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Muriel Livernet

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From left modules to algebras over an operad: application to combinatorial Hopf algebras

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Abstract

The purpose of this paper is two fold: we study the behaviour of the forgetful functor from $\mathbb{S}$-modules to graded vector spaces in the context of algebras over an operad and derive the construction of combinatorial Hopf algebras. As a byproduct we obtain freeness and cofreeness results for those Hopf algebras.

Let $\mathcal{O}$ denote the forgetful functor from $\mathbb{S}$-modules to graded vector spaces. Left modules over an operad $\mathcal{P}$ are treated as $\mathcal{P}$-algebras in the category of $\mathbb{S}$-modules. We generalize the results obtained by Patras and Reutenauer in the associative case to any operad $\mathcal{P}$: the functor $\mathcal{O}$ sends $\mathcal{P}$-algebras to $\mathcal{P}$-algebras. If $\mathcal{P}$ is a Hopf operad the functor $\mathcal{O}$ sends Hopf $\mathcal{P}$-algebras to Hopf $\mathcal{P}$-algebras. If the operad $\mathcal{P}$ is regular one gets two different structures of Hopf $\mathcal{P}$-algebras in the category of graded vector spaces. We develop the notion of unital infinitesimal $\mathcal{P}$-bialgebras and prove freeness and cofreeness results for Hopf algebras built from Hopf operads. Finally, we prove that many combinatorial Hopf algebras arise from our theory, as it is the case for various Hopf algebras defined on the faces of the permutohedra and associahedra.

Résumé

Nous étudions en détail le foncteur oubli de la catégorie des $\mathbb{S}$-modules dans la catégorie des espaces vectoriels gradués. Cela nous permet de généraliser les résultats de Patras et Reutenauer obtenus dans le cadre associatif à toute opérade $\mathcal{P}$: les $\mathcal{P}$-algèbres dans la catégorie des $\mathbb{S}$-modules deviennent des $\mathcal{P}$-algèbres dans la catégorie des espaces vectoriels gradués. Il en est de même pour les $\mathcal{P}$-algèbres de Hopf lorsque l'opérade $\mathcal{P}$ est une opérade de Hopf. De plus, si l'opérade est régulière, alors on obtient deux structures de $\mathcal{P}$-algèbres (de Hopf) dans la catégorie des espaces vectoriels gradués. Comme application, nous montrons qu'un certain nombre d'opérades de Hopf donne lieu à des algèbres de Hopf combinatoires connues. Le fait que ces algèbres de Hopf soient libres ou colibres est une conséquence directe de la théorie des opérades.

Keywords: $\mathbb{S}$-module, operad, twisted bialgebra, free associative algebra, combinatorial Hopf algebra.
Math. classification: 18D50, 16W30, 16A06.

Introduction

An $\mathbb{S}$-module, also named symmetric sequence, is a graded vector space $(V_n)_{n\geq 0}$ together with a right action of the symmetric group S_n on V_n for each n. The present paper is concerned with the study, in an operadic point of view, of the forgetful functor $\mathcal{O}$ from $\mathbb{S}$-modules to graded vector spaces and its applications.

The category **S-mod** of $\mathbb{S}$-modules is a tensor category. Motivated by the study of homotopy invariants, Barratt introduced the notion of *twisted Lie algebras* in [3], which are Lie algebras in the category of $\mathbb{S}$-modules or – in the operad context – *left modules* over the operad $\mathcal{L}$ie. More precisely, a twisted Lie algebra is an $\mathbb{S}$-module $(L_n)_{n\geq 0}$ together with a bilinear operation $[,]$ satisfying, for any $a\in L_p, b\in L_q$ and $c\in L_r$ the relations

$$\begin{aligned}&[b,a]\cdot\zeta_{p,q}=-[a,b],\\&[a,[b,c]]+[c,[a,b]]\cdot\zeta_{p+q,r}+[b,[c,a]]\cdot\zeta_{p,q+r}=0,\end{aligned}$$

where $\zeta_{p,q}$ is the permutation of S_{p+q} given by $\zeta_{p,q}(i)=q+i$ if $1\leq i\leq p$ and $\zeta_{p,q}(i)=i-p$ if $p+1\leq i\leq p+q$. For instance the $\mathbb{S}$-module $(\mathcal{L}\text{ie}(n))_{n\geq 0}$ is a twisted Lie algebra for the bracket induced by the operadic composition

$$\mathcal{L}\text{ie}(2)\otimes\mathcal{L}\text{ie}(n)\otimes\mathcal{L}\text{ie}(m)\to\mathcal{L}\text{ie}(n+m).$$

There exists also a notion of *twisted associative algebras*– associative algebras in the tensor category **S-mod**– and a notion of *twisted associative bialgebras* – associative bialgebras in the category **S-mod**. Stover proved in [24] that a Cartier-Milnor-Moore theorem relates the categories of twisted associative bialgebras and twisted Lie algebras. Following an idea lying in [24], Patras and Reutenauer proved in [21] that two associative bialgebras arise naturally from a twisted associative bialgebra (A,m,Δ): the *symmetrized bialgebra* $\bar{A}=(A,\hat{m},\bar{\Delta})$ and the *cosymmetrized bialgebra* $\hat{A}=(A,\bar{m},\hat{\Delta})$. In [22] Patras and Schocker derived from this construction some known combinatorial Hopf algebras. The first part of this paper is the generalization of these constructions to any operad $\mathcal{P}$. This generalization is performed in two steps: the first step will focus on the algebra constructions and the second step on the coalgebra constructions.

Given an operad $\mathcal{P}$, the following notions are the same

Twisted $\mathcal{P}$-algebras (see e.g. [12]),

Left modules over $\mathcal{P}$ (see e.g. [11]) and

$\mathcal{P}$-algebras in the category of $\mathbb{S}$-modules.

Our first question is the following one: given a $\mathcal{P}$-algebra M in **S-mod** can one endow the graded vector space $\mathcal{O}(M)$ with a $\mathcal{P}$-algebra structure? This question comes from the observation that $\oplus_n \mathcal{P}(n)$ is a $\mathcal{P}$-algebra in **S-mod** but is not *a priori* a $\mathcal{P}$-algebra in the category of graded vector spaces. To convice the reader, one can look at the Lie case: if $\mathcal{P} = \mathcal{L}\text{ie}$ then $\oplus_n \mathcal{L}\text{ie}(n)$ is a twisted Lie algebra, and the bracket is not anti-symmetric since the action of the symmetric group is not trivial. Our first theorem 2.3.1 states that if we apply a symmetrization to the twisted $\mathcal{P}$-algebra structure on M then $\mathcal{O}(M)$ is a $\mathcal{P}$-algebra. If $\mathcal{P} = \mathcal{A}\text{s}$ we recover the definition of the symmetrized product $\hat{m}$ of Patras and Reutenauer. Our second theorem 2.4.3 states that another product can be defined if the operad $\mathcal{P}$ is *regular*, that is, $\mathcal{P}$ is obtained from a non-symmetric operad tensored by the regular representation of the symmetric group. Then in case $\mathcal{P} = \mathcal{A}\text{s}$ we recover the product $\bar{m}$ defined by Patras and Reutenauer.

The second step of the construction involves Hopf operads. We define the notion of Hopf $\mathcal{P}$-algebras in the category **S-mod**, so that in case $\mathcal{P} = \mathcal{A}\text{s}$ we recover the notion of twisted associative bialgebras. We develop the analogues of the constructions of Patras and Reutenauer. In theorem 3.1.3 we prove that a Hopf $\mathcal{P}$-algebra (M, μ_M, Δ_M) in **S-mod** gives rise to a Hopf $\mathcal{P}$-algebra in **grVect**, denoted $\bar{M} = (\mathcal{O}(M), \hat{\mu}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$, analogous to the symmetrized bialgebra construction. This is the *symmetrized* Hopf $\mathcal{P}$-algebra associated to M. If $\mathcal{P}$ is regular then there is another Hopf $\mathcal{P}$-algebra structure $\hat{M} = (\mathcal{O}(M), \bar{\mu}_{\mathcal{O}(M)}, \hat{\Delta}_{\mathcal{O}(M)})$, analogous to the cosymmetrized bialgebra construction (see theorem 3.2.1): this is the *cosymmetrized* Hopf $\mathcal{P}$-algebra associated to M. Thus the theory developed by Patras and Reutenauer is a consequence of the regularity of the operad $\mathcal{A}\text{s}$. The example of a Hopf $\mathcal{P}$-algebra in **S-mod** that we should have in mind throughout the text is the $\mathbb{S}$-module $\mathcal{P}$ itself, if $\mathcal{P}$ is a connected Hopf operad (see section 3.1.4).

The case $\mathcal{P}$ regular has another advantage: we define *unital infinitesimal $\mathcal{P}$-bialgebras*, which when restricted to $\mathcal{P} = \mathcal{A}\text{s}$, are unital infinitesimal bialgebras as defined by Loday and Ronco in [16]. The theorem 4.1.2 asserts that $(\mathcal{O}(M), \bar{\mu}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$ is a unital infinitesimal $\mathcal{P}$-bialgebra if

M is a Hopf $\mathcal{P}$-algebra. It is shown in theorem 4.1.3 that the graded vector space $\oplus_n \mathcal{P}(n)/S_n$ has also a structure of unital infinitesimal $\mathcal{P}$-bialgebra. These results combined with the theorem of Loday and Ronco (see 4.2.1) yield the main theorems of our paper, which have some importance in the study of combinatorial Hopf algebras. First of all if $\mathcal{P} = \mathcal{A}s$ and (A, m, Δ) is a twisted associative bialgebra, then the algebra $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra and consequently is free and cofree (see theorem 4.2.2).

Before going through the applications to combinatorial Hopf algebras, let us summarize the results in a table.

Operad $\mathcal{P}$	$\mathbb{S}$-module M	graded vector space $\mathcal{O}(M)$	Thm
any	$\mathcal{P}$-alg (M, μ)	$\mathcal{P}$-alg $(\mathcal{O}(M), \hat{\mu})$	2.3.1
regular	$\mathcal{P}$-alg (M, μ)	$\mathcal{P}$-alg $(\mathcal{O}(M), \hat{\mu})$ $\mathcal{P}$-alg $(\mathcal{O}(M), \bar{\mu})$	2.4.3
Hopf	Hopf $\mathcal{P}$-alg (M, μ, Δ)	Hopf $\mathcal{P}$-alg $(\mathcal{O}(M), \hat{\mu}, \bar{\Delta})$	3.1.3
Hopf regular	Hopf $\mathcal{P}$-alg (M, μ, Δ)	Hopf $\mathcal{P}$-alg $(\mathcal{O}(M), \hat{\mu}, \bar{\Delta})$ Hopf $\mathcal{P}$-alg $(\mathcal{O}(M), \bar{\mu}, \hat{\Delta})$ u.i. $\mathcal{P}$-bialg $(\mathcal{O}(M), \bar{\mu}, \bar{\Delta})$	 3.2.1 4.1.2

Given a graded vector space H, the question is: how does a Hopf algebra structure arise on H? In the last section, we illustrate with examples that many combinatorial Hopf algebras arise from our theory. We distinguish two cases.

If $\mathcal{P}$ is a Hopf multiplicative operad then $H = \oplus_n \mathcal{P}(n)$ has two structures of Hopf algebras: the symmetrized Hopf algebra $\bar{H}$ is cofree and the cosymmetrized algebra $\hat{H}$ is free. For instance, we prove that the Malvenuto Reutenauer Hopf algebra arises as the symmetrized Hopf algebra associated to the operad $\mathcal{A}s$ and also as the cosymmetrized Hopf algebra associated to the operad $\mathcal{Z}in$ defining Zinbiel algebras. It gives yet another proof for the Hopf algebra of Malvenuto and Reutenauer to be free and cofree independent from its self-duality. We prove that Hopf algebra structures on the faces of the permutohedra given e.g. by Chapoton in [5], Bergeron and Zabrocky in [4] and Patras and Schocker in [22], arise from the operad $\mathcal{C}$TD of commutative tridendriform algebras defined by Loday in [14]. We deduce also some freeness results from our theory.

In the second case, we assume that there exists a Hopf multiplicative regular operad $\mathcal{P}$ such that $H = \oplus_n \mathcal{P}(n)/S_n$. Our theory implies that H is a Hopf $\mathcal{P}$-algebra hence a Hopf algebra and is also a unital infinitesimal $\mathcal{P}$-bialgebra. We prove that the Hopf algebra of planar trees described by Chapoton in [5], and the one of planar binary trees described by Loday and Ronco in [15] arise this way. As a byproduct we obtain freeness results for these Hopf algebras.

The organization of the present paper is as follows. After some preliminaries in section 1 we define in section 2 the notion of algebras over an operad $\mathcal{P}$ in the category of $\mathbb{S}$-modules and in the category of graded vector spaces. We explore the structures of $\mathcal{P}$-algebras on the underlying graded vector space of a $\mathcal{P}$-algebra in **S-mod**. We prove that such a structure always exists and when the operad is regular one has an additional structure. We compare this result to the one obtained by Patras and Reutenauer [21] in the case of twisted associative algebras. In section 3 we study Hopf operads and the consequences on the underlying graded vector space of a Hopf $\mathcal{P}$-algebra in **S-mod**. The section 4 is the study of unital infinitesimal $\mathcal{P}$-bialgebras and states the freeness theorems. We develop in section 5 the application to combinatorial Hopf algebras by means of examples.

Throughout the paper, the ground field is denoted by **k** and all vector spaces are **k**-vector spaces.

Contents

1. $\mathbb{S}$-modules and related functors

1.1. The symmetric group

In this section we develop some material on the symmetric group needed in the paper. The set $\{1,\dots,n\}$ is written $[n]$. For any set of integers S, the set $\{p+s, s\in S\}$ is denoted $p+S$. For sets $S\subset[n]$ and $T\subset[m]$, the set $S\times T$ is the subset $S\cup(T+n)$ of $[n+m]$.

Any permutation $\sigma\in S_n$ is written $(\sigma_1,\dots,\sigma_n)$ with $\sigma_i=\sigma(i)$. There is a natural injection

$$\begin{array}{ccc} S_n\times S_m & \to & S_{n+m} \\ (\sigma,\tau) & \mapsto & \sigma\times\tau=(\sigma_1,\dots,\sigma_n,\tau_1+n,\dots,\tau_m+n). \end{array}$$

The *standardisation* of a sequence of distinct integers $(a_1,\dots,a_p)$ is the unique permutation $\sigma\in S_p$ following the conditions

$$\sigma(i)<\sigma(j)\Leftrightarrow a_i<a_j,\ \forall i,j.$$

For instance

$$\mathrm{st}(2,13,9,4)=(1,4,3,2).$$

Any subset $A = \{a_1 < \ldots < a_p\} \subset [n]$ induces a map

$$\begin{array}{ccc} S_n & \to & S_p \\ \sigma & \mapsto & \sigma|_A = \mathrm{st}(\sigma(a_1), \ldots, \sigma(a_p)). \end{array}$$

For instance

$$(2,6,1,3,5,4)|_{\{1,2,4\}} = \mathrm{st}(2,6,3) = (1,3,2).$$

If A is the empty set then $\sigma|_\emptyset = 1_0 \in S_0$.

A $(p_1, \ldots, p_r)$-*shuffle* is a permutation of $S_{p_1+\ldots+p_r}$ of type

$$(\tau^1_1, \ldots, \tau^1_{p_1}, \ldots, \tau^r_1, \ldots, \tau^r_{p_r})^{-1}$$

with $\tau^k_1 < \ldots < \tau^k_{p_k}$ for all $1 \leq k \leq r$. The set of all $(p_1, \ldots, p_r)$-shuffles is denoted by $\mathrm{Sh}_{p_1,\ldots,p_r}$. For simplicity, a $(p_1, \ldots, p_r)$-shuffle is written as a r-uple $(A_1, \ldots, A_r)$ where $A_1 \sqcup \ldots \sqcup A_r$ is an ordered partition of the set $[p_1 + \ldots + p_r]$. Some of the A_i's may be empty.

