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Existence of strong solutions for nonisothermal Korteweg system

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Abstract

This work is devoted to the study of the initial boundary value problem for a general non isothermal model of capillary fluids derived by J. E Dunn and J. Serrin (1985) in [9, 16], which can be used as a phase transition model.
We distinguish two cases, when the physical coefficients depend only on the density, and the general case. In the first case we can work in critical scaling spaces, and we prove global existence of solution and uniqueness for data close to a stable equilibrium. For general data, existence and uniqueness is stated on a short time interval.
In the general case with physical coefficients depending on density and on temperature, additional regularity is required to control the temperature in L^∞ norm. We prove global existence of solution close to a stable equilibrium and local in time existence of solution with more general data. Uniqueness is also obtained.

Existence de solutions fortes pour le système de Korteweg

Résumé

Ce travail est consacré à l'étude d'un modèle de fluide compressible non isotherme avec un terme de capillarité dérivé par J. E Dunn et J. Serrin (1985) dans [9, 16], qui peut-être utilisé comme un modèle de transition de phase.
Nous distinguons deux cas en fonction que les coefficients physiques ne dépendent que de la densité ou non. Dans le premier cas nous travaillons dans des espaces critiques et prouvons l'existence de solutions fortes proches d'un état d'équilibre. Pour des données grandes on montre l'existence et l'unicité de solutions en temps fini.
Dans le cas général où les coefficients physiques dépendent à la fois de la densité et de la température, des données initiales plus régulières sont nécessaires afin de contrôler la norme L^∞ de la température. Nous prouvons alors l'existence de solutions fortes globales avec données petites ainsi que l'existence de solutions fortes avec données initiales grandes sur un intervalle de temps fini.

Keywords: PDE, Harmonic analysis.
Math. classification: 76N10, 35D05, 35Q05.

1. **Introduction**

1.1. **Derivation of the Korteweg model**

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by van der Waals and Korteweg [14] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [13, 17]. Let us consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$ $(N \geq 2)$, entropy density e, and temperature $\theta = (\frac{\partial e}{\partial s})_\rho$. Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. We note $w = \nabla\rho$ and we suppose that e depends on the density ρ, on the entropy specific s, and on w. In terms of the free energy, this principle takes the form of a generalized Gibbs relation :

$$de = \widetilde{T}ds + \frac{p}{\rho^2}d\rho + \frac{1}{\rho}\phi^* \cdot dw$$

where $\widetilde{T}$ is the temperature, p the pressure, ϕ a vector column of $\mathbb{R}^N$ and ϕ^* the adjoint vector and w stands for $\nabla\rho$. In the same way we can write a differential equation for the intern energy per unit volume, $E = \rho e$,

$$dE = \widetilde{T}dS + gd\rho + \phi^* \cdot dw$$

where $S = \rho s$ is the entropy per unit volume and $g = e - s\widetilde{T} + \frac{p}{\rho}$ is the chemical potential. In terms of the free energy, the Gibbs principle gives us:

$$dF = -Sd\widetilde{T} + gd\rho + \phi^* \cdot dw.$$

In the present paper, we shall make the hypothesis that: $\phi = \kappa w$. The nonnegative coefficient κ is called the capillarity and may depend on both ρ and $\widetilde{T}$. All the thermodynamic quantities are sum of their classic version (it means independent of w) and of one term in $|w|^2$. In this case the free energy F decomposes into a standard part F_0 and an additional term due to gradients of density:

$$F = F_0 + \frac{1}{2}\kappa|w|^2.$$

We denote $v = \frac{1}{\rho}$ the specific volume and $k = v\kappa$. Similar decompositions hold for S, p and g:

$$p = p_0 - \frac{1}{2}K_p|w|^2 \quad \text{where: } K_p = k_v^{'} \text{ and } p_0 = -(f_0)_v^{'}$$

$$g = g_0 + \frac{1}{2}K_g|w|^2 \quad \text{where: } K_e = k - \widetilde{T}k_{\widetilde{T}}^{'} \text{ and } e_0 = f_0 - \widetilde{T}(f_0)_{\widetilde{T}}^{'}.$$

The model deriving from a Cahn-Hilliard like free energy (see the pioneering work by J. E. Dunn and J. Serrin in [9] and also in [1, 4, 10]), the conservation of mass, momentum and energy read:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = \operatorname{div}(K + D) + \rho f, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \operatorname{div}(u(\rho e + \frac{1}{2}\rho|u|^2 + p)) = \operatorname{div}((D + K)\cdot u - Q + W) \\ \hfill + \rho f \cdot u, \end{cases}$$

with:

$$D = (\lambda \operatorname{div} u)I + \mu({}^t\nabla u + \nabla u), \text{ is the diffusion tensor}$$

$$K = (\rho \operatorname{div}\phi)I - \phi w^*, \text{ is the Korteweg tensor}$$

$$Q = -\eta\nabla\widetilde{T}, \text{ is the heat flux.}$$

The term

$$W = (\partial_t\rho + u^* \cdot \nabla\rho)\phi = -(\rho \operatorname{div} u)\phi$$

is the intersticial work which is needed in order to ensure the entropy balance and was first introduced by Dunn and Serrin in [9].
The coefficients (λ, μ) represent the viscosity of the fluid and may depend on both the density ρ and the temperature $\widetilde{T}$. The thermal coefficient η is a given non negative function of the temperature $\widetilde{T}$ and of the density ρ.
Differentiating formally the equation of conservation of the mass, we obtain a law of conservation for w:

$$\partial_t w + \operatorname{div}(uw^* + \rho du) = 0\ .$$

One may obtain an equation for e by using the mass and momentum conservation laws and the relations:

$$\operatorname{div}((-pI + K + D)u) = (\operatorname{div}(-pI + K + D))\cdot u - p\operatorname{div}(u) + (K + D) : \nabla u\ .$$

Multiplying the momentum equation by u yields:

$$(\operatorname{div}(-pI + K + D))\cdot u = (\partial_t(\rho u) + \operatorname{div}(\rho u u^*))\cdot u = \partial_t(\frac{\rho|u|^2}{2}) + \operatorname{div}(\frac{\rho|u|^2}{2}u)\ .$$

We obtain then:

$$\rho(\partial_t e + u^* \cdot \nabla e) + p\,\mathrm{div}u = (K + D) : \nabla u + \mathrm{div}(W - Q)\ .$$

By substituting K, we have (with the summation convention over repeated indices):

$$K : \nabla u = \rho\,\mathrm{div}\phi\,\mathrm{div}u - \phi_i w_j \partial_i u_j\ ,$$

while:

$$-\mathrm{div}W = \mathrm{div}((\rho\mathrm{div}u)\phi) = \rho(\mathrm{div}\phi)(\mathrm{div}u) + (w^* \cdot \phi)\mathrm{div}u + \phi_i\,\rho\,\partial^2_{j,i}u_j.$$

By using $w_j = \partial_j \rho$, we obtain:

$$\begin{aligned} K : \nabla u - \mathrm{div}W &= -(w^* \cdot \phi)\mathrm{div}u - \phi_i \partial_j(\rho \partial_i u_j) \\ &= -(w^* \cdot \phi)\mathrm{div}u - (\mathrm{div}(\rho du)) \cdot \phi. \end{aligned}$$

Finally, the equation for e rewrites:

$$\rho(\partial_t e + u^* \cdot \nabla e) + (p + w^* \cdot \phi)\mathrm{div}u = D : \nabla u - (\mathrm{div}(\rho du)) \cdot \phi - \mathrm{div}Q\ .$$

From now on, we shall denote: $d_t = \partial_t + u^* \cdot \nabla$.

1.2. The case of a generalized Van der Waals law

From now on, we assume that there exist two functions Π_0 and Π_1 such that:

$$\begin{aligned} p_0 &= \widetilde{T}\Pi_1^{'}(v) + \Pi_0^{'}(v), \\ e_0 &= -\Pi_0(v) + \varphi(\widetilde{T}) - \widetilde{T}\varphi^{'}(\widetilde{T}), \end{aligned}$$

and that the capillarity κ doesn't depend on the temperature. Moreover we suppose that the intern specific energy is an increasing function of $\widetilde{T}$:

$$(A) \qquad \Psi^{'}(\widetilde{T}) > 0 \text{ with } \Psi(\widetilde{T}) = \varphi(\widetilde{T}) - \widetilde{T}\varphi^{'}(\widetilde{T}).$$

We then set $\theta = \Psi(\widetilde{T})$ and we search to obtain an equation on θ.

Obtaining an equation for θ :

As:

$$e = -\Pi_0(v) + \theta + \frac{1}{2}\kappa|w|^2,$$

we thus have:

$$d_t e = -\Pi_0^{'}(v)d_t v + d_t\theta + \frac{1}{2}\kappa_v^{'}|w|^2 d_t v + \kappa w^* \cdot d_t w\ .$$

By a direct calculus we find:

$$d_t v = v\mathrm{div}u \quad \text{and} \quad w^* \cdot d_t w = -|w|^2 \mathrm{div}u - \mathrm{div}(\rho\, du)\cdot w\ .$$

Then we have:

$$d_t\theta = d_t e + v(p - \widetilde{T}\Pi_1^{'}(v))\mathrm{div}u + \kappa|w|^2\mathrm{div}u + \kappa\mathrm{div}(\rho du)\cdot w.$$

And by using the third equation of the system, we get an equation on θ:

$$d_t\theta + v\mathrm{div}Q + v\widetilde{T}\Pi_1^{'}(v)\mathrm{div}u = vD : \nabla u + \mathrm{div}(\rho du)\cdot(\kappa w - v\phi) + (\kappa|w|^2 - vw^*\cdot\phi)\mathrm{div}u\ .$$

But as we have $\phi = \kappa w$ and $k = v\kappa$ we conclude that:

$$d_t\theta - v\mathrm{div}(\chi\nabla\theta) + v\Psi^{-1}(\theta)\Pi_1^{'}(v)\mathrm{div}u = vD : \nabla u$$

with: $\chi(\rho,\theta) = \eta(\rho,\widetilde{T})(\Psi^{-1})^{'}(\theta)$.

Obtaining a system for ρ, u, θ:

We obtain then for the momentum equation:

$$d_t u - \frac{\mathrm{div}D}{\rho} + \frac{\nabla p_0}{\rho} = \frac{\mathrm{div}\mathrm{K}}{\rho} + \frac{1}{2}\frac{\nabla(K_p|w|^2)}{\rho} \quad \text{where } K_p = \kappa - \rho\kappa_\rho^{'}.$$

And by a calculus we check that:

$$\mathrm{div}K + \frac{1}{2}\nabla(K_p|w|^2) = \rho\nabla(\kappa\Delta\rho) + \frac{\rho}{2}\nabla(\kappa_\rho^{'}|\nabla\rho|^2).$$

Finally we have obtained the following system:

$$(NK)\quad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u - \dfrac{\mathrm{div}D}{\rho} - \nabla(\kappa\Delta\rho) + \dfrac{\nabla(P_0(\rho) + \Psi^{-1}(\theta)P_1(\rho))}{\rho} \\ \qquad\qquad = \nabla(\dfrac{\kappa_\rho^{'}}{2}|\nabla\rho|^2), \\ \partial_t\theta + u\cdot\nabla\theta - \dfrac{\mathrm{div}(\chi\nabla\theta)}{\rho} + \Psi^{-1}(\theta)\dfrac{P_1(\rho)}{\rho}\mathrm{div}(u) = \dfrac{D:\nabla u}{\rho}, \end{cases}$$

where: $P_0 = \Pi_0^{'}$, $P_1 = \Pi_1^{'}$ and $\widetilde{T} = \Psi^{-1}(\theta)$. We supplement (NK) with initial conditions:

$$\rho_{/t=0} = \rho_0 \geq 0 \;\; u_{/t=0} = u_0, \text{ and } \theta_{/t=0} = \theta_0.$$

2. Mathematical results

Before getting into the heart of mathematical results, let us derive the physical energy bounds of the (NK) system when κ is a constant and where the pressure just depends on the density to simplify. Let $\bar{\rho} > 0$ be a constant reference density, and let Π be defined by:

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s} \frac{P_0(z)}{z^2} dz - \frac{P_0(\bar{\rho})}{\bar{\rho}}\Big)$$

Multiplying the equation of momentum conservation in the system (NK) by ρu and integrating by parts over $\mathbb{R}^N$, we obtain the following estimate:

$$\int_{\mathbb{R}^N} (\frac{1}{2}\rho|u|^2 + \rho\theta + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{\kappa}{2}|\nabla\rho|^2)(t)dx + 2\int_0^t \int_{\mathbb{R}^N} (2\mu D(u) : D(u) + (\lambda+\mu)|\text{div}u|^2)dx \leq \int_{\mathbb{R}^N} (\frac{|m_0|^2}{2\rho} + \rho_0\theta_0 + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa}{2}|\nabla\rho_0|^2)dx.$$

In the case $\kappa > 0$, the above inequality gives a control of the density in $L^\infty(0,\infty,\dot{H}^1(\mathbb{R}^N))$. Hence, in contrast to the non capillary case one can easily pass to the limit in the pressure term. However let us emphasize at this point that the above a priori bounds do not provide any L^∞ control on the density from below or from above. Indeed, even in dimension $N = 2$, H^1 functions are not necessarily locally bounded. Thus, vacuum patches are likely to form in the fluid in spite of the presence of capillary forces, which are expected to smooth out the density. Danchin and Desjardins show in [8] that the isothermal model has weak solutions if there exists c_1 and M_1 such that:

$$c_1 \leq |\rho| \leq M_1 \text{ and } |\rho - 1| << 1.$$

The vacuum is one of the main difficulties to get weak solutions, and the problem remains open. Existence of strong solution with κ, μ and λ constant is known since the work by Hattori an Li in [11], [12] in the whole space $\mathbb{R}^N$. In [8], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces.

Here we want to investigate the well-posedness of the full non isothermal problem in critical spaces, that is, in spaces which are invariant by the scaling of Korteweg's system. Recall that such an approach is now classical for incompressible Navier-Stokes equation and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity. Let us explain precisely the scaling of Korteweg's system. We can easily check that, if (ρ, u, θ) solves (NK), so does $(\rho_\lambda, u_\lambda, \theta_\lambda)$, where:

$$\rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x)\,,\; u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \;\text{ and }\; \theta_\lambda(t,x) = \lambda^2 \theta(\lambda^2 t, \lambda x)$$

provided the pressure laws P_0, P_1 have been changed into $\lambda^2 P_0, \lambda^2 P_1$.

Definition 2.1. We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation:

$$(\rho, u, \theta) \longrightarrow (\rho_\lambda, u_\lambda, \theta_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0, u_0, θ_0) in spaces whose norm is invariant for all $\lambda > 0$ by $(\rho_0, u_0, \theta_0) \longrightarrow (\rho_0(\lambda\cdot), \lambda u_0(\lambda\cdot), \lambda^2\theta_0(\lambda\cdot))$. A natural candidate is the homogeneous Sobolev space

$$\dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N \times \dot{H}^{N/2-2},$$

but since $\dot{H}^{N/2}$ is not included in L^∞, we cannot expect to get L^∞ control on the density as on the vacuum when $\rho_0 \in \dot{H}^{N/2}$. Indeed a control on the vacuum will aloow us to benefit from the parabolicity of the momentum equation, and a control L^∞ on the density is indispensable for considerations of pointwise multipliers of Besov spaces. The same problem occurs in the equation for the temperature when dealing with the non linear term involving $\Psi^{-1}(\theta)$.

This is the reason why, instead of the classical homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^N)$, we will consider homogeneous Besov spaces with the same derivative index $B^s = \dot{B}^s_{2,1}(\mathbb{R}^N)$ (for the corresponding definition we refer to section 4). One of the nice property of B^s spaces for critical exponent s is that $B^{N/2}$ is an algebra embedded in L^∞ (and it's the only Besov spaces with our critical regularity index) . This allows to control the density from below and from above, without requiring more regularity on derivatives of ρ. For similar reasons, we shall take θ_0 in $B^{\frac{N}{2}}$ in the general case where appear non-linear terms in function of the temperature. Since a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria (Section 5). This motivates the following definition:

Definition 2.2. Let $\bar{\rho} > 0$, $\bar{\theta} > 0$. We will note in the sequel:

$$q = \frac{\rho - \bar{\rho}}{\bar{\rho}} \text{ and } \pi = \theta - \bar{\theta}.$$

One can now state the main results of the paper. The first two theorems concern the global existence and uniqueness of solution to the Korteweg's system with *small* initial data. We are interested here by the specific case where Ψ depends linearly on $\widetilde{T}$ (that is $\Psi(\widetilde{T}) = A\widetilde{T}$). In the appendix 6 we treat the general case by explaining the changes to make on the initial temperature data in particular.

