{\otimes}{}_{r_2}\ [L^2(\mathcal{G}_2,\mu_2)\ {}_{s_2}\underset{\nu}{\otimes}{}_{r_1}\ L^2(\mathcal{G}_1,\mu_1)]$$

Notation 3.20. Due to 3.12 and 3.11, one can consider the fibered product

$$\mathfrak{a}\star\mathfrak{a} : L^\infty(\mathcal{G}_2,\mu_2)\ {}_{s_2}\underset{\nu}{\star}{}_{r_2}\ L^\infty(\mathcal{G}_2,\mu_2) \to L^\infty(\mathcal{G},\mu)\ {}_{s}\underset{\nu}{\star}{}_{m}\ L^\infty(\mathcal{G},\mu)$$

Lemma 3.21. *For $\mu^2_{s,m}$-almost any $(g,g')\in\mathcal{G}^2_{s,m}$, one has:*

$$D(g,g') = \delta^{-1}(p_1(p_2(g)^{-1}g'))$$

Proof: For $\mu^2_{s,m}$-almost any $(g,g') \in \mathcal{G}^2_{s,m}$, and any $\xi \in L^2(\mathcal{G}^2_{s,m}, \mu^2_{s,m})$ one has:

$$\begin{aligned} W\xi(g,g') &= W_1W_2(g,g') = W_2\xi(g, p_2(g)^{-1}g') \\ &= \Delta^{-\frac{1}{2}}(g, p_2(g)^{-1}g')\xi(gp_1(p_2(g)^{-1}g'), p_2(g)^{-1}g') \\ &= \delta^{-\frac{1}{2}}(p_1(p_2(g)^{-1}g'))\xi(gp_1(p_2(g)^{-1}g'), p_2(g)^{-1}g') \end{aligned}$$

The lemma follows. □

Proposition 3.22. *The von Neumann algebra generated by the left (respectively the right) leg of $W_{\mathcal{G}_1,\mathcal{G}_2}$ is isomorphic to the crossed product $L^\infty(\mathcal{G}_2,\mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ (resp.$L^\infty(\mathcal{G}_1,\mu_1) \rtimes_{\hat{\mathfrak{a}}} \mathfrak{G}(\mathcal{G}_2)$).*

Proof: For any $f, h, \eta, \eta' \in \mathcal{K}(\mathcal{G})$, in $L^2(\mathcal{G},\mu)$ one has:

$$\begin{aligned} &((i \star \omega_{f,h})(W_{\mathcal{G}_1,\mathcal{G}_2})\eta, \eta') = \\ &= \int_{\mathcal{G}^2_{m,r}} W_{\mathcal{G}_1,\mathcal{G}_2}(\eta \;{}_{s_2}\!\underset{\nu}{\otimes}_{r_1} f)(g,g')\overline{\eta'(g)h(g')}d\mu^2_{m,r}(g,g') \\ &= \int_{\mathcal{G}^2_{m,r}} D(g,g')^{\frac{1}{2}}\eta(gp_1(p_2(g)^{-1}g'))f(p_2(g)^{-1}g')\overline{\eta'(g)h(g')}d\mu^2_{m,r}(g,g') \end{aligned}$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}^2_{m,r}} D(g,g')^{\frac{1}{2}}\eta(gp_1(p_2(g)^{-1}g'))f(p_2(g)^{-1}g')\overline{\eta'(g)h(g')}d\lambda^{m(g)}(g')\times$$
$$\times\, d\lambda^u(g)d\nu(u)$$

$$= \int_{\mathcal{G}}(\int_{\mathcal{G}} D(g,g')^{\frac{1}{2}}\eta(gp_1(p_2(g)^{-1}g'))f(p_2(g)^{-1}g')\overline{h(g')}d\lambda^{m(g)}(g'))\overline{\eta'(g)}d\mu(g)$$

Let's change of variable: $g' \mapsto p_2(g)^{-1}g'$:

$$\begin{aligned} &((i\star\omega_{f,h})(W_{\mathcal{G}_1,\mathcal{G}_2})\eta, \eta') = \\ &\quad = \int_{\mathcal{G}}(\int_{\mathcal{G}} D(g, p_2(g)g')^{\frac{1}{2}}\eta(gp_1(g')f(g')\overline{h}(p_2(g)g')d\lambda^{s(g)}(g'))\overline{\eta'(g)}d\mu(g) \end{aligned}$$

which gives, using Proposition 3.5 and Lemma 3.21, that for μ-almost any $g \in \mathcal{G}$, one has:

$$(i \star \omega_{f,h})(W_{\mathcal{G}_1,\mathcal{G}_2})\eta(g) =$$

$$= \int_{\mathcal{G}_1\times\mathcal{G}_2} D(g, p_2(g)g_1'g_2')^{\frac{1}{2}}\eta(gg_1')f(g_1'g_2')\overline{h}(p_2(g)g_1'g_2')\frac{\delta}{\delta_2}(g_2')d\lambda_2^{s(g_1')}(g_2') \, d\lambda_1^{s(g)}(g_1')$$

$$= \int_{\mathcal{G}_1}(\int_{\mathcal{G}_2}\delta(g_1')^{-\frac{1}{2}}\eta(gg_1')f(g_1'g_2')\overline{h}(p_2(g)g_1'g_2')\frac{\delta(g_2')}{\delta_2(g_2')})d\lambda_2^{s(g_1')}(g_2')d\lambda_1^{s(g)}(g_1')$$

If one denotes $\Theta(g, g_1', g_2') = f(g_1'g_2')\overline{h}(gg_1'g_2')\frac{\delta}{\delta_2}(g_2')$, then:

$$\begin{aligned}&(i\star\omega_{f,h})(W_{\mathcal{G}_1,\mathcal{G}_2})\eta(g) = \\ &= \int_{\mathcal{G}_1}(\int_{\mathcal{G}_2}\Theta(p_2(g), g_1', g_2')d\lambda_2^{s(g_1')}(g_2'))\delta(g_1')^{-\frac{1}{2}}\eta(gg_1')d\lambda_1^{s(g)}(g_1') \\ &= \mathfrak{R}(F_{f,h})\eta(g)\end{aligned}$$

where $F_{f,h}(g_2, g_1) = \int_{\mathcal{G}_2}\theta(g_2, g_1, g_2')d\lambda_2^{s(g_1)}(g_2')$

Hence left leg of $W_{\mathcal{G}_1,\mathcal{G}_2}$ generates the crossed product $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$, analogue computations give that its right leg generates the von Neumann algebra $L^\infty(\mathcal{G}_1,\mu_1)\rtimes_{\hat{\mathfrak{a}}}\mathfrak{G}(\mathcal{G}_2)$. □

Using Proposition 3.22, we shall identify the left (resp.right) leg of $W_{\mathcal{G}_1,\mathcal{G}_2}$ with crossed products.

Corollary 3.23. *Thanks to the existence of $W_{\mathcal{G}_1,\mathcal{G}_2}$, one can define two Hopf bimodule structures, one is $(L^\infty(\mathcal{G}^0,\nu), L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1), m, s, \Gamma)$ for the left leg, and the other $(L^\infty(\mathcal{G}^0,\nu), L^\infty(\mathcal{G}_1,\mu_1)\rtimes_{\hat{\mathfrak{a}}}\mathfrak{G}(\mathcal{G}_2), r, m, \hat{\Gamma})$ for the right one.*

Proof: This is a consequence of Remark 2.4 □

4. The quantum groupoid structures associated with a matched pair

In this chapter we shall describe in full details the Hopf bimodule structures found in the previous one. We shall complete them to obtain measured quantum groupoids structures. In order to simplify notations and using 3.11, we can suppose $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$ is acting on $L^2(G,\mu)$ and is generated by products $\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h))$, where, for any $f\in L^\infty(\mathcal{G}_2,\mu_2)$

and μ-almost any $g \in \mathcal{G}$ one has $\mathfrak{a}(f)(g) = f(p_2(g))$ and for any $\xi \in L^2(\mathcal{G},\mu)$, $h \in \mathcal{K}(\mathcal{G}_1) : (1_{s_2} {}_{s_2}\!\otimes_{r_1} \rho(h))\xi(g) = \int_{\mathcal{G}_1} h(g_1)\xi(gg_1)d\lambda_1^{s(g)}(g_1)$.

4.1. **The coproduct**

Lemma 4.1. *One has:* $\Gamma \circ \mathfrak{a} = (\mathfrak{a}_{s_2} \star_{r_2} \mathfrak{a})\Gamma_{\mathcal{G}_2}$*, so for any* $f \in L^\infty(\mathcal{G}_2,\mu_2)$ *and* $\mu^2_{s,m}$*-almost any* $(g,g') \in \mathcal{G}^2_{s,m}$ *, one gets:*

$$\Gamma(\mathfrak{a}(f))(g,g') = f(p_2(g)p_2(g'))$$

Proof: As $W_{\mathcal{G}_1,\mathcal{G}_2}$ is a unitary, it is easy to see that for any $\eta \in L^2(\mathcal{G}^2_{m,r},\mu^2_{m,r})$ and $\mu^2_{s,m}$-almost any $(g,g') \in \mathcal{G}^2_{s,m}$, one has:

$$W^*_{\mathcal{G}_1,\mathcal{G}_2}\eta(g,g') = \delta(p_1(g'))^{\frac{1}{2}}\eta(gp_1(g')^{-1}, p_2(gp_1(g')^{-1})g')$$

As $D'(g,g') = \delta(p_1(g'))^{\frac{1}{2}}$ is a density, then one has:

$$D'(g,g')D((gp_1(g')^{-1}, p_2(gp_1(g')^{-1})g') = 1,$$

hence for any $f \in L^\infty(\mathcal{G}_2,\mu_2)$, any $\xi \in L^2(\mathcal{G}^2_{s,m},\mu^2_{s,m})$ and $\mu^2_{s,m}$-almost any $(g,g') \in \mathcal{G}^2_{s,m}$, one has:

$$\begin{aligned}\Gamma(\mathfrak{a}(f))\xi(g,g') &= W^*_{\mathcal{G}_1,\mathcal{G}_2}(1_m \otimes_r \mathfrak{a}(f))W_{\mathcal{G}_1,\mathcal{G}_2}\xi(g,g')\\ &= D'(g,g')^{-\frac{1}{2}}(1_m \otimes_r \mathfrak{a}(f))W_{\mathcal{G}_1,\mathcal{G}_2}\xi(gp_1(g')^{-1}, p_2(gp_1(g')^{-1})g')\\ &= \mathfrak{a}(f)(p_2(gp_1(g')^{-1})g')\xi(g,g')\\ &= f(p_2(p_2(gp_1(g')^{-1})g'))\xi(g,g')\\ &= f(p_2(gp_1(g')^{-1}g'))\xi(g,g') = f(p_2(gp_2(g'))\xi(g,g')\\ &= f(p_2(g)p_2(g'))\xi(g,g')\end{aligned}$$

□

A good description of $\Gamma(1_{s_2} {}_{s_2}\!\otimes_{r_1} \mathcal{R}(\mathcal{G}_1))$ is given by an integral.

