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Abstract

In this paper, two important geometric concepts– grapical center and width, are introduced in p-adic numbers field. Based on the concept of width, we give the Heisenberg uncertainty relation on harmonic analysis in p-adic numbers field, that is the relationship between the width of a complex-valued function and the width of its Fourier transform on p-adic numbers field.

1 Introduction

In reference [1], wavelet transform is introduced to the field of p-adic numbers. In references [2] and [5], some theory of wavelet analysis and affine frame on harmonic analysis are introduced to the field of p-adic numbers respectively on the basis of a mapping $\mathbf{P}$: $\mathbf{R}^+ \cup \{0\} \to \mathbf{Q}_p$(field of p-adic numbers)

In this paper, based on the same mapping $\mathbf{P}$ we will give the Heisenberg uncertainty relation in harmonic analysis on p-adic numbers field as

$$\triangle_f \triangle_{\widehat{f}} \geq \frac{1}{4\pi^2}$$

where Δ_f, $\Delta_{\widehat{f}}$ are the widths of function f and its transform $\widehat{f}$ respectively.

The field of the p-adic numbers is defined as the completion of field $\mathbf{Q}$ of rationals with respect to the p-adic metric induced by the p-adic norm $|\cdot|_p$(see [6]). A p-adic numbers $x_p \neq 0$ is uniquely represented in the canonical form

$$x_p = p^{-r} \sum_{k=0}^{\infty} x_k p^k, |x_p|_p = p^r, \tag{1.1}$$

where $r{\in}Z$ and $x_k{\in}Z$ such that $0 \leq x_k \leq p-1, x_0 \neq 0$, For $x_p, y_p \in \mathbf{Q}_p$, we define $x_p < y_p$ either when $|x_p|_p < |y_p|_p$ or when $|x_p|_p = |y_p|_p$, and there exist an integer j such that $x_0 = y_0, \cdots, x_{j-1} = y_{j-1}, x_j < y_j$ from the viewpoint of (1.1). By interval $[a_p, b_p]$, we mean the set defined by $\{x_p \in \mathbf{Q}_p | a_p \leq x_p \leq b_p\}$.

The mapping $\mathbf{P}$: $\mathbf{R}^+ \cup \{0\} \to \mathbf{Q}_p$ are introduced in the references [5] and [2] as

$$\mathbf{P}(0) = 0;\ \mathbf{P}\left(p^r \sum_{k=0}^{\infty} x_k p^{-k}\right) = p^{-r} \sum_{k=0}^{\infty} x_k p^k \in \mathbf{Q}_p\ . \tag{1.2}$$

It is known that if $x_{_R} = p^r \sum_{k=0}^{n} x_k p^{-k} \in \mathbf{R}^+ \cup \{0\}, x_0 \neq 0$ and $0 \leq x_k \leq p-1$, then it has another expression

$$x_{_R} = p^r \sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1)p^{-n} + (p-1) \sum_{k=n+1}^{\infty} p^{-k}. \tag{1.3}$$

that we won't use it in this paper. Let $\mathbf{M}_{_R}$ be the set of numbers that is expressed by formula (1.3) and $\mathbf{M}_p = \mathbf{P}(\mathbb{M}_{_R})$. Let $B_r(a_p) = \{x_p \in \mathbf{Q}_p \parallel |x_p - a_p|_p \leq p^r, r \in Z\}, S_r(a_p) = \{x_p \in \mathbf{Q}_p \parallel |x_p - a_p|_p = p^r, r \in Z\}$. According to (1.2), we reach a conclusion that for an interval $[a_{_R}, b_{_R}]$ in $\mathbf{R}^+ \cup \{0\}$ and its corresponding interval $[a_p, b_p]$ in $\mathbf{Q}_p$

$$\mathbf{P}\{B_r(a_p)\} = [0, p^{r+1}), \tag{1.4}$$

$$\mathbf{P}\{S_r(a_p)\} = [p^r, p^{r+1}), \tag{1.5}$$

$$\mathbf{P}\{[a_p, b_p)\} = [a_{_R}, b_{_R}), \tag{1.6}$$

$$|a_{_R} - b_{_R}| \leq p |a_P - b_p|_p, \tag{1.7}$$

where $\mathbf{P}(a_{_R}) = a_p, \mathbf{P}(b_{_R}) = b_p$(see [3]). Let f be a complex-valued function on $\mathbf{Q}_p$, for $x_p \in \mathbf{Q}_p \setminus \mathbf{M}_p$, let

$$f(x_p) = f(\mathbf{P} \circ \mathbf{P}^{-1}(x_p)) = (f \circ \mathbf{P})(x_{_R}) \stackrel{\text{def}}{=} f_{_R}(x_{_R}), (f_{_R} = f \circ \mathbf{P}), x_{_R} = \mathbf{P}^{-1} x_p\ . \tag{1.8}$$