Theorem 2.3. *Let $N \geq 3$. Assume that all the physical coefficients are smooth functions depending only on the density. Let $\bar{\rho} > 0$ be such that:*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}) > 0, \quad \lambda(\bar{\rho}) + 2\mu(\bar{\rho}) > 0, \quad \eta(\bar{\rho}) > 0 \text{ and } \partial_\rho P_0(\bar{\rho}) > 0.$$

Moreover suppose that:

$$q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ u_0 \in B^{\frac{N}{2}-1},\ \pi_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}.$$

There exists an η depending only on the physical coefficients (that we will precise later) such that if:

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1}} + \|\pi_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}} \leq \eta$$

then (NK) has a unique global solution (q,u,π) in $E^{N/2}$ where E^s is defined by:

$$\begin{aligned} E^s =& [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{s+1,s+2})] \times [C_b(\mathbb{R}_+, B^{s-1})^N \cap \\ & L^1(\mathbb{R}_+, B^{s+1})^N] \times [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s-2}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{s+1,s})]. \end{aligned}$$

Remark 2.4. Above, $\widetilde{B}^{s,t}$ stands for a Besov space with regularity B^s in low frequencies and B^t in high frequencies (see definition 3.4). Here it is crucial to use hybrid Besov spaces. Indeed we have to take in account the behavior in low and high frequencies of the linear part associated to the system. For example, to control the pressure term requires to work in good adapted spaces in frequencies if we want use paraproduct technics. To finish we can observe that our initial data respect the scaling associates to the system for high frequencies.

The case $N = 2$ requires more regular initial data because of technical problems involving some nonlinear terms in the temperature equation.

Theorem 2.5. *Let $N = 2$. Under the assumption of the theorem 2.3 for Ψ and the physical coefficients, let $\varepsilon' > 0$ and suppose that:*

$$q_0 \in \widetilde{B}^{0,1+\varepsilon'},\ u_0 \in \widetilde{B}^{0,\varepsilon'},\ \pi_0 \in \widetilde{B}^{0,-1+\varepsilon'}.$$

There exists an η depending only on the physical coefficients such that if:

$$\|q_0\|_{\widetilde{B}^{0,1+\varepsilon'}} + \|u_0\|_{\widetilde{B}^{0,\varepsilon'}} + \|\pi_0\|_{\widetilde{B}^{0,-1+\varepsilon'}} \leq \eta$$

then (NK) has a unique global solution (q, u, π) in:

$$\begin{aligned} E' =& [C_b(\mathbb{R}_+, \widetilde{B}^{0,1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,3+\varepsilon'})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,\varepsilon'})^2 \cap \\ & L^1(\mathbb{R}_+, \widetilde{B}^{2,2+\varepsilon'})^2] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,-1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,1+\varepsilon'})]. \end{aligned}$$

In the previous theorem we can observe that the initial data are *almost* in the energy space.
In the following two theorems we are interested by the existence and uniqueness of solution in finite time for *large* data. We distinguish always the different cases $N \geq 3$ and $N = 2$.

Theorem 2.6. *Let $N \geq 3$,and Ψ and the physical coefficients be as in theorem 2.3 (where we replace systematically $\mu(\bar{\rho}) > 0$ by $\mu(\rho) > 0$). We suppose that $(q_0, u_0, \pi_0) \in B^{\frac{N}{2}} \times (B^{\frac{N}{2}-1})^N \times B^{\frac{N}{2}-2}$ and that $\rho_0 \geq c$ for some $c > 0$.*
Then there exists a time T such that system (NK) has a unique solution (q, u, π)in F_T

$$\begin{aligned} F_T =& [\widetilde{C}_T(B^{\frac{N}{2}}) \cap L^1_T(B^{\frac{N}{2}+2})] \times [\widetilde{C}_T(B^{\frac{N}{2}-1})^N \cap L^1_T(B^{\frac{N}{2}+1})^N] \\ & \times [\widetilde{C}_T(B^{\frac{N}{2}-2}) \cap L^1_T(B^{\frac{N}{2}})] \ . \end{aligned}$$

We can observe here that we work in larger initial data spaces, indeed we don't consider the behavior in low frequencies of the system. To speak roughly, it explains simply by the fact that L^p_{loc} with $p > 1$ is included in L^1_{loc}. With this in mind, to take in account the low frequencies behavior is not necessary. For the same reasons as previously in the case $N = 2$ we cannot reach the critical level of regularity.

Theorem 2.7. *Let $N = 2$ and $\varepsilon' > 0$. Under the assumptions of theorem 2.3 for Ψ and the physical coefficients we suppose that $(q_0, u_0, \pi_0) \in \widetilde{B}^{1,1+\varepsilon'} \times (\widetilde{B}^{0,\varepsilon'})^2 \times \widetilde{B}^{-1,-1+\varepsilon'}$ and $\rho_0 \geq c$ for some $c > 0$.*

Then there exists a time T such that the system (NK) has a unique solution (q,u,π) in $F_T(2)$ with:

$$\begin{aligned}F_T(2) =&[\widetilde{C}_T(\widetilde{B}^{1,1+\varepsilon'})\cap L^1_T(\widetilde{B}^{3,3+\varepsilon'})]\times[\widetilde{C}_T(\widetilde{B}^{0,\varepsilon'})^2\cap L^1_T(\widetilde{B}^{2,2+\varepsilon'})^2]\\ &\times[\widetilde{C}_T(\widetilde{B}^{-1,-1+\varepsilon'})\cap L^1_T(\widetilde{B}^{1,1+\varepsilon'})]\,.\end{aligned}$$

The paper is structured in the following way, first of all we recall in the section 3 some definitions on Besov spaces and some useful theorem concerning Besov spaces. Next we will concentrate in the section 4 on the global existence and uniqueness of solution for our system (NK) with small initial data. In subsection 4.1 we will give some necessary conditions to get the stability of the linear part associated to the system (NK). In subsection 4.2 we will study the case where the specific intern energy is linear and where the physical coefficients are independent of the temperature. In our proof we will distinguish the case $N\geq 3$ and the case $N=2$ for some technical reasons. In the section 5 we will examine the local existence and uniqueness of solution with general initial data. For the same reasons as in section 4 we will distinguish the cases $N=2$ and $N\geq 3$. To finish with, in the appendix 6 we will treat the general case when the intern energy is only a increasing regular function and when the physical coefficients except the capillarity coefficient depend both the density and the temperature.

3. Littlewood-Paley theory and Besov spaces

3.1. Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any $\varphi\in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C}=\{\xi\in\mathbb{R}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ such that:

$$\sum_{l\in\mathbb{Z}}\varphi(2^{-l}\xi)=1\ \text{ if }\ \xi\neq 0.$$

Denoting $h=\mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u=2^{lN}\int_{\mathbb{R}^N}h(2^l y)u(x-y)dy\ \text{ and }\ S_l u=\sum_{k\leq l-1}\Delta_k u\,.$$

Formally, one can write that: $u = \sum_{k\in\mathbb{Z}} \Delta_k u$. This decomposition is called homogeneous Littlewood-Paley decomposition. Let us point out that the above formal equality holds in $\mathcal{S}'(\mathbb{R}^N)$ modulo polynomials only.

3.2. **Homogeneous Besov spaces and first properties**

Definition 3.1. For $s \in \mathbb{R},$ and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:

$$\|u\|_{B^s} = (\sum_{l\in\mathbb{Z}} 2^{ls}\|\Delta_l u\|_{L^2}).$$

A difficulty due to the choice of homogeneous spaces arises at this point. Indeed, $\|.\|_{B^s}$ cannot be a norm on $\{u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^s} < +\infty\}$ because $\|u\|_{B^s} = 0$ means that u is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces, see [3].

Definition 3.2. Let $s \in \mathbb{R}$. Denote $m = [s - \frac{N}{2}]$ if $s - \frac{N}{2} \notin \mathbb{Z}$ and $m = s - \frac{N}{2} - 1$ otherwise. If $m < 0$, then we define B^s as:

$$B^s = \Big\{ u \in \mathcal{S}'(\mathbb{R}^N) \;/\; \|u\|_{B^s} < \infty \text{ and } u = \sum_{l\in\mathbb{Z}} \Delta_l u \text{ in } \mathcal{S}'(\mathbb{R}^N) \Big\}.$$

If $m \geq 0$, we denote by $\mathcal{P}_m[\mathbb{R}^N]$ the set of polynomials of degree less than or equal to m and we set:

$$B^s = \Big\{ u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \;/\; \|u\|_{B^s} < \infty \;\; and \;\; u = \sum_{l\in\mathbb{Z}} \Delta_l u \\ \text{in } \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \Big\}.$$

Proposition 3.3. *The following properties hold:*

(1) *Derivatives: There exists a universal constant C such that:*

$$C^{-1}\|u\|_{B^s} \leq \|\nabla u\|_{B^{s-1}} \leq C\|u\|_{B^s}.$$

(2) *Algebraic properties: For $s > 0$, $B^s \cap L^\infty$ is an algebra.*

(3) *Interpolation: $(B^{s_1}, B^{s_2})_{\theta,1} = B^{\theta s_2 + (1-\theta)s_1}$.*

3.3. Hybrid Besov spaces and Chemin-Lerner spaces

Hybrid Besov spaces are functional spaces where regularity assumptions are different in low frequency and high frequency, see [8]. They may be defined as follows:

Definition 3.4. Let s, $t \in \mathbb{R}$.We set:

$$\|u\|_{\widetilde{B}^{s,t}} = \sum_{q\leq 0} 2^{qs}\|\Delta_q u\|_{L^2} + \sum_{q>0} 2^{qt}\|\Delta_q u\|_{L^2}\,.$$

Let $m = -[N/2+1-s]$, we then define:

- $\widetilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N)\ /\ \ \|u\|_{\widetilde{B}^{s,t}} < +\infty\}$, if $m<0$
- $\widetilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N]\ \ /\ \|u\|_{\widetilde{B}^{s,t}} < +\infty\}$ if $m \geq 0$.

Let us now give some properties of these hybrid spaces and some results on how they behave with respect to the product.

Proposition 3.5. *Let $s \in \mathbb{R}$.*

- *(i)We have $\widetilde{B}^{s,s} = B^s$.*
- *(ii) If $s \leq t$ then $\widetilde{B}^{s,t} = B^s \cap B^t$, if $s > t$ then $\widetilde{B}^{s,t} = B^s + B^t$.*
- *(iii) If $s_1 \leq s_2$ and $t_1 \geq t_2$ then $\widetilde{B}^{s_1,t_1} \hookrightarrow \widetilde{B}^{s_2,t_2}$.*

Proposition 3.6. *For all s, $t>0$, we have:*

$$\|uv\|_{\widetilde{B}^{s,t}} \leq C(\|u\|_{L^\infty}\|v\|_{\widetilde{B}^{s,t}} + \|v\|_{L^\infty}\|u\|_{\widetilde{B}^{s,t}})\,.$$

For all s_1, s_2, t_1, $t_2 \leq N/2$ such that $\min(s_1+s_2,\, t_1+t_2) > 0$ *we have:*

$$\|uv\|_{\widetilde{B}^{s_1+t_1-\frac{N}{2},s_2+t_2-\frac{N}{2}}} \leq C\|u\|_{\widetilde{B}^{s_1,t_1}}\|v\|_{\widetilde{B}^{s_2,t_2}}\,.$$

For a proof of this proposition see [5]. We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces:

$$L^\rho_T(B^s) := L^\rho(0,T,B^s).$$

Definition 3.7. Let $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ and $s \in \mathbb{R}$. We then denote:

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} = \sum_{l\leq 0} 2^{ls_1}\|\Delta_l u(t)\|_{L^\rho_T(L^2)} + \sum_{l>0} 2^{ls_2}\|\Delta_l u(t)\|_{L^\rho_T(L^2)}\,.$$

We note that thanks to Minkowsky inequality we have:

$$\|u\|_{L^\rho_T(\widetilde{B}^{s_1,s_2})} \leq \|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} \quad \text{and} \quad \|u\|_{L^1_T(\widetilde{B}^{s_1,s_2})} = \|u\|_{\widetilde{L}^1_T(\widetilde{B}^{s_1,s_2})}.$$

From now on, we will denote:

$$\|u\|^-_{\widetilde{L}^\rho_T(B^{s_1})} = \sum_{l\leq 0} 2^{ls_1}\Big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^p} dt\Big)^{1/\rho},$$

$$\|u\|^+_{\widetilde{L}^\rho_T(B^{s_2})} = \sum_{l> 0} 2^{ls_2}\Big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^p} dt\Big)^{1/\rho}.$$

Hence: $\|u\|_{\widetilde{L}^\rho_T(B^{s_1})} = \|u\|^-_{\widetilde{L}^\rho_T(B^{s_1,s_2})} + \|u\|^+_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}$. We then define the space:

$$\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}) = \{u \in L^\rho_T(\widetilde{B}^{s_1,s_2}) \;/\; \|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} < \infty\}\,.$$

We denote moreover by $\widetilde{C}_T(\widetilde{B}^{s_1,s_2})$ the set of those functions of $\widetilde{L}^\infty_T(\widetilde{B}^{s_1,s_2})$ which are continuous from $[0,T]$ to $\widetilde{B}^{s_1,s_2}$. In the sequel we are going to give some properties of this spaces concerning the interpolation and their relationship with the heat equation.

Proposition 3.8. *Let s, t, s_1, $s_2 \in \mathbb{R}$, ρ, ρ_1, $\rho_2 \in [1,+\infty]$. We have:*

(1) *Interpolation:*

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s,t})} \leq \|u\|^\theta_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})}\|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})} \quad \textit{with} \;\; \theta\in[0,1]$$

$$\textit{and} \;\; \frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2},\; s = \theta s_1 + (1-\theta)s_2,\; t = \theta t_1 + (1-\theta)t_2.$$

(2) *Embedding:*

$$\widetilde{L}^\rho_T(\widetilde{B}^{s,t}) \hookrightarrow L^\rho_T(C_0) \;\; \textit{and} \;\; \widetilde{C}_T(B^{\frac{N}{2}}) \hookrightarrow C([0,T]\times\mathbb{R}^N).$$

The $\widetilde{L}^\rho_T(B^s_p)$ spaces suit particulary well to the study of smoothing properties of the heat equation. In [6], J-Y. Chemin proved the following proposition:

Proposition 3.9. *Let $p\in[1,+\infty]$ and $1 \leq \rho_2 \leq \rho_1 \leq +\infty$. Let u be a solution of:*

$$\begin{cases} \partial_t u - \mu\Delta u = f \\ u_{t=0} = u_0\,. \end{cases}$$

Then there exists $C>0$ depending only on N,μ,ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1})}\leq C\|u_0\|_{B^s}+C\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2})}.$$

To finish with, we explain how the product of functions behaves in the spaces of Chemin-Lerner. We have the following properties:

Proposition 3.10.

- *Let $s>0$, $t>0$, $1/\rho_2+1/\rho_3=1/\rho_1+1/\rho_4=1/\rho\leq 1$,*

  $$u\in\widetilde{L}^{\rho_3}_T(\widetilde{B}^{s,t})\cap\widetilde{L}^{\rho_1}_T(L^\infty)\text{ and }v\in\widetilde{L}^{\rho_4}_T(\widetilde{B}^{s,t})\cap\widetilde{L}^{\rho_2}_T(L^\infty).$$

  Then $uv\in\widetilde{L}^{\rho}_T(\widetilde{B}^{s,t})$ and we have:

  $$\|uv\|_{\widetilde{L}^{\rho}_T(\widetilde{B}^{s,t})}\leq C\|u\|_{\widetilde{L}^{\rho_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{\rho_4}_T(\widetilde{B}^{s,t})}+\|v\|_{\widetilde{L}^{\rho_2}_T(L^\infty)}\|u\|_{\widetilde{L}^{\rho_3}_T(\widetilde{B}^{s,t})}.$$

- *If $s_1,s_2,t_1,t_2\leq\frac{N}{2}$, $s_1+s_2>0$, $t_1+t_2>0$, $1/\rho_1+1/\rho_2=1/\rho\leq 1$,*

  $$u\in\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})\text{ and }v\in\widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})$$

  then $uv\in\widetilde{L}^{\rho}_T(\widetilde{B}^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}})$ and:

  $$\|uv\|_{\widetilde{L}^{\rho}_T(\widetilde{B}_2^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}}),}\leq C\|u\|_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})}\|v\|_{\widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})}.$$

For a proof of this proposition see [8].