Proposition 4.2. *Let* h *(resp.*f*) be any element in* $\mathcal{K}(\mathcal{G}_1)$ *(resp.*$\mathcal{K}(\mathcal{G}_2)$*), then:*

i) for all $\xi \in \mathcal{K}(\mathcal{G}^2_{s,m})$ *and* $\mu^2_{s,m}$*-almost any* $(g,g') \in \mathcal{G}^2_{s,m}$*, one has:*

$$\Gamma(\mathfrak{a}(f)(1_{s_2} {}_{s_2}\!\otimes_{r_1} \rho(h))\xi(g,g') =$$

$$= f(p_2(g)p_2(g'))\int_{\mathcal{G}_1} h(g_1)\xi(gp_1(p_2(g')g_1), g'g_1)d\lambda_1^{s(g')}(g_1)$$

ii) with the notations of 3.13, for any $\phi \in \mathcal{K}(\mathcal{G})$:

$$(\omega_\phi \; {}_s\underset{\nu}{\star}_m \; i)(\Gamma(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))) = \mathfrak{R}(\Psi_{f,h})$$

where $\Psi_{f,h} \in \mathcal{K}(\mathcal{G}_2{}_{s_2}\times_{r_1}\mathcal{G}_1)$ is defined for any $(g_2, g_1) \in \mathcal{G}_2{}_{s_2}\times_{r_1}\mathcal{G}_1$ by:

$$\Psi_{f,h}(g_2, g_1) =$$

$$= h(g_1)\int_{\mathcal{G}^{r(g_2)}} \delta(g^{-1})f(p_2(g^{-1})g_2)\phi(g^{-1}p_1(g_2g_1))\overline{\phi(g^{-1})}d\lambda^{r(g_2)}(g)$$

Proof: i) For any $h \in \mathcal{K}(\mathcal{G}_1)$, any $\xi \in \mathcal{K}(\mathcal{G}^2_{s,m})$ and $\mu^2_{s,m}$-almost any $(g, g') \in \mathcal{G}^2_{s,m}$, one has:

$$\begin{aligned}\Gamma(1_{s_2}\otimes_{r_1}\rho(h))\xi(g,g') &= W^*_{\mathcal{G}_1,\mathcal{G}_2}(1_m\otimes_r(1_{s_2}\otimes_{r_1}\rho(h)))W_{\mathcal{G}_1,\mathcal{G}_2}\xi(g,g')\\ &= D'(g,g')^{-\frac{1}{2}}(1_m\otimes_r(1_{s_2}\otimes_{r_1}\rho(h)))W_{\mathcal{G}_1,\mathcal{G}_2}\xi(gp_1(g')^{-1}, p_2(gp_1(g')^{-1})g')\\ &= \int_{\mathcal{G}_1} h(g_1)\xi(gp_1(p_2(g')g_1), g'g_1)d\lambda_1^{s(g')}(g_1)\end{aligned}$$

i) follows immediately.

ii) For any $f \in \mathcal{K}(\mathcal{G}_2), \phi, \phi' \in \mathcal{K}(\mathcal{G})$ and $\mu^2_{s,m}$-almost any $(g, g') \in \mathcal{G}^2_{s,m}$, let us note:

$$X^f_{\phi,\phi'}(g,g') = f(p_2(g)p_2(g'))\int_{\mathcal{G}^1} h(g_1)\phi(gp_1(p_2(g')g_1))\phi'(g'g_1)d\lambda_1^{s(g')}(g_1).$$

Due to i) and 4.1, one has:

$$(\omega_\phi \; {}_s\underset{\nu}{\otimes}_m \; \omega_{\phi'})(\Gamma(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))) =$$

$$= \int_{\mathcal{G}^2_{s,m}}\int_{\mathcal{G}_1} h(g_1)f(p_2(g)p_2(g'))\phi(gp_1(p_2(g')g_1))\phi'(g'g_1)d\lambda_1^{s(g')}(g_1) \overline{\phi(g)\phi'(g')}d\mu^2_{s,m}$$

$$\begin{aligned}&= \int_{\mathcal{G}^2_{s,m}} X^f_{\phi,\phi'}(g,g')\overline{\phi(g)\phi'(g')}d\mu^2_{s,m}(g,g')\\ &= \int_{\mathcal{G}}\Big(\int_{\mathcal{G}^{m(g')}} \delta(g^{-1})X^f_{\phi,\phi'}(g^{-1},g')\overline{\phi(g^{-1})}d\lambda^{m(g')}(g)\Big)\overline{\phi'(g')}d\mu(g') \quad \text{by } \textit{3.17}\end{aligned}$$

If $\Psi_{f,h} \in \mathcal{K}(\mathcal{G}_{2\,s_2}\times_{r_1}\mathcal{G}_1)$ is defined for any $(g_2, g_1) \in \mathcal{G}_{2\,s_2}\times_{r_1}\mathcal{G}_1$ by:

$$\Psi_{f,h}(g_2, g_1) = \\ = h(g_1) \int_{\mathcal{G}^{r(g_2)}} \delta(g^{-1}) f(p_2(g^{-1})g_2)\phi(g^{-1}p_1(g_2 g_1))\overline{\phi(g^{-1})}d\lambda^{r(g_2)}(g)$$

we can write:

$$\int_{\mathcal{G}^{m(g')}} \delta(g^{-1}) X^f_{\phi,\phi'}(g^{-1}, g')\overline{\phi(g^{-1})}d\lambda^{m(g')}(g) = \\ = \int_{\mathcal{G}_1} h(g_1) \int_{\mathcal{G}^{m(g')}} f(p_2(g^{-1})p_2(g'))\phi(g^{-1}p_1(p_2(g')g_1))\times \\ \times \overline{\phi(g^{-1})}d\lambda^{m(g')}(g)\phi'(g'g_1)d\lambda_1^{s(g')}$$

$$= \int_{\mathcal{G}_1} \Psi_{f,h}(p_2(g'), g_1)\phi'(g'g_1)d\lambda_1^{s(g')}(g_1) \qquad \text{as} \;\; \mathrm{m(g') = r(p_2(g'))}$$

hence one has:

$$\omega_{\phi'}\Big((\omega_\phi \; {}_s\underset{\nu}{\star}_m \; i)(\Gamma(1_{s_2}\otimes_{r_1}\rho(h)))\Big) = (\omega_\phi \; {}_s\underset{\nu}{\star}_m \; \omega_{\phi'})(\Gamma(1_{s_2}\otimes_{r_1}\rho(h)))$$

$$= \textstyle\int_{\mathcal{G}} \Big(\int_{\mathcal{G}_1} \Psi_{f,h}(p_2(g'), g_1)\phi'(g'g_1)d\lambda_1^{s(g')}(g_1)\Big)\overline{\phi'(g')}d\mu(g')$$

$$= \textstyle\int_{\mathcal{G}} \Big(\mathfrak{R}(\Psi_{f,h})\phi'(g')\Big)\overline{\phi'(g')}d\mu(g') = \omega_{\phi'}(\mathfrak{R}(\Psi_{f,h}))$$

ii) follows □

Remark 4.3. The formulas of Lemma 4.2 generalize the ones obtained by Stefaan Vaes in [21] 4.20 (and maybe elsewhere).

4.2. **The co-involution**

In this paragraph, a co-involution for $L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ is constructed.

Lemma 4.4. *Using notations of Remark 3.9, let* $\phi : \mathcal{G} \to \mathcal{G}$ *(resp.* $\hat{\phi} : \mathcal{G} \to \mathcal{G}$*) be the function defined by the formula:* $\phi(g) = p_1(g)^{-1}p'_2(g)$ *(resp.* $\hat{\phi}(g) = \phi(g)^{-1}$*), then one has:*

i) $\hat{\phi}(g) = p'_2(g)^{-1}p_1(g) = p'_1(g)p_2(g)^{-1}$

ii) $\phi^2 = \hat{\phi}^2 = id_{\mathcal{G}}$

iii) $s \circ \phi = \hat{m}, m \circ \phi = r, m \circ \hat{\phi} = s, r \circ \phi = m$

Proof: For any $g \in \mathcal{G}_1\mathcal{G}_2 \cap \mathcal{G}_2\mathcal{G}_1$, one has $p_1(g)p_2(g) = p'_2(g)p'_1(g)$ so i) is true; using obvious notations, let us write: $g = g_1g_2 = g'_2g'_1$; one has: $\phi(\phi(g)) = \phi(g_1^{-1}g'_2) = \phi(g_2{g'_1}^{-1}) = g_1g_2 = g$, this gives ii), the assertion iii) is obvious. □

Using Lemma 4.4, one can define a map $\hat{\phi}_s{\times}_m\phi : \mathcal{G}^2_{s,m} \to \mathcal{G}^2_{m,r}$ (resp. $\hat{\phi}_m{\times}_r\phi : \mathcal{G}^2_{m,r} \to \mathcal{G}^2_{s,m}$) such that for almost any $(g, g') \in \mathcal{G}^2_{s,m}$ (resp. $\mathcal{G}^2_{m,r}$), one has: $(\hat{\phi}_s{\times}_m\phi)(g, g') = (\hat{\phi}(g), \phi(g'))$ (respectively $(\hat{\phi}_m{\times}_r\phi)(g, g') = (\hat{\phi}(g), \phi(g'))$.

