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Abstract

$S^H = \{S_t^H, t \geq 0\}$ be a sub-fractional Brownian motion with $H \in (0,1)$. We establish the existence, the joint continuity and the Hölder regularity of the local time L^H of S^H. We will also give Chung's form of the law of iterated logarithm for S^H. This results are obtained with the decomposition of the sub-fractional Brownian motion into the sum of fractional Brownian motion plus a stochastic process with absolutely continuous trajectories. This decomposition is given by Ruiz de Chavez and Tudor [10].

1. Introduction

The intuitive idea of a local time $L(t,x)$ for a process X is that $L(t,x)$ measures the amount of time X spends at the level x during the interval $[0,t]$. We are concerned in this paper with the existence and regularity of the local time of the sub-fractional Brownian motion (Sub-fBm). We will also give Chung's form of the law of iterated logarithm for S^H.
Sub-fractional Brownian motion $S^H = \{S_t^H, t \geq 0\}$ is a centered Gaussian process with covariance function

$$\mathbb{E}[S_s^H S_t^H] = s^H + t^H - \frac{1}{2}[(s+t)^H + |s-t|^H]$$

where $H \in (0,2)$. This process was introduced by Bojdecky et al [8] as an intermediate process between standard Brownian motion and fractional Brownian motion. Recall that fractional Brownian motion (fBm for short) $B^H = \{B_t^H, t \geq 0\}$ is a centered Gaussian process with covariance function

$$\mathbb{E}[B_s^H B_t^H] = \frac{1}{2}(s^H + t^H - |s-t|^H)$$

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where $H \in (0,2)$. Note that both fBm and Sub-fBm are standard Brownian motion for $H = 1$. For $H \neq 1$, Sub-fBm preserves some of main properties of fBm, such as long-range dependence, but its increments are not stationary, they are more weakly correlated on non-overlapping intervals than fBm ones, and their covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. For a more detailed discussion of Sub-fBm and its properties we refer the reader to Bojdecky et al [8]. Some properties of this process have also been studied in Tudor [21] and [22].

In [10] the authors obtain the following equality in law

$$S_t^H \stackrel{d}{=} C_1 X_t^H + B_t^H \tag{1.1}$$

where $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$, $H \in (0,1)$, $X_t^H = \int_0^\infty (1-e^{-\theta t})\theta^{-\frac{H+1}{2}} dW_\theta$ and standard Brownian motion W and fractional Brownian motion B^H are independents. The centered Gaussian process $X^H = \{X_t^H, t \geq 0\}$ is introduced by Lei and Nualart [17] in order to obtain a decomposition of bifractional Brownian motion into the sum of a transformation of X_t^H and a fBm. We will establish our results by using an approach based on the concept of local nondeterminism (LND for simplicity), introduced by Berman [6] to unify and extend his earlier works on the local times of stationnaire Gaussian processes. The joint continuity as well as Hölder conditions in both the space and the (time) set variable of the local time of locally nondeterministic (LND) Gaussian process and fields have been studied by Berman [4] and [6], Pitt [20], Kôno [15], Geman and Horowitz [13], and recently by Csörgo, Lin and Shao [11] and [23]. Recently, Boufoussi, Dozzi and Guerbaz [9] and Guerbaz [14] have studied respectively the local time of the multifractional Brownian motion (mBm) and the local time of the filtered white noises. Th multifractional Brownian motion extend the fBm in the sens that its Hurst parameter is not more constant, but a Hölder function of time. The paper is organized as follows. Section 2 contains a brief review on the local times of Gaussian processes and Berman's concept of local nondeterminism. In section 3 we prove the existence of a square integrable version of the local time, the joint continuity and Hölder regularity in time and in space. Chung's form of the law of iterated logarithm for Sub-fBm is obtained in section 4, which is applied to derive a lower bound for local moduli of continuity of local times of Sub-fBm. Will use

$C, C_1, \ldots$ to denote unspecified positive finite constants which may not necessary be the same at each occurrence.