4. Existence of solutions for small initial data

4.1. Study of the linear part

This section is devoted to the study of the linearization of system (NK) about constant state $(\bar{\rho},0,\bar{\theta})$. This induces us to study the following linear system where $(F,G,H)\in H^\infty\times(H^\infty)^N\times H^\infty$ with $H^\infty=\cap_{s\in\mathbb{R}}\dot{H}^s$:

$$(M')\qquad\begin{cases}\partial_t q+\mathrm{div}u=F\\ \partial_t u-\widetilde{\mu}\Delta u-(\widetilde{\mu}+\widetilde{\lambda})\nabla\mathrm{div}u-\varepsilon\nabla\Delta q-\beta\nabla q-\gamma\nabla=G\\ \partial_t-\alpha\Delta+\delta\mathrm{div}u=H\end{cases}$$

where $\bar{\lambda}$, ε, α, β, γ, δ and $\widetilde{\mu}$ are given real parameters. Note that the linearization of (NK) corresponds to the choice:

$$\widetilde{\mu}=\frac{\bar{\mu}}{\bar{\rho}},\ \widetilde{\lambda}=\frac{\bar{\lambda}}{\bar{\rho}},\ \varepsilon=\bar{\rho}\,\bar{\kappa},\ \beta=P_0'(\bar{\rho})+\bar{T}P_1'(\bar{\rho}),\ \gamma=\frac{P_1(\bar{\rho})}{\bar{\rho}\psi'(\bar{T})},\ \alpha=\frac{\bar{\chi}}{\bar{\rho}},$$
$$\delta=\frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}}\ .$$

We transform the system by setting:

$$d=\Lambda^{-1}\mathrm{div}\,u\ \text{ and }\ \Omega=\Lambda^{-1}\mathrm{curl}\,u$$

where we set: $\Lambda^s h =^{-1} (|\xi|^s\hat{h})$. We finally obtain the following system by projecting on divergence free vector fields and on potential vector fields:

$$(M_1')\qquad \begin{cases}\partial_t q+\Lambda d=F,\\ \partial_t d-\nu\Delta d-\varepsilon\Lambda^3 q-\beta\Lambda q-\gamma\Lambda=\Lambda^{-1}\mathrm{div}\,G,\\ \partial_t-\alpha\Delta+\delta\Lambda d=H,\\ \partial_t\Omega-\widetilde{\mu}\Delta\Omega=\Lambda^{-1}\mathrm{curl}\,G,\\ u=-\Lambda^{-1}\nabla d-\Lambda^{-1}\mathrm{div}\,\Omega.\end{cases}$$

The fourth equation is just a heat equation. Hence we are going to focus on the first three equations. However the fourth equation gives us an idea of which spaces we can work with. The first three equations rewrite as follows:

$$(M_2')\qquad \partial_t\begin{pmatrix}\hat{q}(t,\xi)\\ \hat{d}(t,\xi)\\ \hat{}(t,\xi)\end{pmatrix}+A(\xi)\begin{pmatrix}\hat{q}(t,\xi)\\ \hat{d}(t,\xi)\\ \hat{}(t,\xi)\end{pmatrix}=\begin{pmatrix}\hat{F}(t,\xi)\\ \Lambda^{-1}\mathrm{div}\,\hat{G}(t,\xi)\\ \hat{H}(t,\xi)\end{pmatrix}$$

where we set:

$$A(\xi)=\begin{pmatrix}0 & |\xi| & 0\\ -\varepsilon|\xi|^3-\beta|\xi| & \nu|\xi|^2 & -\gamma|\xi|\\ 0 & \delta|\xi| & \alpha|\xi|^2\end{pmatrix}.$$

The eigenvalues of the matrix $-A(\xi)$ are of the form $|\xi|^2\lambda_\xi$ with λ_ξ being the roots of the following polynomial:

$$P_\xi(X)=X^3+(\nu+\alpha)X^2+\Big(\varepsilon+\nu\alpha+\frac{\gamma\delta+\beta}{|\xi|^2}\Big)X+\Big(\alpha\varepsilon+\frac{\alpha\beta}{|\xi|^2}\Big).$$

For very large ξ, the roots tend to those of the following polynomial (by virtue of continuity of the roots in function of the coefficients):

$$X^3 + (\nu + \alpha)X^2 + (\varepsilon + \nu\alpha)X + \alpha\varepsilon.$$

The roots are $-\alpha$ and $-\frac{\nu}{2}(1 \pm \sqrt{1 - \frac{4\varepsilon}{\nu^2}})$. The system $(M_1^{'})$ is well-posed if and only if for $|\xi|$ tending to $+\infty$ the real part of the eigenvalues associated to $A(\xi)$ stay non positive. Hence, we must have:

$$\varepsilon,\ \nu,\ \alpha \geq 0.$$

Let us now state a necessary and sufficient condition for the global stability of $(M^{'})$.

Proposition 4.1. *The linear system $(M^{'})$ is globally stable if and only if the following conditions are verified:*

$$(*) \qquad \nu,\, \varepsilon,\, \alpha \geq 0, \;\; \alpha\beta \geq 0, \;\; \gamma\delta(\nu + \alpha) + \nu\beta \geq 0, \;\; \gamma\delta + \beta \geq 0.$$

If all the inequalities are strict, the solutions tend to 0 in the sense of distributions and the three eigenvalues $\lambda_1(\xi)$, $\lambda_+(\xi)$, $\lambda_-(\xi)$ have the following asymptotic behavior when ξ tends to 0:

$$\lambda_1(\xi) \sim -(\frac{\alpha\beta}{\beta + \gamma\delta})|\xi|^2,\ \lambda_\pm(\xi) \sim -(\frac{\gamma\delta(\nu + \alpha) + \nu\beta}{2(\gamma\delta + \beta)})|\xi|^2 \pm i|\xi|\sqrt{\gamma\delta + \beta}.$$

Proof. We already know that the system is well-posed if and only if ε, ν, $\alpha \geq 0$. Then We want that all the eigenvalues have a negative real part for all ξ. We have so to distinguish two cases: either all the eigenvalues are real or there are two complex conjugated eigenvalues.

First case:

The eigenvalues are real. A necessary condition for negativity of the eigenvalues is that $P(X) \geq 0$ for $X \geq 0$. We must have in particular:

$$P_\xi(0) = \alpha\varepsilon + \frac{\alpha\beta}{|\xi|^2} \geq 0 \quad \forall \xi \neq 0.$$

This imply that $\alpha\beta \geq 0$ and $\alpha\varepsilon \geq 0$. Hence, given that $\alpha \geq 0$, we must have $\beta \geq 0$ and $\varepsilon \geq 0$. For ξ tending to 0, we have:

$$P_\xi(\lambda) \sim \frac{\lambda(\gamma\delta + \beta) + \alpha\beta}{|\xi|^2}.$$

Making λ tend to infinity, we must have $P_\xi(\lambda) \geq 0$ and so $\gamma\delta + \beta \geq 0$. The converse is trivial.

Second case:

P_ξ has two complex roots $z_\pm = a \pm ib$ and one real root λ, we have:

$$P_\xi(X) = (X - \lambda)(X^2 - 2aX + |z_\pm|^2).$$

A necessary condition to have the real parts negative is in the same way that $P_\xi(X) \geq 0$ for all $X \geq 0$. If $\gamma\delta + \beta > 0$, we are in the case where ξ tends to 0 (and we see that P_ξ is increasing).
We can observe the terms of degree 2 and we get: $\lambda + 2a = -\alpha - \nu$ then λ and α are non positive if and only if $P_\xi(-\alpha - \nu) \leq 0$ (for this it suffices to rewrite P_ξ like $P_\xi(X) = (X - \lambda)(X^2 - 2aX + |z_\pm|^2)$). Calculate:

$$P_\xi(-\alpha - \nu) = -\nu\varepsilon - \nu^2\alpha - \frac{\nu\beta + \nu\gamma\delta + \alpha\gamma\delta}{|\xi|^2}.$$

With the hypothesis that we have made, we deduce that $P_\xi(-\alpha - \nu) \leq 0$ for ξ tending to 0 if and only if $\nu\beta + \nu\gamma\delta + \alpha\gamma\delta \geq 0$.

Behavior of the eigenvalues in low frequencies:

Let us now study the asymptotic behavior of the eigenvalues when ξ tends to 0 and all the inequalities in (A) are strict.
We remark straight away that the condition $\gamma\delta + \beta > 0$ ensures the strict monotonicity of the function: $\lambda \to P_\xi(\lambda)$ for ξ small. Then there's only one real eigenvalue $\lambda_1(\xi)$ and two complex eigenvalues $\lambda_\pm(\xi) = a(\xi) \pm ib(\xi)$.
Let $\varepsilon^- < -\frac{\alpha\beta}{\gamma\delta+\beta} < \varepsilon^+ < 0$. When ξ tends to 0, we have:

$$P_\xi(\lambda) \sim |\xi|^{-2}(\lambda(\gamma\delta + \beta) + \alpha\beta)).$$

Then $P_\xi(\varepsilon^-) < 0$ and $P_\xi(\varepsilon^+) > 0$ and P_ξ has a unique real root included between ε^- and ε^+. These considerations give the asymptotic value of $\lambda_1(\xi)$. Finally, we have:

$$\lambda_1(\xi) + 2a(\xi) = -\alpha - \nu \text{ and } -(a(\xi)^2 + b(\xi)^2)\lambda(\xi) = \alpha\xi + \frac{\alpha\beta}{|\xi|^2} \sim \frac{\alpha\beta}{|\xi|^2},$$

whence the result. □

We summarize this results in the following remark.

Remark 4.2. According to the analysis made in proposition 4.1, we expect the system (NK) to be locally well-posed close to the equilibrium $(\bar{\rho},0,\bar{T})$ if and only if we have:

$$(C)\qquad \mu(\bar{\rho},\bar{\theta})\geq 0,\ \lambda(\bar{\rho},\bar{\theta})+2\mu(\bar{\rho},\bar{\theta})\geq 0,\ \kappa(\bar{\rho})\geq 0,\quad\text{and}\ \ \chi(\bar{\rho},\bar{T})\geq 0.$$

By the calculus we have:

$$\beta=\partial_\rho p_0(\bar{\rho},\bar{T}),\ \gamma=\frac{\partial_T p_0(\bar{\rho},\bar{T})}{\bar{\rho}\,\partial_T e_0(\bar{\rho},\bar{T})},\ \delta=\frac{\bar{T}\,\partial_T p_0(\bar{\rho},\bar{T})}{\bar{\rho}}.$$

We remark that $\gamma\delta\geq 0$ if $\partial_T e_0(\bar{\rho},\bar{T})\geq 0$. In the case where η verifies $\eta(\bar{\rho},\bar{T})>0$, the supplementary condition giving the global stability reduces to:

$$(D)\qquad \partial_\rho p_0(\bar{\rho},\bar{T})\geq 0.$$

Now that we know the stability conditions on the coefficients of the system (M'), we are interested in estimates on this system in the space $E^{\frac{N}{2}}$. We add a condition in this following proposition compared with the proposition 4.1 which is: $\gamma\delta>0$, but it's not so important because in the system (NK) we have in reality $\gamma\delta=\frac{1}{\bar{T}\Psi'(\bar{T})}>0$.

Proposition 4.3. *Under the conditions of proposition 4.1 with strict inequalities and with the condition $\gamma\delta>0$, let $(q,d,)$ be a solution of the system (M') on $[0,T)$ with initial data (q_0,u_0,π_0) such that:*

$$q_0\in\widetilde{B}^{s-1,s},d_0\in B^{s-1},\pi_0\in\widetilde{B}^{s-1,s-2}\ \textit{for some } s\in\mathbb{R}\ .$$

Moreover we suppose that for some $1\leq r_1\leq+\infty$, we have:

$$\begin{aligned}&F\in\widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}}),\\&G\in\widetilde{L}^{r_1}_T(B^{s-3+\frac{2}{r_1}}),\\&H\in\widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-4+\frac{2}{r_1}}).\end{aligned}$$

We then have the following estimate for all $r_1\leq r\leq+\infty$:

$$\begin{aligned}\|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}+\|\pi\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s-2+\frac{2}{r}})}+\|u\|_{\widetilde{L}^r_T(B^{s-1+\frac{2}{r}})}\leq C\|q_0\|_{\widetilde{B}^{s-1,s}}\\+\|u_0\|_{B^{s-1}}+\|\pi_0\|_{\widetilde{B}^{s-1,s-2}}+\|F\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})}+\|G\|_{L^{r_1}_T(B^{s-3+\frac{2}{r_1}})}\\+\|H\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-4+\frac{2}{r_1}})}.\end{aligned}$$

Proof. We are going to separate the case of the low, medium and high frequencies, particulary the low and high frequencies which have a different behavior.

1) Case of low frequencies:

Let us focus on the first three equation of system $(M_1^{'})$ and applying operator Δ_l to the system, we get then by setting:

$$q_l = \Delta_l q,\ d_l = \Delta_l d,\ \pi_l = \Delta_l \pi$$

$$\partial_t q_l + \Lambda d_l = F_l, \tag{4.1}$$

$$\partial_t d_l - \nu\Delta d_l - \varepsilon\Lambda^3 q_l - \beta\Lambda q_l - \gamma\Lambda_l\pi = \Lambda^{-1}\text{div}G_l, \tag{4.2}$$

$$\partial_t \pi_l - \alpha\Delta\pi_l + \delta\Lambda d_l = H_l. \tag{4.3}$$

Throughout the proof, we assume that $\delta \neq 0$: if not we can use proposition 3.9 to have the estimate on and we have just to deal with the first two equations. Denoting by $W(t)$ the semi-group associated to $(4.1 - 4.3)$ we have:

$$\begin{pmatrix} q(t) \\ u(t) \\ \theta(t) \end{pmatrix} = W(t)\begin{pmatrix} q_0 \\ u_0 \\ \theta_0 \end{pmatrix} + \int_0^t W(t-s)\begin{pmatrix} F(s) \\ G(s) \\ H(s) \end{pmatrix} ds\ .$$

We set:

$$f_l^2 = \beta\|q_l\|_{L^2}^2 + \|d_l\|_{L^2}^2 + \frac{\gamma}{\delta}\|\pi_l\|_{L^2}^2 - 2K\langle\Lambda q_l, d_l\rangle$$

for some $K \geq 0$ to be fixed hereafter and $\langle\cdot,\cdot\rangle$ noting the L^2 inner product. To begin with, we consider the case where $F = G = H = 0$.
Then we take the inner product of (4.2) with d_l, of (4.1) with βq_l and of (4.3) with γ_l. We obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|d_l\|_{L^2}^2 + \beta\|q_l\|_{L^2}^2 + \frac{\gamma}{\delta}\|\pi_l\|_{L^2}^2) + \nu\|\nabla d_l\|_{L^2}^2 - \varepsilon\langle\Lambda^3 q_l, d_l\rangle \\ + \frac{\gamma\alpha}{\delta}\|\nabla\pi_l\|_{L^2}^2 = 0.\end{aligned} \tag{4.4}$$

Next, we apply the operator Λ to (4.2) and take the inner product with q_l, and we take the scalar product of (4.1) with Λd_l to control the term

$\frac{d}{dt}\langle\Lambda q_l, d_l\rangle$. Summing the two resulting equalities, we have:

$$\begin{aligned}\frac{d}{dt}\langle\Lambda q_l, d_l\rangle + \|\Lambda d_l\|_{L^2}^2 - \nu\langle\Delta d_l, \Lambda q_l\rangle - \varepsilon\|\Lambda^2 q_l\|_{L^2}^2 - \beta\|\Lambda q_l\|_{L^2}^2 \\ - \gamma\langle\Lambda_l, \Lambda q_l\rangle = 0.\end{aligned}\tag{4.5}$$

We obtain then by summing (4.4) and (4.5):

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2 + (\nu\|\nabla d_l\|_{L^2}^2 - K\|\Lambda d_l\|_{L^2}^2) + (K\beta\|\Lambda q_l\|_{L^2}^2 + K\varepsilon\|\Lambda^2 q_l\|_{L^2}^2) \\ &+ \frac{\gamma\alpha}{\delta}\|\nabla\pi_l\|_{L^2}^2 + K\nu\langle\Delta d_l, \Lambda q_l\rangle + K\gamma\langle\Lambda_l, \Lambda q_l\rangle - \varepsilon\langle\Lambda^3 q_l, d_l\rangle = 0\,.\end{aligned}\tag{4.6}$$

Like indicated, we are going to focus on low frequencies so assume that $l \leq l_0$ for some l_0 to be fixed hereafter. We have then $\forall c,\, b,\, d > 0$:

$$\begin{aligned}&|\langle\Delta d_l, \Lambda q_l\rangle| \leq \frac{b}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{C2^{2l_0}}{2b}\|\Lambda d_l\|_{L^2}^2\,, \\ &|\langle\Lambda^3 q_l, d_l\rangle| = |\langle\Lambda^2 q_l, \Lambda d_l\rangle| \leq \frac{C2^{2l_0}}{2c}\|\Lambda q_l\|_{L^2}^2 + \frac{c}{2}\|\Lambda d_l\|_{L^2}^2\,.\end{aligned}\tag{4.7}$$