Lemma 4.5. *Using the notations of 4.4 and 3.18, one has:*

$$\theta(\hat{\phi}_s{\times}_m\phi) = (\hat{\phi}_m{\times}_r\phi)\theta^{-1}.$$

Proof: For almost any $(g, g') \in \mathcal{G}^2_{m,r}$, due to 4.4 i), one has $p_2(\hat{\theta}(g)) = p_2(g)^{-1}$, so :

$$\begin{aligned}
\theta(\hat{\phi}_s{\times}_m\phi)(g, g') &= \theta(\hat{\phi}(g), \phi(g')) \\
&= (\hat{\phi}(g)p_1(p_2(\hat{\phi}(g))^{-1}\phi(g')), p_2(\hat{\phi}(g))^{-1}\phi(g')) \\
&= (p'_2(g)^{-1}p_1(g)p_1(p_2(g)\phi(g')), p_2(g)\phi(g')) \\
&= (p'_2(g)^{-1}p_1(gp_1(g')^{-1}p'_2(g')), p_2(g)p_1(g')^{-1}p'_2(g')) \\
&= (p'_2(g)^{-1}p_1(gp_1(g')^{-1}), p_2(g)p_1(g')^{-1}p'_2(g'))
\end{aligned}$$

Also one has:

$$(\hat{\phi}_m{\times}_r\phi)\theta^{-1}(g, g') = (\hat{\phi}_m{\times}_r\phi)(gp_1(g')^{-1}, p_2(gp_1(g')^{-1})g')$$

Let us define:

$$X = gp_1(g')^{-1}, \quad Y = p_2(gp_1(g')^{-1})g'$$

this gives that:

$$\begin{aligned}(\hat{\phi}_m \times_r \phi)\theta^{-1}(g, g') &= (p_2'(X)^{-1} p_1(X), p_1(Y)^{-1} p_2'(Y)) \\ &= (p_2'(X)^{-1} p_1(X), p_2(Y) p_1'(Y)^{-1})\end{aligned}$$

as $p_2'(X) = p_2'(g)$, one deduces that:

$$p_2'(X)^{-1} p_1(X) = p_2'(g)^{-1} p_1(g p_1(g')^{-1})$$

Also we can write that:

$$\begin{aligned}Y[p_2(g)p_2(g')]^{-1} &= p_2(g p_1(g')^{-1}) p_1(g') p_2(g') [p_2(g) p_2(g')]^{-1} \\ &= p_2(g p_1(g')^{-1}) p_1(g') p_2(g)^{-1} \\ &= p_2(p_2(g) p_1(g')^{-1}) [p_2(g) p_1(g')^{-1}]^{-1} \\ &= [p_1(p_2(g) p_1(g')^{-1})]^{-1}\end{aligned}$$

this implies that $p_2(Y) = p_2(g)p_2(g')$, and as $Y = p_2(g p_1(g')^{-1}) g'$, therefore $p_1'(Y) = p_1'(g')$, which gives that:

$$p_2(Y) p_1'(Y)^{-1} = p_2(g) p_2(g') p_1'(g')^{-1} .$$

Finally:

$$\begin{aligned}(\hat{\phi}_m \times_r \phi)\theta^{-1}(g, g') &= (p_2'(X)^{-1} p_1(X), p_2(Y) p_1'(Y)^{-1}) \\ &= (p_2'(g)^{-1} p_1(g p_1(g')^{-1}), p_2(g) p_2(g') p_1'(g')^{-1}) \\ &= \theta(\hat{\phi}_s \times_m \phi)(g, g')\end{aligned}$$

which gives the lemma. □

Proposition 4.6. *i) Let $J, \hat{J} : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ be defined for any $\xi \in L^2(\mathcal{G}, \mu)$ and μ-almost any $g \in \mathcal{G}$ by :*

$$J\xi(g) = \overline{\xi(\phi(g))} (\frac{d\mu \circ \phi}{d\mu})^{\frac{1}{2}}(g), \qquad \hat{J}\xi(g) = \overline{\xi(\hat{\phi}(g))} (\frac{d\mu \circ \hat{\phi}}{d\mu})^{\frac{1}{2}}(g)$$

then J and $\hat{J}$ are antilinear involutive isometries.

ii) For all $f \in \mathcal{K}(\mathcal{G}^0)$, one has: $\hat{J}s(f) = m(\overline{f})\hat{J}$ and $Jm(f) = r(\overline{f})J$, hence one can define $\hat{J}_s \otimes_m J : L^2(\mathcal{G}^2_{s,m}, \mu^2_{s,m}) \to L^2(\mathcal{G}^2_{m,r}, \mu^2_{m,r})$, and it's inverse $\hat{J}_m \otimes_r J : L^2(\mathcal{G}^2_{m,r}, \mu^2_{m,r}) \to L^2(\mathcal{G}^2_{s,m}, \mu^2_{s,m})$, which verify:

$$(\hat{J}_m \otimes_r J) W_{\mathcal{G}_1,\mathcal{G}_2} = W^*_{\mathcal{G}_1,\mathcal{G}_2} (\hat{J}_s \otimes_m J)$$

Proof: This is a straightforward consequence of Lemma 4.5. □

Proposition 4.7. *Let R be the map defined for any $x \in L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ by: $R(x) = \hat{J}x^*\hat{J}$, then R is a co-involution for measured quantum groupoid $(L^\infty(\mathcal{G}^0, \nu), L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1), m, s, \Gamma)$, more precisely, for any $f \in \mathcal{K}(\mathcal{G}_2)$, if we define f^{-1} by $f^{-1}(g_2) = f({g_2}^{-1})$ one has: $R(\mathfrak{a}(f)) = \mathfrak{a}(f^{-1})$.*

Proof: Obviously, $Ad(\hat{J})$ is an involutive $*$-antiautomorphism for $\mathcal{L}(L^2(\mathcal{G}))$. Due to proposition 4.6, for any $h, k \in \mathcal{K}(\mathcal{G})$, one has:

$$\begin{aligned} R((i \star \omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2})) &= \hat{J}(i \star \omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2})^* \hat{J} = \hat{J}(i \star \omega_{k,h})(W^*_{\mathcal{G}_1,\mathcal{G}_2})\hat{J} \\ &= (i \star \omega_{Jk,Jh})(\hat{J}_m \otimes_r J) W^*_{\mathcal{G}_1,\mathcal{G}_2} (\hat{J}_m \otimes_r J)) \\ &= (i \star \omega_{Jk,Jh})(W_{\mathcal{G}_1,\mathcal{G}_2}) \end{aligned}$$

so, by restriction, R is an involutive $*$-antiautomorphism of $L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$.

From proposition 4.6, one obtains that $R \circ m = s$.

For all $f \in \mathcal{K}(\mathcal{G}_2)$, all $\xi \in L^2(\mathcal{G}, \mu)$ and μ-almost any $g \in \mathcal{G}_1\mathcal{G}_2 \cap \mathcal{G}_2\mathcal{G}_1$, with $g = g_1 g_2 = g'_1 g'_2$, one has:

$$\begin{aligned} R(\mathfrak{a}(f))\xi(g) &= \hat{J}\mathfrak{a}^*(f)\hat{J}\xi(g) = \overline{\mathfrak{a}(\overline{f})(\hat{\phi}(g))\xi(g)} \\ &= f(p_2({g'_2}^{-1} g_1))\xi(g) = f(p_2(g'_1 g_2^{-1})\xi(g) \\ &= f(g_2^{-1})\xi(g) = \mathfrak{a}(f^{-1})(g)\xi(g) \end{aligned}$$

So $R(\mathfrak{a}(f)) = \mathfrak{a}(f^{-1})$.

Due to Proposition 3.7 of [8], for any $k, k_1, f_2, h_1, h_2 \in \mathcal{K}(\mathcal{G})$, one has:

$$\begin{aligned} &(\Gamma((i \star \omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2}))(h_1 \underset{\nu}{{}_s\otimes_m} k_1), h_2 \underset{\nu}{{}_s\otimes_m} k_2) = \\ &\qquad ((\omega_{h_1,h_2} \star i)(W_{\mathcal{G}_1,\mathcal{G}_2})(\omega_{k_1,k_2} \star i)(W_{\mathcal{G}_1,\mathcal{G}_2})h, k) \end{aligned}$$

Hence, on one hand:

$$\begin{aligned}(\varsigma_{m,s}\Gamma(R((i\star\omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2})))(h_1 \underset{\nu}{{}_m\otimes_s} k_1), h_2 \underset{\nu}{{}_m\otimes_s} k_2)\\ = (\Gamma((i\star\omega_{Jk,Jh})(W_{\mathcal{G}_1,\mathcal{G}_2})))(k_1 \underset{\nu}{{}_s\otimes_m} h_1), k_2 \underset{\nu}{{}_s\otimes_m} h_2)\\ = ((\omega_{k_1,k_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})(\omega_{h_1,h_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})Jk, Jh)\end{aligned}$$

On the other hand, using proposition 4.6 ii), one has:

$$\begin{aligned}((R_m\star_s R)\Gamma((i\star\omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2}))(h_1 \underset{\nu}{{}_m\otimes_s} k_1), h_2 \underset{\nu}{{}_m\otimes_s} k_2)\\ = (\hat{J}h_2 \underset{\nu}{{}_s\otimes_m} \hat{J}k_2, \Gamma((i\star\omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2})^*)(\hat{J}h_1 \underset{\nu}{{}_s\otimes_m} \hat{J}k_1))\\ = (\Gamma((i\star\omega_{h,k})(W_{\mathcal{G}_1,\mathcal{G}_2}))(\hat{J}h_2 \underset{\nu}{{}_s\otimes_m} \hat{J}k_2), \hat{J}h_1 \underset{\nu}{{}_s\otimes_m} \hat{J}k_1)\\ = ((\omega_{\hat{J}h_2,\hat{J}h_1}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})(\omega_{\hat{J}k_2,\hat{J}k_1}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})h, k)\\ = (J(\omega_{h_2,h_1}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2}{}^*)(\omega_{k_2,k_1}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2}{}^*)Jh, k)\\ = (J(\omega_{h_1,h_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})^*(\omega_{k_1,k_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})^*Jh, k)\\ = (Jk, (\omega_{h_1,h_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})^*(\omega_{k_1,k_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})^*Jh)\\ = ((\omega_{k_1,k_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})(\omega_{h_1,h_2}\star i)(W_{\mathcal{G}_1,\mathcal{G}_2})Jk, Jh)\end{aligned}$$

which gives that:

$$(R_m\star_s R)\Gamma = \varsigma_{m,s}\Gamma\circ R$$

□

4.3. The Haar operator valued weights

In this paragraph, we define two invariant operator valued weights on $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$.

Definition 4.8. Let T_2 be the left Haar operator valued weight of $\mathfrak{G}(\mathcal{G}_2)$ and let $T_L = \widetilde{T_2}$ be its dual operator valued weights on $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$ in the sense of 2.9 and let T_R be equal to RT_LR.

Lemma 4.9. *The operator valued weight T_L (resp.T_R) takes its values in the range (resp.source) basis of $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$.*

Proof: Due to Lemma 3.12, $T_L = \mathfrak{a}\circ T_2\circ\mathfrak{a}^{-1}\circ T_{\tilde{\mathfrak{a}}} = m\circ r_2^{-1}\circ T_2\circ\mathfrak{a}^{-1}\circ T_{\tilde{\mathfrak{a}}}$, so T_L takes its values in $m(L^\infty(\mathcal{G}^0,\nu))$, which is the range basis of

$L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$. Due to 4.7, one has $s = R\circ m$, so $T_R = RT_LR$ takes its values in $s(L^\infty(\mathcal{G}^0,\nu))$ which gives the second part of the lemma. $\square$

Hence T_L is an operator valued weight from $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$ to its range basis, and $\phi_L = \nu\circ m^{-1}\circ T_L$ is a dual weight, in the sense of [6] 13.1.