For instance $(\{2,5\}, \{1,3,4\})$ denotes the $(2,3)$-shuffle $(3,1,4,5,2)$. Recall that $\mathrm{Sh}_{p_1,\ldots,p_r}$ constitutes a set of right coset representatives for $S_{p_1} \times \ldots \times S_{p_r} \subset S_{p_1+\ldots+p_r}$, i.e. any $\sigma \in S_{p_1+\ldots+p_r}$ has a unique factorization

$$\sigma = (\sigma_1 \times \ldots \times \sigma_r)\alpha,$$

where $\sigma_i \in S_{p_i}$ and where α is a $(p_1, \ldots, p_r)$-shuffle. More precisely, if $r = 2$

$$\sigma = (\sigma|_{\sigma^{-1}([p])} \times \sigma|_{\sigma^{-1}(p+[q])})(\sigma^{-1}([p]), \sigma^{-1}(p+[q])).$$

1.2. Graded vector spaces and $\mathbb{S}$-modules

1.2.1. Definition

A *graded vector space* A is a collection $\{A_n\}_{n\geq 0}$ of $\mathbf{k}$-vector spaces A_n indexed by the non-negative integers. One can define also A as $A = \oplus_n A_n$. A map $A \to B$ of graded vector spaces is a collection of linear morphisms $A_n \to B_n$. The category of graded vector spaces is denoted **grVect**.

An $\mathbb{S}$-*module* M is a graded vector space together with a right S_n-action $M_n \otimes \mathbf{k}[S_n] \to M_n$ for each $n \geq 0$. A map $M \to N$ of $\mathbb{S}$-modules is a collection $M_n \to N_n$ of morphisms of right S_n-modules. The category of $\mathbb{S}$-modules is denoted **S-mod**.

There is a forgetful functor

$$\mathcal{O} : \textbf{S-mod} \to \textbf{grVect}$$

which forgets the action of the symmetric group.

1.2.2. Tensor product

The category **grVect** is a linear symmetric monoidal category with the following *tensor product*:

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q.$$

The symmetry isomorphism $\tau : A \otimes B \to B \otimes A$ is given by

$$\begin{array}{rccc} \tau : & A_p \otimes B_q & \to & B_q \otimes A_p \\ & a \otimes b & \mapsto & b \otimes a \end{array}$$

The symmetry isomorphism τ induces a left action of the symmetric group S_k on $A^{\otimes k}$, for $A \in \mathbf{grVect}$.

The category **S-mod** is a linear symmetric monoidal category with the following *tensor product*:

$$\begin{aligned} (M \otimes N)(n) &= \bigoplus_{p+q=n} (M(p) \otimes N(q)) \otimes_{\mathbf{k}[S_p \times S_q]} \mathbf{k}[S_n] \\ &= \bigoplus_{p+q=n} (M(p) \otimes M(q)) \otimes \mathbf{k}[\mathrm{Sh}_{p,q}]. \end{aligned}$$

Since a (p,q)-shuffle is uniquely determined by an ordered partition $I \sqcup J$ of $[p+q]$, an element in $(M \otimes N)(p+q)$ can be written $m \otimes n \otimes (I,J)$. For the sequel $m \otimes n$ denotes the element $m \otimes n \otimes ([p], p+[q])$ of $(M \otimes N)(p+q)$. The right action of the symmetric group is given by

$$(m \otimes n \otimes (I,J)) \cdot \sigma = m \cdot \sigma|_{\sigma^{-1}(I)} \otimes n \cdot \sigma|_{\sigma^{-1}(J)} \otimes (\sigma^{-1}(I), \sigma^{-1}(J)) \quad (1.1)$$

The unit for the tensor product is the $\mathbb{S}$-module $\mathbf{1}$ given by

$$\mathbf{1}(n) = \begin{cases} \mathbf{k}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The symmetry isomorphism $\tau : M \otimes N \to N \otimes M$ is given by

$$\tau(m \otimes n \otimes (I,J)) = n \otimes m \otimes (J,I).$$

For any $\sigma \in S_k$, the symmetry isomorphism induces an isomorphism τ_σ of $\mathbb{S}$-modules from $M_1 \otimes \ldots \otimes M_k$ to $M_{\sigma^{-1}(1)} \otimes \ldots \otimes M_{\sigma^{-1}(k)}$ given by

$$\tau_\sigma(m_1 \otimes \ldots \otimes m_k \otimes (I_1, \ldots, I_k)) = \\ m_{\sigma^{-1}(1)} \otimes \ldots \otimes m_{\sigma^{-1}(k)} \otimes (I_{\sigma^{-1}(1)}, \ldots, I_{\sigma^{-1}(k)}). \quad (1.2)$$

As a consequence τ_σ induces a left S_k-action on $M^{\otimes k}$, for any $\mathbb{S}$-module M.

When it is necessary to distinguish the tensor products, we write $\otimes_g$ for the tensor product in **grVect** and $\otimes_{\mathbb{S}}$ for the one in **S-mod**.

The forgetful functor $\mathcal{O}$ does not preserve the tensor product. There are two natural transformations, $\pi^{\mathcal{O}}$ and $\iota^{\mathcal{O}}$

$$\pi^{\mathcal{O}}_{M,N}, \iota^{\mathcal{O}}_{M,N} : \mathcal{O}(M \otimes_{\mathbb{S}} N) \to \mathcal{O}(M) \otimes_g \mathcal{O}(N)$$

defined by, for any $m \in M(p), n \in N(q)$

$$\pi^{\mathcal{O}}_{M,N}(m \otimes n \otimes (I, J)) = \begin{cases} m \otimes n, & \text{if } (I, J) = ([p], p + [q]), \\ 0, & \text{otherwise}; \end{cases}$$

$$\iota^{\mathcal{O}}_{M,N}(m \otimes n \otimes (I, J)) = m \otimes n.$$

When restricted to the full subcategory of vector spaces (the $\mathbb{S}$-modules concentrated in degree 0), these two natural transformations restrict to the identity.

1.3. Endofunctors induced by an $\mathbb{S}$-module

1.3.1. Endofunctors in S-mod

The category of $\mathbb{S}$-modules is endowed with another monoidal structure (which is not symmetric): the *plethysm* $\circ$.

$$(M \circ N)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} (N^{\otimes k})(n),$$

where S_k acts on the left on $(N^{\otimes k})$ by formula (1.2). The left and right unit for the plethysm is the $\mathbb{S}$-module I given by

$$I(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence any $\mathbb{S}$-module M defines a functor

$$\begin{array}{rccc} F_M: & \textbf{S-mod} & \to & \textbf{S-mod} \\ & N & \mapsto & M \circ N \end{array}$$

satisfying

$$\begin{cases} F_I = & \mathrm{Id} \\ F_{M \circ M'} = & F_M F_{M'} \end{cases}$$

1.3.2. **Endofunctors in grVect**

For $M \in \textbf{S-mod}$ and $A \in \textbf{grVect}$, one can use the same definition for the plethysm:

$$(M \circ A)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{S_k} (A^{\otimes k})(n).$$

where the tensor product $A^{\otimes k}$ is taken in **grVect**. Similarly any $\mathbb{S}$-module M defines a functor

$$\begin{array}{rccc} F^g_M: & \textbf{grVect} & \to & \textbf{grVect} \\ & A & \mapsto & M \circ A \end{array}$$

satisfying

$$\begin{cases} F^g_I = & \mathrm{Id} \\ F^g_{M \circ M'} = & F^g_M F^g_{M'} \end{cases}$$

1.3.3. **Example**

Here is an example that emphasizes the fact that the two functors are different even if evaluated at the same underlying vector space. Consider the $\mathbb{S}$-module $\mathcal{C}\mathrm{om}(n) = \mathbf{k}$ with the trivial S_n-action. A vector space V is considered either as a graded vector space concentrated in degree 1 or as an $\mathbb{S}$-module concentrated in degree 1. This gives

$$F^g_{\mathcal{C}\mathrm{om}}(V) = \oplus_{n \geq 0} \mathbf{k} \otimes_{S_n} V^{\otimes_g n} = S(V),$$
$$F_{\mathcal{C}\mathrm{om}}(V) = \oplus_{n \geq 0} \mathbf{k} \otimes_{S_n} V^{\otimes_{\mathbb{S}} n} = T(V).$$

1.4. **Proposition**

Let M, N be two $\mathbb{S}$-modules. The map

$$\begin{array}{ccc} M \circ \mathcal{O}(N) & \to & \mathcal{O}(M \circ N) \\ m \otimes n_1 \otimes \ldots \otimes n_k & \mapsto & \sum_{(T_1,\ldots,T_k)} m \otimes n_1 \otimes \ldots \otimes n_k \otimes (T_1, \ldots, T_k), \end{array}$$

where $n_i \in N(l_i)$ and $(T_1, \ldots, T_k)$ is an ordered partition of $[l_1 + \ldots + l_k]$ with $|T_i| = l_i$, defines a natural transformation

$$\psi_M : F^g_M \, \mathcal{O} \to \mathcal{O} \, F_M$$

*functorial in $M \in$ **S-mod**. Furthermore the following diagram commutes*

$$\begin{array}{ccccc} F^g_{M\circ N}\,\mathcal{O} & & \xrightarrow{\psi_{M\circ N}} & & \mathcal{O}\,F_{M\circ N} \\ & {\scriptstyle F^g_M \psi_N} \searrow & & \nearrow {\scriptstyle \psi_M F_N} & \\ & & F^g_M\,\mathcal{O}\,F_N & & \end{array}$$

Proof. Relation (1.2) implies that

$$m \cdot \sigma \otimes n_1 \otimes \ldots \otimes n_k \otimes (T_1, \ldots, T_k) = \\ m \otimes n_{\sigma^{-1}(1)} \otimes \ldots \otimes n_{\sigma^{-1}(k)} \otimes (T_{\sigma^{-1}(1)}, \ldots, T_{\sigma^{-1}(k)}).$$

Since the sum is taken over all ordered partitions $(T_1, \ldots, T_k)$, the image of $m \cdot \sigma \otimes n_1 \otimes \ldots \otimes n_k$ is

$$\sum_{(U_1,\ldots,U_k)} m \otimes n_{\sigma^{-1}(1)} \otimes \ldots \otimes n_{\sigma^{-1}(k)} \otimes (U_1, \ldots, U_k)$$

with $|U_i| = l_{\sigma^{-1}(i)}$, which is the image of $m \otimes n_{\sigma^{-1}(1)} \otimes \ldots \otimes n_{\sigma^{-1}(k)}$. As a consequence the map is well defined.

The known formula

$$\mathrm{Sh}_{p_1^1,\ldots,p_1^{l_1},\ldots,p_k^1,\ldots,p_k^{l_k}} = \\ (\mathrm{Sh}_{p_1^1,\ldots,p_1^{l_1}} \times \ldots \times \mathrm{Sh}_{p_k^1,\ldots,p_k^{l_k}})\,\mathrm{Sh}_{p_1^1+\ldots+p_1^{l_1},\ldots,p_k^1+\ldots+p_k^{l_k}} \quad (1.3)$$

yields the commutativity of the diagram. □

2. Algebras over an operad

In this section we give the definitions of operads and algebras over an operad and we refer to Fresse [11] for more general theory on operads. We further state the main results of the section: the underlying graded vector space of an algebra over an operad $\mathcal{P}$ in **S-mod** is always a $\mathcal{P}$-algebra and when $\mathcal{P}$ is regular there exists a second $\mathcal{P}$-algebra structure.

2.1. Operads

An *operad* is a monoid in the category of $\mathbb{S}$-modules with respect to the plethysm. Namely, an operad is an $\mathbb{S}$-module $\mathcal{P}$ together with a product $\mu_{\mathcal{P}} : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and a unit $\eta_{\mathcal{P}} : I \to \mathcal{P}$ satisfying

$$\begin{aligned} \mu_{\mathcal{P}}(\mathrm{Id}_{\mathcal{P}} \circ \mu_{\mathcal{P}}) =& \mu_{\mathcal{P}}(\mu_{\mathcal{P}} \circ \mathrm{Id}_{\mathcal{P}}), \\ \mu_{\mathcal{P}}(\mathrm{Id}_{\mathcal{P}} \circ \eta_{\mathcal{P}}) =& \mu_{\mathcal{P}}(\eta_{\mathcal{P}} \circ \mathrm{Id}_{\mathcal{P}}) = \mathrm{Id}_{\mathcal{P}}. \end{aligned}$$

As a consequence the functors $F_{\mathcal{P}}$ and $F^g_{\mathcal{P}}$ are monads in the category **S-mod** and **grVect**.

The product $\mu_{\mathcal{P}}$ is expressed in terms of maps called compositions

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \mathcal{P}(l_1) \otimes \ldots \otimes \mathcal{P}(l_n) & \to & \mathcal{P}(l_1 + \ldots + l_n) \\ \mu \otimes \nu_1 \otimes \ldots \otimes \nu_n & \mapsto & \mu(\nu_1, \ldots, \nu_n) \end{array}$$

which are morphisms of right $S_{l_1+\ldots+l_n}$-modules and which factors through the quotient by the action of the symmetric group S_n.

The operad $\mathcal{A}\mathrm{s}$ is the $\mathbb{S}$-module $(\mathbf{k}[S_n])_{n\geq 0}$. For $\sigma \in S_n$, $\tau_i \in S_{l_i}$ the composition $\mu_{\mathcal{A}\mathrm{s}}(\sigma; \tau_1, \ldots, \tau_n)$ is the permutation of $S_{l_1+\ldots+l_n}$ obtained by substituting the block $\tau_i + l_{\sigma^{-1}(1)} + l_{\sigma^{-1}(2)} \ldots + l_{\sigma^{-1}(\sigma(i)-1)}$ for the integer σ_i. For instance

$$\mu_{\mathcal{A}\mathrm{s}}((3,2,1,4);(2,1),(1,3,2),(1),(2,3,1)) = (\underbrace{6,5}_{\tau_1+4}, \underbrace{2,4,3}_{\tau_2+1}, \underbrace{1}_{\tau_3}, \underbrace{8,9,7}_{\tau_4+6}).$$

2.2. Algebras over an operad

Let $\mathbf{C}$ denotes either the category of $\mathbb{S}$-modules or the category of graded vector spaces. For any $\mathbb{S}$-module M, the functor $F^{\mathbf{C}}_M$ denotes the functor F_M or F^g_M.

Let $\mathcal{P}$ be an operad. A *$\mathcal{P}$-algebra* or an *algebra over $\mathcal{P}$* is an algebra over the monad $F^{\mathbf{C}}_{\mathcal{P}}$, that is an object M of $\mathbf{C}$ together with a product $\mu_M : F^{\mathbf{C}}_{\mathcal{P}}(M) \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} F^{\mathbf{C}}_{\mathcal{P}\circ\mathcal{P}}(M) & \xrightarrow{F^{\mathbf{C}}_{\mathcal{P}}(\mu_M)} & F^{\mathbf{C}}_{\mathcal{P}}(M) \\ \downarrow{\scriptstyle F^{\mathbf{C}}_{\mu_{\mathcal{P}}}(M)} & & \downarrow{\scriptstyle \mu_M} \\ F^{\mathbf{C}}_{\mathcal{P}}(M) & \xrightarrow{\mu_M} & M \end{array} \qquad \begin{array}{ccc} F^{\mathbf{C}}_{I}(M) & \xrightarrow{=} & M \\ \downarrow{\scriptstyle F^{\mathbf{C}}_{\eta_{\mathcal{P}}}(M)} & \nearrow{\scriptstyle \mu_M} & \\ F^{\mathbf{C}}_{\mathcal{P}}(M) & & \end{array}$$

For $p \in \mathcal{P}(n)$ and $m_1, \ldots, m_n \in M$ the product $\mu_M(p \otimes m_1 \otimes \ldots \otimes m_n)$ is usually written

$$p(m_1, \ldots, m_n) \in M.$$

In the category of graded vector spaces one gets the usual definition of an algebra over an operad. In the category of $\mathbb{S}$-modules, $\mathcal{P}$-algebras are also

-Left modules over $\mathcal{P}$ in the terminology of Fresse [11],

- If $\mathcal{P} = \mathcal{A}$s or $\mathcal{P} = \mathcal{L}$ie, *twisted associative or twisted Lie algebras* in the terminology of Barratt [3],

- *Twisted $\mathcal{P}$-algebras* in the terminology of Livernet and Patras [12].