Moreover we have: $\|\nabla d_l\|_{L^2}^2 = \|\Lambda d_l\|_{L^2}^2$. Finally we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}f_l^2 + [\nu - (K + \frac{C2^{2l_0}}{2b}K\nu + \frac{c\varepsilon}{2})]\|\Lambda d_l\|_{L^2}^2 + [\frac{\gamma\alpha}{\delta} - \frac{K\gamma}{2d}]\|\Lambda\pi_l\,\|_{L^2}^2 \\ + K[\beta + \varepsilon C2^{2l_0} - \nu\frac{b}{2} - \varepsilon\frac{C2^{2l_0}}{2c} - \gamma\frac{d}{2}]\|\Lambda q_l\|_{L^2}^2 \leq 0.\end{aligned}$$

Then we choose (b, c, d) such that:

$$b = \frac{\beta}{2\nu}\ ,\ c = \frac{\nu}{\varepsilon}\ ,\ d = \frac{\beta}{2\gamma}\ ,$$

which is possible if $\gamma > 0$ as $\nu > 0,\ \varepsilon > 0$. In the case where $\gamma \leq 0$, we recall that γ and δ have the same sign, we have then no problem because with our choice the first and third following inequalities will be satisfied and if $\gamma \leq 0$ in the second equation the term $\gamma\frac{d}{2}$ is positive by taking $d > 0$. So we assume from now on that $\gamma > 0$ and so with this choice, we want that:

$$\begin{aligned}&\frac{\nu}{2} - K(1 + C2^{2l_0}\frac{\nu^2}{\beta}) > 0, \\ &\frac{\beta}{2} + \varepsilon C2^{2l_0} - C2^{2l_0}\frac{\varepsilon^2}{2\nu} > 0, \\ &\frac{\gamma\alpha}{\delta} - K\frac{\gamma}{2} > 0.\end{aligned}$$

We recall that in your case $\nu > 0,\ \beta > 0,\ \alpha > 0$ and $\gamma > 0,\ \delta > 0$. So it suffices to choose K and l_0 such that:

$$K < \min\Big(\frac{\nu}{2(1+C2^{2l_0}\frac{\nu^2}{2\beta})},\frac{2\alpha}{\delta}\Big) \quad \text{and} \quad 2^{2l_0} < \min\Big(\frac{\beta\nu}{6C\varepsilon^2},\frac{1}{6\varepsilon C}\Big).$$

Finally we conclude by using Proposition 3.3 part (ii) with a $c^{'}$ small enough. We get:

$$\frac{1}{2}\frac{d}{dt}f_l^2 + c^{'}2^{2l}f_l^2 \leq 0 \ \text{ for } l \leq l_0\,. \tag{4.8}$$

2) Case of high frequencies:

We are going to work with $l \geq l_1$ where we will determine l_1 hereafter. We set:

$$f_l^2 = \varepsilon B\|\Lambda q_l\|_{L^2}^2 + B\|d_l\|_{L^2}^2 + \|\Lambda^{-1}\pi_l\|_{L^2}^2 - 2K\langle\Lambda q_l, d_l\rangle,$$

and we choose B and K later on. Then we take the inner product of (4.2) with d_l:

$$\frac{1}{2}\frac{d}{dt}\|d_l\|_{L^2}^2 + \nu\|\nabla d_l\|_{L^2}^2 - \varepsilon\langle\Lambda^3 q_l, d_l\rangle - \beta\langle\Lambda q_l, d_l\rangle - \gamma\langle\Lambda\pi_l, d_l\rangle = 0. \tag{4.9}$$

Moreover we have by taking the scalar product of (4.1) with $\Lambda^2 q_l$:

$$\frac{1}{2}\frac{d}{dt}\|\Lambda q_l\|_{L^2}^2 + \langle\Lambda^2 d_l, \Lambda q_l\rangle = 0. \tag{4.10}$$

And in the same way with (4.3), we have:

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\pi_l\|_{L^2}^2 + \alpha\|\pi_l\|_{L^2}^2 + \delta\langle d_l, \Lambda^{-1}\pi_l\rangle = 0. \tag{4.11}$$

After we sum (4.9), (4.10) and (4.11) to get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(B\|d_l\|_{L^2}^2 + \varepsilon B\|\Lambda q_l\|_{L^2}^2 + \|\Lambda^{-1}\pi_l\|_{L^2}^2) + B\nu\|\nabla d_l\|_{L^2}^2 \\ + \alpha\|\pi_l\|_{L^2}^2 - B\beta\langle\Lambda q_l, d_l\rangle - B\gamma\langle\Lambda\pi_l, d_l\rangle + \delta\langle d_l, \Lambda^{-1}\pi_l\rangle = 0.\end{aligned} \tag{4.12}$$

Then like previously we can play with $\langle\Lambda q_l, d_l\rangle$ to obtain a term in $\|\Lambda q_l\|_{L^2}^2$. We have then again the following equation:

$$\begin{aligned}\frac{d}{dt}\langle\Lambda q_l, d_l\rangle + \|\Lambda d_l\|_{L^2}^2 - \nu\langle\Delta d_l, \Lambda q_l\rangle - \varepsilon\|\Lambda^2 q_l\|_{L^2}^2 - \beta\|\Lambda q_l\|_{L^2}^2 \\ - \gamma\langle\Lambda\pi_l, \Lambda q_l\rangle = 0.\end{aligned} \tag{4.13}$$

We sum all these expressions and get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}f_l^2+\big[B\nu\|\nabla d_l\|_{L^2}^2-K\|\Lambda d_l\|_{L^2}^2\big]+\alpha\|\pi_l\|_{L^2}^2+K\big[\beta\|\Lambda q_l\|_{L^2}^2\\ +\varepsilon\|\Lambda^2 q_l\|_{L^2}^2\big]-B\beta\langle\Lambda q_l,d_l\rangle-B\gamma\langle\Lambda\pi_l,d_l\rangle+\delta\langle d_l,\Lambda^{-1}\pi_l\rangle\\ +K\nu\langle\Delta d_l,\Lambda q_l\rangle+\gamma K\langle\Lambda\pi_l,\Lambda q_l\rangle=0.\end{aligned}\tag{4.14}$$

The main term in high frequencies will be: $\|\Lambda^2 q_l\|_{L^2}^2$. The other terms may be treated by mean of Young's inequality:

$$|\langle\Lambda q_l,d_l\rangle|\leq\frac{1}{2a\,c2^{2l_1}}\|\Lambda^2 q_l\|_{L^2}^2+\frac{a}{2}\|\Lambda d_l\|_{L^2}^2.$$

We do as before with the other terms in the second line of (4.14) and we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2+(B\nu-K)\|\Lambda d_l\|_{L^2}^2]+\alpha\|\pi_l\|_{L^2}^2+K(\frac{\beta}{c2^{2l_1}}+\varepsilon)\|\Lambda^2 q_l\|_{L^2}^2\leq\\ &B\gamma[\frac{1}{2a}\|\pi_l\|_{L^2}^2+\frac{a}{2}\|\Lambda d_l\|_{L^2}^2]+K[\frac{\nu b}{2}\|\Lambda^2 q_l\|_{L^2}^2+\frac{2\nu}{b}\|\Lambda d_l\|_{L^2}^2+\frac{\gamma}{2c'}\|\pi_l\|_{L^2}^2\\ &\qquad+\frac{\gamma c'}{2}\|\Lambda^2 q_l\|_{L^2}^2]+B\beta[\frac{1}{2d}\frac{1}{c2^{2l_1}}\|\Lambda^2 q_l\|_{L^2}^2+\frac{d}{2}\frac{1}{c2^{2l_1}}\|\Lambda d_l\|_{L^2}^2]\\ &\qquad\qquad+\delta[\frac{1}{2e}\frac{1}{c2^{2l_1}}\|\pi_l\|_{L^2}^2+\frac{e}{2}\frac{1}{c2^{2l_1}}\|\Lambda d_l\|_{L^2}^2].\end{aligned}$$

We obtain then for some $a,\ b,\ c',\ d,\ e$ to be chosen:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}f_l^2+\big[B\nu-(K+B\gamma\frac{a}{2}+K\nu\frac{2}{b}+B\beta\frac{d}{2c2^{2l_1}}+\delta\frac{e}{2}\frac{1}{c2^{2l_1}})\big]\\ \times\|\Lambda d_l\|_{L^2}^2+\big[\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c'}+\delta\frac{1}{2e}\frac{1}{c2^{2l_1}})\big]\|\pi_l\|_{L^2}^2\\ +[\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c'}{2}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}}]\|\Lambda^2 q_l\|_{L^2}^2\leq 0\,.\end{aligned}\tag{4.15}$$

We claim that a, b, c', d, e, l_1, K may be chosen so that:

$$B\nu-(K+B\gamma\frac{a}{2}+K\nu\frac{2}{b}+B\beta\frac{d}{2}+\delta\frac{e}{2}\frac{1}{2^{2l_1}})>0,\tag{4.16}$$

$$\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c'}+\delta\frac{1}{2e}\frac{1}{c2^{2l_1}})>0,\tag{4.17}$$

$$\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c'}{2}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}}>0.\tag{4.18}$$

We want at once that for (4.16) and (4.18):

$$\nu - \gamma\frac{a}{2} - \beta\frac{d}{2} > 0\ , \tag{4.19}$$

$$\varepsilon - \nu\frac{b}{2} - \gamma\frac{c^{'}}{2} > 0\ . \tag{4.20}$$

So we take:

$$\begin{aligned} &e = 1,\ a = 2h\delta,\ d = 2h,\ h = \frac{\nu}{2(\gamma\delta+\beta)},\ b = 2\beta h^{'},\\ &c^{'} = 2\delta(\nu+\alpha)h^{'} \quad \text{and} \quad h^{'} = \frac{\varepsilon}{2(\gamma\delta(\nu+\alpha)+\nu\beta)}\ . \end{aligned}$$

With this choice, we get (4.19) and (4.20). In what follows it suffices to choice B, K small enough and l_1 large enough. We have then:

$$f_l \simeq \text{Max}(1, 2^l)\|q_l\|_{L^2} + \|d_l\|_{L^2} + \min(1, 2^l)\|\pi_l\|_{L^2}$$

One can now conclude that there exists some $c^{'}$ such that for all $l \leq l_0$ or $l \geq l_1$:

$$\frac{1}{2}\frac{d}{dt}f_l^2 + c^{'}2^{2l}f_l^2 \leq 0.$$

3) Case of Medium frequencies:

For $l_0 \leq l \leq l_1$, there is only a finite number of terms to treat. So it suffices to find a C such that for all theses terms:
(B)

$$\|q_l\|_{L^r_T(L^2)} \leq C,\ \|d_l\|_{L^r_T(L^2)} \leq C,\ \|\pi_l\|_{L^r_T(L^2)} \leq C \quad \text{for all} \quad T \in [0, +\infty] \quad \text{and} \quad r \in [1, +\infty],$$

with C large enough independent of T. And this is true because the system is globally stable: indeed according to proposition 4.1, we have:

$$\left\| W(t)\begin{pmatrix} a\\ b\\ c\end{pmatrix}\right\|_{L^2} \leq Ce^{-c_1(\xi)t}\left\|\begin{pmatrix} a\\ b\\ c\end{pmatrix}\right\|_{L^2} \quad \forall a, b, c \in L^2$$

with $c_1(\xi) = \min_{2^{l_0}\leq|\xi|\leq 2^{l_1}}(\text{Re}(\lambda_1(\xi)),\ \text{Re}(\lambda_2(\xi)),\ \text{Re}(\lambda_3(\xi)))$ where the $\lambda_i(\xi)$ correspond to the eigenvalues of the system. We have then by using the estimate in low and high frequencies in part 4.1 and the continuity of

$c_1(\xi)$ the fact that there exist c_1 such that: $c_1(\xi) \geq c_1 > 0$. So that we have for $l_0 \leq l \leq l_1$:

$$\Big(\int_0^T \begin{pmatrix} \|q_l(t)\|_{L^2}^r \\ \|u_l(t)\|_{L^2}^r \\ \|\pi_l(t)\|_{L^2}^r \end{pmatrix} dt\Big)^{\frac{1}{r}} \leq C\Big(\int_0^T e^{-c_1 rs} ds\Big)^{\frac{1}{r}} \begin{pmatrix} \|(q_0)_l\|_{L^2} \\ \|(u_0)_l\|_{L^2} \\ \|(\pi_0)_l\|_{L^2} \end{pmatrix}.$$

And so we have the result (B).

4) Conclusion:

By using Duhamel formula for W and in taking C large enough we have for all l:

$$\begin{aligned} \max(1,2^l)\|q_l(t)\|_{L^2} + \|d_l(t)\|_{L^2} + \min(1,2^{-l})\|\pi_l(t)\|_{L^2} \leq C e^{-c2^{2l}t} \\ \times \big(\max(1,2^l)\|(q_0)_l\|_{L^2} + \|(d_0)_l\|_{L^2} + \min(1,2^{-l})\|(\pi_0)_l\|_{L^2}\big) \\ + C\int_0^t e^{-c2^{2l}(t-s)}\big(\max(1,2^{2l})\|F_l\|_{L^2} + \|G_l\|_{L^2} + \min(1,2^{-l})\|H_l\|_{L^2}\big) ds\,. \end{aligned}$$

Now we take the L^r norm in time and we sum in multiplying by $2^{l(s-1+\frac{2}{r})}$ for the low frequencies and we sum in multiplying by $2^{l(s+\frac{2}{r})}$ for the high frequencies. This yields:

$$\begin{aligned} \|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})} + \|\pi\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s-2+\frac{2}{r}})} + \|d\|_{\widetilde{L}^r_T(B^{s-1+\frac{2}{r}})} \leq \|q_0\|_{\widetilde{B}^{s-1,s}} \\ + \|\pi_0\|_{\widetilde{B}^{s-1,s-2}} + \|d_0\|_{B^{s-1}} + \sum_{l\leq 0} 2^{l(s-1+\frac{2}{r})} \int_0^T \Big(\int_0^T e^{c(t-\tau)}(\|F_l(\tau)\|_{L^2} \\ + \|G_l(\tau)\|_{L^2} + \|H_l(\tau)\|_{L^2}) d\tau\Big)^r dt\Big)^{\frac{1}{r}} + \sum_{l\geq 0} 2^{l(s+\frac{2}{r})}\Big(\int_0^T \Big(\int_0^T e^{c(t-\tau)} \\ \times (\|\nabla F_l(\tau)\|_{L^2} + \|G_l(\tau)\|_{L^2} + \|\Lambda^{-1}H_l(\tau)\|_{L^2}) d\tau\Big)^r dt\Big)^{\frac{1}{r}}. \end{aligned}$$

Bounding the right hand-side may be done by taking advantage of convolution inequalities. To complete the proof of proposition 4.3, it suffices to use that $u = -\Lambda^{-1}\nabla d - \Lambda^{-1}\text{div}\Omega$ and to apply proposition 3.9. □

4.2. Global existence for temperature independent coefficients

This section is devoted to the proof of theorem 2.3. Let us first recall the spaces in which we work with:

$$E^s = [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s}) \cap L^1(\mathbb{R}_+\widetilde{B}^{s+1,s+2})] \times [C_b(\mathbb{R}_+, B^{s-1})^N \cap L^1(\mathbb{R}_+, B^{s+1})^N] \times [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s-2}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{s+1,s})].$$

In what follows, we assume that $N \geq 3$.