Proposition 4.10. *For any $f\in\mathcal{K}(\mathcal{G}_2), h\in\mathcal{K}(\mathcal{G}_1)$ and μ-almost any $y\in\mathcal{G}$, one has:*

$$i) T_L(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))(y) = \int_{\mathcal{G}_2} f(x)h(s(x))d\lambda_2^{m(y)}(x)$$

ii) With the notations of remark 3.13, for any $F\in\mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$ and μ-almost any $y\in\mathcal{G}$, one has:

$$T_L(\mathfrak{R}(F))(y) = \int_{\mathcal{G}_2} F(x,s(x))d\lambda_2^{m(y)}(x)$$

iii) Let $\Phi_L = \nu\circ m^{-1}\circ T_L$ be the lifted n.sf.f weight of T_L , then one has:

$$(i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L))\Gamma(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h))) = T_L(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h))).$$

Proof: For any $h\in\mathcal{K}(\mathcal{G}_1)$ and μ-almost any $y\in\mathcal{G}$, one has:

$$T_L(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))(y) =$$

$$= (m\circ r_2^{-1}\circ T_2\circ\mathfrak{a}^{-1}\circ T_{\tilde{\mathfrak{a}}})(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h))(y)$$

$$= (r_2^{-1}\circ T_2(f\mathfrak{a}^{-1}(1_{s_2}\otimes_{r_1}\hat{T}^c_{\mathcal{G}_1}(\rho(h))(m(y)) \qquad \text{by [6], } \textit{9.6}$$

$$= (r_2^{-1}\circ T_2(f\mathfrak{a}^{-1}(1_{s_2}\otimes_{r_1}s_1(h_{|\mathcal{G}^0}))(m(y))$$

$$= (r_2^{-1}\circ T_2(fs_2(h_{|\mathcal{G}^0}))(m(y)) \qquad \text{by } \textit{3.12}$$

$$= \int_{\mathcal{G}_2} fh_{|\mathcal{G}^0}(s(x))d\lambda_2^{m(y)}(x) = \int_{\mathcal{G}_2} f(x)h(s(x))d\lambda_2^{m(y)}(x)$$

which gives i). The assertion ii) is an easy consequence of i)

iii) For any $\phi\in\mathcal{K}(\mathcal{G})$, one has:

$$\omega_\phi((i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L)\Gamma(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))) =$$

$$= \Phi_L((\omega_\phi \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} i)\Gamma(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))$$

$$= \Phi_L(\mathfrak{R}(\Psi_{f,h}))) \qquad \text{by } \textit{4.2}$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\Psi_{f,h}(x,s(x))d\lambda_2^u(x)d\nu(u) \qquad \text{by } \textit{ii)}$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}} h(s(x))f(p_2(g^{-1})x)\delta(g^{-1})\phi\overline{\phi}(g^{-1})d\lambda^{r(x))}(g)d\lambda_2^u(x)\times$$
$$\times\, d\nu(u)$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}} h(s(x))f(p_2(g^{-1})x)\delta(g^{-1})|\phi|^2(g^{-1})d\lambda^u(g)d\lambda_2^u(x)d\nu(u)$$
$$\text{as} \quad \mathrm{r(x) = u}$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2} h(s(x))f(p_2(g^{-1})x)d\lambda_2^u(x)\int_{\mathcal{G}}\delta(g^{-1})|\phi|^2(g^{-1})d\lambda^u(g)d\nu(u)$$

$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}}\Big(\int_{\mathcal{G}_2} h(s(x))f(p_2(g^{-1})x)d\lambda_2^u(x)\Big)\delta(g^{-1})|\phi|^2(g^{-1})d\lambda^u(g)\times$$
$$\times\, d\nu(u)$$

$$= \int_{\mathcal{G}}\Big(\int_{\mathcal{G}_2} h(s(p_2(g^{-1})^{-1}x))f(x)d\lambda_2^{m(g^{-1})}(x)\Big)\delta(g^{-1})|\phi|^2(g^{-1})d\mu(g)$$
$$\text{by making the change of variable} : \mathrm{x} \mapsto \mathrm{p_2(g^{-1})x}$$

$$= \int_{\mathcal{G}}\Big(\int_{\mathcal{G}_2} h(s(x))f(x)d\lambda_2^{m(g^{-1})}(x)\Big)\delta(g^{-1})\phi(g^{-1})\overline{\phi(g^{-1})}d\mu(g)$$

$$= \int_{\mathcal{G}}\Big(\int_{\mathcal{G}_2} h(s(x))f(x)d\lambda_2^{m(g)}(x)\Big)\phi(g)\overline{\phi(g)}d\mu(g)$$
$$\text{by making the change of variable} : \mathrm{g} \mapsto \mathrm{g^{-1}}$$

$$= \omega_\phi(T_L(\mathfrak{a}(f)(1_{s_2}\otimes_{r_1}\rho(h)))) \qquad \text{by } \textit{4.2 and i)}$$

which ends the proof. □

Now let's prove the left invariance of T_L, using technics similar to [22] 4.12.

Proposition 4.11. *For any positive Borel function Φ on $\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$, one has:*

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\Phi(p_2(xh),h^{-1})d\lambda_1^{s(x)}(h)d\mu_2(x) = \\ = \int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\Phi(x,h)k_L(x,h)d\lambda_1^{s(x)}(h)d\mu_2(x)$$

where $k_L(x,h) = \frac{\delta}{\delta_1}(p_1(xh))\delta_1(h)^{-1}\delta_2(p_2(xh))\delta_2(x)^{-1}$.

Proof:

For any $H \in \mathcal{K}(\mathcal{G}_r\times_{r_1}\mathcal{G}_1)$, due to Fubini's theorem, Proposition 3.5 and the left invariance for λ_1 one has:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H({g_2}^{-1}g_1,k)\delta(g_1)\delta_1(g_1)^{-1}\delta_2(g_2)^{-1}d\lambda_1^{r(g_2)}(g_1)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$
$$= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H(g_2g_1,k)\delta(g_1)\delta_1(g_1)^{-1}d\lambda_1^{s(g_2)}(g_1)d\lambda_1^{r(g_2)}(k)d\mu_2(g_2)$$
$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H(g_2g_1,k)\delta(g_1)\delta_1(g_1)^{-1}d\lambda_1^{s(g_2)}(g_1)d\lambda_1^{r(g_2)}(k)d\lambda_2^{u}(g_2)\times \\ \times d\nu(u)$$
$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H(g_2g_1,k)\delta(g_1)\delta_1(g_1)^{-1}d\lambda_1^{s(g_2)}(g_1)d\lambda_1^{u}(k)d\lambda_2^{u}(g_2)d\nu(u)$$
$$= \int_{\mathcal{G}^0}\int_{\mathcal{G}_1}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} H(g_2g_1,k)\delta(g_1)\delta_1(g_1)^{-1}d\lambda_1^{s(g_2)}(g_1)d\lambda_2^{u}(g_2)d\lambda_1^{u}(k))d\nu(u)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} H(g_2g_1,k)\delta(g_1)\delta_1(g_1)^{-1}d\lambda_1^{s(g_2)}(g_1)d\lambda_2^{r(k)}(g_2)d\mu_1(k)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}} H(g,k)d\lambda^{r(k)}(g)d\mu_1(k) = \int_{\mathcal{G}_1}\int_{\mathcal{G}} H(kg,k)d\lambda^{s(k)}(g)d\mu_1(k)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}} \delta_1(k)^{-1}H(k^{-1}g,k^{-1})d\lambda^{r(k)}(g)d\mu_1(k)$$

To simplify notations let's define: $h(g_1,g_2,k) = \delta_2(g_2)^{-1}\delta_1(k)^{-1}\frac{\delta}{\delta_1}(g_1)$, then due to proposition 3.5, and Fubini's theorem, this gives:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H({g_2}^{-1}g_1,k)\frac{\delta}{\delta_1}(g_1)\delta_2(g_2)^{-1}d\lambda_1^{r(g_2)}(g_1)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} \delta_1(k)^{-1}H(k^{-1}g_2g_1,k^{-1})\frac{\delta}{\delta_1}(g_1)d\lambda_1^{s(g_2)}(g_1)d\lambda_2^{r(k)}(g_2)d\mu_1(k)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} \delta_1(k)^{-1}H(k^{-1}g_2g_1,k^{-1})\frac{\delta}{\delta_1}(g_1)d\lambda_1^{s(g_2)}(g_1)d\lambda_2^{r(k)}(g_2)d\lambda_1^{u}(k)\times$$
$$\times\, d\nu(u)$$
$$= \int_{\mathcal{G}_1}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} \delta_1(k)^{-1}H(k^{-1}g_2g_1,k^{-1})\frac{\delta}{\delta_1}(g_1)d\lambda_1^{s(g_2)}(g_1)d\lambda_2^{u}(g_2)d\lambda_1^{u}(k)\times$$
$$\times\, d\nu(u)$$
$$= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} \delta_1(k)^{-1}H(k^{-1}g_2g_1,k^{-1})\frac{\delta}{\delta_1}(g_1)d\lambda_1^{s(g_2)}(g_1)d\lambda_1^{u}(k)d\lambda_2^{u}(g_2)\times$$
$$\times\, d\nu(u)$$
$$= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1\times\mathcal{G}_1} \delta_1(k)^{-1}H(k^{-1}g_2g_1,k^{-1})\frac{\delta}{\delta_1}(g_1)d\lambda_1^{s(g_2)}(g_1)d\lambda_1^{r(g_2)}(k)d\mu_2(g_2)$$
$$= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1\times\mathcal{G}_1} h(g_1,g_2,k)H(k^{-1}{g_2}^{-1}g_1,k^{-1})d\lambda_1^{r(g_2)}(g_1)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$

In the integral relative to g_1, let's use the left invariance, this gives:

$$\int_{\mathcal{G}_1} h(g_1,g_2,k)H(k^{-1}{g_2}^{-1}g_1,k^{-1})d\lambda_1^{r(g_2)}(g_1) =$$
$$= \int_{\mathcal{G}_1} h(g_1,g_2,k)H(p_2'(k^{-1}{g_2}^{-1})p_1'(k^{-1}{g_2}^{-1})g_1,k^{-1})d\lambda_1^{r(g_2)}(g_1)$$
$$= \int_{\mathcal{G}_1} h(p_1'(k^{-1}{g_2}^{-1})^{-1}g_1,g_2,k)H(p_2'(k^{-1}{g_2}^{-1})g_1,k^{-1})d\lambda_1^{r(p_1'(k^{-1}{g_2}^{-1}))}(g_1)$$

but $p_2'(k^{-1}{g_2}^{-1}) = p_2(g_2k)^{-1}$ and $p_1'(k^{-1}{g_2}^{-1}) = p_1(g_2k)^{-1}$, hence:

$$\int_{\mathcal{G}_1} h(g_1,g_2,k)H(k^{-1}{g_2}^{-1}g_1,k^{-1})d\lambda_1^{r(g_2)}(g_1) =$$
$$= \int_{\mathcal{G}_1} h(p_1(g_2k)g_1,g_2,k)H(p_2(g_2k)^{-1}g_1,k^{-1})d\lambda_1^{r(p_2(g_2k))}(g_1)$$