In the sequel we dedicate the word *twisted* to the only case $\mathcal{P} = \mathcal{A}$s: a *twisted algebra* is an algebra over the operad $\mathcal{A}$s in the category **S-mod**.

Any free $\mathcal{P}$-algebra in the category $\mathbf{C}$ writes $F^{\mathbf{C}}_{\mathcal{P}}(M)$ for some $M \in \mathbf{C}$. As a consequence $\mathcal{P}$ is the free $\mathcal{P}$-algebra in **S-mod** generated by the $\mathbb{S}$-module I.

2.3. Relating $\mathcal{P}$-algebras in S-mod and in grVect

2.3.1. Theorem

*Let $M \in$ **S-mod** be an algebra over an operad $\mathcal{P}$. The graded vector space $\mathcal{O}(M)$ is a $\mathcal{P}$-algebra for the product $\hat{\mu}_{\mathcal{O}(M)}$ given by the composition*

$$F^g_{\mathcal{P}}\,\mathcal{O}(M) \xrightarrow{\psi_{\mathcal{P}}(M)} \mathcal{O}\,F_{\mathcal{P}}(M) \xrightarrow{\mathcal{O}(\mu_M)} \mathcal{O}(M)\ .$$

That is

$$\hat{\mu}_{\mathcal{O}(M)}(p \otimes m_1 \otimes \ldots \otimes m_n) = \mu_M(p \otimes m_1 \otimes \ldots \otimes m_n) \cdot q_{l_1,\ldots,l_n} \qquad (2.1)$$

with $p \in \mathcal{P}(n), m_i \in M(l_i)$ *and* $q_{l_1,\dots,l_n}$ *is the sum of all* $(l_1,\dots,l_n)$*-shuffles.*

Proof. One has to prove the commutativity of the following two diagrams:

$$\begin{array}{ccc} F^g_{\mathcal{P}\circ\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow{F^g_{\mathcal{P}}\hat{\mu}_{\mathcal{O}(M)}} & F^g_{\mathcal{P}}\,\mathcal{O}(M) \\ {\scriptstyle F^g_{\mu_{\mathcal{P}}}\,\mathcal{O}(M)}\downarrow & & \downarrow{\scriptstyle \hat{\mu}_{\mathcal{O}(M)}} \\ F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow{\hat{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \end{array} \qquad \begin{array}{ccc} F^g_{I}\,\mathcal{O}(M) & \xrightarrow{=} & \mathcal{O}(M) \\ {\scriptstyle F^g_{\eta_{\mathcal{P}}}\,\mathcal{O}(M)}\downarrow & \nearrow{\scriptstyle \hat{\mu}_{\mathcal{O}(M)}} & \\ F^g_{\mathcal{P}}\,\mathcal{O}(M) & & \end{array}$$

The second diagram is commutative because ψ_N is functorial in N, so

$$\begin{aligned} \psi_I &= \mathrm{Id}, \\ \psi_{\mathcal{P}}(F^g_{\eta_{\mathcal{P}}}\,\mathcal{O}) &= (\mathcal{O}\,F_{\eta_{\mathcal{P}}})\psi_I, \end{aligned}$$

and because $\mu_M(F_{\eta_{\mathcal{P}}}(M)) = \mathrm{Id}_M$.

Since M is a $\mathcal{P}$-algebra $\mu_M(F_{\mu_{\mathcal{P}}}M) = \mu_M(F_{\mathcal{P}}\mu_M)$.
The commutativity of the first diagram is a consequence of the computation

$$\begin{array}{ll} \hat{\mu}_{\mathcal{O}(M)}F^g_{\mu_{\mathcal{P}}}\,\mathcal{O}(M) & \\ = \mathcal{O}(\mu_M)\psi_{\mathcal{P}}(M)(F^g_{\mu_{\mathcal{P}}}\,\mathcal{O}(M)) & \text{by definition,} \\ = \mathcal{O}(\mu_M)(\mathcal{O}\,F_{\mu_{\mathcal{P}}})\psi_{\mathcal{P}\circ\mathcal{P}}(M) & \text{by functoriality of } \psi, \\ = \mathcal{O}(\mu_M)\,\mathcal{O}(F_{\mathcal{P}}\mu_M)\psi_{\mathcal{P}\circ\mathcal{P}}(M) & M \text{ is a } \mathcal{P}-\text{algebra,} \\ = \mathcal{O}(\mu_M)\,\mathcal{O}(F_{\mathcal{P}}\mu_M)(\psi_{\mathcal{P}}F_{\mathcal{P}})(F^g_{\mathcal{P}}\psi_{\mathcal{P}}) & \text{by proposition 1.4,} \\ = \mathcal{O}(\mu_M)\psi_{\mathcal{P}}(M)(F^g_{\mathcal{P}}\,\mathcal{O}(\mu_M))(F^g_{\mathcal{P}}\psi_{\mathcal{P}}) & \psi \text{ is a natural transformation,} \\ = \hat{\mu}_{\mathcal{O}(M)}(F^g_{\mathcal{P}}\hat{\mu}_{\mathcal{O}(M)}) & \text{by definition.} \end{array}$$ □

As pointed out in section 2.2 any free $\mathcal{P}$-algebra in **S-mod** is a $\mathcal{P}$-algebra and satisfies the conditions of the theorem. Hence any free $\mathcal{P}$-algebra in **S-mod** gives rise to a $\mathcal{P}$-algebra in **grVect**. In particular the graded vector space $\oplus_{n\geq 0}\mathcal{P}(n)$ is a $\mathcal{P}$-algebra.

2.3.2. **Example**

We apply formula (2.1) for the examples of the commutative operad and the associative operad.

The commutative operad $\mathcal{C}\text{om}$ is the trivial S_n-module $\mathbf{k}$ for all n. Let e_n be a generator of $\mathcal{C}\text{om}(n)$. The composition is

$$\mu_{\mathcal{C}\text{om}}(e_n \otimes e_{l_1} \otimes \ldots \otimes e_{l_n}) = e_{l_1+\ldots+l_n}.$$

The graded vector space $\oplus_n \mathcal{C}\text{om}(n)$ is isomorphic to $\mathbf{k}[X]$. The commutative product on $\mathbf{k}[X]$ induced by the composition $\mu_{\mathcal{C}\text{om}}$ is

$$X^n \hat{\cdot} X^m = \binom{n+m}{n} X^{n+m}, \tag{2.2}$$

since the number of (n, m)-shuffles is $\binom{n+m}{n}$.

The associative operad was defined in section 2.1. The associative product on the space $\oplus_n \mathbf{k}[S_n]$ induced by the composition $\mu_{\mathcal{A}\text{s}}$ is

$$\sigma \hat{*} \tau = \sum_{\xi \in \text{Sh}_{p,q}} (\sigma \times \tau) \cdot \xi, \tag{2.3}$$

where $\sigma \in S_p$ and $\tau \in S_q$. This is the product defined by Malvenuto and Reutenauer in [18].

2.3.3. **Remark**

Let $\mathcal{P}$ be an operad and M be a $\mathcal{P}$-algebra in **S-mod** such that the action of S_n on $M(n)$ is trivial. There is another $\mathcal{P}$-algebra structure on $\mathcal{O}(M)$ given by

$$\mu^{t,g}_{\mathcal{O}(M)}(p \otimes m_1 \otimes \ldots \otimes m_n) = \mu_M(p \otimes m_1 \otimes \ldots \otimes m_n), \tag{2.4}$$

since the formula (1.2) together with the trivial action imply the S_n-invariance.

If $\mathcal{P} = \mathcal{C}\text{om}$ then $\mathbf{k}[X]$ is a commutative algebra for the product

$$X^n \cdot X^m = X^{n+m}.$$

In characteristic 0 the two commutative products on $\mathbf{k}[X]$ are isomorphic but it is no more the case in characteristic p.

2.4. **Regular operads**

In this section we prove that any algebra over a regular operad $\mathcal{P}$ gives rise to two structures of $\mathcal{P}$-algebra on its underlying graded vector space. This is the generalization to operads of the result of Patras and Reutenauer in

[21] in the associative case. Note that this generalization holds only for regular operads.

2.4.1. **Definition**

The forgetful functor $\mathcal{O}$: **S-mod** $\to$ **grVect** has a left adjoint, the symmetrization functor $\mathcal{S}$: **grVect** $\to$ **S-mod** which associates to a graded vector space $(V_n)_n$ the $\mathbb{S}$-module $(V_n \otimes \mathbf{k}[S_n])_n$, where the action of the symmetric group is the right multiplication. An $\mathbb{S}$-module M is *regular* if there exists a graded vector space $\tilde{M}$ such that $M = \mathcal{S}\tilde{M}$. For instance, the $\mathbb{S}$-module I is regular, since $I = \mathcal{S}I$. Let $\mathbf{S\text{-}mod_r}$ be the subcategory of **S-mod** of regular modules (and regular morphisms). A *regular operad* $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ is an operad in the category $\mathbf{S\text{-}mod_r}$, i.e. $\mu(\nu_1, \ldots, \nu_k) \in \tilde{\mathcal{P}}$ as soon as $\mu, \nu_i \in \tilde{\mathcal{P}}$.

Indeed, there is also a plethysm in the category **grVect**:

$$V \circ^g W = \oplus_k V_k \otimes W^{\otimes k}.$$

Note that

$$\mathcal{S}V \circ \mathcal{S}W = \mathcal{S}(V \circ^g W).$$

A *non-symmetric operad* is a monoid in the category **grVect** with respect to the plethysm $\circ^g$. The operad $\mathcal{S}\tilde{\mathcal{P}}$ is regular if and only if $\tilde{\mathcal{P}}$ is a non-symmetric operad.

2.4.2. **Proposition**

Let $M = \mathcal{S}\tilde{M}$ be a regular module and N be an $\mathbb{S}$-module. The map

$$\begin{array}{ccc} \tilde{M} \circ \mathcal{O}(N) & \to & \mathcal{O}(M \circ N) \\ m \otimes n_1 \otimes \ldots \otimes n_k & \mapsto & m \otimes n_1 \otimes \ldots \otimes n_k \end{array}$$

defines a natural transformation

$$\psi^r_M : F^g_M\, \mathcal{O} \to \mathcal{O}\, F_M$$

*functorial in $M \in$ **S-mod$_\mathbf{r}$**. Furthermore for M and N in **S-mod$_\mathbf{r}$** the following diagram commutes*

$$\begin{array}{ccc} F^g_{M\circ N}\,\mathcal{O} & \xrightarrow{\psi^r_{M\circ N}} & \mathcal{O}\,F_{M\circ N} \\ & {\scriptstyle F^g_M\psi^r_N}\searrow \quad \nearrow{\scriptstyle \psi^r_M F_N} & \\ & F^g_M\,\mathcal{O}\,F_N & \end{array}$$

Proof. Since any element in M writes $m\cdot\sigma$ for $m \in \tilde{M}$, define $\psi^r_M(m\cdot\sigma\otimes n_1\otimes\ldots\otimes n_k)$ to be $m\otimes n_{\sigma^{-1}(1)}\otimes\ldots\otimes n_{\sigma^{-1}(k)}$. It is straightforward to verify the statement with this definition of ψ^r_M. □

Adapting the proof of theorem 2.3.1 by using $\psi^r_{\mathcal{P}}$ in place of $\psi_{\mathcal{P}}$, we prove the following theorem:

2.4.3. **Theorem**

*Let $M \in$ **S-mod** be an algebra over a regular operad $\mathcal{P}$. The graded vector space $\mathcal{O}(M)$ is a $\mathcal{P}$-algebra for the product $\bar{\mu}_{\mathcal{O}(M)}$ given by the composition*

$$F^g_{\mathcal{P}}\,\mathcal{O}(M) \xrightarrow{\psi^r_{\mathcal{P}}(M)} \mathcal{O}\,F_{\mathcal{P}}(M) \xrightarrow{\mathcal{O}(\mu_M)} \mathcal{O}(M)\ ,$$

that is, for all $p \in \tilde{\mathcal{P}}(k)$ and $m_1,\ldots,m_k \in M$,

$$\bar{\mu}_{\mathcal{O}(M)}(p\otimes m_1\otimes\ldots\otimes m_k) = \mu_M(p\otimes m_1\otimes\ldots\otimes m_k).$$

Hence $\hat{\mu}_{\mathcal{O}(M)}$ and $\bar{\mu}_{\mathcal{O}(M)}$ endow $\mathcal{O}(M)$ with two structures of $\mathcal{P}$-algebra. □

If M is a trivial $\mathbb{S}$-module $\bar{\mu}_{\mathcal{O}(M)}$ coincides with $\mu^{t,g}_{\mathcal{O}(M)}$.

Since any free $\mathcal{P}$-algebra is a $\mathcal{P}$-algebra, the theorem holds for any free $\mathcal{P}$-algebra $F_{\mathcal{P}}(M)$. In particular the graded vector space $\oplus_{n\geq 0}\mathcal{P}(n)$ is endowed with two structures of $\mathcal{P}$-algebra.

2.5. Multiplicative operads

An operad $\mathcal{P}$ is *multiplicative* if there exists a morphism of operads $\mathcal{A}s \to \mathcal{P}$. Any algebra in **S-mod** over a multiplicative operad $\mathcal{P}$ is a twisted algebra and thus its underlying graded vector space is endowed with two associative products. It holds in particular for the graded vector space $\oplus_n\mathcal{P}(n)$. For instance $\mathcal{C}om$ is a multiplicative operad and we recover the two associative (and commutative) structures found in example 2.3.2 and remark 2.3.3.

3. Hopf algebras over a Hopf operad

In this section, we generalize the results of Patras and Reutenauer in [21] obtained in the associative case to any Hopf operad. We prove that any Hopf $\mathcal{P}$-algebra in **S-mod** yields a Hopf $\mathcal{P}$-algebra in **grVect** and two Hopf $\mathcal{P}$-algebras in **grVect** if $\mathcal{P}$ is regular.

3.1. Hopf operads–the general case

From now on **C** is either the category **grVect** or the category **S-mod**. The definitions and propositions related to Hopf operads and Hopf algebras over a Hopf operad can be found in [12]. Here we recall only what is needed for our purpose.

3.1.1. Definition

Let **Coalg** be the category of coassociative counital coalgebras, that is vector spaces V endowed with a coassociative coproduct $\Delta : V \to V \otimes V$ and a counit $\epsilon : V \to \mathbf{k}$. A *Hopf operad* $\mathcal{P}$ is an operad in **Coalg**, i.e. $\mu_{\mathcal{P}}$ and $\eta_{\mathcal{P}}$ are morphisms of coassociative counital coalgebras. A Hopf operad amounts to the following data: for each n a coproduct $\delta(n) : \mathcal{P}(n) \to \mathcal{P}(n) \otimes \mathcal{P}(n)$ and a counit $\epsilon(n) : \mathcal{P}(n) \to \mathbf{k}$ preserving the operadic composition and the action of the symmetric group . We use Sweedler's notation, that is,

$$\delta(\mu) = \sum_{(1),(2)} \mu_{(1)} \otimes \mu_{(2)}.$$

One has maps in **S-mod** and in **grVect**

$$\tau_{M,N} : F_{\mathcal{P}}(M \otimes N) \to F_{\mathcal{P}}(M) \otimes F_{\mathcal{P}}(N)$$

and

$$\tau^g_{X,Y} : F^g_{\mathcal{P}}(X \otimes Y) \to F^g_{\mathcal{P}}(X) \otimes F^g_{\mathcal{P}}(Y)$$

defined by

$$\tau_{M,N}(\mu\otimes m_1\otimes n_1\otimes\cdots\otimes m_k\otimes n_k\otimes(A_1,B_1,\ldots,A_k,B_k)) = \sum_{(1),(2)}(\mu_{(1)}\otimes m_1\ldots m_k\otimes \mathrm{st}(A_1,\ldots,A_k))\otimes(\mu_{(2)}\otimes n_1\ldots n_k\otimes \mathrm{st}(B_1,\ldots,B_k)) \otimes(\cup A_i,\cup B_i). \quad (3.1)$$

and

$$\tau^g_{X,Y}(\mu\otimes x_1\otimes y_1\otimes\cdots\otimes x_k\otimes y_k) = \sum_{(1),(2)}(\mu_{(1)}\otimes x_1\otimes\ldots\otimes x_k)\otimes(\mu_{(2)}\otimes y_1\otimes\ldots\otimes y_k). \quad (3.2)$$

As a consequence if M and N are $\mathcal{P}$-algebras in the category $\mathbf{C}$, then $M\otimes N$ is a $\mathcal{P}$-algebra for the following product

$$F^{\mathbf{C}}_{\mathcal{P}}(M\otimes N)\xrightarrow{\tau^{\mathbf{C}}_{M,N}} F^{\mathbf{C}}_{\mathcal{P}}(M)\otimes F^{\mathbf{C}}_{\mathcal{P}}(N)\xrightarrow{\mu_M\otimes\mu_N} M\otimes N\ .$$

3.1.2. **Definition**

Let $\mathcal{P}$ be a Hopf operad. A $\mathcal{P}$-algebra M is a *Hopf $\mathcal{P}$-algebra* if M is endowed with a coassociative coproduct and a counit

$$\Delta_M : M\to M\otimes M \qquad \epsilon_M : M\to \mathbf{1}$$

which are morphisms of $\mathcal{P}$-algebras. For $\mathcal{P}=\mathcal{A}$s, a Hopf $\mathcal{A}$s-algebra is named a *twisted bialgebra*.