2. Preliminaries

We recall some aspects of local times and we refer to the paper of Geman and Horowitz [13] for an insightful survey local times. Let $X = \{X(t), t \geq 0\}$ be a real valued separable random process with Borel sample functions. For any Borel set B of the real line, the occupation measure of X is defined as follows

$$\mu(A, B) = \lambda\{s \in A : X(s) \in B\} \;\; \forall \;\; A \in \mathcal{B}(\mathbb{R}^+),$$

and λ is the Lebesgue measure on $\mathbb{R}^+$. If $\mu(A, .)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that X has local times on A and define its local time, $L(A, .)$, as the Radon-Nikodym derivative of $\mu(A, .)$. Here x is the so-called space variable, and A is the time variable. The existence of jointly continuous local time reveals information on the fluctuation of the sample paths of process itself [1, Chap 8]. There are several approach for proving the joint continuity of the local times, one of them is the Fourier analytic method developed by Berman to extend his early works on the local times of stationary Gaussian processes. The main tool used in Berman's approach (see Berman [6]) is the local nondeterminism. We give a brief review of the concept of local nondeterminism, more informations on the subject can be found in [6]. Let J be an open interval on t axis. Assume that $\{X(t), t \geq 0\}$ is a zero mean Gaussian process without singularities in any interval of length δ, for some $\delta > 0$, and without fixed zeros; i.e. there exists $\delta > 0$ such that

$$(\mathcal{P}) \begin{cases} \mathbb{E}(X(t) - X(s))^2 & > 0, \text{whenever} \quad 0 < |t - s| < \delta \\ \mathbb{E}(X(t))^2 & > 0, \text{for} \quad t \in J. \end{cases}$$

To introduce the concept of local nondeterminism, Berman defined the relative conditioning error,

$$V_m = \frac{Var\{X(t_m) - X(t_{m-1})/X(t_1), \ldots, X(t_{m-1})\}}{Var\{X(t_m) - X(t_{m-1})\}}; \tag{2.1}$$

where, for $m \geq 2, t_1, \ldots, t_m$ are arbitrary points in J ordered according to their indices, i.e. $t_1 < t_2 < \cdots < t_m$. We say that the process X is locally nondeterministic (LND) on J if for every $m \geq 2$,

$$\liminf_{c \searrow 0^+, 0<t_m - t_1 < c} V_m > 0. \tag{2.2}$$

This condition means that a small increment of the process is not almost relatively predictable on the basis of a finite number of observations from the immediate past. Berman has proved, for Gaussian processes, that the local nondeterminism as characterized as follows.

Proposition 2.1. *X is LND if and only if for every integer $m \geq 2$, there exists positive constants C and δ (both may depend on m) such that*

$$Var\left(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]\right) \geq C_m \sum_{j=1}^{m} u_j^2 Var[X(t_j) - X(t_{j-1})], \tag{2.3}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in J with $t_m - t_1 < \delta, t_0 = 0$ and $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$.

The proof of this proposition is given in [6], Lemmas 2.1 and 8.1.

3. Local time of sub-fractional Brownian motion

The propose of this section is to present sufficient conditions for the existence of the local times of sub-fractional Brownian motion. Furthermore, using the local nondeterminism approach, we show that the local times have a jointly continuous version.

3.1. Square integrability

Theorem 3.1. *Assume $0 < H < 1$. On each (time)-interval $[a, b] \subset [0, +\infty[$, the Sub-fBm S^H admits a local time $L^H([a, b], x)$ which satisfies*

$$\int_{\mathbb{R}} L^H([a, b], x)^2 dx < \infty.$$

For the proof of Theorem 3.1, we need the following lemma. This result on the regularity of the increments of the Sub-fBm will be the key for the existence and the regularity of local times.

Lemma 3.2. *There exists $\delta > 0$ and, for any integer $m \geq 1$, there exists $M_m > 0$, such that*

$$\mathbb{E}[S_t^H - S_s^H]^m \geq M_m |t - s|^{mH}$$

for all, s, t *such that* $|t-s| < \delta$.