From (1.7), we know that the inverse mapping $\mathbf{P}^{-1}$ is continuous on $\mathbf{Q}_p \backslash \mathbf{M}_p$

2 A Haar measure on Q and integration

In this section, a Haar measure is constructed by using the mapping $\mathbf{P}$ of $\mathbf{R}^+ \cup \{0\}$ into $\mathbf{Q}_p \backslash \mathbf{M}_p$ and the Lebesque measure on $\mathbf{R}^+ \cup \{0\}$. The symbol

$\sum$ is the set of all compact subsets of $\mathbf{Q}_p$, and S is the $\sigma - ring$ generated by $\sum$.

Definition 2.1: Let $E \in S$, and put $\mathbf{E}_p = \mathbf{E} \backslash \mathbf{M}_p$, and $\mathbf{E}_R = \mathbf{P}^{-1}(\mathbf{E}_p)$. If $\mathbf{E}_R$ is a measurable set on $\mathbf{R}^+ \cup \{0\}$, then we call $\mathbf{E}$ a measurable set on $\mathbf{Q}_p$, and define a set function $\mu_p(\mathbf{E})$ on $\mathbf{S}$:

$$\mu_p(\mathbf{E}) = \frac{1}{p}\mu(\mathbf{E}_R) \tag{2.1}$$

where $\mu(\mathbf{E}_R)$ is the Lebesque measure on $\mathbf{E}_R$. This $\mu_p(\mathbf{E})$ is called the measure on $\mathbf{E}$.

By the Definition 2.1, some examples can be given immediately:

(1) Let $a_p, b_p \in \mathbf{Q}_p$, then $\mu_p\{[a_p, b_p]\} = (b_R - a_R)/p$ (see (1.7))

(2) $\mu_p\{B_r(0)\} = p^r$ (see (1.4))

(3) $\mu_p\{S_r(0)\} = p^r(1 - \frac{1}{p})$ (see (1.5))

(4) Let $\{B_{r_i}(a_i)\}_i$ be disjoint discs covering $\mathbf{E}$, by the definition of measure μ_p and definition of Lebesque exterior measure on $\mathbf{R}^+ \cup \{0\}$, it is evident that

$$\mu_p(\mathbf{E}) = \inf_{r_i \in Z} \mu_p\{\cup_i B_{r_i}(a_i)\} \tag{2.2}$$

(5) $\mu_p(\mathbf{M}_p) = 0$.

It is obvious that μ_p, by Definition 2.1, is countably additive. In order to prove that μ_p is a Haar measure, we will give the following lemma.

Lemma 2.2: *If $\alpha \in \mathbf{Q}_p$, then*

$$\mu_p\{B_r(\alpha)\} = \mu_p\{B_r(0)\}. \tag{2.3}$$

PROOF: 1° Let $\alpha = p^{-r_1}$, for $r_1 > r, r_1, r \in Z$, and put $x = p^{-r_1} + p^{-r}\sum_{0 \le x_k < \infty} x_k p^k, x_0 \neq 0, 0 \le x_k < p$. Then $\mathbf{E}$ is the set of all these p-adic numbers when x_k change for $k = 0, 1, ..., p-1$. We write $\mathbf{E}_p = \{x_p | x_p \in \mathbf{E} \backslash \mathbf{M}_p\}$. For $x_p \in \mathbf{E}_p$, let

$$\mathbf{P}^{-1}(x_p) = p^{r_1} + p^r \sum_{0 \le K < \infty} x_k p^{-k} \tag{2.4}$$

then $\mathbf{M}_R = [p^{r_1}, p^{r_1} + p^{r+1})$ is the set of all real numbers as presented in (2.4) (see(1.4)). Hence

$$\mu_p(\alpha + B_r(0)) = \mu_p(B_r(\alpha)) = \frac{1}{p}\mu(\mathbf{E}_R) = p^r = \mu_p(B_r(0)) \tag{2.5}$$