Finally:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1} H({g_2}^{-1}g_1,k)\frac{\delta}{\delta_1}(g_1)\delta_2(g_2)^{-1}d\lambda_1^{r(g_2)}(g_1)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)=$$
$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\int_{\mathcal{G}_1}\Theta(g_1,g_2,k)d\lambda_1^{r(p_2(g_2k))}(g_1)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$

where

$$\Theta(g_1,g_2,k)=h(p_1(g_2k)g_1,g_2,k)H(p_2(g_2k)^{-1}g_1,k^{-1})$$

This equality can be extended to all positive Borel functions H. Let us take $H=((\frac{\delta_1}{\delta}h_1)\times(\delta_2h_2))\rho^{-1}\times h_3$, where ρ is the homeomorphism $(g_1,g_2)\to g_2^{-1}g_1$ from $\mathcal{G}_{1r_1}\times_{r_2}\mathcal{G}_2$ onto its image. Let us choose h_1 such that $\varphi_1:u\mapsto\int_{\mathcal{G}_1}h_1(g_1)d\lambda^u(g_1)$ never vanishes, so one gets:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}(\varphi_1\circ r)h_2(g_2)h_3(k)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)=$$
$$=\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}(\varphi_1\circ r)h_2(p_2(g_2k))\frac{\delta}{\delta_1}(p_1(g_2k))\delta_2(g_2)^{-1}\delta_2(p_2(g_2k))\delta_1(k^{-1})\times$$
$$\times\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$

So for any positive Borel function Φ on $\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$, one has:

$$\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\Phi(g_2,k)d\lambda_1^{s(g_2)}(k)d\mu_2(g_2)=$$
$$=\int_{\mathcal{G}_2}\int_{\mathcal{G}_1}\Phi(p_2(g_2k),k^{-1})k_L(g_2,k)\lambda_1^{s(g_2)}(k)d\mu_2(g_2)$$

Let $\gamma:\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1\to\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$ be defined by $\gamma(g_2,k)=\gamma(p_2(g_2k),k^{-1}$, as $g_2k=p_1(g_2k)p_2(g_2k)$ hence $p_2(g_2k)k^{-1}=p_1(g_2k)^{-1}g_2$ and so one has: $p_2(p_2(g_2k)k^{-1}=g_2$ and γ is a symmetry. Applying the last formula to $\Phi\circ\gamma$ leads to the proposition. □

Lemma 4.12. *The map:* $\Lambda_{\Phi_L}\mathcal{R}(F)\to(\delta_1k_L)^{\frac{1}{2}}F$, *for* $F\in\mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$, *realizes an isomorphism between* H_{Φ_L} *and* $L^2(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1,\mu^2_{s_2,r_1})$.

Proof: For $F \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$, using 4.11 one has:

$$\begin{aligned}(\Lambda_{\Phi_L}\mathcal{R}(F), \Lambda_{\Phi_L}\mathcal{R}(F)) &= \Phi_L(\mathcal{R}(F^{\#} \star F)) = \int_{\mathcal{G}_2} F^{\#} \star F(x, s(x)) d\mu_2(x) \\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} F^{\#}(x,h)F(p_2(xh), h^{-1}) d\lambda_1^{s(x)}(h)) d\mu_2(u) \\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} \overline{\delta_1^{\frac{1}{2}}F}\delta_1^{\frac{1}{2}}F(p_2(xh), h^{-1}) d\lambda_1^{s(x)}(h) d\mu_2(x)) \\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} \overline{\delta_1^{\frac{1}{2}}F}\delta_1^{\frac{1}{2}}F(x,h)k_L(x,h) d\lambda_1^{s(x)}(h) d\mu_2(x) \\ &= \mu^2_{s_2,r_1}((\delta_1 k_L)^{\frac{1}{2}}F\overline{(\delta_1 k_L)^{\frac{1}{2}}F})\end{aligned}$$

□

So one can compute the GNS construction of Φ_L the Hilbert space of which is $L^2(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1, \mu^2_{s_2,r_1})$,

Lemma 4.13. *For $\mu^2_{s_2,r_1}$-almost any (x,h) in $\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$, one has:*

$$k_L(p_2(xh), h^{-1}) = k_L^{-1}(x,h).$$

Proof: Using the notations of proposition 4.11, for $\mu^2_{s_2,r_1}$-almost any (x,h) in $\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$, one has:

$$k_L(p_2(xh), h^{-1}) = k_L(\gamma(x,h)) = \frac{d\mu^2_{s_2,r_1} \circ \gamma}{d\mu^2_{s_2,r_1}}(\gamma(x,h)).$$

But γ is a symmetry, hence:

$$k_L(p_2(xh), h^{-1}) = \frac{d\mu^2_{s_2,r_1}}{d\mu^2_{s_2,r_1} \circ \gamma}(x,h) = k_L^{-1}(x,h).$$

□

Lemma 4.14. *For any $F \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$, one has: $\Delta_{\Phi_L}F = k_L^{-1}F$.*

Proof: For any $F, G, H \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$, one has

$$\begin{aligned}(\Delta_{\Phi_L}\Lambda_{\Phi_L}\mathfrak{R}(F),\Lambda_{\Phi_L}\mathfrak{R}(G^{\#})) &= (S_{\Phi_L}\Lambda_{\Phi_L}\mathfrak{R}(G^{\#}), S_{\Phi_L}\Lambda_{\Phi_L}\mathfrak{R}(F))\\ &= (\Lambda_{\Phi_L}\mathfrak{R}(G), \Lambda_{\Phi_L}\mathfrak{R}(F^{\#})) = \Phi_L(\mathfrak{R}(F)\mathfrak{R}(G))\\ &= \Phi_L(\mathfrak{R}(F\star G)) = \int_{\mathcal{G}_0}\int_{\mathcal{G}_2} F\star G(x, s(x))d\lambda_2^u(x)d\nu(u)\\ &= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} F(x,h)G(p_2(xh), h^{-1})d\lambda_1^{s(x)}(h)d\lambda_2^u(x)d\nu(u)\\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} F(x,h)G(p_2(xh), h^{-1})d\lambda_1^{s(x)}(h)d\mu_2(x)\end{aligned}$$

Due proposition 4.11 and lemma 4.13, one has

$$\begin{aligned}&(\Delta_{\Phi_L}\Lambda_{\Phi_L}\mathfrak{R}(F), \Lambda_{\Phi_L}\mathfrak{R}(G^{\#})) =\\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} F(p_2(xh), h^{-1})k_L(x,h)G(x,h)d\lambda_1^{s(x)}(h)d\mu_2(x)\\ &= \int_{\mathcal{G}_2}\int_{\mathcal{G}_1} F(p_2(xh), h^{-1})k_L^{-1}(p_2(xh), h^{-1})G(x,h)d\lambda_1^{s(x)}(h)d\mu_2(x)\\ &= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2}\int_{\mathcal{G}_1} G(x,h)(k_L^{-1}F)(p_2(xh), h^{-1})d\lambda_1^{s(x)}(h)d\lambda_2^u(x)d\nu(u)\\ &= \int_{\mathcal{G}^0}\int_{\mathcal{G}_2} G\star(k_L^{-1}F)(x, s(x))d\lambda_2^u(x)d\nu(u) = \Phi_L(\mathfrak{R}(G\star d_LF))\\ &= \Phi_L(\mathfrak{R}(G)\mathfrak{R}(k_L^{-1}F))) = \Phi_L(\mathfrak{R}(G^{\#})^{\#}\mathfrak{R}(k_L^{-1}F)))\\ &= (\Lambda_{\Phi_L}\mathfrak{R}(k_L^{-1}F), \Lambda_{\Phi_L}\mathfrak{R}(G^{\#}))\end{aligned}$$

and the lemma follows. □

Remark 4.15. As $T_R = RT_LR$ there also exists another density d_R such that $\Delta_{\Phi_R}F = d_RF$.

Proposition 4.16. *For any $F \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$, one has*

$$\sigma_t^{\phi_L}(\mathfrak{R}(F)) = \mathfrak{R}(\tau^{it}F)$$

where $\tau(x,h) = \frac{\delta}{\delta_1}(p_1(xh))\delta_2(p_2(xh)x^{-1})\delta_1(h)^{-1}$, $\mu_{s,r}^2$-almost everywhere.

Proof:
For any $F, G \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$ and $\mu_{s,r}^2$-almost any $(g_2, g_1) \in \mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1$,

one has:

$$\begin{aligned}
&\sigma_t^{\phi_L}(\mathfrak{R}(F))G(g_2,g_1) = \Delta_{\Phi_L}^{it}\mathfrak{R}(F)\Delta_{\Phi_L}^{-it}G(g_2,g_1) = \\
&= {k_L}^{-it}(g_2,g_1)\mathfrak{R}(F)\Delta_{\Phi_L}^{-it}G(g_2,g_1) \\
&= {k_L}^{-it}(g_2,g_1)\int_{\mathcal{G}_1} F(p_2(g_2g_1),h)\delta_1(h)^{-\frac{1}{2}}{k_L}^{it}(g_2,g_1h)G(g_2,g_1h)d\lambda^{s(g_1)}(h) \\
&= \int_{\mathcal{G}_1}(\frac{k_L(g_2,g_1h)}{k_L(g_2,g_1)})^{it}F(p_2(g_2g_1),h)\delta_1(h)^{-\frac{1}{2}}G(g_2,g_1h)d\lambda^{s(g_1)}(h)
\end{aligned}$$

But

$$\begin{aligned}
\frac{k_L(g_2,g_1h)}{k_L(g_2,g_1)} &= \frac{\frac{\delta}{\delta_1}(p_1(g_2g_1h))\delta_1(g_1h)^{-1}\delta_2(p_2(g_2g_1h))\delta_2(g_2)^{-1}}{\frac{\delta}{\delta_1}(p_1(g_1g_2))\delta_1(g_1)^{-1}\delta_2(p_2(g_2g_1))\delta_2(g_2^{-1})} \\
&= \frac{\delta}{\delta_1}(p_1(g_1g_2)^{-1}(p_1(g_2g_1h))\delta_2(p_2(g_2g_1h)p_2(g_2g_1)^{-1})\delta_1(h)^{-1}
\end{aligned}$$

Moreover, almost everywhere

$$p_1(g_1g_2)^{-1}p_1(g_2g_1h) = p_1(g_1g_2)^{-1}p_1(p_1(g_2g_1)p_2(g_2g_1)h) = p_1(p_2(g_2g_1)h)$$

and

$$\begin{aligned}
p_2(g_2g_1h)p_2(g_2g_1)^{-1} &= p_2(p_1(g_2g_1)p_2(g_2g_1)h)p_2(g_2g_1)^{-1} \\
&= p_2(p_2(g_2g_1)h)p_2(g_2g_1)^{-1}
\end{aligned}$$