3.1.3. **Theorem**

The underlying graded vector space of any Hopf $\mathcal{P}$-algebra M in **S-mod** *is a Hopf $\mathcal{P}$-algebra in* **grVect**. *More precisely, the $\mathcal{P}$-algebra product on $\mathcal{O}(M)$ is $\hat{\mu}_{\mathcal{O}(M)}$ and the coproduct*

$$\bar{\Delta}_{\mathcal{O}(M)}:\ \mathcal{O}(M)\xrightarrow{\mathcal{O}(\Delta_M)}\mathcal{O}(M\otimes M)\xrightarrow{\pi^{\mathcal{O}}_{M,M}}\mathcal{O}(M)\otimes\mathcal{O}(M)$$

is a morphism of $\mathcal{P}$-algebras. This Hopf $\mathcal{P}$-algebra is denoted $\bar{M}$ and named the symmetrized Hopf $\mathcal{P}$-algebra associated to M.

In particular, if for $m \in M(p)$ one writes

$$\Delta(m) = \sum_{\substack{(1),(2)\\ S\sqcup T=[p]}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)} \otimes (S,T)$$

then

$$\bar{\Delta}(m) = \sum_{(1),(2)} \sum_{i=0}^{p} m_{(1)}^{([i],i+[p-i])} \otimes m_{(2)}^{([i],i+[p-i])} \tag{3.3}$$

Proof. One has to prove that the following diagram is commutative:

$$\begin{array}{ccc}
F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow{\hat{\mu}_{\mathcal{O}(M)}} & \mathcal{O}(M) \\
\Big\downarrow{\scriptstyle F^g_{\mathcal{P}}\bar{\Delta}_{\mathcal{O}(M)}} & & \\
F^g_{\mathcal{P}}(\mathcal{O}(M)\otimes\mathcal{O}(M)) & & \Big\downarrow{\scriptstyle \bar{\Delta}_{\mathcal{O}(M)}} \\
\Big\downarrow{\scriptstyle \tau^g_{\mathcal{O}(M),\mathcal{O}(M)}} & & \\
F^g_{\mathcal{P}}\,\mathcal{O}(M)\otimes F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow[\hat{\mu}_{\mathcal{O}(M)}\otimes\hat{\mu}_{\mathcal{O}(M)}]{} & \mathcal{O}(M)\otimes\mathcal{O}(M)
\end{array}$$

Recall that

$$\begin{aligned}
\bar{\Delta}_{\mathcal{O}(M)} &= \pi^{\mathcal{O}}_{M,M}\,\mathcal{O}(\Delta_M),\\
\hat{\mu}_{\mathcal{O}(M)} &= \mathcal{O}(\mu_M)\psi_{\mathcal{P}}(M),\\
\Delta_M\mu_M &= (\mu_M\otimes\mu_M)\tau_{M,M}F_{\mathcal{P}}\Delta_M.
\end{aligned}$$

The functoriality and naturality of $\pi^{\mathcal{O}}$ and $\psi_{\mathcal{P}}$ imply

$$\begin{aligned}
\bar{\Delta}_{\mathcal{O}(M)}\hat{\mu}_{\mathcal{O}(M)} &= \pi^{\mathcal{O}}_{M,M}\,\mathcal{O}(\Delta_M)\,\mathcal{O}(\mu_M)\psi_{\mathcal{P}}(M)\\
&= \pi^{\mathcal{O}}_{M,M}\,\mathcal{O}(\mu_M\otimes\mu_M)\,\mathcal{O}(\tau_{M,M})\,\mathcal{O}(F_{\mathcal{P}}\Delta_M)\psi_{\mathcal{P}}(M)\\
&= (\mathcal{O}(\mu_M)\otimes\mathcal{O}(\mu_M))\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}\,\mathcal{O}(\tau_{M,M})\psi_{\mathcal{P}}(M\otimes M)F^g_{\mathcal{P}}\,\mathcal{O}(\Delta_M).
\end{aligned}$$

Therefore, the commutativity of the previous diagram follows from the commutativity of the following diagram

$$\begin{array}{ccc}
F^g_{\mathcal{P}}\,\mathcal{O}(M\otimes M) & \xrightarrow{\psi_{\mathcal{P}}(M\otimes M)} & \mathcal{O}\,F_{\mathcal{P}}(M\otimes M) \\
\downarrow{\scriptstyle F^g_{\mathcal{P}}(\pi^{\mathcal{O}}_{M,M})} & & \downarrow{\scriptstyle \mathcal{O}\,\tau_{M,M}} \\
F^g_{\mathcal{P}}(\mathcal{O}(M)\otimes\mathcal{O}(M)) & & \mathcal{O}(F_{\mathcal{P}}(M)\otimes F_{\mathcal{P}}(M)) \\
\downarrow{\scriptstyle \tau^g_{\mathcal{O}(M),\mathcal{O}(M)}} & & \downarrow{\scriptstyle \pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}} \\
F^g_{\mathcal{P}}\,\mathcal{O}(M)\otimes F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow[\psi_{\mathcal{P}}(M)\otimes\psi_{\mathcal{P}}(M)]{} & \mathcal{O}\,F^g_{\mathcal{P}}(M)\otimes\mathcal{O}\,F^g_{\mathcal{P}}(M)
\end{array}$$

Let us compute the composition $R=\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}\,\mathcal{O}\,\tau_{M,M}\psi_{\mathcal{P}}(M\otimes M)$ applied to

$X=\mu\otimes(m_1\otimes n_1\otimes(A_1,B_1))\otimes\ldots\otimes(m_k\otimes n_k\otimes(A_k,B_k))\in F^g_{\mathcal{P}}\,\mathcal{O}(M\otimes M)$,

where $m_i\in M(l_i)$ and $n_i\in M(r_i)$ and $A_i\sqcup B_i$ is an ordered partition of $[l_i+r_i]$.

$$Y:=\psi_{\mathcal{P}}(M\otimes M)(X)=\sum_{(T_1,\ldots,T_k)}X\otimes(T_1,\ldots,T_k),$$

where the sum is taken over all ordered partitions of $[l_1+r_1+\ldots+l_k+r_k]$ such that $|T_i|=l_i+r_i$. For $U_i=T_i(A_i)$ and $V_i=T_i(B_i)$ one has

$$((A_1,B_1)\times\ldots\times(A_k,B_k))(T_1,\ldots,T_k)=(U_1,V_1,\ldots,U_k,V_k).$$

As a consequence

$$Y=\sum_{(U_1,V_1\ldots,U_k,V_k)}\mu\otimes m_1\otimes n_1\otimes\ldots\otimes m_k\otimes n_k\otimes(U_1,V_1,\ldots,U_k,V_k),$$

where the sum is taken over all ordered partitions of $[l_1+r_1+\ldots+l_k+r_k]$ such that $|U_i|=l_i,|V_i|=r_i$ and $\mathrm{st}(U_i)=A_i,\mathrm{st}(V_i)=B_i$. Set $\bar m=m_1\otimes\ldots\otimes m_k$ and $\bar n=n_1\otimes\ldots\otimes n_k$. By the definition of τ (see (3.1)),

$$Z:=\mathcal{O}\,\tau_{M,M}(Y)=\sum_{(U_1,V_1\ldots,U_k,V_k)}$$
$$(\mu_{(1)}\otimes\bar m\otimes\mathrm{st}(U_1,\ldots,U_k))\otimes(\mu_{(2)}\otimes\bar n\otimes\mathrm{st}(V_1,\ldots,V_k))\otimes(\cup U_i,\cup V_i).$$

Furthermore $\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}(Z)=0$ except in case $(\cup U_i,\cup V_i)=\mathrm{Id}$. But

$$(\cup U_i,\cup V_i)=\mathrm{Id}\Leftrightarrow(A_i,B_i)=\mathrm{Id},\forall i.$$

For, if $(\cup U_i, \cup V_i) = \mathrm{Id}$ then $T_i(A_i) < T_i(B_i)$ and $A_i < B_i$ which is equivalent to $(A_i, B_i) = \mathrm{Id}$.

As a consequence if $\forall i,\ (A_i, B_i) = \mathrm{Id}$, then

$$R(X) = \sum_{\substack{(U_1,\ldots,U_k)\\(V_1,\ldots,V_k)}} (\mu_{(1)} \otimes \bar{m} \otimes (U_1, \ldots, U_k)) \otimes (\mu_{(2)} \otimes \bar{n} \otimes (V_1, \ldots, V_k)),$$

where the sum is taken over all shuffles $(U_1, \ldots, U_k)$ and $(V_1, \ldots, V_k)$. If there exists i such that $(A_i, B_i) \neq \mathrm{Id}$, then $R(X) = 0$.

The composition $L = (\psi_{\mathcal{P}}(M) \otimes \psi_{\mathcal{P}}(M))\tau^g_{\mathcal{O}(M),\mathcal{O}(M)} F^g_{\mathcal{P}}(\pi^{\mathcal{O}}_{M,M})$ is easier to compute. First of all, if there exists i such that $(A_i, B_i) \neq \mathrm{Id}$, then $L(X) = 0$. Assume that $(A_i, B_i) = \mathrm{Id}, \forall i$. Then

$$W = \tau^g_{\mathcal{O}(M),\mathcal{O}(M)} F^g_{\mathcal{P}}(\pi^{\mathcal{O}}_{M,M})(X) = \sum_{(1),(2)} (\mu_{(1)} \otimes \bar{m}) \otimes (\mu_{(2)} \otimes \bar{n})$$

and $(\psi_{\mathcal{P}}(M) \otimes \psi_{\mathcal{P}}(M))(W) = R(X)$. Thus $R(X) = L(X), \forall X$ and the diagram is commutative. $\square$

3.1.4. Connected operads and connected Hopf operads

An operad is *connected* if $\mathcal{P}(0) = \mathbf{k}$ and $\mathcal{P}(1) = \mathbf{k}$. Let 1_0 denote the unit of $\mathbf{k} \in \mathcal{P}(0)$. If $\mathcal{P}$ is a connected operad, for any subset S of $[n]$, there exists a map

$$\begin{array}{ccc} \mathcal{P}(n) & \to & \mathcal{P}(|S|) \\ \mu & \mapsto & \mu|_S = \mu(x_1, \ldots, x_n) \end{array}$$

where

$$\begin{cases} x_i = 1_1 & \text{if } i \in S, \\ x_i = 1_0 & \text{if } i \notin S. \end{cases}$$

For $\mathcal{P} = \mathcal{A}s$ one recovers the definition given in section 1.1 for the symmetric group.

A *connected Hopf operad* is a Hopf operad which is connected and such that $\epsilon(0) : \mathbf{k} = \mathcal{P}(0) \to \mathbf{k}$ is the identity isomorphism. As a consequence

$$\epsilon(\mu) = \mu|_{\emptyset}.$$

Recall from [12] that if $\mathcal{P}$ is a connected Hopf operad then $\mathcal{P}$ is a Hopf $\mathcal{P}$-algebra in **S-mod** for the coproduct given by

$$\begin{aligned}\Delta(\mu) = &\sum_{\substack{(1),(2)\\ S\sqcup T=[n]}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \otimes (S,T)\\ &=1\otimes \mu + \mu\otimes 1 + \sum_{\substack{S\sqcup T=[n]\\ S,T\neq\emptyset}} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \otimes (S,T).\end{aligned}$$

Indeed the map Δ is the unique $\mathcal{P}$-algebra morphism such that $\Delta(1_1) = 1_0\otimes 1_1 + 1_1\otimes 1_0$.

It happens in many examples that $\mathcal{P}$ is not connected and $\mathcal{P}(0) = 0$. Nevertheless it is sometimes possible to define a $\mathcal{P}$-algebra structure on $(\mathcal{P}_+\otimes\mathcal{P}_+)_-$ where

$$\begin{cases}\mathcal{P}_+(0) = & \mathbf{k}\\ \mathcal{P}_+(n) = & \mathcal{P}(n),\ n>0\end{cases}$$

and

$$\begin{cases}(\mathcal{P}_+\otimes\mathcal{P}_+)_-(0) = & 0\\ (\mathcal{P}_+\otimes\mathcal{P}_+)_-(n) = & (\mathcal{P}_+\otimes\mathcal{P}_+)(n),\ n>0.\end{cases}$$

(see for instance [13]). In that case, Δ is defined as the unique $\mathcal{P}$-algebra map such that $\Delta(1_1) = 1_0\otimes 1_1 + 1_1\otimes 1_0$. These kind of operads will be treated as connected operads.

If $\mathcal{P}$ is connected, one has two examples of Hopf $\mathcal{P}$-algebras in **grVect**:

- $F^g_{\mathcal{P}}(V)$ for V in **grVect** where the product is given by $F^g_{\mu_{\mathcal{P}}}(V)$ and the coproduct is given by $F^g_{\Delta}(V)$.
- For any $\mathbb{S}$-module M the free $\mathcal{P}$-algebra $F_{\mathcal{P}}(M)$ is a Hopf $\mathcal{P}$-algebra in **S-mod**. The symmetrized Hopf $\mathcal{P}$-algebra $\overline{F_{\mathcal{P}}(M)}$ is a Hopf $\mathcal{P}$-algebra in **grVect** by theorem 3.1.3.

3.2. Regular Hopf operads

Let $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ be a regular operad. Assume $(\mathcal{P},\delta)$ is a Hopf operad. The operad $\mathcal{P}$ is a *regular Hopf operad* if $\delta(\tilde{\mathcal{P}}) \subset \tilde{\mathcal{P}}\otimes\tilde{\mathcal{P}}$. For instance $\mathcal{A}s$ is a regular Hopf operad. We prove that for any regular Hopf operad a Hopf $\mathcal{P}$-algebra in the category **S-mod** gives rise to two structures of Hopf $\mathcal{P}$-algebra in the category **grVect**. This is a generalization to regular operads of a theorem announced by Stover in [24] and proved by Patras and Reutenauer in [21] in the context of twisted bialgebras. Note that the hypothesis regular is needed to obtain two structures of Hopf $\mathcal{P}$-algebras.

3.2.1. **Theorem**

Let $\mathcal{P}$ be a regular Hopf operad. Let (M, μ_M, Δ_M) be a Hopf $\mathcal{P}$-algebra in **S-mod**. *The product $\bar{\mu}_{\mathcal{O}(M)}$ together with the coproduct $\hat{\Delta}_{\mathcal{O}(M)}$ defined as the composite*

$$\mathcal{O}(M) \xrightarrow{\mathcal{O}(\Delta_M)} \mathcal{O}(M \otimes M) \xrightarrow{\iota^g_{M,M}} \mathcal{O}(M) \otimes \mathcal{O}(M)$$

endows $\mathcal{O}(M)$ with a structure of Hopf $\mathcal{P}$-algebra which is cocommutative if (M, Δ_M) is. This Hopf $\mathcal{P}$-algebra is denoted $\hat{M}$ and named the cosymmetrized Hopf $\mathcal{P}$-algebra associated to M.