2° Let $\alpha = p^{-r_2}$, for $r_2 \leq r, r_2, r \in Z$. then $\alpha + B_r(0) = B_r(\alpha) = B_r(0)$, by $\alpha \in B_r(0)$. So that

$$\mu_p(B_r(\alpha)) = \mu_p(B_r(0)) \tag{2.6}$$

3° Let $\alpha = p^{-r_3} \sum_{0 \leq k < \infty} \alpha_k p^k$, and put $\alpha^n = p^{-r_3} \sum_{0 \leq k \leq n} \alpha_k p^k$, applying to the result of 1° and 2° repeatedly in this case, we have

$$\mu_p(\alpha^n + B_r(0)) = \mu_p(B_r(\alpha^n)) = \mu_p(B_r(0)) \tag{2.7}$$

However

$$\lim_{n \to \infty} \mu_p(B_r(\alpha^n)) = \lim_{n \to \infty} \frac{1}{p} \mu\{\mathbf{P}^{-1}(B_{rp}(\alpha^n))\} \tag{2.8}$$

where $B_{rp}(\alpha^n) = B_r(\alpha^n) \backslash M_p$. By the continuity of the mapping $\mathbf{P}^{-1}$ (see(1.7)), we obtain

$$\lim_{n \to \infty} \frac{1}{p} \mu\{\mathbf{P}^{-1}(B_{rp}(\alpha^n))\} = \frac{1}{p}\{\mathbf{P}^{-1}\mu\{\mathbf{P}^{-1}(B_{rp}(\alpha))\} = \mu_p\{B_r(\alpha)\} \tag{2.9}$$

The part 3° follows from (2.7), (2.8) and (2.9). □

Theorem 2.3: *(The translation invariance of the measure μ_p) Let $\mathbf{E} \in \mathbf{S}$ and let $\alpha \in \mathbf{Q}_p$, then*

$$\mu_p(\alpha + \mathbf{E}) = \mu_p(\mathbf{E}) \tag{2.10}$$

Proof: Let $\{B_{r_i}(a_i)\}_{i=1}^{\infty}$ be disjoint discs covering $\mathbf{E}$, then $\{B_{r_i}(a_i + \alpha)\}_{i=1}^{\infty}$ are disjoint discs covering $\alpha + E$. By the formula (2.2) in the example 4, we have

$$\mu_p(\alpha + E) = \inf_{r_i \in Z} \mu_p\{\cup\{\alpha + B_{r_i}(a_i)\}\} \tag{2.11}$$

Applying the lemma 2.2 to the right side of the above formula, then

$$\begin{aligned}
& \inf_{r_i \in Z} \mu_p\{\cup_i B_{r_i}(\alpha + a_i)\} \\
= & \inf_{r_i \in Z} \sum_i \mu_p\{B_{r_i}(\alpha + a_i)\} \\
= & \inf_{r_i \in Z} \sum_i \mu_p\{B_{r_i}(a_i)\} \\
= & \inf_{r_i \in Z} \mu_p\{\cup_i B_{r_i}(a_i)\} \\
= & \mu_p(E)
\end{aligned}$$

Therefore, μ_p is a Haar measure. $\square$

According to the above definition of Haar measure, we can define the integration over measurable sets $\mathbf{E}$ in $\mathbf{Q}_p$ (firstly define the integration of simple functions, then regard the limit of integration of simple functions as the definition of the integration of general functions (see [4]))

$$\int_E f(x_p)\mathrm{d}\mu_p \tag{2.12}$$

By the theorem 2.3, the definition of measure and (1.8), we have the following theorem

Theorem 2.4: *Suppose $f(x_p)$ is a complex-valued function on $\mathbf{Q}_p$, then $f(x_p)$ is integrable over the interval $[a_p, b_p]$ ($a_p, b_p \in \mathbf{Q}_p$), if and only if the real function $f_R(x_R)$ defined on $\mathbf{R}^+ \cup \{0\}$ is integrable over the interval $[a_R, b_R]$, and*

$$\int_{[a_p,b_p]} f(x_p)\mathrm{d}\mu(x_p) = \frac{1}{p}\int_{a_R}^{b_R} f(x_R)\mathrm{d}\mu(x_R) \tag{2.13}$$

where $f_R(x_R)$ is defined by (1.8), and $\mathbf{P}(x_R) = x_p, \mathbf{P}(a_R) = a_p, \mathbf{P}(b_R) = b_p, a_p, b_p \bar{\in} \mathbf{M}_p$

Corollary 2.5: *If $f(x_p)$ is a bounded continuous function on the interval $[a_p, b_p] \subset \mathbf{Q}_p$, then $f(x_p)$ is integrable over $[a_p, b_p]$, where $[a_p, b_p]$ can be $\mathbf{Q}_p$.*

Notice that under the condition of theorem $f_R(x_R)$ is a bounded piecewise continuous function on $\mathbf{R}^+ \cup \{0\}$ by (1.4), By the theorem 2.4, $f(x_p)$ is integrable.