So

$$\frac{k_L(g_2,g_1h)}{k_L(g_2,g_1)} = \tau(p_2(g_2g_1),h)$$

where

$$\tau(x,h) = \frac{\delta}{\delta_1}(p_1(xh))\delta_2(p_2(xh)x^{-1})\delta_1(h)^{-1}$$

The lemma follows. □

Proposition 4.17. *T_L is a left invariant operator valued weight on von Neumann algebra $L^\infty(\mathcal{G}_2,\mu_2)\rtimes_{\mathfrak{a}}\mathfrak{G}(\mathcal{G}_1)$ in the sense of definition 2.1 iii).*

Proof: For any $F \in \mathcal{K}(\mathcal{G}_{2s_2}\times_{r_1}\mathcal{G}_1)$ and μ-almost any $y \in \mathcal{G}$, using the fact that $\tau(x, s(x)) = 1$ gives:

$$\begin{aligned}
(i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L))\Gamma(\sigma_t^{\Phi_L}(\mathfrak{R}(F))(y) &= (i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L))\Gamma(\mathfrak{R}(\tau^{it}F))(y)\\
&= T_L(\mathfrak{R}(\tau^{it}F))(y)\\
&= \int_{\mathcal{G}_2} (\tau^{it}F)(x, s(x))d\lambda_2^{m(y)}(x)\\
&= \int_{\mathcal{G}_2} F(x, s(x))d\lambda_2^{m(y)}(x)\\
&= T_L(\mathfrak{R}(F))(y)\\
&= (i \underset{L^\infty(\mathcal{G}^0,\mu)}{{}_m\star_s} \Phi_L))\Gamma(\mathfrak{R}(F))(y)
\end{aligned}$$

Hence $(i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L))\Gamma\sigma_t^{T_L} = (i \underset{L^\infty(\mathcal{G}^0,\nu)}{{}_m\star_s} \Phi_L))\Gamma$, so using [19] Theorem 6.2 and proposition 4.10 iii), T_L is left invariant. □

Theorem 4.18. *$(L^\infty(\mathcal{G}^0, \nu), L^\infty(\mathcal{G}_2, \mu_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1), m, s, \Gamma, T_L, T_R, \nu)$ is a measured quantum groupoid.*

Proof: Since T_L is left invariant, then $T_R = RT_LR$ is automatically right invariant and if $\Phi_R = \nu \circ s^{-1} \circ T_R$ is the lifted weight, then using 4.14 and 4.15, σ^{Φ_R} and σ^{Φ_L} commute as these are multiplication by functions, the theorem follows. □

Remark 4.19. Theorem 4.18 is a generalisation of the bicrossed product construction ([3], [22]....)

5. **Two families of examples**

In this chapter we describe two families of examples coming from case 3.2.2 and case 3.3.

5.1. **A matched pair of groups action on a space**

Let us use the notations of example 3.2.2, so $\mathcal{G} = X \times G$ where G is a group matched pair acting on a locally compact space X.

Let p_1^G and p_2^G be the almost everywhere defined functions associated with the matched pair G_1, G_2 ([3] 3.2), then for $\nu \times dg$ almost any $(x,g) \in X \times G$, one has:

$$p_1(x,g) = (x, p_1^G(g)) \quad p_2(x,g) = (x.p_1^G(g), p_2^G(g)) \quad m(x,g) = xp_1^G(g)$$

Let a_1 be the action of the quantum group $L^\infty(G_1)$ on the von Neumann algebra $L^\infty(G_2)$ coming from the usual bicrossed product construction, and Γ_1 usual coproduct for the crossed product $L^\infty(\,G_2) \rtimes_{a_1} L^\infty(G_1)$. So, due to Proposition 4.2 i), for any $h \in \mathcal{K}(G_1)$, any $\xi \in \mathcal{K}(G \times G)$ and $dg \times dg$-almost any $(g,g') \in G \times G$:

$$\Gamma_1(1 \otimes \rho(h))\xi(g,g') = \int_{\mathcal{G}_1} h(g_1)\xi(gp_1(p_2(g')g_1), g'g_1)dg_1$$

Thanks to 2.3, one easily sees that $L^2(X \times G_2) \underset{L^\infty(X,\nu)}{{}_{s_2}\otimes_{r_1}} L^2(X \times G_1)$ is isomorphic to $L^2(X \times G_2 \times G_1, \nu \times dg_2 \times dg_1)$ (and then to $L^2(X \times G_2) \otimes L^2(G_1)$) by the application θ such that for any $f \in \mathcal{K}(X \times G_2 \times X \times G_1)$ and $\nu \times dg_2 \times dg_1$-almost any (x, g_2, g_1) in $X \times G_2 \times G_1$:

$$\theta(f)(x, g_2, g_1) = f(x, g_2, x.g_2, g_1)$$

This leads to a spatial isomorphism between von Neumann algebra $L^\infty(X \times \mathcal{G}_2) \underset{L^\infty(X,\nu)}{{}_{s_2}\star_{r_1}} L^\infty(X \times \mathcal{G}_1)$ and $L^\infty(X \times G_2 \times G_1, \nu \times dg_2 \times dg_1)$ with the same formula as for θ.

So the action

$$\mathfrak{a} : L^\infty(X \times G_2) \to L^\infty(X \times G_2) \underset{L^\infty(X,\nu)}{{}_{s_2}\star_{r_1}} L^\infty(X \times G_1)$$

can be identified with a one to one homomorphism

$$L^\infty(X \times G_2) \to L^\infty(X \times G_2) \otimes L^\infty(G_1)$$

Remark 5.1. By similar arguments, in that case $L^2(\mathcal{G}^2_{s,m}, \mu^2_{s,m})$ can be identified with the space $L^2(X \times G \times G)$ using the map Σ such that, for any$f \in \mathcal{K}(\mathcal{G}^2_{s,m})$, one has: $\Sigma(f)(x,g,g') = f(x,g,x.gp_1^G(g')^{-1}, g')$ (observation: we have $s(x,g) = m(x.gp_1^G(g')^{-1}, g')$).

Proposition 5.2. *i) The action $\mathfrak{a}$ given by 3.12, can be identified with a usual action of G_1 on $L^\infty(X \times G_2)$; if one denotes $\tilde{\mathfrak{a}}$ this action, for any $(x, g_1, g_2) \in X \times G_1 \times G_2$ such that $g_2 g_1 \in G_1 G_2$ and any $f \in L^\infty(X \times G_2)$:*

$$\tilde{\mathfrak{a}}(f)(x, g_2, g_1) = f(x.p_1^G(g_2 g_1), p_2^G(g_2 g_1))$$

ii) The crossed product $L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ is isomorphic to the usual crossed product $L^\infty(X \times G_2) \rtimes_{\tilde{\mathfrak{a}}} L^\infty(G_1)$

iii) Using the identification of remark 5.1, for almost any $(x, g, g') \in X \times G \times G$ and any $f \in L^\infty(X \times G_2)$, one has

$$\Gamma(\mathfrak{a}(f))(x, g, g') = f(x.p_1^G(g), p_2^G(g) p_2^G(g'))\,;$$

moreover, for any $h \in L^\infty(X), k \in L^\infty(G_1)$, one has:

$$\Gamma(1_{s_2} \otimes_{r_1} \rho(h \otimes k)) = M(h)(1 \otimes \Gamma_1(1 \otimes \rho_1(k)))$$

where $M(h)$ is (the multiplication by) map $M(h)(x, g, g') = h(x.g p_2^G(g'))$

iv) For any $f \in \mathcal{K}(X \times G_2)$, any $h \in \mathcal{K}(X \times G_1)$ and almost any $(x, g) \in X \times G$, one has:

$$T_L(\mathfrak{a}(f)(1_{s_2} \otimes_{r_1} \rho(h))(x, g) = \int_{G_2} f(x p_1^G(g), g_2) h(x p_1^G(g) g_2, e) dg_2$$

$$T_R(\mathfrak{a}(f)(1_{s_2} \otimes_{r_1} \rho(h))(x, g) = \int_{G_2} f(x g g_2, g_2^{-1}) h(xg, e) dg_2$$

Proof: i) One easily sees that $\tilde{\mathfrak{a}}$ is an action. For any $h \in L^\infty(X \times G_2)$, any function $f \in \mathcal{K}(X \times G_2 \times X \times G_1)$ and $\nu \times dg_2 \times dg_1$ almost any (x, g_2, g_1) in $X \times G_2 \times G_1$, one has:

$$\begin{aligned}
\theta(\mathfrak{a}(h)f)(x, g_2, g_1) &= \mathfrak{a}(h)f)(x, g_2, x.g_2, g_1) \\
&= h(p_2((x, g_2)(x.g_2, g_1)) f(x, g_2, x.g_2, g_1) \\
&= h(p_2(x, g_2 g_1) \theta(f)(x, g_2, g_1) \\
&= h(x.p_1^G(g_2 g_1), p_2^G(g_2 g_1)) \theta(f)(x, g_2, g_1) \\
&= \tilde{\mathfrak{a}}(h) \theta(f)(x, g_2, g_1)
\end{aligned}$$

One deduces that $Ad(\theta) \circ \mathfrak{a} = \tilde{\mathfrak{a}}$, which gives i).

ii) The crossed product $L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ is generated, in the von Neumann algebra $\mathcal{L}(L^2(X \times G_2) \underset{L^\infty(X,\nu)}{{}_{s_2}\otimes_{r_1}} L^2(X \times G_1))$, by $\mathfrak{a}(L^\infty(X \times G_2))$ and $1 \underset{L^\infty(X,\nu)}{{}_{s_2}\otimes_{r_1}} \mathfrak{G}(\mathcal{G}_1)'$.