Proof of theorem 2.3:

We shall use a contracting mapping argument, for the function ψ defined as follows:

$$\psi(q,u,\pi) = W(t,\cdot) * \begin{pmatrix} q_0 \\ u_0 \\ \pi_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} F(q,u,\pi) \\ G(q,u,\pi) \\ H(q,u,\pi) \end{pmatrix} ds \,. \tag{4.21}$$

In what follows we set: $\rho = \bar{\rho}(1+q)$, $\theta = \bar{\theta} + \pi$, $\widetilde{T} = \Psi^{-1}(\theta)$. The non linear terms F, G, H are defined as follows:

$$\begin{aligned} &F = -\text{div}(qu), \\ &G = -u.\nabla u + \nabla(\frac{K^{'}_\rho}{2}|\nabla\rho|^2\ ) + \Big[\frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}}\Big]\Delta u + \Big[\frac{\zeta(\rho)}{\rho} - \\ &\qquad \frac{\zeta(\bar{\rho})}{\bar{\rho}}\Big]\nabla\text{div}\, u + (\nabla((K(\rho) - K(\bar{\rho}))\Delta\rho) + \Big[\frac{P_0^{'}(\rho) + \widetilde{T}P_1^{'}(\rho)}{\rho} \\ &\quad - \frac{P_0^{'}(\bar{\rho}) + \bar{T}P_1^{'}(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho + \Big[\frac{P_1(\rho)}{\rho\Psi'(\pi)} - \frac{P_1(\bar{\rho})}{\bar{\rho}\Psi'(\bar{T})}\Big]\nabla\theta + \frac{\lambda^{'}(\rho)\nabla\rho\text{div}u}{\rho} \\ &\qquad\qquad + \frac{(du + \nabla u)\mu^{'}(\rho)\nabla\rho}{\rho}\,, \end{aligned} \tag{4.22}$$

where we note: $\zeta = \lambda + \mu$, and:

$$H = \Big(\frac{\text{div}(\chi(\rho)\nabla\theta)}{\rho} - \frac{\bar{\chi}}{\bar{\rho}}\Delta\theta\Big) + \Big[\frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}} - \frac{\widetilde{T}P_1(\rho)}{\rho}\Big]\text{div}u - u^*.\nabla\theta + \frac{D:\nabla u}{\rho}\,. \tag{4.23}$$

1) First step, uniform bounds:

Let:

$$\eta=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B^{\frac{N}{2}-1}}+\|\pi_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\;.$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. According to proposition 4.3, we have:

$$\|W(t,\cdot)*\begin{pmatrix}q_0\\u_0\\\pi_0\end{pmatrix}\|_{E^{\frac{N}{2}}}\leq C\eta\;. \tag{4.24}$$

Hence:

$$\begin{aligned}\|\psi(q,u,\pi)\|_{E^{\frac{N}{2}}}\leq\;& C\eta+\|F(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\\&+\|G(q,u,\pi)\|_{L^1(B^{\frac{N}{2}-1})}+\|H(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\;.\end{aligned} \tag{4.25}$$

Moreover we suppose for the moment that:

$$(\mathcal{H})\qquad\qquad \|q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}\leq 1/2\;.$$

We will use the different theorems on the paradifferential calculus to obtain estimates on

$$\|F(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},\;\|G(q,u,\pi)\|_{L^1(B^{\frac{N}{2}-1})},\;\|H(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}.$$

• Let us first estimate $\|F(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$. According to proposition 3.10, we have:

$$\|\mathrm{div}(qu)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|qu\|_{L^1(B^{\frac{N}{2}})}+\|qu\|_{L^1(B^{\frac{N}{2}+1})},$$

and:

$$\|qu\|_{L^1(B^{\frac{N}{2}})}\leq\|q\|_{L^2(B^{\frac{N}{2}})}\|u\|_{L^2(B^{\frac{N}{2}})},$$

$$\|qu\|_{L^1(B^{\frac{N}{2}+1})}\leq\|q\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}+\|q\|_{L^2(B^{\frac{N}{2}+1})}\|u\|_{L^2(B^{\frac{N}{2}})}.$$

Because $\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}\hookrightarrow B^{\frac{N}{2}}$ and $\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}\hookrightarrow B^{\frac{N}{2}+1}$ (from proposition 3.5), we get:

$$\begin{aligned}\|\mathrm{div}(qu)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|u\|_{L^1(B^{\frac{N}{2}+1})}+\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\\\times\|u\|_{L^2(B^{\frac{N}{2}})}.\end{aligned}$$

• We have to estimate $\|G(q,u,\pi)\|_{L^1(B^{\frac{N}{2}-1})}$. We see straight away that: $[\frac{\mu(\rho)}{\rho}-\frac{\mu(\bar{\rho})}{\bar{\rho}}]\Delta u = K(q)\Delta u$ for some smooth function K such that $K(0)=0$. Hence by propositions 6.5, 3.10 and 3.5:

$$\begin{aligned}\Big\|[\frac{\mu(\rho)}{\rho}-\frac{\mu(\bar{\rho})}{\bar{\rho}}]\Delta u\Big\|_{L^1(B^{\frac{N}{2}-1})} &\leq C\|K(q)\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})},\\ &\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}.\end{aligned}$$

In the same way we have:

$$\begin{aligned}&\|[\frac{\zeta(\rho)}{\rho}-\frac{\zeta(\bar{\rho})}{\bar{\rho}}]\nabla\mathrm{div}u\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})},\\ &\|\nabla(K(\rho)-K(\bar{\rho}))\Delta q)\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+2})},\\ &\|[\frac{P_0'(\rho)}{\rho}-\frac{P_0'(\bar{\rho})}{\bar{\rho}}]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}.\end{aligned}$$

After it remains two terms to treat:

$$\begin{aligned}\|\Big[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}&\leq C\ \|[\frac{P_1'(\rho)}{\rho}-\frac{P_1'(\bar{\rho})}{\bar{\rho}})]\bar{\theta}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\\ &+\|\frac{P_1'(\bar{\rho})}{\bar{\rho}}\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})}+\|\pi(\frac{P_1'(\rho)}{\rho}-\frac{P_1'(\bar{\rho})}{\bar{\rho}})\nabla q\|_{L^1(B^{\frac{N}{2}-1})},\end{aligned}$$

$$\begin{aligned}\|\Big[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}&\leq C\|q\|_{L^\infty(B^{\frac{N}{2}-1})}\|q\|_{L^1(B^{\frac{N}{2}+1})}\\ &+\|\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})}+\|K_1(q)\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})},\end{aligned}$$

According to proposition 6.5, we have:

$$\begin{aligned}&\|\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\leq\|\pi\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}-1})}\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})},\\ &\|K_1(q)\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|\pi\nabla q\|_{L^1(B^{\frac{N}{2}-1})}.\end{aligned}$$

Therefore:

$$\begin{aligned}\|[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}&\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}\\ &+(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})})\|\pi\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}-1})}\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}.\end{aligned}$$

In the same spirit:

$$\|(\frac{P_1(\rho)}{\rho}-\frac{P_1(\bar{\rho})}{\bar{\rho}})\nabla\theta\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\pi\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}\,,$$
$$\leq\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|\pi_{BF}\|_{L^2(B^{\frac{N}{2}})}+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\pi_{HF}\|_{L^1(B^{\frac{N}{2}})}\,,$$

where we have: $\pi_{BF}=\sum_{l\leq 0}\Delta_l\pi$ and $\pi_{HF}=\sum_{l>0}\Delta_l\pi$. Next we have the following term:

$$\|u^*.\nabla u\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|u\|_{L^\infty(B^{\frac{N}{2}-1})}\|u\|_{L^1(B^{\frac{N}{2}+1})}.$$

And finally we have the terms coming from $\text{div}(D)$ which are of the form:

$$\|\frac{\lambda'(\rho)\nabla\rho\,\text{div}u}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}\leq\|L(q)\nabla\rho\,\text{div}u\|_{L^1(B^{\frac{N}{2}-1})}$$
$$+\|\frac{\lambda'(\bar{\rho})}{\bar{\rho}}\nabla\rho\,\text{div}u\|_{L^1(B^{\frac{N}{2}-1})},$$

where we have set:

$$L(x_1)=\frac{\lambda'(\bar{\rho}(1+x_1))}{\bar{\rho}(1+x_1)}-\frac{\lambda'(\bar{\rho})}{\bar{\rho}}.$$

Afterwards we can apply proposition 6.5 to get:

$$\|\nabla\rho\,\text{div}u\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(B^{\frac{N}{2}})},$$
$$\|L(q)\nabla\rho\text{div}u\|_{L^1(B^{\frac{N}{2}-1})}\leq\|L(q)\|_{L^\infty(B^{\frac{N}{2}})}\|\nabla\rho\,\text{div}u\|_{L^1(B^{\frac{N}{2}-1})}.$$

As we assumed that (H) is satisfied, we have by using proposition 6.5:

$$\|L(q)\|_{L^\infty(B^{\frac{N}{2}})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\,.$$

So we have:

$$\|\frac{\lambda'(\rho)\nabla\rho\,\text{div}u}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}$$
$$\leq C\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\times(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}).$$

In the same way we have by using propositions 3.10, 6.5 and 3.5:

$$\|\frac{(du+\nabla u)\nabla\rho\,\mu'(\rho)}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}\leq$$
$$C\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(B^{\frac{N}{2}})}\times(1+\|q\|_{L^\infty(B^{\frac{N}{2}})}).$$

$$\|\nabla(\frac{K'_\rho}{2}|\nabla\rho|^2)\|_{L^1(B^{\frac{N}{2}-1})} \leq C\|(\frac{K'_\rho}{2}-\frac{K'_{\bar{\rho}}}{2})|\nabla\rho|^2)\|_{L^1(B^{\frac{N}{2}})} + \|\frac{K'_{\bar{\rho}}}{2}|\nabla\rho|^2\|_{L^1(B^{\frac{N}{2}})}$$
$$\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|q\|^2_{L^2(B^{\frac{N}{2}+1})} + \|q\|^2_{L^2(B^{\frac{N}{2}+1})}.$$

Bounding $\|H(q,u,\pi)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$ goes along the same lines, we just give one example.

$$\|\frac{\mathrm{div}(\chi(\rho)\nabla\theta)}{\rho}-\frac{\bar{\chi}}{\bar{\rho}}\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$$
$$\leq \|K(q)\mathrm{div}(K_1(q)\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$$
$$+\|\mathrm{div}(K_1(q)\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}+\|K(q)\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$$
$$\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\pi\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}\big(2+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\big)\ .$$

Finally by using (4.24), (4.25) and all the previous bound, we get:

$$\|\psi(q,u,\pi)\|_{E^{\frac{N}{2}}} \leq C((C+1)\eta+R)^2. \tag{4.26}$$

Let c be such that $\|\cdot\|_{B^{\frac{N}{2}}} \leq c$ implies that: $\|\cdot\|_{L^\infty} \leq 1/3$. Then we choose R and η such that:

$$R \leq \inf((4C)^{-1},c,1),\ \text{and}\ \ \eta \leq \frac{\inf(R,c)}{C+1}\ .$$

So $(\mathcal{H})$ is verified and we have by using (4.26): $\psi(B(0,R)) \subset B(0,R)$.

2) Second step: Property of contraction

We consider (q'_1,u'_1,π'_1), (q'_2,u'_2,π'_2) in $B(0,R)$ where we note: $\theta_i=\pi_i+\bar{\theta}$, $\widetilde{T}_i=\Psi^{-1}(\theta_i)$ and we set:

$$(\delta q=q'_2-q'_1,\ \delta u=u'_2-u'_1,\ \delta\pi=\pi'_2-\pi'_1)\ .$$

We have according to proposition 4.3 and (4.21):

$$\|\psi_{(q_L,u_L,\pi_L)}(q_2^{'},u_2^{'},\pi_2^{'})-\psi_{(q_L,u_L,\pi_L)}(q_1^{'},u_1^{'},\pi_1^{'})\|_{E^{\frac{N}{2}}}\leq$$
$$C\big(\|F(q_2,u_2,\pi_2)-F(q_1,u_1,\pi_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|G(q_2,u_2,\pi_2)-G(q_1,u_1,\pi_1)\|_{L^1(B^{\frac{N}{2}-1})}$$
$$+\|H(q_2,u_2,\pi_2)-H(q_1,u_1,\pi_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\big).$$

As an example, let us bound $\|F(q_2,u_2,\pi_2)-F(q_1,u_1,\pi_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$. We have:

$$\|F(q_2,u_2,\pi_2)-F(q_1,u_1,\pi_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$\leq\|\mathrm{div}((q_2-q_1)u_2)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|\mathrm{div}(q_1(u_2-u_1))\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$
$$\leq C\big(\|\delta q\|_{L^2(B^{\frac{N}{2}})}\|u_2\|_{L^2(B^{\frac{N}{2}})}+\|\delta q\|_{L^\infty(B^{\frac{N}{2}})}\|u_2\|_{L^1(B^{\frac{N}{2}+1})}$$
$$+\|\delta q\|_{L^2(B^{\frac{N}{2}+1})}\|u_2\|_{L^2(B^{\frac{N}{2}})}+\|q_1\|_{L^2(B^{\frac{N}{2}})}\|\delta u\|_{L^2(B^{\frac{N}{2}})}$$
$$+\|q_1\|_{L^\infty(B^{\frac{N}{2}})}\|\delta u\|_{L^1(B^{\frac{N}{2}+1})}+\|q_1\|_{L^2(B^{\frac{N}{2}+1})}\|\delta u\|_{L^2(B^{\frac{N}{2}})}.\big)$$

Bounding the following terms $\|G(q_2,u_2,\pi_2)-G(q_1,u_1,\pi_1)\|_{L^1(B^{\frac{N}{2}-1})}$ and $\|H(q_2,u_2,\pi_2)-H(q_1,u_1,\pi_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$ goes along the same lines. The details are left to the reader. So we get in using the proposition 4.3:

$$\|\Psi(q_2^{'},u_2^{'},\pi_2^{'})-\Psi(q_1^{'},u_1^{'},\pi_1^{'})\|_{E^{\frac{N}{2}}}\leq C\,\|(\delta q,\delta u,\delta\pi)\|_{E^{\frac{N}{2}}}\Big(\|(q_1^{'},u_1^{'},\pi_1^{'})\|_{E^{\frac{N}{2}}}$$
$$+\|(q_2^{'},u_2^{'},\pi_2^{'})\|_{E^{\frac{N}{2}}}+2\|(q_L,u_L,\pi_L)\|_{E^{\frac{N}{2}}}\Big).$$

If one chooses R small enough, we end up with:

$$\|\Psi(q_2^{'},u_2^{'},\pi_2^{'})-\Psi(q_1^{'},u_1^{'},\pi_1^{'})\|_{E^{\frac{N}{2}}}\leq\frac{3}{4}\,\|(\delta q,\delta u,\delta\pi)\|_{E^{\frac{N}{2}}}.$$

We thus have the property of contraction and so by the fixed point theorem, there exists a solution to (NK).

3)Uniqueness of the solution:

Uniqueness stems from arguments similar to those which have been used in the proof of contraction. □

We treat now the specific case of $N=2$, where we need more regularity for the initial data because we cannot use the proposition 3.10 in the case $N=2$ with the previous initial data. Indeed we cannot treat some non-linear terms such as $\|\pi\operatorname{div}u\|_{L^1(\widetilde{B}^{0,-1})}$ or $\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{0,-1})}$ because if we want to use proposition 3.10, we are in the case $s_1+s_2=0$. This is the reason why more regularity is required.

Proof of theorem 2.5

The proof is similar to the previous one except that we have changed the functional space, in which the fixed point theorem is applied. Arguing as before, we get:

$$\begin{aligned}\|\psi(q,u,\pi)\|_{E'}\leq\ C\eta+&\|F(q,u,\pi)\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})}+\|G(q,u,\pi)\|_{L^1(\widetilde{B}^{0,\varepsilon'})}\\&+\|H(q,u,\pi)\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}\,.\end{aligned}$$

if $\|q_0\|_{\widetilde{B}^{0,1+\varepsilon'}}+\|u_0\|_{\widetilde{B}^{0,\varepsilon'}}+\|\pi_0\|_{\widetilde{B}^{0,-1+\varepsilon'}}\leq\eta$. Let us estimate

$$\|F(q,u,\pi)\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})},\ \|G(q,u,\pi)\|_{L^1(\widetilde{B}^{0,\varepsilon'})}\ \text{and}\ \|H(q,u,\pi)\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})},$$

we just give two examples, the other estimates are left to the reader.

$$\|\operatorname{div}(qu)\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})}\leq\|qu\|_{L^1(B^1)}+\|qu\|_{L^1(B^{2+\varepsilon'})},$$

and:

$$\begin{aligned}&\|qu\|_{L^1(B^1)}\leq C\|q\|_{L^2(B^1)}\|u\|_{L^2(B^1)},\\&\|qu\|_{L^1(B^{2+\varepsilon'})}\leq C\|q\|_{L^\infty(B^1)}\|u\|_{L^1(B^{2+\varepsilon'})}+\|q\|_{L^2(B^{2+\varepsilon'})}\|u\|_{L^2(B^1)}.\end{aligned}$$

We do similarly for $\|G(q,u,\pi)\|_{L^1(\widetilde{B}^{0,\varepsilon'})}$. The new difficulty appears on the last term $\|H(q,u,\pi)\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}$. In fact it's only for this term that additional regularity is needed. Proposition 3.10 enables us to get:

$$\begin{aligned}&\|\pi\operatorname{div}u\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}\leq C\|\pi\|_{L^\infty(\widetilde{B}^{0,-1+\varepsilon'})}\|u\|_{L^1(B^2)}.\\&\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}\leq C\|\pi\|_{L^1(\widetilde{B}^{1,1+\varepsilon'})}\|u\|_{L^\infty(B^0)}.\end{aligned}$$

To conclude we follow the previous proof. Uniqueness goes along the lines of the proof of uniqueness in dimension $N\geq3$. □

5. **Local theory for large data**

In this part we are interested in results of existence in finite time for general initial data with density bounded away from zero. We focus on the case where the coefficients depend only on the density with linear specific energy, and next we will treat the general case. As a first step, we shall study the linear part of the system (NK) about non constant reference density and temperature, that is:

$$(N)\qquad \begin{cases} \partial_t q + \mathrm{div} u = F, \\ \partial_t u - \mathrm{div}(a\nabla u) - \nabla(b\mathrm{div} u) - \nabla(c\Delta q) = G, \\ \partial_t \pi - \mathrm{div}(d\nabla\pi) = H, \end{cases}$$

5.1. **Study of the linearized equation**

We want to prove a priori estimates for system (N) with the following hypotheses on a, b, c, d:

$$0 < c_1 \le a < M_1 < \infty, \quad 0 < c_2 \le a + b < M_2 < \infty,$$
$$0 < c_3 \le c < M_3 < \infty, \quad 0 < c_4 \le d < M_4 < \infty.$$

We remark that the last equation is just a heat equation with variable coefficients so that one can apply the following proposition proved in [7].