Proof. We use the decomposition of the Sub-fBm given by Ruiz de Chavez and Tudor [10] :

$$S_t^H \stackrel{d}{=} C_1 X_t^H + B_t^H \tag{3.1}$$

where $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$, $H \in (0,1)$, $X_t^H = \int_0^\infty (1-e^{-\theta t})\theta^{-\frac{H+1}{2}} dW_\theta$ and standard Brownian motion W and fractional Brownian motion B^H are independents.

$$\mathbb{E}[S_t^H - S_s^H]^2 = \mathbb{E}[C_1(X_t^H - X_s^H) + (B_t^H - B_s^H)]^2.$$

Using the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, we obtain

$$\begin{aligned} \mathbb{E}[S_t^H - S_s^H]^2 &\geq \frac{1}{2}\mathbb{E}[B_t^H - B_s^H]^2 - C_1^2\mathbb{E}[X_t^H - X_s^H]^2 \\ &\geq \frac{C_H}{2}|t-s|^{2H} - C_1^2\mathbb{E}[X_t^H - X_s^H]^2. \end{aligned} \tag{3.2}$$

Moreover, we have

$$\mathbb{E}[X_t^H - X_s^H]^2 = \int_0^\infty (e^{-\theta s} - e^{-\theta t})^2 \theta^{-(H+1)} d\theta.$$

Making use of the theorem on finite increments for the function $v \mapsto e^{-\theta v}$, for $v \in (s,t)$, there exists $\alpha \in (s,t)$ such that

$$\begin{aligned} \mathbb{E}[X_t^H - X_s^H]^2 &= |t-s|^2 \int_0^\infty e^{-2\alpha\theta}\theta^{1-H} d\theta \\ &\leq |t-s|^2 \int_0^\infty e^{-2s\theta}\theta^{1-H} d\theta \\ &\leq K|t-s|^2 \end{aligned} \tag{3.3}$$

where $K = \sup_{s\in[a,b]} \int_0^\infty e^{-2s\theta}\theta^{1-H} d\theta$. This last inequality and (3.2) imply that

$$\begin{aligned} \mathbb{E}[S_t^H - S_s^H]^2 &\geq \frac{C_H}{2}|t-s|^{2H} - C_1^2 K|t-s|^2 \\ &= [\frac{C_H}{2} - C_1^2 K|t-s|^{2(1-H)}]|t-s|^{2H}. \end{aligned} \tag{3.4}$$

Since $0 < H < 1$, we can choose δ small enough such that for all $s, t \geq 0$ and $|t-s| < \delta$ we have

$$\frac{C_H}{2} - C_1^2 K|t-s|^{2(1-H)} > 0.$$

Indeed, it suffices to choose $\delta < [(\frac{C_H}{2C_1^2 K}) \wedge 1]^{\frac{1}{2(1-H)}}$ and to take $M = \frac{C_H}{2} - C_1^2 K \delta^{2(1-H)}$. Finally,

$$\mathbb{E}[S_t^H - S_s^H]^2 \geq M|t-s|^{2H},$$

for all s, t such that $|t-s| < \delta$. Since S^H is a centered Gaussian process then we obtain the result. $\square$

Proof of Theorem 3.1. Fix $T > 0$. It is well known (see Berman [4]) that, for a jointly measure zero-mean Gaussian process $X = \{X_t, t \in [0,T]\}$ with bounded variance, the variance condition

$$\int_0^T \int_0^T \left(\mathbb{E}[X_t - X_s]^2\right)^{-\frac{1}{2}} dsdt < \infty$$

is sufficient for the local time $L(t,u)$ of X exists on $[0,T]$ almost surely and be square integrable as a function of u. For any $[a,b] \subset [0,+\infty[$ and for $I = [a', b'] \subset [a,b]$ such that $|b' - a'| < \delta$, according to Lemma 3.2 we have,

$$\int_{a'}^{b'} \int_{a'}^{b'} \left(\mathbb{E}[S_t^H - S_s^H]^2\right)^{-\frac{1}{2}} dsdt < \int_{a'}^{b'} \int_{a'}^{b'} |t-s|^{-H} dsdt.$$

The last integral is finite because $0 < H < 1$. Then according to Geman and Horowitz [13, Theorem 22.1], the conclusion of the theorem holds for any interval $I \subset [a,b]$ with length $|I| < \delta$. Finally, since $[a,b]$ is finite interval, we can obtain the local time on $[a,b]$ by a standard patch-up procedure i.e. we partition $[a,b]$ into $\cup_{i=1}^n [a_{i-1}, a_i]$ such that $|a_i - a_{i-1}| < \delta$ and define $L^H([a,b],x) = \sum_{i=1}^n L^H([a_{i-1}, a_i], x)$ where $a_0 = a$ and $a_n = b$. $\square$

3.2. LND Property of Sub-fBm

In order to study joint continuity of local time we prove the LND of Sub-fBm.