$\mathfrak{G}(\mathcal{G}_1)'$ is generated in $\mathcal{L}(L^2(\mathcal{G}_1)$ by the image of the right regular representation of $\mathcal{G}_1$, as $\mathcal{G}_1 = X \times G_1$, it is the usual crossed product of $L^\infty(X)$ by the right action of G_1. So, if one denotes by a_1 this action and by ρ_1 the right regular representation of the group G_1, the von Neumann algebra $\mathfrak{G}(\mathcal{G}_1)'$ is generated in $\mathcal{L}(L^2(X \times G_1))$ by the products $a_1(\phi)(1 \otimes \rho_1(\phi_1))$, for $\phi \in L^\infty(X)$ and $\phi_1 \in L^\infty(G_1)$. But for $\nu \times dg_2 \times dg_1$ almost any (x, g_2, g_1) in $X \times G_2 \times G_1$ and any $f \in \mathcal{K}(X \times G_2 \times X \times G_1)$ one has:

$$\begin{aligned}
&\theta\big((1_{s_2} \otimes_{r_1} [a_1(\phi)(1_{s_2} \otimes_{r_1} \rho_1(\phi_1))])f\big)(x, g_2, g_1) = \\
&\qquad = (1_{s_2} \otimes_{r_1} [a_1(\phi)(1_{s_2} \otimes_{r_1} \rho_1(\phi_1))])f(x, g_2, x.g_2, g_1) \\
&\qquad = \int_{G_1} (\phi \otimes \phi_1)((x.g_2).g_1, g_1')f(x, g_2, xg_2, g_1g_1')dg_1' \\
&\qquad = \phi(x.(g_2g_1)) \int_{G_1} \phi_1(g_1')\theta(f)(x, g_2, g_1g_1')dg_1'
\end{aligned}$$

Let $k \in L^\infty(X \times G_2)$ be defined for any $(y, g_2) \in X \times G_2$ by:

$$k(y, g_2) = \phi(y.g_2)$$

then one has:

$$\begin{aligned}
&\theta\big((1_{s_2} \otimes_{r_1} [a_1(\phi)(1_{s_2} \otimes_{r_1} \rho_1(\phi_1))])f\big)(x, g_2, g_1) = \\
&\qquad = k(xp_1(g_2g_1), p_2(g_2g_1)) \int_{G_1} \phi_1(g_1')\theta(f)(x, g_2, g_1g_1')dg_1' \\
&\qquad = \tilde{\mathfrak{a}}(k)(x, g_2, g_1) \int_{G_1} \phi_1(g_1')\theta(f)(x, g_2, g_1g_1')dg_1'
\end{aligned}$$

Hence $Ad(\theta) \circ (1_{s_2} \otimes_{r_1} [a_1(\phi)(1_{s_2} \otimes_{r_1} \rho_1(\phi_1))]) = \tilde{\mathfrak{a}}(k)(1 \otimes \rho_1(\phi_1))$, as $Ad(\theta) \circ \mathfrak{a} = \tilde{\mathfrak{a}}$, this proves that $Ad(\theta)(L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1))$ is included in $L^\infty(X \times G_2) \rtimes_{\tilde{\mathfrak{a}}} L^\infty(G_1)$ and contains $\tilde{\mathfrak{a}}(L^\infty(X \times G_2))$ and also $1 \otimes \rho_1(L^\infty(G_1))$ (using $\phi = 1$), so $Ad(\theta)$ realizes a spatial isomorphism between $L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ and $L^\infty(X \times G_2) \rtimes_{\tilde{\mathfrak{a}}} L^\infty(G_1)$.

iii) Due to Proposition 4.2, for almost any $(x, g, g') \in X \times G \times G$ and any $f \in L^\infty(X \times G_2)$, one has:

$$\begin{aligned}\Gamma(\mathfrak{a}(f))((x,g),(xgp_1^G(g')^{-1},g')) &= f(p_2(x,g)p_2(xgp_1^G(g')^{-1},g')))\\ &= f((x.p_1^G(g),p_2^G(g))((x.gp_1^G(g')^{-1}).p_1^G(g'),p_2^G(g')))\\ &= f((x.p_1^G(g),p_2^G(g))(x.g,p_2^G(g')))\\ &= f(x.p_1^G(g),p_2^G(g)p_2^G(g'))\end{aligned}$$

Moreover, for any $h \in L^\infty(X), k \in L^\infty(G_1)$, for almost any $(x, g, g') \in X \times G \times G$, as $s(xg^G p_1(g')^{-1}, g') = x.gp_2^G(g')$ any $\xi \in L^\infty(X), \xi' \in L^\infty(G \times G)$, as one has:

$$\begin{aligned}&\Sigma\Gamma(1_{s_2} \otimes_{r_1} \rho(h \otimes k))\Sigma^*\xi(x,g,g') =\\ &\Gamma(1_{s_2} \otimes_{r_1} \rho(h \otimes k))\Sigma^*\xi((x,g),(xgp_1^G(g')^{-1},g')) =\\ &= \int_{G_1} h(z)k(g_1)\times\\ &\quad \times \Sigma^*\xi((x,g)p_1(p_2(xgp_1^G(g')^{-1},g')(z,g_1)),(xgp_1^G(g')^{-1},g')(z,g_1))dg_1\end{aligned}$$

in which $z = x.gp_2^G(g')$. Hence one has:

$$\begin{aligned}&\Sigma\Gamma(1_{s_2} \otimes_{r_1} \rho(h \otimes k))\Sigma^*\xi(x,g,g') =\\ &= \int_{G_1} h(z)k(g_1)\times\\ &\qquad \times \Sigma^*\xi((x,g)p_1((xg,p_2^G(g'))(z,g_1)),(xgp_1^G(g')^{-1},g')(z,g_1))dg_1\\ &= \int_{G_1} h(z)k(g_1)\Sigma^*\xi((x,g)p_1(xg,p_2^G(g')g_1),(xgp_1^G(g')^{-1},g'g_1))dg_1\\ &= \int_{G_1} h(z)k(g_1)\Sigma^*\xi((x,g)(xg,p_1^G(p_2^G(g')g_1)),(xgp_1^G(g')^{-1},g'g_1))dg_1\\ &= \int_{G_1} h(z)k(g_1)\Sigma^*\xi((x,p_1^G(p_2^G(g')g_1)),(xgp_1^G(g')^{-1},g'g_1))dg_1\\ &= \int_{G_1} h(z)k(g_1)\xi(x,p_1^G(p_2^G(g')g_1),g'g_1))dg_1\\ &= h(x.gp_2^G(g'))\int_{G_1} k(g_1)\xi(x,p_1^G(p_2^G(g')g_1),g'g_1))dg_1\\ &= h(x.gp_2^G(g'))(1 \otimes \Gamma_1(1 \otimes \rho_1(k)))\xi(x,g,g')\\ &= M(h)(1 \otimes \Gamma_1(1 \otimes \rho_1(k)))\xi(x,g,g')\end{aligned}$$

where $M(h)$ is multiplication operation by the function $M(h)(x, g, g') = h(x.gp_2^G(g'))$.

iv) Using proposition 4.10 for almost any $(x, g) \in X \times G$, any $f \in \mathcal{K}(X \times G_1)$ and any $h \in \mathcal{K}(X \times G_2)$, one has:

$$\begin{aligned} T_L(\mathfrak{a}(f)(1_{s_2} \otimes_{r_1} \rho(h))(x, g) &= \int_{X \times G_2} f(y, g_2) h(s(x, g_2)) d\lambda_2^{m(x,g)}(y, g_2) \\ &= \int_{G_2} f(xp_1^G(g), g_2) h(s(xp_1^G(g), g_2)) dg_2 \\ &= \int_{G_2} f(xp_1^G(g), g_2) h(xp_1^G(g) g_2, e) dg_2 \end{aligned}$$

Similar computations give the second equality. □

Remark 5.3. As any usual crossed product, $L^\infty(X \times G_2) \rtimes_{\tilde{\mathfrak{a}}} L^\infty(G_1)$ is isomorphic to $\widehat{\mathfrak{G}}((X \times G_2) \times G_1)$, but the measured quantum groupoid structure we obtain for this crossed product in 5.2 is not isomorphic to the natural one of $\mathfrak{G}((X \times G_2) \times G_1)$, recalled in 2.3, because for one the basis is $L^\infty(X)$ and for the other the basis is $L^\infty(X \times G_2)$.

5.2. The case of a principal and transitive groupoid

Let's use notations similar to 3.3, so we suppose that $\mathcal{G}$ is a transitive and principal groupoid, hence of the form $X_1 \times X_2 \times X_1 \times X_2$ where the X_i's are Hausdorff locally compact, $\mathcal{G}_1 = \bigsqcup_{x_2 \in X_2} X_1 \times \{x_2\} \times X_1 \times \{x_2\}$, and $\mathcal{G}_2 = \bigsqcup_{x_1 \in X_1} \{x_1\} \times X_2 \times \{x_1\} \times X_2$. For any (x_1, x_2, y_1, y_2) in $\mathcal{G}$, one easily sees that:

$$p_1(x_1, x_2, y_1, y_2) = (x_1, x_2, y_1, x_2) \quad , \quad p_2(x_1, x_2, y_1, y_2) = (y_1, x_2, y_1, y_2)$$
$$m(x_1, x_2, y_1, y_2) = (y_1, x_2)$$

One can identify $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) with $X_1 \times X_1 \times X_2$ (resp. $X_2 \times X_2 \times X_1$), using the map $(x_1, x_2, y_1, x_2) \mapsto (x_1, y_1, x_2)$ (resp. $(x_1, x_2, x_1, y_2) \mapsto (x_2, y_2, x_1)$); due to lemma 3.7, the Haar system of $\mathcal{G}_1$ is $(\delta_{x_1} \otimes \nu_1 \otimes \delta_{x_2})_{(x_1, x_2)}$.

So $L^2(\mathcal{G}_1, \mu_1)$ (resp. $L^2(\mathcal{G}_2, \mu_2)$) can be identified with $L^2(X_1 \times X_1 \times X_2, \nu_1 \times \nu_1 \times \nu_2)$ (resp. $L^2(X_2 \times X_2 \times X_1, \nu_2 \times \nu_2 \times \nu_1)$).

This gives a spatial isomorphism between $L^\infty(\mathcal{G}_1, \mu_1)$ (resp. $L^\infty(\mathcal{G}_2, \mu_2)$) and $L^\infty(X_1 \times X_1 \times X_2, \nu_1 \times \nu_1 \times \nu_2)$ (resp. $L^\infty(X_2 \times X_2 \times X_1, \nu_2 \times \nu_2 \times \nu_1)$).

Lemma 3.10 gives an obvious isomorphism between the two von Neumann algebras $L^\infty(\mathcal{G}_2,\mu_2) \underset{L^\infty(X_1\times X_2,\nu_1\times\nu_2)}{{}_{s_2}\otimes_{r_2}} L^\infty(\mathcal{G}_1,\mu_1)$ and $L^\infty(\mathcal{G},\nu_1\times\nu_2\times\nu_1\times\nu_2)$, coming from the map: $((x_2,y_2,x_1),(x_1,y_1,y_2)) \mapsto (x_1,x_2,y_1,y_2)$.