Proposition 5.1. *Let π solution of the heat equation:*

$$\partial_t \pi - \mathrm{div}(d\nabla\pi) = H,$$

we have so for all index τ such that $-\frac{N}{2} - 1 < \tau \le \frac{N}{2} - 1$ the following estimate for all $\alpha \in [1, +\infty]$:

$$\|\pi\|_{\widetilde{L}^{\alpha}_T(B^{\tau+\frac{2}{\alpha}})} \le \|\pi_0\|_{B^{\tau}} + \|H\|_{\widetilde{L}^1_T(B^{\tau})} + \|\nabla d\|_{\widetilde{L}^{\infty}_T(B^{\frac{N}{2}-1})} \|\nabla\pi\|_{\widetilde{L}^1_T(B^{\tau+1})}.$$

We are now interested by the first two equations of the system (N).

$$(N')\qquad \begin{cases} \partial_t q + \mathrm{div} u = F \\ \partial_t u - \mathrm{div}(a\nabla u) - \nabla(b\mathrm{div} u) - \nabla(c\Delta q) = G \end{cases}$$

where we keep the same hypothesis on a, b and c. Here at the difference with the case of global solution, it is crucial to study a linear part with variable coefficient a, b, c ; indeed we could not use some arguments of boobstrap linked with the smalness hypothesis. We have then the following estimate of the solution in the spaces of Chemin-Lerner:

Proposition 5.2. *Let* $1 \leq r_1 \leq r \leq +\infty$, $0 \leq s \leq 1$, $(q_0,u_0) \in B^{\frac{N}{2}+s} \times (B^{\frac{N}{2}-1+s})^N$, *and* $(F,G) \in \widetilde{L}^{r_1}_T(B^{\frac{N}{2}-2+s+2/r_1}) \times (\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+2/r_1}))^N$. *Suppose that* ∇a , ∇b , ∇c *belong to* $\widetilde{L}^2_T(B^{\frac{N}{2}})$ *and that* $\partial_t c \in L^1_T(L^\infty)$. *Let* $(q,u) \in (\widetilde{L}^r_T(B^{\frac{N}{2}+s+2/r}) \cap \widetilde{L}^2_T(B^{\frac{N}{2}+s+1})) \times ((\widetilde{L}^r_T(B^{\frac{N}{2}+s-1+2/r}))^N \cap (\widetilde{L}^2_T(B^{\frac{N}{2}+s})^N)$ *be a solution of the system* (N'). *Then there exists a constant* C *depending only on* r, r_1, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\kappa}$, c_1, c_2, M_1 *and* M_2 *such that:*

$$\begin{aligned}\|(\nabla q,u)\|_{\widetilde{L}^r_T(B^{\frac{N}{2}-1+s+\frac{2}{r}})}(1-C\|\nabla c\|_{L^2_T(L^\infty)}) \leq \|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}}} \\ +\|(\nabla F,G)\|_{\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+\frac{2}{r_1}})}+\|\nabla q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s})}\|\partial_t c\|_{L^1_T(L^\infty)} \\ +\|(\nabla q,u)\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+s})}\big(\|\nabla a\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}+\|\nabla b\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}+\|\nabla c\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\big)\,.\end{aligned}$$

Proof. Like previously we are going to show estimates on q_l and u_l. So we apply to the system the operator Δ_l , and we have then:

$$\partial_t q_l + \mathrm{div} u_l = F_l \tag{5.1}$$
$$\partial_t u_l - \mathrm{div}(a\nabla u_l) - \nabla(b\,\mathrm{div}u_l) - \nabla(c\Delta q_l) = G_l + R_l \tag{5.2}$$

where we denote:

$$R_l = \mathrm{div}([a,\Delta_l]\nabla u) - \nabla([b,\Delta_l]\mathrm{div}u_l) - \nabla([c,\Delta_l]\Delta q).$$

Performing integrations by parts and using (5.1) we have:

$$\begin{aligned}-\int_{\mathbb{R}^N} u_l\nabla(c\Delta q_l)dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} c|\nabla q_l|^2 dx - \int_{\mathbb{R}^N}\big(\mathrm{div}u_l\,(\nabla q_l.\nabla c) \\ +\frac{|\nabla q|^2}{2}\partial_t c + c.\nabla q_l.\nabla F_l\big)\,dx.\end{aligned}$$

Next, we take the inner product of (5.2) with u_l and we use the previous equality, we have then:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_l\|^2_{L^2} + \int_{\mathbb{R}^N} c|\nabla q_l|^2 dx) + \int_{\mathbb{R}^N}(a|\nabla u_l|^2 + b|\mathrm{div}u_l|^2)dx = \\ \int_{\mathbb{R}^N}((G_l+R_l).u_l\,dx + \int_{\mathbb{R}^N}\big((\mathrm{div}u_l(\nabla c.\nabla q_l) + \frac{|\nabla q_l^2|}{2}\partial_t c \\ + c\nabla q_l.\nabla F_l)\big)\,dx\,.\end{aligned} \tag{5.3}$$

In order to recover some terms in Δq_l we take the inner product of the gradient of (5.1) with u_l, the inner product of (5.2) with ∇q_l and we sum, we obtain then:

$$\frac{d}{dt}\int_{\mathbb{R}^N}\nabla q_l.u_l dx+\int_{\mathbb{R}^N}c(\Delta q_l)^2dx=\int_{\mathbb{R}^N}((G_l+R_l).\nabla q_l+|\text{div}u_l|^2 \\ +u_l.\nabla F_l-a\nabla u_l:\nabla^2 q_l-b\Delta q_l\text{div}u_l)dx. \tag{5.4}$$

Let $\alpha>0$ small enough. We define:

$$k_l^2=\|u_l\|_{L^2}^2+\int_{\mathbb{R}^N}(\bar{\kappa}c|\nabla q_l|^2+2\alpha\nabla q_l.u_l)dx\ . \tag{5.5}$$

By using the previous inequality (5.3), (5.4) and the fact that $a_1b_1\leq\frac{1}{2}a_1^2+\frac{1}{2}b_1^2$, we have by summing:

$$\frac{1}{2}\frac{d}{dt}k_l^2+\frac{1}{2}\int_{\mathbb{R}^N}(a|\nabla u_l|^2+\alpha b|\Delta q_l|^2)dx\leq C(\|G_l\|_{L^2}+\|R_l\|_{L^2}) \\ \times(\alpha\|\nabla q_l\|_{L^2}+\|u_l\|_{L^2})+\|\nabla F_l\|_{L^2}(\alpha\|u_l\|_{L^2}+\|c\nabla q_l\|_{L^2}) \\ +\frac{1}{2}\|\partial_t c\|_{L^\infty}\|\nabla q_l\|_{L^2}^2+\|\nabla c\|_{L^\infty}\|\nabla q_l\|_{L^2}\|\nabla u_l\|_{L^2}\ . \tag{5.6}$$

For small enough α, we have according (5.5):

$$\frac{1}{2}k_l^2\leq\|u_l\|^2+\int_{\mathbb{R}^N}\bar{\kappa}c|\nabla q_l|^2dx\leq\frac{3}{2}k_l^2\ , \tag{5.7}$$

Hence, according to (5.6) and (5.7):

$$\frac{1}{2}\frac{d}{dt}k_l^2+K2^{2l}k_l^2\leq\ k_l\,(\|G_l\|_{L^2}+\|R_l\|_{L^2}+\|\nabla F_l\|_{L^2})\|\partial_t c\|_{L^\infty}\|\nabla q_l\|_{L^2} \\ +2^{2l}k_l^2\|\nabla c\|_{L^2}\ .$$

By integrating with respect to the time, we obtain:

$$k_l(t)\leq e^{-K2^{2l}t}k_l(0)+C\int_0^t e^{-K2^{2l}(t-\tau)}(\|\partial_t c\|_{L^\infty}\|\nabla q_l(\tau)\|_{L^2}+\|\nabla F_l(\tau)\|_{L^2} \\ +\|G_l(\tau)\|_{L^2}+\|R_l(\tau)\|_{L^2}+2^l k_l(\tau)\|\nabla c(\tau)\|_{L^2})d\tau\ .$$

Hence, convolution inequalities imply that:

$$\|k_l\|_{L^r([0,T])}\leq\ (2^{-\frac{2l}{r}}k_l(0)+(2^{-2l(1+1/r-1/r_1)}\|(\nabla F_l,G_l)\|_{L_T^{r_1}(L^2)} \\ +2^{-\frac{2l}{r}}\|R_l\|_{L_T^1(L^2)}+2^{-\frac{2l}{r}}\|\nabla q_l\|_{L_T^\infty(L^2)}\|\partial_t c\|_{L_T^1(L^\infty)}+\|\nabla c\|_{L_T^2(L^\infty)} \\ \times\|k_l\|_{L^r([0,T])}\ . \tag{5.8}$$

Finally, by multiplying by $2^{(\frac{N}{2}-1+s+\frac{2}{r})l}$ and using (5.7), we end up with:

$$\|(\nabla q,u)\|_{L^r_T(B^{\frac{N}{2}-1+s+2/r})}(1-C\|\nabla c\|_{L^2(L^\infty)})\leq \|(\nabla F,G)\|_{\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+2/r_1})}$$
$$\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1+s}}+\|\nabla q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s})}\|\partial_t c\|_{L^1_T(L^\infty)}$$
$$+\sum_{l\in\mathbb{Z}}2^{l(\frac{N}{2}+s-1)}\|R_l\|_{L^1_T(L^2)}\,.$$

Finally, applying lemma 6.4 on the appendix to bound the remainder term completes the proof. □

5.2. Local existence Theorem for temperature independent coefficients

We will now prove the local existence of a solution for general initial data with a linear specific intern energy and coefficients independent of the temperature. The functional space we shall work with is larger than previously, the reason why is that the low frequencies don't play an important role as far as one is interested in *local* results. Moreover at the difference with the proof of global existence of strong solutions, we change of strategy to resolve the local existence of strong solutions. Indeed we will construct Cauchy sequence of approximate solutions using an iterative scheme. Indeed the linear system $(N)'$ has variable function coefficients a, b, c which are in reality some function depending of the solution q, so it appears more easy to use an iterative scheme. Moreover we recall that we have studied a linear part with variable coefficients to can get the uniform bound estimates (indeed no arguments of boobstrap linked with the smalness is possible in this case).

Proof of the theorem 2.6:

In what follows, $N\geq 3$ is assumed. Let: $q^n=q^0+\bar{q}^n,\ \rho^n=\bar{\rho}(1+q^n),\ u^n=u^0+\bar{u}^n,\ \pi^n=\pi^0+\bar{\pi}^n$ and $\theta^n=\bar{\theta}+\pi^n$ where (q^0,u^0,π^0) stands for the solution of:

$$\begin{cases}\partial_t q^0-\Delta q^0=0,\\ \partial_t u^0-\Delta u^0=0,\\ \partial_t \pi^0-\Delta\pi^0=0,\end{cases}$$

supplemented with initial data:

$$q^0(0)=q_0\ ,\ u^0(0)=u_0\ ,\ \pi^0(0)=\pi_0.$$

Let $(\bar{q}_n,\bar{u}_n,\bar{\pi}_n)$ be the solution of the following system:
(N_1)

$$\begin{cases}\partial_t\bar{q}_{n+1}+\mathrm{div}(\bar{u}_{n+1})=F_n,\\ \partial_t\bar{u}_{n+1}-\mathrm{div}\Big(\dfrac{\mu(\rho^n)}{\rho^n}\nabla\bar{u}^{n+1}\Big)-\nabla\Big(\dfrac{\zeta(\rho^n)}{\rho^n}\mathrm{div}(\bar{u}^{n+1})\Big)\\ \qquad\qquad\qquad\qquad\qquad\qquad-\nabla(K(\rho^n)\Delta\bar{q}^{n+1})=G_n,\\ \partial_t\bar{\pi}_{n+1}-\mathrm{div}\Big(\dfrac{\chi(\rho^n)}{1+q^n}\bar{\pi}_{n+1}\Big)=H_n,\\ (\bar{q}_{n+1},\bar{u}_{n+1},\bar{\pi}_{n+1})_{t=0}=(0,0,0),\end{cases}$$

where:

$$\begin{aligned}F_n=&-\mathrm{div}(q^nu^n)-\Delta q^0-\mathrm{div}(u^0),\\ G_n=&-(u^n)^*.\nabla u^n+\nabla(\frac{K^{'}_{\rho^n}}{2}|\nabla\rho^n|^2)-\nabla(\frac{\mu(\rho^n)}{\rho^n})\mathrm{div}u^n+\nabla(\frac{\zeta(\rho^n)}{\rho^n})\mathrm{div}u^n\\ &+\frac{\lambda^{'}(\rho^n)\nabla\rho^n\mathrm{div}u^n}{1+q^n}+\frac{(du^n+\nabla u^n)\,\mu^{'}(\rho^n)\nabla\rho^n}{1+q^n}+\Big[\frac{P_1(\rho^n)}{\rho^n\psi'(\widetilde{T}^n)}\Big]\nabla\theta^n\\ &+\frac{[P_0^{'}(\rho^n)+\widetilde{T}^nP_1^{'}(\rho^n)]\nabla q^n}{1+q^n}-\Delta u^0+\mathrm{div}\Big(\frac{\mu(\rho^n)}{1+q^n}\nabla u^0\Big)\\ &+\nabla\Big(\frac{\mu(\rho^n)+\lambda(\rho^n)}{1+q^n}\mathrm{div}(u^0)\Big)+\nabla(K(\rho^n)\Delta q^0),\end{aligned}$$

$$\begin{aligned}H_n=&\nabla(\frac{1}{1+q^n}).\nabla\theta^n\chi(\rho^n)-\frac{\widetilde{T}^nP_1(\rho^n)}{\rho^n}\mathrm{div}u^n-(u^n)^*.\nabla\theta^n+\frac{D_n:\nabla u^n}{\rho^n}\\ &-\Delta\theta_0+\mathrm{div}(\frac{\chi(\rho^n)}{1+q^n}\nabla\theta^0).\end{aligned}$$

1) First Step , Uniform Bound

Let ε be a small parameter and choose T small enough so that in using the estimate of the heat equation stated in proposition 3.9 we have:

$$(\mathcal{H}_\varepsilon)\qquad \begin{aligned}&\|\pi^0\|_{L^1_T(B^{\frac{N}{2}})}+\|u^0\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^0\|_{L^1_T(B^{\frac{N}{2}+2})}\leq\varepsilon,\\&\|\pi^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})}+\|u^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})}+\|q^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})}\leq A_0.\end{aligned}$$

We are going to show by induction that:

$$(\mathcal{P}_n)\qquad \|(\bar{q}^n,\bar{u}^n,\bar{\pi}^n)\|_{F_T}\leq\varepsilon.$$

As $(\bar{q}_0,\bar{u}_0,\bar{\pi}_0)=(0,0,0)$ the result is true for $n=0$. We suppose now $(\mathcal{P}_n)$ true and we are going to show $(\mathcal{P}_{n+1})$.
To begin with we are going to show that $1+q^n$ is positive. Using the fact that $B^{\frac{N}{2}}\hookrightarrow L^\infty$ and that we take ε small enough , we have for $t\in[0,T]$:

$$\begin{aligned}\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq C\|\mathrm{div}\bar{u}^n\|_{L^1_T(B^{\frac{N}{2}})}+\|\mathrm{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})}\\+\|\mathrm{div}u^0\|_{L^1_T(B^{\frac{N}{2}})},\leq C\big(2\varepsilon+\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\big),\end{aligned}$$

and:

$$\begin{aligned}\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\leq\|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\\+\|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}})}.\end{aligned}$$

Hence:

$$\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq C_1(2\varepsilon+(A_0+\varepsilon)\varepsilon).$$