Theorem 3.3. *Assume* $0 < H < 1$. *Then the Sub-fBm* S^H *is LND on* $[0,T]$.

Proof. It is sufficient to prove that the sub-fBm S^H satisfies Proposition 2.1.

$$S^H(t) \stackrel{d}{=} C_1 X^H(t) + B^H(t)$$

then

$$S^H(t) - S^H(s) = B^H(t) - B^H(s) + C_1(X^H(t) - X^H(s)).$$

By using the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, we obtain

$$\begin{aligned} Var\left(\sum_{j=1}^{m} u_j[S^H(t_j) - S^H(t_{j-1})]\right) \\ \geq \frac{1}{2} Var\left(\sum_{j=1}^{m} u_j[B^H(t_j) - B^H(t_{j-1})]\right) \\ - C_1^2 Var\left(\sum_{j=1}^{m} u_j[X^H(t_j) - X^H(t_{j-1})]\right). \end{aligned} \tag{3.5}$$

According to Kôno et al.[16], the fBm B^H is local nondeterministic on $[0,T]$, then by Proposition 2.1, there exists two constants $\delta_m > 0$ and $C_m > 0$ such that for any $t_0 = 0 < t_1 < t_2 < \cdots < t_m < T$, with $t_m - t_1 < \delta_m$, we have

$$\begin{aligned} Var\left(\sum_{j=1}^{m} u_j[S^H(t_j) - S^H(t_{j-1})]\right) \geq \\ \frac{C_m}{2} \sum_{j=1}^{m} u_j^2 Var\left(B^H(t_j) - B^H(t_{j-1})\right) \\ - mC_1^2 \sum_{j=1}^{m} u_j^2 Var\left(X^H(t_j) - X^H(t_{j-1})\right). \end{aligned} \tag{3.6}$$

Moreover, we have

$$\mathbb{E}[X^H(t) - X^H(s)]^2 \leq K|t-s|^2. \tag{3.7}$$

This last inequality imply that (3.6) becomes

$$
\begin{aligned}
& Var\left(\sum_{j=1}^{m} u_j[S^H(t_j)-S^H(t_{j-1})]\right) \\
\geq & \frac{C_m}{2}\sum_{j=1}^{m} u_j^2|t_j-t_{j-1}|^{2H} - mC_1^2\sum_{j=1}^{m} u_j^2|t_j-t_{j-1}|^2 \\
\geq & [\frac{C_m}{2} - mC_1^2\delta_m^{2(1-H)}K]\sum_{j=1}^{m} u_j^2|t_j-t_{j-1}|^{2H}. \qquad (3.8)
\end{aligned}
$$

In addition we have

$$
\begin{aligned}
\mathbb{E}[S^H(t)-S^H(s)]^2 & \leq 2(\mathbb{E}[B^H(t)-B^H(s)]^2 + C_1^2\mathbb{E}[X^H(t)-X^H(s)]^2) \\
& \leq K|t-s|^2 + |t-s|^{2H} \\
& \leq (K\delta_m^{2(1-H)}+1)|t-s|^{2H} \\
& \leq C(\delta_m, H)|t-s|^{2H}. \qquad (3.9)
\end{aligned}
$$

Therefore it suffices now to choose

$$\tilde{\delta}_m < \left(\frac{C_m}{2mC_1^2K}\right)^{\frac{1}{2(1-H)}} \wedge \delta_m$$

and to consider

$$\tilde{C_m} = \frac{1}{C(\delta_m, H)}\left(\frac{C_m}{2} - mC_1^2\tilde{\delta}_m^{2(1-H)}K\right)$$

and the theorem is proved. □

3.3. Joint continuity and Hölder regularity

Let $T>0$ and $\mathcal{H}([0,T])$ be the family of interval $I\subset[0,T]$ of length at most δ (the constant appearing in Lemma 3.2). In this paragraph we will apply some results of Berman on LND process to prove the joint continuity of local times of the Sub-fBm. The main result is the following.