Note that the coproduct $\hat{\Delta}_{\mathcal{O}(M)}$ is always defined: for $m \in M(p)$, if the coproduct in M writes

$$\Delta_M(m) = \sum_{\substack{(1),(2)\\ S \sqcup T = [p]}} m^{(S,T)}_{(1)} \otimes m^{(S,T)}_{(2)} \otimes (S,T),$$

then the induced coproduct in $\mathcal{O}(M)$ writes

$$\hat{\Delta}_{\mathcal{O}(M)}(m) = \sum_{\substack{(1),(2)\\ S \sqcup T = [p]}} m^{(S,T)}_{(1)} \otimes m^{(S,T)}_{(2)}. \tag{3.4}$$

Proof. The proof is similar to the proof of theorem 3.1.3. The commutativity of the diagram

$$\begin{array}{ccc}
F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow{\quad\bar{\mu}_{\mathcal{O}(M)}\quad} & \mathcal{O}(M) \\
\Big\downarrow{\scriptstyle F^g_{\mathcal{P}}\hat{\Delta}_{\mathcal{O}(M)}} & & \\
F^g_{\mathcal{P}}(\mathcal{O}(M) \otimes \mathcal{O}(M)) & & \Big\downarrow{\scriptstyle \hat{\Delta}_{\mathcal{O}(M)}} \\
\Big\downarrow{\scriptstyle \tau^g_{\mathcal{O}(M),\mathcal{O}(M)}} & & \\
F^g_{\mathcal{P}}\,\mathcal{O}(M) \otimes F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow[\bar{\mu}_{\mathcal{O}(M)} \otimes \bar{\mu}_{\mathcal{O}(M)}]{} & \mathcal{O}(M) \otimes \mathcal{O}(M)
\end{array}$$

is a consequence of the commutativity of the diagram

$$\begin{array}{ccc}
F^g_{\mathcal{P}}\,\mathcal{O}(M\otimes M) & \xrightarrow{\psi^r_{\mathcal{P}}(M\otimes M)} & \mathcal{O}\,F_{\mathcal{P}}(M\otimes M) \\
\downarrow{\scriptstyle F^g_{\mathcal{P}}(\iota^{\mathcal{O}}_{M,M})} & & \downarrow{\scriptstyle \mathcal{O}\,\tau_{M,M}} \\
F^g_{\mathcal{P}}(\mathcal{O}(M)\otimes\mathcal{O}(M)) & & \mathcal{O}(F_{\mathcal{P}}(M)\otimes F_{\mathcal{P}}(M)) \\
\downarrow{\scriptstyle \tau^g_{\mathcal{O}(M),\mathcal{O}(M)}} & & \downarrow{\scriptstyle \iota^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}} \\
F^g_{\mathcal{P}}\,\mathcal{O}(M)\otimes F^g_{\mathcal{P}}\,\mathcal{O}(M) & \xrightarrow[\psi^r_{\mathcal{P}}(M)\otimes\psi^r_{\mathcal{P}}(M)]{} & \mathcal{O}\,F^g_{\mathcal{P}}(M)\otimes\mathcal{O}\,F^g_{\mathcal{P}}(M)
\end{array}$$

We first evaluate $R=\iota^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}\,\mathcal{O}\,\tau_{M,M}\psi^r_{\mathcal{P}}(M\otimes M)$ at

$X=\mu\otimes(m_1\otimes n_1\otimes(A_1,B_1))\otimes\ldots\otimes(m_k\otimes n_k\otimes(A_k,B_k))\in F^g_{\mathcal{P}}\,\mathcal{O}(M\otimes M)$,

where $\mu\in\tilde{\mathcal{P}}(k), m_i\in M(l_i), n_i\in M(r_i)$ and $A_i\sqcup B_i$ is an ordered partition of $[l_i+r_i]$. Set $\bar{m}=m_1\otimes\ldots\otimes m_k$ and $\bar{n}=n_1\otimes\ldots\otimes n_k$ and

$$(A_1,B_1)\times\ldots\times(A_k,B_k)=(U_1,V_1,\ldots,U_k,V_k).$$

$$R(X)=\sum_{(1),(2)}(\mu_{(1)}\otimes\bar{m}\otimes\mathrm{st}(U_1,\ldots,U_k))\otimes(\mu_{(2)}\otimes\bar{n}\otimes\mathrm{st}(V_1,\ldots,V_k)),$$

and $\mathrm{st}(U_1,\ldots,U_k)=\mathrm{Id}$ and $\mathrm{st}(V_1,\ldots,V_k)=\mathrm{Id}$.

The composite $L:=(\psi^r_{\mathcal{P}}(M)\otimes\psi^r_{\mathcal{P}}(M))\tau^g_{\mathcal{O}(M),\mathcal{O}(M)}F^g_{\mathcal{P}}(\iota^{\mathcal{O}}_{M,M})$ evaluated at X gives

$$L(X)=\sum_{(1),(2)}(\mu_{(1)}\otimes\bar{m})\otimes(\mu_{(2)}\otimes\bar{n}).$$

Thus $R(X)=L(X),\forall X$ and the diagram is commutative.

It is clear that if Δ_M is cocommutative, so is $\hat{\Delta}_{\mathcal{O}(M)}$. □

As a consequence, if $\mathcal{P}$ is a regular Hopf operad then any Hopf $\mathcal{P}$-algebra M in **S-mod** gives rise to two structures of Hopf $\mathcal{P}$-algebra in **grVect**. In particular this result holds for $\oplus_n\mathcal{P}(n)$ and for the underlying graded vector space of any free $\mathcal{P}$-algebra.

3.3. Application to multiplicative Hopf operads

In a first step we establish that the corresponding Hopf structures in case $\mathcal{P}=\mathcal{A}s$ coincide with the ones discovered by Stover [24] and proved by Patras and Reutenauer in [21]. In a second step we apply the above results to multiplicative Hopf operads.

3.3.1. The associative case

Recall that the operad $\mathcal{A}s$ is a regular Hopf operad. Hence the underlying graded vector space of a twisted bialgebra is endowed with two structures of Hopf algebra. Let M be a twisted bialgebra with product m and coproduct Δ.

The Hopf algebra $\bar{M} = (\mathcal{O}(M), \hat{m}_{\mathcal{O}(M)}, \bar{\Delta}_{\mathcal{O}(M)})$ is described for $a \in M(p), b \in M(q)$ by

$$\begin{aligned} \hat{m}_{\mathcal{O}(M)}(a,b) &= m(a,b) \cdot q_{a,b}, \\ \bar{\Delta}_{\mathcal{O}(M)}(a) &= \sum_{i=0}^{p} a_{(1)}^{([i],i+[p-i])} \otimes a_{(2)}^{([i],i+[p-i])}, \end{aligned}$$

which is the *symmetrized bialgebra* associated to the twisted bialgebra M as in [21, proposition 15].

The Hopf algebra $\hat{M} = (\mathcal{O}(M), \bar{m}, \hat{\Delta}_{\mathcal{O}(M)})$ is described for $a \in M(p), b \in M(q)$ by

$$\begin{aligned} \bar{m}_{\mathcal{O}(M)}(a,b) &= m(a,b), \\ \hat{\Delta}_{\mathcal{O}(M)}(a) &= \sum_{S \sqcup T = [p]} a_{(1)}^{(S,T)} \otimes a_{(2)}^{(S,T)}, \end{aligned}$$

which is the *cosymmetrized bialgebra* associated to the twisted bialgebra M as in [21, definition 8].

A *multiplicative Hopf operad* is a Hopf operad $\mathcal{P}$ together with an operad morphism $\mathcal{A}s \to \mathcal{P}$ which commutes with the Hopf structure. As a consequence any Hopf $\mathcal{P}$-algebra is a Hopf $\mathcal{A}s$-algebra. The result below is a consequence of the previous sections.

3.3.2. Corollary

Let $\mathcal{P}$ be a multiplicative Hopf operad. The underlying graded vector space of any Hopf $\mathcal{P}$-algebra is endowed with two different structures of Hopf algebra. □

4. Unital infinitesimal $\mathcal{P}$-bialgebras

In this section we give some comparison between $\bar{\Delta}$ and $\bar{\mu}$ when the operad is regular, in view of generalizing the theory of *unital infinitesimal* *bialgebra* developed by Loday and Ronco in [17]. This yields the definition of unital infinitesimal $\mathcal{P}$-bialgebras. As a consequence we prove that any Hopf algebra over a multiplicative Hopf operad is isomorphic to a cofree coassociative algebra. Moreover, if $\mathcal{P}$ is regular then this isomorphism respects the $\mathcal{P}$-algebra structure. We study the associative case in detail.

From now on a connected Hopf operad $\mathcal{P}$ is given.

4.1. Unital infinitesimal $\mathcal{P}$-bialgebras

In this section, We prove that the underlying graded vector space of a Hopf $\mathcal{P}$-algebra is a unital infinitesimal $\mathcal{P}$-bialgebra (theorem 4.1.2). We prove also in theorem 4.1.3 that the same result holds for $F^g_{\mathcal{P}}(V)$ when V is a graded vector space such that $V(0) = 0$.

A *connected coalgebra* M in **S-mod** or **grVect** is a coalgebra such that $M(0) = \mathbf{k}$ and such that the counit $\epsilon : \mathbf{k} = M(0) \to \mathbf{k}$ is the identity isomorphism. That is for $M \in$ **S-mod** the coproduct writes

$$\Delta(m) = 1 \otimes m + m \otimes 1 + \sum_{\substack{S \sqcup T = [p] \\ S,T \neq \emptyset}} m_{(1)}^{(S,T)} \otimes m_{(2)}^{(S,T)} \otimes (S,T),$$

and for $V \in$ **grVect** it writes, $\forall v \in V_r$

$$\Delta(v) = 1 \otimes v + v \otimes 1 + \sum_{\substack{p+q=r \\ p,q>0}} m_{(1),p} \otimes m_{(2),q}.$$

4.1.1. Definition

Assume $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ is a regular Hopf operad. A *unital infinitesimal $\mathcal{P}$-bialgebra* M is a $\mathcal{P}$-algebra in **grVect** endowed with a connected coalgebra structure $\Delta : M \to M \otimes M$ satisfying the *infinitesimal relation*:

$$\Delta\mu(m_1, \ldots, m_k) = \sum_{j=1}^{k} \mu(m_1 \otimes 1, \ldots, m_{j-1} \otimes 1, \Delta(m_j), 1 \otimes m_{j+1}, \ldots, 1 \otimes m_k) - \sum_{j=1}^{k-1} \mu(m_1 \otimes 1, \ldots, m_j \otimes 1, 1 \otimes m_{j+1}, \ldots, 1 \otimes m_k), \quad (4.1)$$

for $\mu \in \tilde{\mathcal{P}}(k)$. Note that the operad needs to be regular since the infinitesimal relation is not S_k-equivariant.

For instance if $\mathcal{P} = \mathcal{A}s$, a unital infinitesimal $\mathcal{A}s$-bialgebra is the definition of Loday and Ronco in [17] of a *unital infinitesimal bialgebra* since the previous relation amounts to

$$\Delta(ab) = \Delta(a)(1 \otimes b) + (a \otimes 1)\Delta(b) - a \otimes b.$$

Let M be a Hopf $\mathcal{P}$-algebra in **S-mod** with $\mathcal{P}$ regular. Theorems 3.1.3 and 3.2.1 assert that the underlying graded vector space of M is endowed with two structures of Hopf $\mathcal{P}$-algebras in **grVect**. One is given by $(\hat{\mu}, \bar{\Delta})$ and the other one by $(\bar{\mu}, \hat{\Delta})$. The next theorem explores the relation between $\bar{\mu}$ and $\bar{\Delta}$.

4.1.2. Theorem

Let $\mathcal{P}$ be a connected regular Hopf operad and M be a connected Hopf $\mathcal{P}$-algebra in **S-mod**. *The product $\bar{\mu} := \bar{\mu}_{\mathcal{O}(M)}$ and coproduct $\bar{\Delta} = \bar{\Delta}_{\mathcal{O}(M)}$ endow $\mathcal{O}(M)$ with a structure of unital infinitesimal $\mathcal{P}$-bialgebra.*

Proof. Recall that

$$\begin{aligned}\bar{\Delta} &= \pi^{\mathcal{O}}_{M,M}\, \mathcal{O}(\Delta) \\ \bar{\mu} &= \mathcal{O}(\mu_M)\psi^r_{\mathcal{P}}(M).\end{aligned}$$

Following the proof of theorem 3.1.3 one has

$$\bar{\Delta}\bar{\mu} = (\mathcal{O}\,\mu_M \otimes \mathcal{O}\,\mu_M)\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}\,\mathcal{O}\,\tau_{M,M}\psi^r_{\mathcal{P}}(M\otimes M)F^g_{\mathcal{P}}\,\mathcal{O}\,\Delta.$$

Let $X = \mu \otimes m_1 \otimes \ldots \otimes m_k \in F^g_{\mathcal{P}}\,\mathcal{O}(M)$ with $m_i \in M(h_i)$.

$$Y := \psi^r_{\mathcal{P}}(M\otimes M)F^g_{\mathcal{P}}\,\mathcal{O}\,\Delta(X) =$$
$$\sum \mu\otimes(m_{1(1)}^{(S_1,T_1)}\otimes m_{1(2)}^{(S_1,T_1)}\otimes(S_1,T_1))\otimes\ldots\otimes(m_{k(1)}^{(S_k,T_k)}\otimes m_{k(2)}^{(S_k,T_k)}\otimes(S_k,T_k)),$$

where the sum is taken over all ordered partitions $S_i \sqcup T_i$ of $[h_i]$ for all i. In order to compute $\mathcal{O}\,\tau_{M,M}(Y)$, we write $(S_1,T_1)\times\ldots\times(S_k,T_k)$ as $(U_1,V_1,\ldots,U_k,V_k)$ which is an ordered partition of $[h_1+\ldots+h_k]$ and $U_i = |S_i|, V_i = |T_i|$. It is obvious that $\mathrm{st}(U_1,\ldots,U_k) = \mathrm{Id}$ and that the same equality holds for the sequence of V_i's. Furthermore if $S=\cup U_i$ and $T=\cup V_i$, then $S=S_1\times\ldots\times S_k$ and $T=T_1\times\ldots\times T_k$. As a consequence

$$\mathcal{O}\,\tau_{M,M}(Y)= \sum_{(S,T)}(\mu_{(1)}\otimes m_{1(1)}^{(S_1,T_1)}\otimes\ldots\otimes m_{k(1)}^{(S_k,T_k)})\otimes (\mu_{(2)}\otimes m_{1(2)}^{(S_1,T_1)}\otimes\ldots\otimes m_{k(2)}^{(S_k,T_k)})\otimes(S,T),$$

where the sum is taken over all ordered partitions (S,T) of $[h_1+\ldots+h_k]$ and where $S=S_1\times\ldots\times S_k$ and $T=T_1\times\ldots\times T_k$ with $S_i,T_i\subset[h_i]$. The map $\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}$ is non zero on an ordered partition (S,T) if and only if there exists r such that $S=[r]$. For any such r there exists j such that $S_k=[h_k]$ for $k<j$ and $S_k=\emptyset$ for $k>j$. Since M is connected $m_{(1)}^{\emptyset,[h]}\otimes m_{(2)}^{\emptyset,[h]}=1\otimes m$. As a consequence

$$\begin{aligned}(\mathcal{O}\,\mu_M\otimes\mathcal{O}\,\mu_M)\pi^{\mathcal{O}}_{F_{\mathcal{P}}(M),F_{\mathcal{P}}(M)}\,\mathcal{O}\,\tau_{M,M}(Y)=&\\ \sum_{j=1}^{k}\sum_{\alpha=1}^{h_j-1}\mu_{(1)}(m_1,\ldots,m_{j-1},m_{j(1)}^{([\alpha],\alpha+[h_j-\alpha])},1,\ldots,1)&\\ \otimes\,\mu_{(2)}(1,\ldots,1,m_{j(2)}^{([\alpha],\alpha+[h_j-\alpha])},m_{j+1},\ldots,m_k)&\\ +\sum_{j=0}^{k}\mu_{(1)}(m_1,\ldots,m_j,1,\ldots,1)\otimes\mu_{(2)}(1,\ldots,1,m_{j+1},\ldots,m_k).&\end{aligned}$$