Using this identification one has:

$$\mathfrak{a} : L^\infty(X_2^2\times X_1,\nu_2\times\nu_2\times\nu_1) \to L^\infty(X_2^2\times X_1,\nu_2\times\nu_2\times\nu_1) \underset{L^\infty(X_1\times X_2,\nu_1\times\nu_2)}{{}_{s_2}\otimes_{r_1}} L^\infty(X_1^2\otimes X_2,,\nu_1\times\nu_1\times\nu_2))$$

and for any $f\in\mathcal{K}(\mathcal{G}_2)$, any $(x_1,x_2,y_1,y_2)\in\mathcal{G}$ one has:

$$\mathfrak{a}(f)\big((x_2,y_2,x_1),(x_1,y_1,y_2)\big) = f(p_2(x_1,x_2,y_1,y_2)) = f(x_2,y_2,y_1)$$

This formula can be interpreted just using the natural shift action of the groupoid $X_1\times X_1$ on the fibered set (X_1, id_{X_1}) given for any elements x_1,y_1 in X_1 by $x_1 \underset{\mathfrak{s}_1}{.} (x_1,y_1) = y_1$. To prove this let's give some definitions:

Definition 5.4. i) Let $\underset{\mathfrak{S}_1}{.}$ be the action of $X_1\times X_1$ on the fibered set $(X_2\times X_2\times X_1, pr_3)$, where pr_3 is the projection on X_1, defined for any $(x_2,y_2,x_1)\in X_2\times X_2\times X_1$ and any $y_1\in X_1$, by

$$(x_2,y_2,x_1)\underset{\mathfrak{S}_1}{.}(x_1,y_1) = (x_2,y_2,y_1)$$

Let $(\tilde{\mathfrak{a}}, pr_3)$ be the corresponding action of $\mathcal{G}(X_1\times X_1)$ on $L^\infty(X_2\times X_2\times X_1,\nu_2\times\nu_2\times\nu_1)$

ii) Let $\Sigma : L^2(X_2\times X_2\times X_1,\nu_2\times\nu_2\times\nu_1) \underset{L^\infty(X_1\times X_2,\nu_1\times\nu_2)}{{}_{s_2}\otimes_{r_1}} L^2(X_1\times X_1\times X_2,\nu_1\times\nu_1\times\nu_2) \to L^2(X_2\times X_2\times X_1,\nu_2\times\nu_2\times\nu_1) \underset{L^\infty(X_1,\nu_1)}{{}_{pr_3}\otimes_{r}} L^2(X_1\times X_1,\nu_1\times\nu_1)$ be the isometric isomorphism given for any $\phi\in\mathcal{K}((X_2\times X_2\times X_1)\times(X_1\times X_1\times X_2)$ and almost any $(x_1,x_2,y_1,y_2)\in\mathcal{G}$ by:

$$\Sigma(\phi)\big((x_2,y_2,x_1),(x_1,y_1)\big) = \phi\big((x_2,y_2,x_1)),(x_1,y_1,y_2)\big)$$

iii) Let $\Sigma' : L^2(X_2\times X_2\times X_1,\nu_2\times\nu_2\times\nu_1) \underset{L^\infty(X_1,\nu_1)}{{}_{pr_3}\otimes_{r}} L^2(X_1\times X_1,\nu_1\times\nu_1) \to L^2(X_2\times X_2\times X_1\times X_1,\nu_2\times\nu_2\times\nu_1\times\nu_1)$ be the isometric isomorphism given for any $\phi \in \mathcal{K}((X_2 \times X_2 \times X_1) \times (X_1 \times X_1))$ and almost any $(x_1, x_2, y_1, y_2) \in \mathcal{G}$ by:

$$\Sigma'(\phi)(x_2, y_2, x_1, y_1) = \phi((x_2, y_2, x_1), (x_1, y_1))$$

Theorem 5.5. *Using the previous notations one has:*

i) $\tilde{\mathfrak{a}} = Ad\Sigma \circ \mathfrak{a}$,

ii) $\theta = Ad\Sigma'\Sigma$ *realizes a spatial isomorphism between* $L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ *and* $L^\infty({X_2}^2, \nu_2^{\otimes 2}) \otimes \mathbb{C}1_{\mathcal{L}(L^2(X_1,\nu_1))} \otimes \mathcal{L}(L^2(X_1, \nu_1))$*:* $L^\infty(\mathcal{G}_2) \rtimes_{\mathfrak{a}} \mathfrak{G}(\mathcal{G}_1)$ *is isomorphic to* $L^\infty(X_2 \times X_2, \nu_2 \times \nu_2) \otimes \mathcal{L}(L^2(X_1, \nu_1))$ *and to* $\mathfrak{G}(X_2 \times X_2) \otimes \hat{\mathfrak{G}}'(X_1 \times X_1)$ *,*

iii) If $\tau \colon L^2(X_2^3 \times X_1^3, \nu_2^3 \times \nu_1^3) \to L^2(X_2^2 \times X_1 \times X_2 \times X_1^2, \nu_2^2 \times \nu_1 \otimes \nu_2 \times \nu_1^2)$ *is the map which flips the third and fourth factor, one has:*

$$(\theta_s \star_m \theta)\Gamma\theta^* = Ad\tau(\Gamma_{X_2^2} \otimes \widehat{\Gamma}'_{X_1^2})$$

iv) $$\theta T_L \theta^* = (T_{X_2^2} \otimes \widehat{T_{X_1^2}}') \qquad \theta T_R \theta^* = (T^{-1}_{X_2^2} \otimes \widehat{T_{X_1^2}}').$$

Proof:

i) The first assertion is obvious.

ii) For any $f_1, h_1, k_1 \in \mathcal{K}(X_1)$, any $f_2, g_2 \in \mathcal{K}(X_2)$, any $\xi \in \mathcal{K}(X_2 \times X_2 \times X_1 \times X_1)$, and almost any $(x_2, y_2, x_1, y_1) \in X_2 \times X_2 \times X_1 \times X_1$, due to i) one has:

$$\theta[\mathfrak{a}(f_2 \otimes g_2 \otimes f_1)]\xi(x_2, y_2, x_1, y_1) = \\ = (f_2 \otimes g_2 \otimes 1 \otimes f_1)(x_2, y_2, x_1, y_1)\xi(x_2, y_2, x_1, y_1)$$

so

$$\theta[\mathfrak{a}(f_2 \otimes g_2 \otimes f_1)] = f_2 \otimes g_2 \otimes 1 \otimes f_1$$

Let's denote T_{ψ_1}, for any $\psi_1 \in \mathcal{K}(X_1 \times X_1)$, the integral (compact) operator defined for any $\xi_1 \in \mathcal{K}(X_1)$ and almost any x_1 in X_1 by

$$(T_{\psi_1}\xi_1)(x_1) = \int_{X_1} \psi_1(x_1, z_1)\xi_1(z_1)d\nu_1(z_1)$$

A straightforward calculation gives that

$$\theta[\mathfrak{a}(f_2 \otimes g_2 \otimes f_1)(1 \; {}_{s_2}\underset{\nu}{\otimes}{}_{r_1} \; \rho(h_1 \otimes k_1 \otimes h_2)] = f_2 \otimes g_2 h_2 \otimes 1 \otimes T_{f_1 h_1 \otimes k_1} \tag{5.1}$$

The assertion ii) follows.

iii) One easily sees that the coproduct $\Gamma_{\mathcal{G}_2} : L^\infty(\mathcal{G}_2) \to L^\infty(\mathcal{G}_2)_s \star_r L^\infty(\mathcal{G}_2)$ is given for any $f_2, g_2 \in X_2$ and any $f_1 \in \mathcal{K}(X_1)$ by:

$$\Gamma_{\mathcal{G}_2}(f_2 \otimes g_2 \otimes f_1) = (f_2 \otimes 1 \otimes f_1)_s \otimes_r (1 \otimes g_2 \otimes 1)$$

Moreover, using 3.12, one has

$$\begin{aligned}(\theta \circ m)(f_1 \otimes f_2) &= (\theta \circ \mathfrak{a} \circ r_2)(f_1 \otimes f_2))) = \hat{\mathrm{E}}\theta(\mathfrak{a}(f_2 \otimes 1 \otimes f_1) \\ &= f_2 \otimes 1 \otimes 1 \otimes f_1\end{aligned}$$

$$\begin{aligned}(\theta \circ s)(f_1 \otimes f_2) &= (\theta \circ \mathfrak{a} \circ s_2)(f_1 \otimes f_2) = \theta(\mathfrak{a}(1 \otimes f_2 \otimes f_1) \\ &= 1 \otimes f_2 \otimes 1 \otimes f_1\end{aligned}$$

This gives an isometric isomorphism between the Hilbert space $L^2(X_2^2 \times X_1^2, \nu_2^{\otimes 2} \otimes \nu_1^{\otimes 2}) \underset{L^\infty(X_1 \times X_2, \nu_1 \otimes \nu_2)}{{}_{\theta\circ s}\otimes_{\theta\circ m}} L^2(X_2^2 \times X_1^2, \nu_2^{\otimes 2} \otimes \nu_1^{\otimes 2})$ onto the Hilbert space $L^2(X_2^2 \times X_1 \times X_2 \times X_1^2, \nu_2^{\otimes 2} \otimes \nu_1 \otimes \nu_2 \otimes \nu_1^{\otimes 2})$, if Ψ is this map, for any $\xi^1, \xi^2 \in \mathcal{K}(X_2^2 \times X_1^2)$ one has:

$$\Psi(\xi^1 {}_{\theta\circ s}\otimes_{\theta\circ m} \xi^2)(x_2, y_2, x_1, z_2, y_1, z_1) = \xi^1(x_2, y_2, x_1, y_1)\xi^2(y_2, z_2, z_1, y_1)$$

Hence, using 4.1:

$$(\theta_s \star_m \theta)\Gamma\theta^*(f_2 \otimes g_2 \otimes 1 \otimes 1) = (f_2 \otimes 1 \otimes 1 \otimes 1)_{\theta\circ s} \otimes_{\theta\circ m} (1 \otimes g_2 \otimes 1 \otimes 1)$$

due to (2) and 2.3.1, this gives:

$$(\theta_s \star_m \theta)\Gamma\theta^*(f_2 \otimes g_2 \otimes 1 \otimes 1) = Ad\tau(\Gamma_{X_2^2} \otimes \widehat{\Gamma}'_{X_1^2})(f_2 \otimes g_2 \otimes 1 \otimes 1)$$

Quite simple computations also imply that for any $h_1, k_1 \in \mathcal{K}(X_1)$,

$$\begin{aligned}&(\theta_s \star_m \theta)\Gamma(1 \; {}_{s_2}\underset{\nu}{\otimes}{}_{r_1} \; \rho(h_1 \otimes k_1 \otimes 1)) = \\ &\qquad = Ad\tau(\Gamma_{X_2^2} \otimes \widehat{\Gamma}'_{X_1^2})\theta^*(1 \; {}_{s_2}\underset{\nu}{\otimes}{}_{r_1} \; \rho(h_1 \otimes k_1 \otimes 1))\end{aligned}$$

which gives iii).

iv) Due to (1), 4.10 and 2.3.1, for any $f_1, g_1 \in \mathcal{K}(X_1)$ any $f_2, g_2 \in \mathcal{K}(X_2)$, one has

$$\theta T_L \theta^*(f_2 \otimes g_2 \otimes 1 \otimes T_{h_1 \otimes k_1}) = (T_{X_2^2} \otimes \widehat{T}'_{X_1^2})(f_2 \otimes g_2 \otimes 1 \otimes T_{h_1 \otimes k_1}),$$

which gives $\theta T_L \theta^* = T_{X_2^2} \otimes \widehat{T}'_{X_1^2}$.

And for similar reasons: $\theta T_R \theta^* = T^{-1}_{X_2^2} \otimes \widehat{T}'_{X_1^2}$. □

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