Theorem 3.4. *Assume* $0<H<1$. *Then the Sub-fBm* S^H *has, almost surely, a jointly continuous local time* $\{L(t,x), t\in[0,T], x\in\mathbb{R}\}$. *It satisfies for any compact* $U\subset\mathbb{R}$

(i)

$$\sup_{x\in U}\frac{L(t+h,x)-L(t,x)}{|h|^{\lambda}}<+\infty \ a.s., \tag{3.10}$$

where $\lambda < 1-H$ *and* $|h|<\eta$, η *being a small random variable almost surely positive and finite,*

(ii) *for any* $I \in \mathcal{H}([0,T])$,

$$\sup_{x,y\in U, x\neq y}\frac{L(I,x)-L(I,y)}{|x-y|^{\alpha}}<+\infty \ a.s., \tag{3.11}$$

where $\alpha < 1 \wedge \frac{1-H}{2H}$.

The proof of Theorem 3.4 relies on the following upper bounds for the moments of the local times.

Lemma 3.5. *Assume* $0<H<1$ *and let* δ *be the constant appearing in Lemma 3.2. For any even integer* $m\geq 2$ *there exists a positive and finite constant* C_m *such that, for any* $t\in[0,+\infty[$, *any* $h\in(0,\delta)$, *any* $x,y\in\mathbb{R}$ *and any* $\xi<1\wedge\frac{1-H}{2H}$

$$\mathbb{E}[L(t+h,x)-L(t,x)]^m \leq C_m\frac{h^{m(1-H)}}{\Gamma(1+m(1-H))}, \tag{3.12}$$

$$\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^m \leq C_m|y-x|^{m\xi} \times\frac{h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi)))}. \tag{3.13}$$

Proof. We will proof only (3.13), the proof of (3.12) is similar. It follows from (25.7) in Geman and Horowitz [13](see also Boufoussi et al. [9]) that for any $x,y\in\mathbb{R}, t, t+h\in[0,+\infty[$ and for every even integer $m\geq 2$,

$$\begin{aligned}&\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^m\\&=(2\pi)^{-m}\int_{[t,t+h]^m}\int_{\mathbb{R}^m}\prod_{j=1}^{m}[e^{-iyu_j}-e^{-ixu_j}]\\&\times\mathbb{E}\left(e^{i\sum_{j=1}^m u_jS^H_{s_j}}\right)\prod_{j=1}^m du_j\prod_{j=1}^m ds_j.\end{aligned}$$

Using the elementary inequality $|1-e^{i\theta}| \leq 2^{1-\xi}|\theta|^{\xi}$ for all $0<\xi<1$ and any $\theta \in \mathbb{R}$, we obtain

$$\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^m \leq (2^{\xi}\pi)^{-m} m! |y-x|^{m\xi}$$
$$\times \int_{t<t_1<\cdots<t_m<t+h} \int_{\mathbb{R}^m} \prod_{j=1}^{m} |u_j|^{\xi} \mathbb{E}[\exp{(i\sum_{j=1}^{m} u_j S^H_{t_j})}] \prod_{j=1}^{m} du_j \prod_{j=1}^{m} dt_j, \quad (3.14)$$

where in order to apply the LND property of S^H, we replaced the integration over the domain $[t, t+h]$ by over the subset $t<t_1<\cdots<t_m<t+h$. We deal now with the inner multiple integral over the u's. Change the variable of integration by mean of the transformation

$$u_j = v_j - v_{j+1}, j = 1,2,\ldots,m-1; u_m = v_m.$$

Then the linear combination in the exponent in (3.14) is transformed according to

$$\sum_{j=1}^{m} u_j S^H_{t_j} = \sum_{j=1}^{m} v_j (S^H_{t_j} - S^H_{t_{j-1}}),$$

where $t_0 = 0$. Since S^H is a Gaussian process, the characteristic function in (3.14) has the form

$$\exp\left(-\frac{1}{2}Var\left[\sum_{j=1}^{m} v_j (S^H_{t_j} - S^H_{t_{j-1}})\right]\right). \quad (3.15)$$

Since $|x-y|^{\xi} \leq |x|^{\xi} + |y|^{\xi}$ for all $0<\xi<1$, it follows that

$$\begin{aligned}\prod_{j=1}^{m} |u_j|^{\xi} &= \prod_{j=1}^{m-1} |v_j - v_{j+1}|^{\xi} |v_m|^{\xi} \\ &\leq \prod_{j=1}^{m-1} (|v_j|^{\xi} + |v_{j+1}|^{\xi}) |v_m|^{\xi}. \quad (3.16)\end{aligned}$$

Moreover, the last product is at most equal to a finite sum of 2^{m-1} terms of the form $\prod_{j=1}^{m} |x_j|^{\xi\varepsilon_j}$, where $\varepsilon_j = 0, 1$ or 2 and $\sum_{j=1}^{m} \varepsilon_j = m$.