On the other hand let us compute the right hand side of the equation (4.1):

$$\sum_{j=1}^{k}\mu(m_1\otimes 1,\ldots,m_{j-1}\otimes 1,\Delta(m_j),1\otimes m_{j+1},\ldots,1\otimes m_k) -\sum_{j=1}^{k-1}\mu(m_1\otimes 1,\ldots,m_j\otimes 1,1\otimes m_{j+1},\ldots,1\otimes m_k)=$$

$$\sum_{j=1}^{k} \mu(m_1 \otimes 1, \ldots, m_{j-1} \otimes 1, \Delta'(m_j), 1 \otimes m_{j+1}, \ldots, 1 \otimes m_k) +$$

$$\sum_{j=1}^{k} \mu(m_1 \otimes 1, \ldots, m_{j-1} \otimes 1, 1 \otimes m_j + m_j \otimes 1, 1 \otimes m_{j+1}, \ldots, 1 \otimes m_k)$$

$$- \sum_{j=1}^{k-1} \mu(m_1 \otimes 1, \ldots, m_j \otimes 1, 1 \otimes m_{j+1}, \ldots, 1 \otimes m_k)$$

where

$$\Delta'(m_j) = \sum_{\alpha=1}^{h_j-1} m_{j(1)}^{([\alpha],\alpha+[h_j-\alpha])} \otimes m_{j(2)}^{([\alpha],\alpha+[h_j-\alpha])}.$$

Thus the left and right hand sides of the equation (4.1) are equal and the theorem is proved. □

4.1.3. **Theorem**

Let $\mathcal{P} = \mathcal{S}\tilde{\mathcal{P}}$ be a connected regular Hopf operad. Let V be a graded vector space with $V(0) = 0$. The free $\mathcal{P}$-algebra in **grVect** *$F^g_{\mathcal{P}}(V)$ is a unital infinitesimal $\mathcal{P}$-bialgebra. The product is given by the usual product on free $\mathcal{P}$-algebras and the coproduct is given for $x = \mu \otimes v_1 \otimes \ldots \otimes v_k \in F^g_{\mathcal{P}}(V)$ with $\mu \in \tilde{\mathcal{P}}(k)$ by*

$$\bar{\Delta}(x) = 1 \otimes x + x \otimes 1 + \sum_{i=1}^{k-1} (\mu_{(1)}|_{[i]} \otimes v_1 \ldots \otimes v_i) \otimes (\mu_{(2)}|_{i+[k-i]} \otimes v_{i+1} \ldots \otimes v_k).$$

Proof. When $\mathcal{P}$ is regular $F^g_{\mathcal{P}}(V) = \oplus_n \tilde{\mathcal{P}}(n) \otimes V^{\otimes_g n}$, hence it is enough to prove the formula (4.1) for

$$\bar{\Delta}\mu(\nu_1 \otimes \bar{v}_1, \ldots, \nu_k \otimes \bar{v}_k)$$

with $\mu, \nu_i \in \tilde{\mathcal{P}}$ and $\bar{v}_i \in V^{\otimes l_i}$. The computation is straightforward. □

4.2. Rigidity for twisted bialgebras

Loday and Ronco proved a theorem of rigidity for unital infinitesimal bialgebras. Recall from [17] that the fundamental example of a unital infinitesimal bialgebra is given by $T^{fc}(V) = F^g_{\mathcal{A}s}(V)$ where V is a graded vector space concentrated in degree 1 and where the product is given by the concatenation and the coproduct is given by the deconcatenation. Recall also that for a connected coalgebra C, with a coproduct Δ and a counit ϵ, the space of primitive elements is defined by

$$\mathrm{Prim}_{\Delta}(C) = \{x \in \mathrm{Ker}\,\epsilon | \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Here is the statement of the theorem:

4.2.1. **Theorem** [17]

Any connected unital infinitesimal Hopf bialgebra H is isomorphic to $T^{fc}(\mathrm{Prim}(H))$.

Let (A, m, Δ) be a connected twisted bialgebra and $\bar{A} = (A, \hat{m}, \bar{\Delta})$ the symmetrized bialgebra and $\hat{A} = (A, \bar{m}, \hat{\Delta})$ the cosymmetrized bialgebra as in paragraph 3.3.1. The triple $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra by theorem 4.1.2 and then, by theorem 4.2.1 is isomorphic to $T^{fc}(\mathrm{Prim}_{\bar{\Delta}}(A))$. Hence $\hat{A}$ is a free associative algebra and $\bar{A}$ is a cofree coassociative coalgebra. Assume furthermore that $\mathbf{k}$ is of characteristic 0 and Δ is cocommutative. Then $\hat{A}$ is a cocommutative Hopf algebra, and by the theorem of Cartier-Milnor-Moore, it is the universal enveloping algebra of its primitive elements. If each A_n is finite dimensional, since $\hat{A}$ is free as an associative algebra, by lemma 22 in [21] the space of primitive elements is a free Lie algebra.

These results are summed up in the following theorem:

4.2.2. **Theorem**

Let (A, m, Δ) be a connected twisted bialgebra. The associated symmetrized bialgebra $\bar{A}$ is a cofree coassociative algebra. The associated cosymmetrized bialgebra $\hat{A}$ is a free associative algebra.

If $\mathbf{k}$ is of characteristic 0, if Δ is cocommutative and if A_n is finite dimensional for all n, there is an isomorphism of Lie algebras

$$\mathrm{Prim}_{\hat{\Delta}}(A) = F^{g}_{Lie}(\mathrm{Prim}_{\bar{\Delta}(A)}).$$

This isomorphism is functorial in A. □

Using the results of Loday and Ronco we have improved the results of Patras and Reutenauer. Furthermore, if $\mathcal{P}$ is a connected multiplicative Hopf operad then it provides connected twisted bialgebras: indeed, any Hopf $\mathcal{P}$-algebra in **S-mod** is a twisted bialgebra. For instance $\mathcal{P}$ and more generally $F_{\mathcal{P}}(M)$ with M an $\mathbb{S}$-module such that $M(0) = 0$ are connected twisted bialgebras.

4.2.3. **Remark**

If (A, m, Δ) is a connected twisted bialgebra then $(A, \hat{m}, \bar{m}, \bar{\Delta})$ is a connected *2-associative bialgebra* in the terminology of Loday and Ronco in [17], that is $(A, \hat{m}, \bar{\Delta})$ is a Hopf algebra and $(A, \bar{m}, \bar{\Delta})$ is a unital infinitesimal bialgebra. By the structure theorem in [17], one gets that $\mathrm{Prim}_{\bar{\Delta}}(A)$ is a B_∞-algebra and A is the enveloping 2-as bialgebra of its primitive elements.

Assume $\mathcal{P}$ and V satisfy the conditions of theorem 4.1.3. Assume $\mathcal{P}$ is multiplicative and $A = F^g_{\mathcal{P}}(V)$ is finite dimentional in each degree. Then $A2 = (A^*, {}^t\Delta, {}^t\bar{\Delta}, {}^tm)$ where m is the associative product induced by the multiplicative structure of $\mathcal{P}$ is also a 2-associative bialgebra. If $A2$ is connected then it is the enveloping 2-as bialgebra of its primitive elements.

5. Application to combinatorial Hopf algebras

In this section, we would like to apply our previous results to combinatorial Hopf algebras. The idea is the following: given a graded vector space $H = \oplus_n H(n)$, how does a Hopf algebra structure arise on H? We present two cases coming from the two examples detailed in section 3.1.4.

Case 1. The space $H(n)$ is endowed with a right S_n-action. We denote by $H^{\mathbb{S}}$ the associated $\mathbb{S}$-module. Assume there exists a connected multiplicative Hopf operad structure on $\mathcal{P}_H = H^{\mathbb{S}}$. From section 3.1.4, we obtain our first result: there exists a $\mathcal{P}_H$-algebra product μ and a coalgebra coproduct Δ such that $(H^{\mathbb{S}}, \mu, \Delta)$ is a Hopf $\mathcal{P}_H$-algebra. The graded vector space H has a Hopf $\mathcal{P}_H$-algebra structure which is the symmetrized Hopf $\mathcal{P}_H$-algebra $\overline{H^{\mathbb{S}}}$ by theorem 3.1.3.

The second result is a direct consequence of theorem 4.2.2: since the operad $\mathcal{P}_H$ is multiplicative, there is a twisted product $m : H^{\mathbb{S}} \otimes H^{\mathbb{S}} \to H^{\mathbb{S}}$. As a consequence $(H^{\mathbb{S}}, m, \Delta)$ is a twisted bialgebra. The associated symmetrized Hopf algebra $(H, \hat{m}, \bar{\Delta})$ is cofree and the associated cosymmetrized Hopf algebra $\bar{H} = (H, \bar{m}, \hat{\Delta})$ is free. In case Δ is cocommutative, under the hypothesis of theorem 4.2.2, $\bar{H}$ is the enveloping algebra of the free Lie algebra generated by $\mathrm{Prim}_{\bar{\Delta}}(H)$. Furthermore, by remark 4.2.3 the 2-associative bialgebra $(H, \hat{m}, \bar{m}, \bar{\Delta})$ is the 2-associative enveloping bialgebra of its primitive elements: $\mathrm{Prim}_{\bar{\Delta}}(H)$ is endowed with a B_∞-structure.

Case 1 applies also when H is a free $\mathcal{P}$-algebra in **S-mod** generated by an $\mathbb{S}$-module M, with $\mathcal{P}$ a multiplicative Hopf operad and $M(0) = 0$.

Case 2. Assume $\mathcal{P}_H^r = \mathcal{S}H$ is a connected regular Hopf operad. The graded vector space H is the free graded $\mathcal{P}_H^r$-algebra generated by the graded vector space I. As a consequence, the graded vector space (H, μ, Δ) is a Hopf $\mathcal{P}_H^r$-algebra, where μ is the $\mathcal{P}_H^r$-product and where

$$\Delta(h) = \sum_{S \sqcup T = [|h|]} h_{(1)}|_S \otimes h_{(2)}|_T$$

comes from the regular Hopf operad $\mathcal{P}_H^r$. Also $(H, \mu, \bar{\Delta})$ with

$$\bar{\Delta}(h) = \sum_{i=0}^{|h|} h_{(1)}|_{[i]} \otimes h_{(2)}|_{i+[|h|-i]}$$

is a unital infinitesimal $\mathcal{P}_H^r$-bialgebra by theorem 4.1.3.

Again, by remark 4.2.3, if H_n is finite dimensional, then $(H^*, {}^t\Delta, {}^t\bar{\Delta}, {}^t m)$ in which m is the associative product, is a 2-associative bialgebra, which is the 2-associative enveloping bialgebra of its primitive elements.

Case 2 applies also when H is a free $\mathcal{P}$-algebra in **grVect** generated by a graded vector space V, with $\mathcal{P}$ a Hopf regular operad and $V(0) = 0$.

We illustrate by some examples that many combinatorial Hopf algebras arise either from case 1 or from case 2.

5.1. The Hopf algebra $T(V)$

Let us apply **Case 1** for $H = F_{\mathcal{C}\mathrm{om}}(V)$ where V is considered as an $\mathbb{S}$-module concentrated in degree 1. That is $H = T(V)$. As a twisted bialgebra, $T(V)$ is endowed with the concatenation product and with the following coproduct

$$\Delta(v_1 \otimes \ldots \otimes v_k) = \sum_{S \sqcup T = [k]} \bar{v}|_S \otimes \bar{v}|_T \otimes (S, T)$$

where $\bar{v}|_S = v_{s_1} \otimes \ldots \otimes v_{s_j}$ for $S = \{s_1 < \ldots < s_j\}$. It is cocommutative. The symmetrized Hopf algebra structure on $T(V)$ is the shuffle product together with the deconcatenation, whereas the cosymmetrized Hopf algebra structure on $T(V)$ is the dual structure: the product is the concatenation and the coproduct is the unshuffle coproduct. In characteristic 0 it is the enveloping algebra of the free Lie algebra generated by V.

5.2. **The Malvenuto-Reutenauer Hopf algebra**

This Hopf algebra, denoted H_{MR} has been extensively studied in [18], in [9] under the name of free quasisymmetric functions or in [1]. The graded vector space considered is $A = \oplus_n \mathbf{k}[S_n]$. It is the underlying graded vector space of the operad $\mathcal{A}s$ and **Case 1** applies. Recall that the operad $\mathcal{A}s$ gives rise to a cocommutative twisted bialgebra:

$$\begin{aligned} m(\sigma,\tau) =& \sigma \times \tau, \\ \Delta(\sigma) =& \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T \otimes (S,T). \end{aligned}$$

The Hopf algebra H_{MR} is the symmetrized Hopf algebra $(A, \hat{m}, \bar{\Delta})$. That is, for $\sigma \in S_n, \tau \in S_m$

$$\begin{aligned} \hat{m}(\sigma,\tau) =& \sum_{\xi \in \mathrm{Sh}_{p,q}} (\sigma \times \tau) \cdot \xi, \\ \bar{\Delta}(\sigma) =& \sum_{i=0}^{n} \mathrm{st}(\sigma_1, \ldots, \sigma_i) \otimes \mathrm{st}(\sigma_{i+1}, \ldots, \sigma_n). \end{aligned}$$

which is not commutative nor cocommutative.

The cosymmetrized Hopf algebra $\hat{A} = (A, \bar{m}, \hat{\Delta})$ is given by

$$\begin{aligned} \bar{m}(\sigma,\tau) =& \sigma \times \tau, \\ \hat{\Delta}(\sigma) =& \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T. \end{aligned}$$

The latter Hopf algebra is different from the former one or its dual since it is a cocommutative Hopf algebra.

From **Case 1** we get that H_{MR} is cofree and that $\hat{A}$ is free as an associative algebra: it is generated by the *connected permutations*, the ones which don't write $\sigma \times \tau$ for $\sigma \in S_n, \tau \in S_m$, $n, m > 0$. In characteristic

0, $\hat{A}$ is isomorphic to the enveloping algebra of the free Lie algebra generated by the connected permutations (compare with theorems 20 and 21 in [21]). Furthermore, H_{MR} together with $\bar{m}$ is a 2-associative bialgebra and it is isomorphic to the 2-associative enveloping bialgebra generated by the connected permutations: in [9] and in [1] a basis of the space of primitive elements of H_{MR}, indexed by the connected permutations is given.

In paragraph 5.3.4 we prove that H_{MR} is free as an associative algebra, without using the self-duality of H_{MR}.

5.3. Hopf algebra structures on the faces of the permutohedron

Recall that $\mathcal{C}$om is a Hopf operad. Let $\overline{\mathcal{C}\mathrm{om}}(n) = \begin{cases} \mathcal{C}\mathrm{om}(n) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$. The $\mathbb{S}$-module $\mathcal{C}\mathrm{omp} = \mathcal{A}\mathrm{s} \circ \overline{\mathcal{C}\mathrm{om}}$ has for linear basis the faces of the n-permutohedra. Indeed

$$\begin{aligned} \mathcal{A}\mathrm{s} \circ \overline{\mathcal{C}\mathrm{om}}(n) &= \oplus_{k \geq 0} \mathcal{A}\mathrm{s}(k) \otimes_{S_k} (\overline{\mathcal{C}\mathrm{om}})^{\otimes k}(n) \\ &= \sum_{(I_1, \ldots, I_k) = [n]} \mathbf{k} \end{aligned}$$

where the sum is taken over all *set compositions* (or *ordered set partitions*) $(I_1, \ldots, I_k)$ of $[n]$ such that $I_j \neq \emptyset, \forall 1 \leq j \leq k$. The action of the symmetric group is given, for $\sigma \in S_n$, by

$$(I_1, \ldots, I_k) \cdot \sigma = (\sigma^{-1}(I_1), \ldots, \sigma^{-1}(I_k)).$$

Chapoton described some Hopf algebra structures on the graded vector space $\mathcal{O}(\mathcal{C}\mathrm{omp})$ in [5] and in [6], whereas Patras and Schocker described a twisted bialgebra structure on $\mathcal{C}$omp in [22]. Chapoton described a (differential graded) operad structure on $\mathcal{C}$omp in [8] and Loday described a (filtered) one in [14].