Finally we thus have:

$$\begin{aligned}\|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}-\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq 1+q^n\\\leq\|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}+\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)},\end{aligned}$$

whence if ε is small enough:

$$\frac{c}{2\bar{\rho}}\leq 1+q^n\leq 1+\frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}}.$$

In order to bound $(\bar{q}^n,\bar{u}^n,\bar{\pi}^n)$ in F_T, we shall use proposition 5.2. For that we must check that the different hypotheses of this proposition adapted to our system (N_1) are satisfied, so we study the following terms:

$$a^n=\frac{\mu(\rho^n)}{1+q^n}\ ,\ b^n=\frac{\zeta(\rho^n)}{1+q^n}\ ,\ c^n=K(\rho_n)\ ,\ d^n=\frac{\chi(\rho^n)}{1+q^n}.$$

By using $(\mathcal{P}_n)$ and by continuity of μ and the fact that μ is positive on $\big[\bar{\rho}(1+\min(q_0))-\alpha,\bar{\rho}(1+\max(q_0))+\alpha\big]$, we have:

$$0<c_1\leq a^n=\frac{\mu(\rho^n)}{1+q^n}\leq M_1\;.$$

We proceed similarly for the others terms. Next, notice that hypotheses of proposition 5.2 are verified:

$$\begin{aligned}
&\|\nabla a^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq\|\frac{\mu(\rho^n)}{1+q^n}-\mu(\bar{\rho})\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.\\
&\|\nabla b^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq\|\frac{\zeta(\rho^n)}{1+q^n}-\zeta(\bar{\rho})\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\\
&\|\nabla c^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\;.
\end{aligned}$$

To end on our hypotheses we have to control $\partial_t c^n$ in norm $\|\cdot\|_{L^1_T(L^\infty)}$. As $B^{\frac{N}{2}}\hookrightarrow L^\infty$, it actually suffices to bound $\|\partial_t c^n\|_{L^1_T(B^{\frac{N}{2}})}$. We have:

$$\partial_t c^n=K^{'}(\rho^n)\partial_t q^n=K^{'}(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1})-\mathrm{div}(u^n))\;.$$

And we have by using the propositions 3.10 and 6.5:

$$\begin{aligned}
&\|K^{'}(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1})-\mathrm{div}(u^n))\|_{L^1_T(B^{\frac{N}{2}})}\\
&\leq\|K^{'}(\rho^n)\mathrm{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})}+\|K^{'}(\rho^n)\mathrm{div}(u^n)\|_{L^1_T(B^{\frac{N}{2}})},\\
&\leq C(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})})\\
&\leq C(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\\
&\qquad\qquad+\|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}})})\;.
\end{aligned}$$

We now use proposition 5.1 to get the bound on $\bar{\pi}^n$, so we obtain by taking $\tau=\frac{N}{2}-2$:

$$\begin{aligned}
\|\pi^n\|_{L^1_T(B^{\frac{N}{2}})\cap L^\infty_T(B^{\frac{N}{2}-2})}\leq C\big(&\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})}\\
&+\|\nabla(\frac{\chi(\rho^n)}{\rho^n})\|_{L^\infty_T(B^{\frac{N}{2}-1})}\|\pi^n\|_{L^1_T(B^{\frac{N}{2}})}\big).
\end{aligned}\tag{5.9}$$

So we need to bound d^n in $L^\infty_T(B^{\frac{N}{2}})$:

$$\|\nabla d^n\|_{L^\infty_T(B^{\frac{N}{2}-1})}\leq C\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}.$$

Finally, applying propositions 5.2 and 5.1, we conclude that:

$$\|(\bar{q}^{n+1},\bar{u}^{n+1},\bar{\pi}^{n+1})\|_{F_T}(1-C(\|a^n\|_{L^2_T(B^{\frac{N}{2}+1})}+\|b^n\|_{L^2_T(B^{\frac{N}{2}+1})}$$
$$+\|c^n\|_{L^2_T(B^{\frac{N}{2}+1})}+\|d^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\partial_t c^n\|_{L^1_T(B^{\frac{N}{2}})})) \leq \quad (5.10)$$
$$\|(\nabla F_n,G_n)\|_{L^1_T(B^{\frac{N}{2}-1})}+\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})}.$$

Bounding the right-hand side may be done by applying propositions 3.10 and 6.5. For instance, we have:

$$\|F_n\|_{L^1_T(B^{N/2})}\leq\|\mathrm{div}(q^nu^n)\|_{L^1_T(B^{N/2})}+\|\mathrm{div}u^0\|_{L^1_T(B^{N/2})}+\|\Delta q^0\|_{L^1_T(B^{N/2})}.$$

Since:

$$\|u^nq^n\|_{L^1_T(B^{N/2+1})}\leq C\big(|q^n\|_{L^\infty_T(B^{N/2})}\|u^n\|_{L^1_T(B^{N/2+1})}+\|q^n\|_{L^2_T(B^{N/2+1})}$$
$$\times\|u^n\|_{L^2_T(L^\infty)}\big),$$

we can conclude that:

$$\|F_n\|_{L^1_T(B^{N/2})}\leq C(A_0+\varepsilon+\sqrt{\varepsilon})^2.$$

Bounding $\|G_n\|_{L^1_T(B^{\frac{N}{2}-1})}$ and $\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})}$ goes along the same lines. After an easy calculation, we obtain by using (5.10) and the different previous inequalities:

$$\|(\bar{q}_{n+1},\bar{u}_{n+1},\bar{\pi}_{n+1})\|_{F_T}(1-C2\sqrt{\varepsilon}(A_0+\sqrt{\varepsilon}))$$
$$\leq C_1\big(\varepsilon(A_0+\sqrt{\varepsilon})^2+T(A_0+\sqrt{\varepsilon})\big).$$

By taking T and ε small enough we have $(\mathcal{P}_{n+1})$, so we have shown by induction that (q^n,u^n,π^n) is bounded in F_T.

Second Step: Convergence of the sequence and uniqueness

We will show that (q^n,u^n,π^n) is a Cauchy sequence in the Banach space F_T, hence converges to some $(q,u,\pi)\in F_T$. Let:

$$\delta q^n=q^{n+1}-q^n,\ \delta u^n=u^{n+1}-u^n,\ \delta\pi^n=\pi^{n+1}-\pi^n.$$

The system verified by $(\delta q^n, \delta u^n, \delta\pi^n)$ reads:

$$\begin{cases} \partial_t \delta q^n + \mathrm{div}\delta u^n = F_n - F_{n-1}, \\ \partial_t \delta u^n - \mathrm{div}\Big(\dfrac{\mu(\rho^n)}{\rho^n}\nabla\delta u^n\Big) - \nabla\Big(\dfrac{\zeta(\rho^n)}{\rho^n}\mathrm{div}(\delta u^n)\Big) - \nabla(K(\rho^n)\Delta\delta q^n) = \\ \hfill G_n - G_{n-1} + G_n^{'} - G_{n-1}^{'}, \\ \partial_t \delta\pi^n - \mathrm{div}\Big(\dfrac{\chi(\rho^n)}{1+q^n}\nabla\delta\pi^n\Big) = H_n - H_{n-1} + H_n^{'} - H_{n-1}^{'}, \end{cases}$$

where we define:

$$G_n^{'} = -\mathrm{div}((\frac{\mu(\rho^{n+1})}{\rho^{n+1}} - \frac{\mu(\rho^n)}{\rho^n})\nabla u^{n+1}) - \nabla((K(\rho^{n+1}) - K(\rho^n))\Delta q^{n+1})$$
$$- \nabla((\frac{\zeta(\rho^{n+1})}{\rho^{n+1}} - \frac{\zeta(\rho^n)}{\rho^n})\mathrm{div}(u^{n+1})).$$

In the same way we have:

$$H_n^{'} = \mathrm{div}((\frac{\chi(\rho^{n+1})}{1+q^{n+1}} - \frac{\chi(\rho^n)}{1+q^n})\nabla\theta^{n+1}).$$

Applying propositions 5.1, 5.2, and using (P_n), we get:

$$\|(\delta q^n, \delta u^n, \delta\pi^n)\|_{F_T} \leq C(\|F_n - F_{n-1}\|_{L^1_T(B^{N/2})} + \|G_n - G_{n-1}$$
$$+ G_n^{'} - G_{n-1}^{'}\|_{L^1_T(B^{N/2-1})} + \|H_n - H_{n-1} + H_n^{'} - H_{n-1}^{'}\|_{L^1_T(B^{N/2-2})}),$$

And by the same type of estimates as before, we get:

$$\|(\delta q^n, \delta u^n, \delta\pi^n)\|_{F_T} \leq C\sqrt{\varepsilon}(1+A_0)^3\|(\delta q^{n-1}, \delta u^{n-1}, \delta\pi^{n-1})\|_{F_T}.$$

So by taking ε enough small we have that (q^n, u^n, π^n) is Cauchy sequence, the limit (q, u, π) is in F_T and we verify easily that this is a solution of the system. Uniqueness stems from similar arguments . □

Proof of the theorem 2.7

In the special case $N = 2$, we need to take more regular initial data for the same reasons as in theorem 2.5. Indeed some terms like $\Psi(\theta)\mathrm{div}u$ or $u^*.\nabla\theta$ can't be controlled without more regularity.
The proof is similar to the previous proof of theorem 2.6 except that we have changed the functional space $F_T(2)$, in which the fixed point theorem

is going to be applied. As we explain above we can use the paraproduct because we have more regularity. For instance, one may write:

$$\|u^*.\nabla\theta\|_{L^1_T(\widetilde{B}^{-1,-1+\varepsilon'})} \leq C\|\pi\|_{L^1_T(\widetilde{B}^{0,1+\varepsilon'})}\|u\|_{L^\infty_T(B^0)}.$$

□

6. Appendix

In this appendix, we are interested in generalizing our results of existence and uniqueness for the general case when the intern specific energy is a general regular strictly increasing function and when all the physical coefficients except the capillarity coefficient depend both the density and the temperature. In this general case, we have to control the temperature in norm L^∞ in the goal to use the parabolic effect on the temperature, Moreover estimating the non linear terms depending of the temperature in Besov spaces requires a control L^∞ on the temperature that's why we need to take more regular initial data to preserve the L^∞ bound. For these reasons, it seems judicious to choose the initial temperature in $B^{\frac{N}{2}}$, in consequence we have to adapt the other initial data. We can then observe that in the general case we can not reach critical initial data for the scaling of the system as we want that π_0 belongs to L^∞.

6.1. Existence of a solution in the general case with small initial data

In this section we are interested in the general case with small initial data, and we want get global strong solution. As we need more regular initial data, we have to obtain new estimates in Besov spaces on the linear system (M').

Proposition 6.1. *Under conditions of proposition 4.1 with strict inequality, let* (q,u,π) *be a solution of the system* (M') *on* $[0,T)$ *with initial conditions* (q_0,u_0,π_0) *such that:*

$$q_0 \in \widetilde{B}^{s-1,s+1}, u_0 \in \widetilde{B}^{s-1,s}, \pi_0 \in \widetilde{B}^{s-1,s}.$$

Moreover we suppose $1 \leq r_1 \leq +\infty$ *and:* $F \in \widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-1+\frac{2}{r_1}})$, $G \in \widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})$, $H \in \widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})$.

We then have the following estimate for all $r\in[r_1,+\infty]$:

$$\|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+1+\frac{2}{r}})}+\|u\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}+\|\pi\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}\leq$$
$$C\big(\|q_0\|_{\widetilde{B}^{s-1,s+1}}+\|u_0\|_{\widetilde{B}^{s-1,s}}+\|\pi_0\|_{\widetilde{B}^{s-1,s}}+\|F\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-1+\frac{2}{r_1}})}$$
$$+\|G\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})}+\|H\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})}\big).$$

Proof. The proof is similar to that of proposition 4.3. Low frequencies are treated as in proposition 4.3 because we don't change the regularity index for the low frequencies. On the other hand in the case of high frequencies the regularity index has changed so that we have to adapt the proof, it is left to the reader. For the medium frequencies we can proceed as in proposition 4.3. ☐

In the following theorem we are interested in showing the global existence of solution for Korteweg's system with general conditions and small initial data.

Theorem 6.2. *Let $N\geq 2$. Assume that Ψ be a regular function depending on θ. Assume that all the coefficients are smooth functions of ρ and θ except κ which depends only on the density. Take $(\bar{\rho},\bar{T})$ such that:*

$$\kappa(\bar{\rho})>0,\ \mu(\bar{\rho},\bar{T})>0,\ \lambda(\bar{\rho},\bar{T})+2\mu(\bar{\rho},\bar{T})>0,\ \eta(\bar{\rho},\bar{T})>0\ \partial_\rho P_0(\bar{\rho},\bar{T})>0.$$

Moreover suppose that:

$$q_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1},\ u_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ \pi_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}.$$

There exists an ε_1 depending only on the physical coefficients such that if:

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}}+\|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\pi_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\leq\varepsilon$$

then (NHV) has a unique global solution (ρ,u,π) in:

$$F^{\frac{N}{2}}=[C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})\cap L^1(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})]\times[C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})^N$$
$$\cap L^1(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})^N][C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})\cap L^1(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})].$$

Proof. The principle of the proof is similar to the proof of theorem 2.3 and we use the same notation. We define the map ψ as before with the same F, G and H except that our coefficients depends on the density and the temperature. We have ti verify that ψ maps a ball $B(0,R)$ into itself. We just give an example of estimates in the space $F^{\frac{N}{2}}$:

$$\|\,[\,\frac{\widetilde{T}\bar{\rho}\,P_1'(\rho)}{\rho}-\bar{T}P_1'(\bar{\rho})\,]\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq C\;\|L_1(q)L_2(\pi)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|L_1(q)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|L_2(\pi)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

where L_1 and L_2 are regular function in the sense of proposition 6.5. And we have:

$$\|L_1(q)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq C\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})},$$
$$\|L_2(\pi)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq C\|\pi\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}.$$

Then:

$$\|\,[\,\frac{\widetilde{T}\bar{\rho}\,P_1'(\rho)}{\rho}-\bar{T}P_1'(\bar{\rho})\,]\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq C\;\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}$$
$$\|\pi\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}+\big(\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}+\|\pi\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\big)$$
$$\times\|q\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})}.$$

The end is left to the reader. Uniqueness in $F^{\frac{N}{2}}$ goes along the same lines of the contraction. □

6.2. **Local existence theorem in the general case**

In the following theorem we consider the case of local existence and uniqueness for large data in the general case. As previously we need more regular initial data.

Theorem 6.3. *Under the hypotheses of theorem 6.2 (where we replace systematically $\mu(\bar{\rho})>0$ by $\mu(\rho)>0$), we suppose that:*

$$(q_0,u_0,\pi_0)\in\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}\times(B^{\frac{N}{2}})^N\times B^{\frac{N}{2}}\quad\textit{and}\;\;\rho_0\geq c\;\;\textit{for some}\;\;c>0.$$

Then there exists a time T such that the system has a unique solution in :

$$F_T'=[\widetilde{C}_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})\cap L^1_T(\widetilde{B}^{\frac{N}{2}+2,\frac{N}{2}+3})]\times[\widetilde{C}_T(B^{\frac{N}{2}})^N\cap L^1_T(B^{\frac{N}{2}+2})^N]$$
$$\times\,[\widetilde{C}_T(B^{\frac{N}{2}})\cap L^1_T(B^{\frac{N}{2}+2})]\;.$$

Proof. We proceed exactly like in theorem 2.6 except that we ask more regularity for the initial data. The principle of the proof being similar to the proof of theorem 2.6, we use the same notation. We will just give some examples of estimates in the space $F_T^{'}$ to get uniform bound on the sequel (q^n, u^n, θ^n), we have then:

$$\begin{aligned}\|\frac{[P_0^{'}(\rho^n)+\widetilde{T}^n P_1^{'}(\rho^n)]\nabla q^n}{1+q^n}\|_{L^1_T(B^{\frac{N}{2}})} &\leq CT\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\\ &\quad+T\|\pi^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\big(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\big)\,,\end{aligned}$$

$$\begin{aligned}\|\text{div}(\frac{\mu(\rho^n,\theta^n)}{1+q^n}\nabla u^0)\|_{L^1_T(B^{\frac{N}{2}})} &\leq C\big(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\pi^n\|_{L^\infty_T(B^{\frac{N}{2}})}\big)\\ \times\|u^0\|_{L^1_T(B^{\frac{N}{2}+2})}&+\|u^0\|_{L^2_T(B^{\frac{N}{2}+1})}\big(\sqrt{T}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}+\|\pi^n\|_{L^2_T(B^{\frac{N}{2}+1})}\big)\,.\end{aligned}$$

The end is left to the reader. Uniqueness in $F_T^{'}$ goes along the same lines of the contraction. □

6.3. Composition and commutator propositions

This part consists in one commutator lemma which enables us to conclude in proposition 5.2. Moreover we give the proof of proposition 6.5 on the composition of function in hybrid spaces adapted from Bahouri-Chemin in [2].