Let us write for simply $\sigma_j^2 = \mathbb{E}\left(S^H_{t_j} - S^H_{t_{j-1}}\right)^2$. Combining the result of Proposition 2.1, (3.15) and (3.16), we get that the integral in (3.14) is dominated by the sum over all possible of $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1,2\}^m$ of the following

$$\int_{t<t_1<\ldots<t_m<t+h}\int_{\mathbb{R}^m}\prod_{j=1}^m |v_j|^{\xi\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m v_j^2\sigma_j^2\right)\prod_{j=1}^m dt_j dv_j,$$

where C_m is the constant given in Proposition 2.1. The change of variable $x_j = v_j\sigma_j$ converts the last integral to

$$\int_{t<t_1<\cdots<t_m<t+h}\prod_{j=1}^m \sigma_j^{-1-\xi\varepsilon_j} dt_1\cdots dt_m$$
$$\times\int_{\mathbb{R}^m}\prod_{j=1}^m |x_j|^{\xi\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m x_j^2\right)\prod_{j=1}^m dx_j.$$

Let us denote

$$J(m,\xi)=\int_{\mathbb{R}^m}\prod_{j=1}^m |x_j|^{\xi\varepsilon_j}\exp\left(-\frac{C_m}{2}\sum_{j=1}^m x_j^2\right)\prod_{j=1}^m dx_j.$$

Consequently

$$\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^m$$
$$\leq C_m J(m,\xi)|y-x|^{m\xi}\int_{t<t_1<\cdots<t_m<t+h}\prod_{j=1}^m \sigma_j^{-1-\xi\varepsilon_j} dt_1\cdots dt_m. \quad (3.17)$$

According to Lemma 3.2, for h sufficient small, namely $0<h<\inf(\delta,1)$, we have

$$\mathbb{E}\left(S_{t_i}^H - S_{t_j}^H\right)^2 \geq C|t_i-t_j|^{2H} \text{ for all } t_i,t_j\in[t,t+h]. \quad (3.18)$$

It follows that the integral on the right hand side of (3.17) is bounded, up to a constant, by

$$\int_{t<t_1<\cdots<t_m<t+h}\prod_{j=1}^m (t_j-t_{j-1})^{-H(1+\xi\varepsilon_j)} dt_1\cdots dt_m. \quad (3.19)$$

Since, $(t_j-t_{j-1})<1$, for all $j\in\{2,\ldots,m\}$, we have

$$(t_j-t_{j-1})^{-H(1+\xi\varepsilon_j)} < (t_j-t_{j-1})^{-H(1+2\xi)} \ \forall\varepsilon_j\in\{0,1,2\}.$$

Since by hypothesis $\xi<\frac{1}{2H}-\frac{1}{2}$, the integral in (3.19) is finite. Moreover, by an elementary calculation(cf. Ehm [12]), for all $m\geq 1, h>0$ and

$b_j < 1$,

$$\int_{t<s_1<\cdots<s_m<t+h} \prod_{j=1}^{m} (s_j - s_{j-1})^{-b_j} ds_1 \cdots ds_m = h^{m-\sum_{j=1}^{m} b_j} \frac{\prod_{j=1}^{m} \Gamma(1-b_j)}{\Gamma(1+h-\sum_{j=1}^{m} b_j)},$$

where $s_0 = t$. It follows that (3.19) is dominated by

$$C_m \frac{h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi))},$$

where $\sum_{j=}^{m} \varepsilon_j = m$. Consequently

$$\mathbb{E}[L(t+k,y) - L(t,y) - L(t+k,x) + L(t,x)]^m \leq C_m \frac{|y-x|^{m\xi} h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi))}.$$