The aim of this section is to apply our operadic point of view **Case 1** in order to relate these structures.

A set composition of $[n]$ is written as a word in the alphabet $\{,\} \cup \{i, 1 \leq i \leq n\}$. For instance $(14, 2, 35)$ is the set composition $(\{1,4\},\{2\},\{3,5\})$ of $[5]$.

5.3.1. **Operad structures on the faces of the permutohedron.**

Both operads built by Loday in [14] and Chapoton in [8] are quadratic binary operads. They are generated by the commutative operation represented by the set composition (12) and by the operation represented by the set composition $(1,2)$ in $\mathcal{C}\mathrm{omp}(2)$. Let $w_f = (12) + (1,2) + (2,1)$ and $w_g = (1,2) + (2,1)$. The composition in the operad $\mathcal{C}\mathrm{TD}$ described by Loday is given by the following inductive formula

$$\begin{matrix} (12)(\emptyset, P) = 0, & (12)(P,\emptyset) = 0, \\ (1,2)(\emptyset, P) = 0, & (1,2)(P,\emptyset) = P, \end{matrix}$$

$$\begin{aligned} (12)(P,Q) =& (P_1 \cup Q_1, w_f((P_2,\dots,P_k),(Q_2,\dots,Q_l))), \\ (1,2)(P,Q) =& (P_1, w_f((P_2,\dots,P_k),Q)), \end{aligned}$$

with $P = (P_1,\dots,P_k)$ a set composition of $[n]$ and $Q = (Q_1,\dots,Q_l)$ a set composition of $[m]$ considered as a set composition of $n+[m]$. By convention, $w_f(\emptyset,\emptyset) = 0$. The degree of the set composition P is $n-k$. Set compositions of degree 0 are in 1-to-1 correspondance with permutations.

The operad $\mathcal{C}\mathrm{TD}$ is filtered by the degree of set compositions but not graded. It is not regular and algebras (in the category of vector spaces) over this operad are named *commutative tridendriform algebras* by Loday, that is vector spaces endowed with a product $\prec$ and a commutative product $\cdot$ satisfying the relations

$$\begin{aligned} (x \prec y) \prec z =& x \prec (y \prec z + z \prec y + y \cdot z), \\ (x \cdot y) \prec z =& x \cdot (y \prec z), \\ (x \cdot y) \cdot z =& x \cdot (y \cdot z). \end{aligned}$$

The composition in the operad Π described by Chapoton has the same definition except that w_f is replaced by w_g. It is graded by the degree of set compositions. Algebras (in the category of graded vector spaces) over Π are described in [8].

These operads are not connected in the strict sense, since the composition with $\emptyset \in \mathcal{C}\mathrm{omp}(0)$ is not always defined. The equalities involving the emptyset above, are needed for an inductive definition and are needed in order to build a coproduct $\mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp} \otimes \mathcal{C}\mathrm{omp}$, as was explained in the paragraph 3.1.4 on connected operads.

5.3.2. **Proposition**

The $\mathbb{S}$*-module* $\mathcal{C}\mathrm{omp}$ *is a* $\mathcal{C}\mathrm{TD}$*-Hopf algebra for the coproduct*

$$\Delta(P_1,\ldots,P_k) = \sum_{l=0}^{k} \mathrm{st}(P_1,\ldots,P_l)\otimes \mathrm{st}(P_{l+1},\ldots,P_k)\otimes(\cup_{1\leq j\leq l}P_j,\cup_{l+1\leq h\leq k}P_h).$$

The $\mathbb{S}$*-module* $\mathcal{C}\mathrm{omp}$ *is a* Π*-Hopf algebra for the same coproduct.*

Proof. Let $\mathcal{X}$ denote either the operad $\mathcal{C}\mathrm{TD}$ or the operad Π. Let w denote either $w_f \in \mathcal{C}\mathrm{TD}(2)$ or $w_g \in \Pi(2)$. Note that for any set composition P

$$w(P,\emptyset) = w(\emptyset,P) = P. \tag{5.1}$$

One needs first to define the $\mathcal{X}$-algebra structure on $\mathcal{C}\mathrm{omp}\otimes\mathcal{C}\mathrm{omp}$. This trick is due to Loday and Ronco: for $x\in\mathcal{X}(2)$,

$$x(P_1\otimes P_2, Q_1\otimes Q_2) = \begin{cases}\emptyset\otimes x(P_2,Q_2), & \text{if } P_1=Q_1=\emptyset,\\ x(P_1,Q_1)\otimes w(P_2,Q_2)\otimes(P_1\cup Q_1, P_2\cup Q_2), & \text{otherwise.}\end{cases} \tag{5.2}$$

The coproduct $\Delta : \mathcal{C}\mathrm{omp}\to\mathcal{C}\mathrm{omp}\otimes\mathcal{C}\mathrm{omp}$ is the unique $\mathcal{X}$-algebra morphism mapping (1) to $(1)\otimes\emptyset+\emptyset\otimes(1)$.

Let $(I_{l_1},\ldots,I_{l_k})$ be the set composition of $[l_1+\ldots+l_k]$ defined by

$$I_{l_j} = l_1+\ldots+l_{j-1}+[l_j], \forall 1\leq j\leq k.$$

Let $n = l_1+\ldots+l_k$. We prove the formula for such a set composition by induction on k. If $k=1$, the set composition is just (n). For $n=1$ the formula is proved. If $n>1$ then $(n) = (12)((1),(n-1))$. By induction one has

$$\Delta(n) = (12)(\Delta(1),\Delta(n-1)) = (12)((1)\otimes\emptyset+\emptyset\otimes(1),(n-1)\otimes\emptyset+\emptyset\otimes(n-1)) = (n)\otimes\emptyset+\emptyset\otimes(n),$$

because $(12)(P,Q)=0$ if P or Q is empty and because of relation (5.2) and (5.1).

If $k > 1$, then $X = (I_{l_1}, \dots, I_{l_k}) = (1,2)((I_{l_1}), (I_{l_2}, \dots, I_{l_k}))$. By induction

$$\Delta(X) = (1,2)(\Delta(I_{l_1}), \Delta(I_{l_2}, \dots, I_{l_k})) =$$
$$(1,2)(I_{l_1} \otimes \emptyset + \emptyset \otimes I_{l_1}, \sum_{j=1}^{k} (I_{l_2}, \dots, I_{l_j}) \otimes (I_{l_{j+1}}, \dots, I_{l_k})) =$$
$$\emptyset \otimes X + \sum_{j=1}^{k} (I_{l_1}, I_{l_2}, \dots, I_{l_j}) \otimes (I_{l_{j+1}}, \dots, I_{l_k})),$$

because $(1,2)(P, \emptyset) = P$ and $(1,2)(\emptyset, P) = 0$ and because of relations (5.2) and (5.1).

For any set composition $P = (P_1, \dots, P_k)$ of $[n]$, there exists $\sigma \in S_n$ such that

$$(P_1, \dots, P_k) = (I_{l_1}, \dots, I_{l_k}) \cdot \sigma.$$

One can choose for σ the shuffle associated to the set composition P. The coproduct Δ is a morphism of $\mathbb{S}$-modules. One gets the conclusion with formula (1.1). $\square$

In [12] we proved that the space of primitive elements with respect to Δ is a suboperad of the initial operad. The space of primitive elements is clearly the vector space generated by the set compositions (n), for $n > 0$. Then $\mathrm{Prim}_\Delta(\mathcal{C}\mathrm{TD})$ is the operad $\mathcal{C}\mathrm{om}$ (compare with [14]).

5.3.3. Twisted bialgebras associated to the faces of the permutohedron

The operation w_f (resp. w_g) is associative and commutative. As a consequence, the operads $\mathcal{C}\mathrm{TD}$ and Π are Hopf multiplicative operads and give rise to twisted connected commutative (non cocommutative) bialgebras $H_f = (\mathcal{C}\mathrm{omp}, w_f, \Delta)$ and $H_g = (\mathcal{C}\mathrm{omp}, w_g, \Delta)$.

i) The twisted bialgebra H_f. Patras and Schocker [22] defined a twisted bialgebra structure on $\mathcal{C}\mathrm{omp}$ denoted $\mathcal{T} = (\mathcal{C}\mathrm{omp}, \star, \delta)$ which is the following. The product $\star$ is the concatenation of set compositions and the coproduct δ is defined for a set composition P of $[n]$ by

$$\delta(P) = \sum_{S \sqcup T = [n]} \mathrm{st}(P \cap S) \otimes \mathrm{st}(P \cap T) \otimes (S, T),$$

where $(P_1, \dots, P_k) \cap S = (P_1 \cap S, \dots, P_k \cap S)$ and if $P_i \cap S = \emptyset$ the i-th term is omitted. For instance $(14, 2, 35) \cap \{1, 3, 5\} = (1, 35)$. It is clear that $\star$ is the dual of Δ and one can check that δ is the dual of w_f. It gives an operadic interpretation of their structure:

The dual of the twisted bialgebra defined by Patras and Schocker is the free commutative tridendriform algebra on one generator in the category **S-mod**.

Applying **Case 1** one gets that the symmetrized Hopf algebra $\bar{\mathcal{T}}$ associated to $\mathcal{T}$ is cofree, and that the associated cosymmetrized Hopf algebra $\hat{\mathcal{T}}$ is free generated by *reduced set compositions*: a set composition which is non reduced is the concatenation of two non trivial set compositions. For instance $(13, 24, 6, 5)$ is non reduced since it is the concatenation of $(13, 24)$ and $(2, 1)$. Moreover if the field $\mathbf{k}$ is of characteristic 0, then $\hat{\mathcal{T}}$ is isomorphic to the enveloping algebra of the free Lie algebra generated by reduced set compositions. (Compare with proposition 10 and corollary 13 in [22]). Applying remark 4.2.3 one gets that $(\mathcal{T}, \bar{\star}, \hat{\star}, \bar{\delta})$ is the 2-associative enveloping bialgebra on its primitive elements.

The Hopf algebra structure given by Chapoton in [5] is $(\mathcal{C}\text{omp}, \bar{w}_f, \hat{\Delta})$ which is the dual of the symmetrized Hopf algebra $(\mathcal{T}, \hat{\star}, \bar{\delta})$. It is also the Hopf algebra *NCQSym* of Bergeron et Zabrocky in [4] and we recover that it is a free algebra.

ii) The twisted bialgebra H_g. This twisted bialgebra gives rise to two Hopf algebras, which are $(\mathcal{C}\text{omp}, \hat{w}_g, \bar{\Delta})$ and $(\mathcal{C}\text{omp}, \bar{w}_g, \hat{\Delta})$. One can check that the latter Hopf algebra is the one described by Chapoton in [6]. Again it is a free associative algebra because $(\mathcal{C}\text{omp}, \bar{w}_g, \bar{\Delta})$ is a unital infinitesimal bialgebra. The space of primitive elements $\text{Prim}_{\bar{\Delta}}(\mathcal{C}\text{omp})$ is generated by reduced set compositions. One can check by an inductive argument, that the Hopf algebra described by Chapoton is a free associative algebra generated by reduced set compositions.

5.3.4. From set compositions to permutations.

Let $\mathcal{C}\text{omp}_0$ be the sub $\mathbb{S}$-module of $\mathcal{C}\text{omp}$ of set compositions of degree 0. The vector space $\mathcal{C}\text{omp}_0(n)$ is isomorphic to $\mathbf{k}[S_n]$ but the right S_n-action is given by $\sigma \cdot \tau = \tau^{-1}\sigma$. The $\mathbb{S}$-module $\mathcal{C}\text{omp}_0$ is a sub-operad of Π. It is the operad $\mathcal{Z}\text{in}$, as noticed by Chapoton in [8]. In the category of vector spaces, an algebra over $\mathcal{Z}\text{in}$ is a Zinbiel algebra, that is, a vector space Z

together with a product $\prec$ satisfying the relation

$$(x \prec y) \prec z = x \prec (y \prec z + z \prec y), \ \forall x, y, z \in Z.$$

As a consequence there are surjective morphisms of Hopf operads

$$\mathcal{C}\mathrm{TD} \to \mathcal{Z}\mathrm{in}, \qquad \Pi \to \mathcal{Z}\mathrm{in}\,.$$

The operad $\mathcal{Z}$in is consequently a multiplicative operad and $\mathcal{C}\mathrm{omp}_0$ is a commutative twisted bialgebra. The product and coproduct are given, for $\sigma \in \mathcal{C}\mathrm{omp}_0(p)$ and $\tau \in \mathcal{C}\mathrm{omp}_0(q)$ by

$$m_{\mathcal{Z}}(\sigma, \tau) = \sum_{x \in \mathrm{Sh}_{p,q}} (\sigma \times \tau) x$$

$$\Delta_{\mathcal{Z}}(\sigma) = \sum_{i=0}^{p} \sigma|_{[i]} \otimes \sigma|_{i+[p-i]} \cdot (\cup_{1 \le j \le i} \{\sigma(j)\}, \cup_{i+1 \le k \le p} \{\sigma(k)\})$$

The morphisms above induce surjective morphisms of twisted bialgebras

$$H_f \to \mathcal{C}\mathrm{omp}_0, \qquad H_g \to \mathcal{C}\mathrm{omp}_0\,.$$

The cosymmetrized Hopf algebra associated to $\mathcal{C}\mathrm{omp}_0$ is clearly H_{MR}, and since it is a cosymmetrized algebra associated to a twisted bialgebra it is free on $\mathrm{Prim}_{\bar{\Delta}_{\mathcal{Z}}}(\mathcal{C}\mathrm{omp}_0)$. But $\bar{\Delta}_{\mathcal{Z}}(\sigma) = \sum_{\rho \times \tau = \sigma} \rho \otimes \tau$. As a consequence, H_{MR} is free generated by the connected permutations and cofree (see section 5.2). We recover the results obtained in e.g. [23], [9] and [1].

Considering the graded linear duals, one has an embedding of cocommutative twisted bialgebras

$$(\mathcal{C}\mathrm{omp}_0)^* \hookrightarrow (H_f)^* = \mathcal{T}.$$

The symmetrized Hopf algebra associated to $(\mathcal{C}\mathrm{omp}_0)^*$ is the dual of the Malvenuto-Reutenauer Hopf algebra H^*_{MR} (which is isomorphic to H_{MR}). By functoriality in theorem 4.2.2, we obtain in characteristic 0 an embedding of enveloping algebras at the level of associated cosymmetrized algebras (compare with Theorem 17 in [22]).

5.4. Hopf algebra structures on the faces of the associahedron

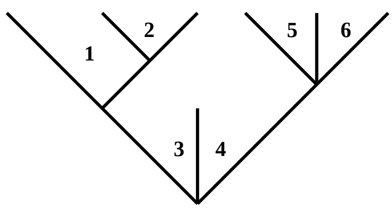
In his thesis, Chapoton considered various Hopf algebra structures on the faces of the associahedra, or Stasheff polytopes, filtered in [5], graded in [6]. He considered also filtered and graded operad structures on these objects in [7]. The filtered operadic structure coincides with the one defined by Loday and Ronco in [16], under the name of *tridendriform operad*. In this section, we apply **Case 2** to obtain Hopf algebra structures on the faces of the associahedra.