Lemma 6.4. *Let $0 \leq s \leq 1$. Suppose that $A \in \widetilde{L}^2_T(B^{\frac{N}{2}+1})$ and $B \in \widetilde{L}^2_T(B^{\frac{N}{2}-1+s})$. Then we have the following result:*

$$\|\partial_k[A,\Delta_l]B\|_{L^1_T(L^2)} \leq Cc_l 2^{-l(\frac{N}{2}-1+s)}\|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})},$$

with $\sum_{l\in\mathbb{Z}} c_l = 1$.

Proof. We have the following decomposition:

$$uv = T_u v + T_v^{'} u$$

where: $T_u v = \sum_{l\in\mathbb{Z}} S_{l-1}u\Delta_l v$ and: $T_v^{'}u = \sum_{l\in\mathbb{Z}} S_{l+2}v\Delta_l u$. We then have:

$$\begin{aligned}\partial_k[A,\Delta_l]B = \partial_k T^{'}_{\Delta_l B}A - \partial_k\Delta_l T^{'}_B A + [T_A,\Delta_l]\partial_k B + T_{\partial_k A}\Delta_l B&\\ - \Delta_l T_{\partial_k A}B.&\end{aligned} \tag{6.1}$$

From now on, we will denote by $(c_l)_{l\in\mathbb{Z}}$ a sequence such that: $\sum_{l\in\mathbb{Z}} c_l \leq 1$. Now we are going to treat each term of (6.1). According to the properties of quasi-orthogonality and the definition of T' we have:

$$\partial_k T'_{\Delta_l B}A = \sum_{m\geq l-2} \partial_k(S_{m+2}\Delta_l B\Delta_m A).$$

Next, by using Bernstein inequalities, we have:

$$\begin{aligned}
\|\partial_k T'_{\Delta_l B}A\|_{L^1_T(L^2)} &\leq C\sum_{m\geq l-2} 2^m \|\Delta_l B\|_{L^2_T(L^\infty)}\|\Delta_m A\|_{L^2_T(L^2)},\\
&\leq C2^{l\frac{N}{2}}\|\Delta_l B\|_{L^2_T(L^2)}\sum_{m\geq l-2} 2^{-m\frac{N}{2}}(2^{m(\frac{N}{2}+1)}\|\Delta_m A\|_{L^2_T(L^2)}),\\
&\leq C2^{-l(N/2-1+s)}(2^{l(\frac{N}{2}-1+s)}\|\Delta_l B\|_{L^2_T(L^2)})\sum_{m\geq l-2}(2^{m(\frac{N}{2}+1)}\|\Delta_m A\|_{L^2_T(L^2)}),\\
&\leq Cc_l 2^{-l(N/2-1+s)}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.
\end{aligned}$$

Classic estimates on the paraproduct yield:

$$\|T'_B A\|_{L^1_T(B^{\frac{N}{2}+s})} \leq C\|B\|_{L^2_T(B^{\frac{N}{2}-1+s})}\|A\|_{L^2_T(B^{\frac{N}{2}+1})}.$$

After by using the spectral localization we have:

$$\|\partial_k\Delta_l T'_B A\|_{L^1_T(L^2)} \leq Cc_l 2^{-l(\frac{N}{2}-1+s)}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.$$

According to the properties of orthogonality of Littlewood-Payley decomposition we have:

$$[T_A,\Delta_l]\partial_k B = \sum_{|m-l|\leq 4}[S_{m-1}A,\Delta_l]\Delta_m\partial_k B\ .$$

By applying Taylor formula, we obtain for $x\in\mathbb{R}^N$:

$$\begin{aligned}
[S_{m-1}A,\Delta_l]\Delta_m\partial_k B(x) = 2^{-l}\int_{\mathbb{R}^N}\int_0^1 h(y)(y.S_{m-1}\nabla A(x-2^{-l}\tau y))&\\
\times\Delta_m\partial_k B(x-2^{-l}y)d\tau dy\ .&
\end{aligned}$$

By an inequality of convolution we have:

$$\|[S_{m-1}A,\Delta_l]\Delta_m\partial_k B\|_{L^2} \leq C2^{-l}\|\nabla A\|_{L^\infty}\|\Delta_m\partial_k B\|_{L^2}\ .$$

So we get:

$$\|[T_A,\Delta_l]\partial_k B\|_{L^1_T(L^2)} \leq Cc_l 2^{-l(\frac{N}{2}-1+s)}\|\nabla A\|_{L^2_T(L^\infty)}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\ .$$

Finally we have:

$$T_{\partial_k A}\Delta_l B = \sum_{|l-m|\leq 4} S_{m-1}\partial_k A \Delta_l \Delta_m B.$$

Hence:

$$\|T_{\partial_k A}\Delta_l B\|_{L^1_T(L^2)} \leq \|\partial_k A\|_{L^2_T(L^\infty)}\|\Delta_l B\|_{L^2_T(L^2)}\,.$$

And the classic estimates on the paraproduct give:

$$\|T_{\partial_k A}\Delta_l B\|_{L^1_T(B^{\frac{N}{2}-1})} \leq C c_l 2^{-l(\frac{N}{2}-1+s)}\|\partial_k A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\,.$$

The proof is complete. □

We give here an estimate on the composition of functions in the space $\widetilde{L}^\rho_T(B^s_p)$.

Proposition 6.5. *Let $s>0$, $p\in[1,+\infty]$ and $u_1, u_2\cdots, u_d \in \widetilde{L}^\rho_T(B^s_p)\cap L^\infty_T(L^\infty)$.*
(i) Let $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u_1,u_2,\cdots,u_d)\in \widetilde{L}^\rho_T(B^s_p)$. More precisely, there exists a constant C depending only on s, p, N and F such that:

$$\|F(u_1,u_2,\cdots,u_d)\|_{\widetilde{L}^\rho_T(B^s_p)} \leq C(\|u_1\|_{L^\infty_T(L^\infty)},\cdots,\|u_d\|_{L^\infty_T(L^\infty)}) \times(\|u_1\|_{\widetilde{L}^\rho_T(B^s_p)}+\cdots+\|u_d\|_{\widetilde{L}^\rho_T(B^s_p)}).$$

(ii) Let $u\in \widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})$, s_1, $s_2>0$ then we have $F(u)\in \widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})$ and

$$\|F(u)\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} \leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.$$

(iii) If v, $u\in \widetilde{L}^\rho_T(B^s_p)\cap L^\infty_T(L^\infty)$ and $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ then $G(u)-G(v)$ belongs to $\widetilde{L}^\rho_T(B^s_p)$ and there exists a constant C depending only of s,p,N and G such that:

$$\|G(u)-G(v)\|_{\widetilde{L}^\rho_T(B^s_p)} \leq C(\|u\|_{L^\infty_T(L^\infty)},\|v\|_{L^\infty_T(L^\infty)})\big(\|v-u\|_{\widetilde{L}^\rho_T(B^s_p)}\times (1+\|u\|_{L^\infty_T(L^\infty)}+\|v\|_{L^\infty_T(L^\infty)})+\|v-u\|_{L^\infty_T(L^\infty)}(\|u\|_{\widetilde{L}^\rho_T(B^s_p)}+\|v\|_{\widetilde{L}^\rho_T(B^s_p)})\big).$$

(iv) If v, $u\in \widetilde{L}^\rho_T(B^{s_1,s_2}_p)\cap L^\infty_T(L^\infty)$ and $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ then $G(u)-G(v)$ belongs to $\widetilde{L}^\rho_T(B^{s_1,s_2}_p)$ and it exists a constant C depending only of

s, p, N and G such that:

$$\begin{aligned}\|G(u)-G(v)\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})} \leq\ & C(\|u\|_{L^{\infty}_{T}(L^{\infty})},\|v\|_{L^{\infty}_{T}(L^{\infty})})\big(\|v-u\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})}\\ &\times(1+\|u\|_{L^{\infty}_{T}(L^{\infty})}+\|v\|_{L^{\infty}_{T}(L^{\infty})})+\|v-u\|_{L^{\infty}_{T}(L^{\infty})}(\|u\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})}\\ &+\|v\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})})\big).\end{aligned}$$

The proof is an adaptation of a theorem by J.-Y. Chemin and H. Bahouri in [2].

Proof. To show (i) we use the 'first linéarisation' method introduced by Y. Meyer in [15], which amounts to write that:

$$F(u_1,u_2,\cdots,u_d)=\sum_{p\in\mathbb{Z}}(F(S_{p+1}u_1,\cdots,S_{p+1}u_d)-F(S_pu_1,\cdots,S_pu_d)).$$

According to Taylor formula, we have:

$$F(S_{p+1}u_1,\cdots,S_{p+1}u_d)-F(S_pu_1,\cdots,S_pu_d)=m^1_pu^p_1+\cdots+m^d_pu^p_d$$

with $u^p_i=\Delta_pu_i$ and

$$m^i_p=\int_0^1\partial_iF(S_pu_1+su^p_1,\cdots,S_pu_i+su^p_i,\cdots,S_pu_d+su^p_d)ds.$$

Observe that:

$$\|m^i_p\|_{L^\infty}\leq\|\nabla F\|_{L^\infty}.$$

We have: $\Delta_pF(u_1,u_2,\cdots,u_d)=\Delta^1_p+\Delta^2_p$ where we have decomposed the sum into two parts:

$$\Delta^{(1)}_p=\sum_{q\geq p}\Big(\Delta_p(u^q_1m^1_q)+\cdots+\Delta_p(u^q_dm^1_q)\Big)$$

$$\Delta^{(2)}_p=\sum_{q\leq p-1}\Big(\Delta_p(u^q_1m^1_q)+\cdots+\Delta_p(u^q_dm^1_q)\Big).$$

Now we bound $\|\Delta^{(1)}_p\|_{L^{\rho}_{T}(L^p)}$ in this way:

$$\begin{aligned}\|\Delta^{(1)}_p\|_{L^{\rho}_{T}(L^p)}\leq&\sum_{q\geq p}\big(\|u^q_1\|_{L^{\rho}_{T}(L^p)}\|m^1_q\|_{L^{\infty}_{T}(L^{\infty})}+\cdots+\|u^q_d\|_{L^{\rho}_{T}(L^p)}\\ &\times\|m^1_q\|_{L^{\infty}_{T}(L^{\infty})}\big)\leq\sum_{q\geq p}\|\nabla F\|_{L^\infty}2^{-qs}c_q(\|u_1\|_{\widetilde{L}^{\rho}_{T}(B^s_p)}+\cdots+\|u_d\|_{\widetilde{L}^{\rho}_{T}(B^s_p)})\end{aligned}$$

with $(c_q) \in l^1(\mathbb{Z})$. Therefore, since $s > 0$:

$$\sum_{p\in\mathbb{Z}} 2^{ps}\|\Delta_p^{(1)}\|_{L^\rho_T(L^p)} \leq C\|\nabla F\|_{L^\infty}(\|u_1\|_{\widetilde{L}^\rho_T(B^s_p)} + \cdots + \|u_d\|_{\widetilde{L}^\rho_T(B^s_p)}).$$

To bound $\|\Delta_p^{(2)}\|_{L^\rho_T(L^p)}$ we use the fact that the support of the Fourier transform of $\Delta_p^{(2)}$ is included in the shell $2^p C$, so that according to Bernstein inequality:

$$\begin{aligned}\|\Delta_p^{(2)}\|_{L^\rho_T(L^p)} &\leq \sum_{q\leq p-1}(\|\Delta_p(u_1^q m_q^1)\|_{L^\rho_T(L^p)} + \cdots + \|\Delta_p(u_d^q m_q^1)\|_{L^\rho_T(L^p)}),\\ &\leq C2^{-p([s]+1)}\sum_{q\leq p-1}(\|\partial^{[s]+1}(u_1^q m_q^1)\|_{L^\rho_T(L^p)} + \cdots + \|\partial^{[s]+1}(u_d^q m_q^1)\|_{L^\rho_T(L^p)}).\end{aligned}$$

Moreover we have according to Faà-di-Bruno formula:

$$\partial^k m_q^i = \int_0^1 \sum_{\substack{l_1+\cdots+l_m=k\\ l_m\neq 0}} A^k_{l_1\cdots l_m} F^{m+1}(S_q(u)+su^q)\prod_{n=1}^m \partial_{l_n}(S_q(u)+su^q))ds.$$

Hence we get for all $k \in \mathbb{N}$:

$$\|\partial^k m_q^i\|_{L^\infty_T(L^\infty)} \leq C_{u_i,k}2^{qk},$$

with: $C_{u_i,k} = C(1 + \|u_i\|_{L^\infty_T(L^\infty)})$. We have then:

$$\|\Delta_p^{(2)}\|_{L^\rho_T(L^p)} \leq C2^{-p([s]+1)}\sum_{q\leq p-1} c_q 2^{q(-s+[s]+1)}C_{u_1,\cdots,u_d}(\|u_1\|_{\widetilde{L}^\rho_T(B^s_p)} + \cdots + \|u_d\|_{\widetilde{L}^\rho_T(B^s_p)}).$$

Hence the result:

$$\sum_{p\in\mathbb{Z}} 2^{ps}\|\Delta_p^{(2)}\|_{L^\rho_T(L^p)} \leq C_{u_1,\cdots,u_d}(\|u_1\|_{\widetilde{L}^\rho_T(B^s_p)} + \cdots + \|u_d\|_{\widetilde{L}^\rho_T(B^s_p)}).$$

So the first part of the proof is complete. For proving (ii) we proceed in the same way as before. We get:

$$F(u) = \sum_{q\in\mathbb{Z}} m_q u_q.$$

And we have for $p > 0$: $\Delta_p F(u) = \Delta_p^1 + \Delta_p^2$ so:

$$\|\Delta_p^1\|_{L^\rho_T(L^2)} \leq \sum_{q\geq p}\|F'\|_{L^\infty}2^{-qs_2}c_q\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.$$

Hence by using convolution inequality:

$$\sum_{p>0} 2^{ps_2}\|\Delta_p^1\|_{L^\rho_T(L^2)} \leq C\|F^{'}\|_{L^\infty}\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.$$

After we get for all $s>0$:

$$\begin{aligned}\|\Delta_p^2\|_{L^\rho_T(L^2)} &\leq C2^{-p([s]+1)}\sum_{q\leq p-1}\|\partial^{[s]+1}(m_q u_q)\|_{L^\rho_T(L^2)},\\ &\leq C2^{-p([s]+1)}\sum_{q\leq p-1}2^{q([s]+1-s(q))}c_q\|u_q\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.\end{aligned}$$

with $s(q)=s_1$ or s_2. So we obtain:

$$\begin{aligned}&\sum_{p>0}2^{ps_2}\|\Delta_p^2\|_{L^\rho_T(L^2)} \leq \sum_{p>0,\,q\leq 0} c_q 2^{q([s]+1-s_1)}\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}\\ &2^{-p([s]+1-s_2)}+\sum_{p>0}2^{-p([s]+1-s_2)}\sum_{0<q\leq p-1}c_q 2^{q([s]+1-s_2)}\\ &\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.\end{aligned} \tag{6.2}$$

We have to choose s, so for the first term of (6.2) we just need that: $[s]+1-s_2>0$ and $[s]+1-s_1>0$ and for the second term of (6.2) we just have a inequality of convolution. We can take then $s=1+\max(s_1,s_2)$. We do the same for $p<0$ and we have:

$$\begin{aligned}\sum_{p\leq 0}2^{ps_1}\|\Delta_p^1\|_{L^\rho_T(L^2)} \leq &\sum_{p\leq 0}2^{ps_1}\sum_{q\geq p}\|F^{'}\|_{L^\infty}2^{-qs_1}c_q\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}\\ &+\sum_{p\leq 0}2^{ps_1}\sum_{p\leq q\leq 0}\|F^{'}\|_{L^\infty}2^{-qs_2}c_q\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}.\end{aligned}$$

We conclude by a inequality of convolution. And for the term Δ_p^2 we get:

$$\sum_{p\leq 0}2^{ps_1}\|\Delta_p^2\|_{\widetilde{L}^\rho(L^2)} \leq \sum_{p\leq 0}2^{-p([s]+1-s_1)}\sum_{q\leq p-1}c_q 2^{q([s]+1-s_1)}\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}.$$

For proving (iii) and (iv), one just has to use the following identity:

$$G(v)-G(u)=(v-u)\int_0^1 H(u+\tau(v-u))d\tau+G^{'}(0)(v-u)$$

where $H(w)=G^{'}(w)-G^{'}(0)$, and we conclude by using (i), (ii) and proposition 3.10. □

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