□

Proof of Theorem 3.4. Since $L(0,x) = 0$ for all $x \in \mathbb{R}$, hence if we replace t and $t+h$ by 0 and t respectively in 3.13, we obtain

$$\mathbb{E}[L(t,y) - L(t,x)]^m \leq \tilde{C}_m |y-x|^{m\xi}. \tag{3.20}$$

The jointly continuity of the local time straightforward from (3.12), (3.13) and (3.20) and classical parameter Kolmogorov's theorem (c.f. Berman [5], Theorem 5.1).
The Hölder condition (i) of Theorem 3.1 follows of (3.13) and one parameter Kolmogorov's theorem (see also the proof Theorem 2 in Pitt [20]).
We turn out to the proof of (ii). According to Theorem 3.1 in Berman [7], the inequalities (3.12), (3.13) and (3.20) imply that (ii) holds for any $\delta < 1 - H(1+\xi)$, for all $0 < \xi < 1 \wedge \frac{1-H}{2H}$,. Letting ξ tends to zero, we obtain the desired result. □

As a classical consequence, we have the following result on the Hausdorff dimension of the level set. We refer to Adler [1] and Baraka et al.[3] for definition and results for the fractional Brownian motion.

Proposition 3.6. *With probability one, for any interval $I \subset [0,T]$, we have*

$$dim\{t \in I / S_t^H = x\} = 1 - H, \tag{3.21}$$

for all x such that $L(t,x) > 0$.

Proof. According to (3.9) and Kolmogorov's theorem, the Sub-fBm is β-Hölder for every $\beta < H$. Moreover, the Sub-fBm has a jointly continuous local time, then Theorem 8.7.3 in Adler [1] completes the proof of the upper bound, i.e $dim\{t \in I/S_t^H = x\} \leq 1-H$, a.s. Now by (i) of Theorem 3.4, the jointly continuous local time of the Sub-fBm satisfies an uniform Hölder of any order smaller than of $1-H$. Then the Theorem 8.7.4 of Adler [1] implies that $dim\{t \in I/S_t^H = x\} \geq 1-H$, a.s. for all x such that $L(t,x) > 0$. This completes the proof. □

4. Chung's law for the Sub-fBm and pointwise Hölder exponent of local time

The main result of this section is that the Sub-fBm satisfies the same form of Chung's law of iterated logarithm (LIL) as the fBm. For an excellent summary on LIL, we refer to the survey paper of Li and Shao [18].

Theorem 4.1. *Assume $0 < H < 1$. Then the following Chung's law of iterated logarithm hold for the sub-fBm:*

$$\liminf_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|S^H(t) - S^H(s)|}{(\delta/\log|\log(\delta)|)^H} = C(H), a.s. \tag{4.1}$$

where $C(H)$ is the constant appearing in the Chung's law of fBm.

Proof. Conserving the same notations as above, we can write

$$S^H(t) - S^H(s) = B^H(t) - B^H(s) + C_1(X^H(t) - X^H(s))$$

According to Monrad and Rootzen [19], the fBm B^H satisfies (4.1). Then (4.1) will be proved if we show that

$$\lim_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|X^H(t) - X^H(s)|}{(\delta/\log|\log(\delta)|)^H} = 0, a.s. \tag{4.2}$$

According to (3.3) there exists a positive constant K such that

$$\sup_{s\in[t,t+\delta]} \mathbb{E}[X^H(t) - X^H(s)]^2 \leq K\delta^2. \tag{4.3}$$

Hence, according to Theorem 2.1 in Adler [2, page 43], and a symmetry argument, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in[t,t+\delta]}|X^H(t)-X^H(s)|\geq u\right)\\
&\qquad\leq 2\mathbb{P}\left(\sup_{s\in[t,t+\delta]}(X^H(t)-X^H(s))\geq u\right)\\
&\qquad\leq 4\exp\left(-\frac{\left(u-\mathbb{E}(\sup_{s\in[t,t+\delta]}(X^H(t)-X^H(s)))\right)^2}{K\delta^2}\right). \qquad (4.4)
\end{aligned}$$

For the sake of simplicity, let $\Lambda=\sup_{s\in[t,t+\delta]}(X^H(t)-X^H(s))$. By (4.4), we obtain

$$\begin{aligned}
\mathbb{E}(\Lambda) &\leq \int_0^{+\infty}\mathbb{P}\left(\sup_{s\in[t,t+\delta]}|X^H(t)-X^H(s)|>x\right)dx\\
&\leq 4\int_0^{+\infty}\exp\left(-\frac{[x-\mathbb{E}(\Lambda)]^2}{K\delta^2}\right)dx\\
&= \frac{4\sqrt{K}\delta}{\sqrt{2}}\int_{-\frac{\sqrt{2}\mathbb{E}(\Lambda)}{\sqrt{K}\delta}}^{+\infty}e^{-\frac{y^2}{2}}dy\\
&\leq 4\sqrt{K\pi}\delta.
\end{aligned}$$