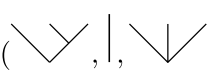
5.4.1. **Planar trees**

The set of planar trees with $n+1$ leaves is denoted by T_n. The set $T_n = \cup_{k=0}^{n-1} T_{n,k}$ is graded by k where $n-k$ is the number of internal vertices. For instance $T_{n,0}$ is the set of planar binary trees. The Stasheff polytope of dimension $n-1$ has its faces of dimension $0 \leq k \leq n-1$ indexed by $T_{n,k}$. The aim of this section is to provide the vector space $\oplus\, \mathbf{k}[T_n]$ with Hopf structures.

Given some planar trees $t_1, \dots, t_k$ the planar tree $\vee(t_1, \dots, t_k)$ is the one obtained by joining the roots of the trees $t_1, \dots, t_k$ to an extra root, from left to right. If t_i has degree l_i then $\vee(t_1, \dots, t_k)$ has degree $l_1 + \dots + l_k + k - 1$.

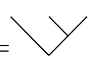
One can label the n sectors delimited by a tree t in T_n from left to right as in the following example:

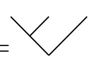
$t =$ 

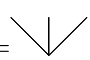
$= \vee($  $).$

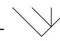
5.4.2. **The operad of tridendriform algebras** [16], [7]

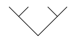
The operad $\mathcal{T}$riDend is a regular operad whose underlying $\mathbb{S}$-module is $\mathcal{T}\mathrm{riDend}(n) = \mathcal{S}\,\mathbf{k}[T_n]$. It is a quadratic binary operad generated by 3 operations

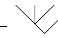
$\prec :=$  ,

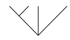
$\succ :=$  ,

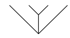
$\cdot :=$  , satisfying the relations

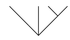
$\left\{\begin{array}{l}(x \prec y) \prec z = x \prec (y * z) =\end{array}\right.$  $+$  $+$ ,

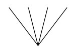
$(x \succ y) \prec z = x \succ (y \prec z) =$ ,

$(x * y) \succ z = x \succ (y \succ z) =$  $+$  $+$ ,

$\left\{\begin{array}{l}(x \succ y) \cdot z = x \succ (y \cdot z) =\end{array}\right.$ ,

$(x \prec z) \cdot z = x \cdot (y \succ z) =$ ,

$(x \cdot y) \prec z = x \cdot (y \prec z) =$ ,

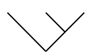
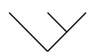
$\left\{(x \cdot y) \cdot z = x \cdot (y \cdot z) =\right.$ ,

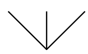
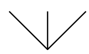
where

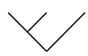
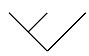
$$x * y = x \cdot y + x \prec y + x \succ y \tag{5.3}$$

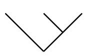
is associative.

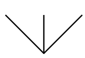
One can also give an inductive formula for the composition in $\mathcal{T}$riDend by the following

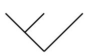
 $(|, y) = y,$  $(x, |) = 0,$

 $(|, y) = 0,$  $(x, |) = 0,$

 $(|, y) = 0,$  $(x, |) = x,$

 $(\vee(x_1, \dots, x_k), y) = \vee\, (x_1, \dots, x_{k-1}, x_k * y),$

 $(\vee(x_1, \dots, x_k), \vee(y_1, \dots, y_l)) = \vee\, (x_1, \dots, x_{k-1}, x_k * y_1, y_2, \dots, y_l),$

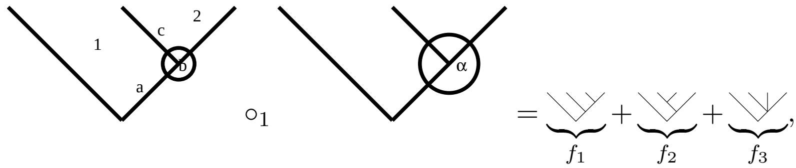
 $(x, \vee(y_1, \dots, y_l)) = \vee\, (x * y_1, y_2, \dots, y_l).$

In [7], Chapoton describes the composition $x \circ_i y$ for trees x and y:

$$x \circ_i y = \sum_{(f_l, f_r)} x \circ_i^{(f_l, f_r)} y,$$

where y is inserted in the sector i of x following the maps (f_l, f_r): the left (right) edge of the sector i of x is a set of edges and vertices ordered from bottom to top and denoted by x_l^i, (x_r^i). The left (right) most edge of y has several vertices: the ordered set of these vertices is denoted by y_l (y_r).

The map f_l (f_r) is an increasing map from y_l (y_r) to x_l^i (x_r^i). For instance



where

$$x_l^1 = y^l = \emptyset,$$
$$x_r^1 = \{a < b < c\},\ y_r = \{\alpha\},$$
$$f_1 = (\mathrm{Id}, \alpha \mapsto a),\ f_2 = (\mathrm{Id}, \alpha \mapsto c) \ \text{ and } \ f_3 = (\mathrm{Id}, \alpha \mapsto b).$$

Moreover one can define a Hopf structure on this operad following the same pattern than proposition 5.3.2: for $x \in \mathcal{T}\mathrm{riDend}(2)$,

$$x(t_1 \otimes t_2, s_1 \otimes s_2) = \begin{cases} | \otimes x(t_2, s_2), & \text{if } t_1 = s_1 = |, \\ x(t_1, s_1) \otimes t_2 * s_2, & \text{otherwise.} \end{cases}$$

By induction on the degree of a tree t one can prove that

5.4.3. **Proposition**

The $\mathbb{S}$-module $(\mathbf{k}[T_n] \otimes \mathbf{k}[S_n])_n$ is a tridendriform Hopf algebra for the coproduct

$$\Delta(\vee(t_1, \dots, t_k)) = | \otimes \vee(t_1, \dots, t_k) +$$
$$\sum_{S_i \sqcup T_i = [l_i]} \vee(t_{1(1)}^{S_1}, \dots, t_{k(1)}^{S_k}) \otimes t_{1(2)}^{T_1} * \dots * t_{k(2)}^{T_k} \otimes (S_1 \bar{\times} \dots \bar{\times} S_k, T_1 \bar{\times} \dots \bar{\times} T_k)$$

where $|t_i| = l_i$,

$$\Delta(t_i) = \sum_{S_i \sqcup T_i = [l_i]} t_{i(1)}^{S_i} \otimes t_{i(2)}^{T_i} \otimes (S_i, T_i)$$

and for any $U_i \subset [l_i]$,

$U_1 \bar{\times} \dots \bar{\times} U_k = \{l_1 + \dots + l_i + i, 1 \leq i \leq k-1\} \cup_{1 \leq i \leq k} U_i + l_1 + \dots + l_{i-1} + i - 1.$ □

5.4.4. **Hopf structures**

Let us apply **Case 2** to the connected Hopf operad $\mathcal{T}$riDend which is multiplicative with the product $*$ introduced in equation (5.3).

The graded vector space $\mathcal{T}\mathrm{ree} = \oplus_n \mathbf{k}[T_n]$ is the free graded tridendriform algebra over one generator, hence it is a Hopf tridendriform algebra in **grVect**. Let μ be the tridendriform product and Δ be the coproduct. By theorem 4.1.3, $(\mathcal{T}\mathrm{ree}, \mu, \bar{\Delta})$ is a unital tridendriform bialgebra. The description of Δ gives the description of $\bar{\Delta}$: if $\bar{\Delta}(t_k) = \sum t_{k(a)} \otimes t_{k(b)}$, then

$$\bar{\Delta}(\vee(t_1, \ldots, t_k)) = | \otimes \vee(t_1, \ldots, t_k) + \sum \vee(t_1, \ldots, t_{k-1}, t_{k(a)}) \otimes t_{k(b)}.$$

As a consequence a basis of $\mathrm{Prim}_{\bar{\Delta}}(\mathcal{T}\mathrm{ree})$ is given by the planar trees of type $\vee(t_1, \ldots, t_{k-1}, |)$.

The Hopf algebra $(\mathcal{T}\mathrm{ree}, *, \Delta)$ *is the free associative algebra spanned by the set of trees of the form* $\vee(t_1, \ldots, t_{k-1}, |)$.

Recall that the product $*$ is defined by induction: for $x = \vee(x_1, \ldots, x_k)$ and $y = \vee(y_1, \ldots, y_l)$

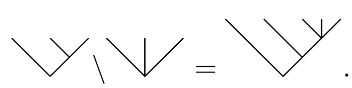
$$x * y = \vee(x_1, \ldots, x_{k-1}, x_k * y) + \vee(x * y_1, y_2, \ldots, y_l) + \vee(x_1, \ldots, x_{k-1}, x_k * y_1, y_2, \ldots, y_l).$$

and

$$\Delta(t) = | \otimes t + \sum \vee(t_{1(1)}, \ldots, t_{k(1)}) \otimes t_{1(2)} * \ldots * t_{k(2)}.$$

The Hopf structure defined by Chapoton in [5] is essentially the same: the product is the same and the coproduct is $\tau\Delta$, where τ is the symmetry isomorphism. Hence it is a free associative algebra spanned by the set of trees of the form $\vee(|, t_2, \ldots, t_k)$.

The graded linear dual of $\mathcal{T}$ree is a 2-associative bialgebra: it is free as an associative algebra for the product ${}^t\bar{\Delta}$, cofree as a coalgebra for ${}^t*$ and it is the enveloping 2-as bialgebra of its primitive elements (with respect to ${}^t*$). The product of two trees ${}^t\bar{\Delta}(t, s)$ is the tree obtained by gluing the tree s on the right most leave of t. it is usually denoted $t \backslash s$. For instance



5.4.5. **Some operad morphisms**

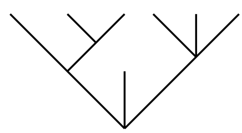
There are morphisms of Hopf operads

$$\begin{array}{ccc} \mathcal{T}\text{riDend} & \xrightarrow{\psi} & \mathcal{C}\text{TD} \\ {\scriptstyle \pi_{\mathcal{T}\text{riDend}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}\text{TD}}} \\ \mathcal{D}\text{end} & \xrightarrow{\psi_0} & \mathcal{Z}\text{in} \end{array} \tag{5.4}$$

The vertical maps are projection onto cells of degree 0. The map $\pi_{\mathcal{C}\text{TD}}$ has been explained in paragraph 5.3.4. The map $\pi_{\mathcal{T}\text{riDend}}$ is the projection onto the dendriform operad, which is a regular operad generated by planar binary trees (see e.g. [15]). The morphism ψ sends $\prec$ to the set composition $(1,2)$, $\succ$ to the set composition $(2,1)$ and $\cdot$ to the set composition (12). Indeed we can describe ψ at the level of trees. There is a map ϕ from set compositions to trees, described by induction as follows. Let $P = (P_1,\dots,P_k)$ be a set composition of $[n]$. If $P_1 = \{l_1 < \dots < l_j\}$ then it splits $[n]$ into $j+1$ intervals I_s possibly empty: for $0 \le s \le j,\ I_s =]l_s, l_{s+1}[$ with $l_0 = 0$ and $l_{j+1} = n+1$. The map ϕ is defined by

$$\begin{cases} \phi(\emptyset) = & |, \\ \phi(P) = & \vee(\phi(P \cap I_0),\dots,\phi(P \cap I_j)). \end{cases}$$

For instance if $P = (34,1,56,2)$ then $I_0 = \{1,2\}, I_1 = \emptyset$ and $I_2 = \{5,6\}$ and

$$\phi(34,1,56,2) = \vee(\phi(1,2),\phi(\emptyset),\phi(12)) =$$  .

Note that the function θ from the permutohedra to the associahedra defined by Tonks in [25] (see also [5] and [22]) satisfies

$$\theta(P_1,\dots,P_k) = \phi(P_k,P_{k-1},\dots,P_2,P_1).$$

The morphism ψ is the transpose of ϕ. It is an operad morphism, whereas the transpose of θ is not an operad morphism.

The morphism ψ_0 is the transpose of ϕ_0 which is an operad morphism. Loday and Ronco defined in [15] a function from $\mathbf{k}[Y_n]$ (the vector space generated by planar binary trees with n vertices) to $\mathbf{k}[S_n]$ in order to embed $\mathbf{k}[Y_n]$ as a Hopf subalgebra of the (graded linear dual of the)

Malvenuto-Reutenauer Hopf algebra. It is also a transpose of a set morphism $S_n \to Y_n$. If $\alpha : S_n \to S_n$ is the involution defined by

$$\alpha(\sigma_1, \ldots, \sigma_n) = (\sigma_n, \ldots, \sigma_1)^{-1},$$

then the set morphism defined by Loday and Ronco is $\phi_0\alpha$.

5.4.6. **Consequences on Hopf algebra morphisms**

The operad $\mathcal{T}$riDend is a regular operad. A *tridendriform 2-bialgebra* is a 4-uple (H, μ, Δ, δ) where (H, μ, Δ) is a Hopf tridendriform bialgebra in **grVect** and (H, μ, δ) is a unital infinitesimal tridendriform bialgebra. The diagram (5.4) is a diagram of tridendriform 2-bialgebras:

- The operad $\mathcal{T}$riDend induces the tridendriform 2-bialgebra

  $$(\mathcal{T}\text{ree} := \oplus_n \mathbf{k}[T_n], \mu_T, \Delta_T, \bar{\Delta}_T)$$

  explained in paragraph 5.4.4.

- Using the surjective operad morphism $\Pi_{\mathcal{T}\text{riDend}} : \mathcal{T}\text{riDend} \to \mathcal{D}\text{end}$ one gets the tridendriform 2-bialgebra structure on the vector space spanned by planar binary trees denoted by

  $$(PBT := \oplus_n \mathbf{k}[Y_n], \mu_Y, \Delta_Y, \bar{\Delta}_Y),$$

  where Y_n is the set of planar binary trees with $n+1$ leaves as in [15].

- The underlying $\mathbb{S}$-modules of the operads $\mathcal{C}$TD and $\mathcal{Z}$in are tridendriform algebras in **S-mod** then by theorems 3.1.3 and 4.1.2 one gets tridendriform 2-bialgebras on the underlying graded vector spaces. These structures are denoted respectively

  $$(\mathcal{C}\text{omp}, \mu_C, \Delta_C, \bar{\Delta}_C) \text{ and } (\oplus_n \mathbf{k}[S_n], \mu_S, \Delta_S, \bar{\Delta}_S).$$

As a consequence we obtain a diagram of Hopf algebras (and unital infinitesimal bialgebras as well):

$$\begin{array}{ccc}
(\oplus_n \mathbf{k}[T_n], *_T, \Delta_T) & \xrightarrow{\psi} & (\mathcal{C}\mathrm{omp}, *_C, \Delta_C) = NCQSym \\
\downarrow \pi_{\mathcal{T}\mathrm{riDend}} & & \downarrow \pi_{\mathcal{C}\mathrm{TD}} \\
(\oplus_n \mathbf{k}[Y_n], *_Y, \Delta_Y) & \xrightarrow{\psi_0} & (\oplus_n \mathbf{k}[S_n], *_S, \Delta_S) = H_{MR}
\end{array}$$

where the horizontal arrows are injective morphisms of Hopf algebras and vertical arrows are surjective morphisms of Hopf algebras. Note that the Hopf algebra structure on the planar binary tree $(\oplus_n \mathbf{k}[Y_n], *_Y, \tau\Delta_Y)$ where τ is the symmetry isomorphism is the one described by Loday and Ronco in [15]. Note also that the graded linear dual of this diagram is a diagram of 2-associative bialgebras. It extends the results obtained by Palacios and Ronco in [20].

5.4.7. **Conclusion**

For the last decade, many results of freeness and cofreeness of combinatorial Hopf algebras have appeared in the litterature (see the references cited throughout the paper and recently [2], [10], [19]). The present paper illustrates that these freeness results are a consequence of an operadic structure on the Hopf algebra H itself or its symmetrization $\mathcal{S}H$. Namely, either the Hopf algebra H is an $\mathbb{S}$-module and one can find an operad structure on H in order to apply **Case 1**; or the Hopf algebra is not an $\mathbb{S}$-module and one can find an operad structure on $\mathcal{S}H$ in order to apply **Case 2**.

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Muriel Livernet
Université Paris 13,
CNRS, UMR 7539 LAGA,
99, Avenue Jean-Baptiste Clément,
F-93430 Villetaneuse, France.
livernet@math.univ-paris13.fr