It follows that

$$(u-\mathbb{E}(\Lambda))^2\geq\frac{1}{2}u^2-(\mathbb{E}(\Lambda))^2\geq\frac{1}{2}u^2-16K\pi\delta.$$

Consequently, (4.4) becomes

$$\mathbb{P}\left(\sup_{s\in[t,t+\delta]}|X^H(t)-X^H(s)|\geq u\right)\leq C\exp\left(-\frac{u^2}{2K\delta^2}\right). \qquad (4.5)$$

Since $H<1$, there exists $0<\xi<1-H$. Consider $\delta_n=n^{1/(2(\xi+H-1))}$ and $u_n=\delta_n^{H+\xi}$. Therefore, according to (4.5), we have

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{s\in[t,t+\delta_n]}|X^H(t)-X^H(s)|\geq u_n\right)\leq C\sum_{n=1}^{\infty}\exp\left(-\frac{1}{K}n\right)<\infty.$$

It follows from the Borel-Cantelli lemma that there exists $n_0=n(\omega)$ such that for all $n\geq n_0$, $\sup_{s\in[t,t+\delta_n]}|X^H(t)-X^H(s)|\leq\delta_n^{H+\xi}$ almost surely. Furthermore, for $\delta_{n+1} \leq \delta \leq \delta_n$, we have almost surely

$$\begin{aligned}
\sup_{s\in[t,t+\delta]} |X^H(t) - X^H(s)| &\leq \sup_{s\in[t,t+\delta_n]} |X^H(t) - X^H(s)| \\
&\leq \delta_n^{H+\xi} \\
&\leq \delta^{H+\xi} \left(\frac{\delta_n}{\delta_{n+1}}\right)^{H+\xi} \\
&\leq 2^\theta \delta^{H+\xi}, \, a.s.,
\end{aligned}$$

where $\theta = \dfrac{H+\xi}{2(1-H-\xi)}$.

Hence,

$$\lim_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|X^H(t) - X^H(s)|}{(\delta/\log|\log(\delta)|)^H} \leq 2^\theta \lim_{\delta\to 0} \delta^\xi (\log|\log(\delta)|)^H = 0 \ \ a.s.$$

Consequently (4.2) is proved. This completes the proof of the Theorem. □

Remark 4.2. The main interest of the previous proof is that it can be used to generalized many other LIL known for the fBm to the Sub-fBm. For example, we have the LIL given in Li and Shao [18, equation (7.5)], for the fBm to the Sub-fBm as follows

$$\limsup_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|S^H(t) - S^H(s)|}{\delta^H (\log|\log(\delta)|)^{\frac{1}{2}}} = C(H), \text{ a.s.}$$

$$\text{where } C(H) = \sqrt{\frac{2\pi}{H\Gamma(2H)\sin(\pi H)}}.$$

The Chung laws are known to be linked to the optimality of the moduli of continuity of local times of stochastic processes. More precisely:

Lemma 4.3. *The following lower bounds for the moduli of continuity of local times holds*

$$\frac{1}{2C(H)} \leq \limsup_{\delta\to 0} \sup_{x\in\mathbb{R}} \frac{L(t+\delta,x) - L(t,x)}{\delta^{1-H}(\log\log(\delta^{-1}))^H}, \, a.s. \tag{4.6}$$

Proof. Combining (4.1) and the following elementary computation

$$\begin{aligned}\delta &= \int_{\mathbb{R}} L([t,t+\delta],x)dx \\ &\leq \sup_{x\in\mathbb{R}} L([t,t+\delta],x) \sup_{s,s'\in[t,t+\delta]} |S^H(s)-S^H(s')| \\ &\leq 2\sup_{x\in\mathbb{R}} L([t,t+\delta],x) \sup_{s\in[t,t+\delta]} |S^H(t)-S^H(s)|\end{aligned}$$

we obtain the lemma. □

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