3 The indefinite integral and derivative of complex-valued function in $\mathbf{Q}_p$

Definition 3.1: Let f be a complex-valued function defined in $\mathbf{Q}_p$ and for $\forall x_p \in \mathbf{Q}_p$, f is integrable on interval $[a_p, b_p]$, then

$$f(x_p) = \int_0^{x_p} g\mathrm{d}x_p \tag{3.1}$$

is called on indefinite integral of g.

Definition 3.2: Let f be a complex-valued function defined in $\mathbf{Q}_p$, if there exist an integrable complex-valued function g such that

$$f(x_p) = \int_0^{x_p} g\mathrm{d}x_p, \quad x_p \in \mathbf{Q}_p \tag{3.2}$$

then $g(x_p)$ is called the derivative of f, which we will denote as $f'(x_p)$.

In formula (3.2), let $f = 1$ then

$$\mu([0, x_p]) = \int_0^{x_p} \mathrm{d}\mu \tag{3.3}$$

The equation (3.3) follows that

$$\bar{\mu}'(x) \stackrel{\text{def}}{=} \mu'([0, x_p]) = 1 \tag{3.4}$$

Theorem 3.3: *For complex-valued functions f, h on $\mathbf{Q}_p$, if $(f)_R(x_R)$ and $(h)_R(x_R)$ are absolutely continuous, then*

$$\begin{gathered} f'_R(x_R) = (f')_R(x_R) \\ (f(x_p)h(x_p))' = f'(x_p)h(x_p) + f(x_p)h'(x_p) \end{gathered} \tag{3.5}$$

Proof: Let $f' = g$ and $g(x_p) = g(\mathbf{P}(x_R)) = (g \circ \mathbf{P})(x_R) = g_R(x_R)$. By definition (3.2) and theorem 2.4, we have

$$\begin{aligned} f(x_p) &= \int_0^{x_p} g(x_p)\mathrm{d}x_p \\ &= \int_0^{x_R} g_R(x_R)\mathrm{d}x_R \\ &= f_R(x_R) \end{aligned}$$

and therefore

$$(f_R)'(x_R) = g_R(x_R) = g(x_p) = f'(x_p) = (f')_R(x_R) \tag{3.6}$$

and

$$\begin{aligned} (f(x_p)h(x_p))' &= (f_R(x_R)h_R(x_R))' \\ &= (f_R)'(x_R)h_R(x_R) + f_R(x_R)(h_R)'(x_R) \\ &= (f')_R(x_R)h_R(x_R) + f_R(x_R)(h')_R(x_R) \\ &= f'(x_p)h(x_p) + f(x_p)h'(x_p) \end{aligned}$$

$\square$

From (3.6) it follows that

Corollary 3.4: *If a complex-valued function $h(x_{_R})$ is absolutely continuous on $\mathbf{R}^+ \cup \{0\}$, then $f(x_p) \stackrel{\text{def}}{=} (h\mathbf{P}^{-1})(x_p)$ is derivable on $\mathbf{Q}_p \setminus \mathbf{M}_p$.*

Corollary 3.5: *A locally constant function is derivable on $\mathbf{Q}_p \setminus \mathbf{M}_p$, and its derivative is equal to 0.*

Similarly, we can prove

Theorem 3.6: *If f is derivable on $[a_p, b_p]$,then*

$$\int_{a_p}^{b_p} f'(x_p)\mathrm{d}\mu = f(b_p) - f(a_p) \tag{3.7}$$

4 Center and width of the graph of f

In this section, we will introduce the concepts of center and width of complex-valued function graph in the filed of p-adic numbers $\mathbf{Q}_p$.

Definition 4.1: Let f be a complex-valued function of p-adic variable. We define the center $t_{_f}$ of the graph $\{(x_p, f(x_p))|x_p \in \mathbf{Q}_p\}$ by

$$\left.\begin{aligned} t_{_f}^{(R)} &\stackrel{\text{def}}{=} \int_{Q_p\setminus M_p} \mathbf{P}^{-1}(x_p)|f(x_p)|^2\mathrm{d}x_p \Big/ \int_{Q_p\setminus M_p} |f(x_p)|^2\mathrm{d}x_p \\ t_{_f} &= \mathbf{P}(t_{_f}^{(R)}) \end{aligned}\right\} \tag{4.1}$$

if the integral (4.1) exists.

Definition 4.2: For a complex-valued function of p-adic variable,we define the width of f by

$$\triangle_f = \left(\int_{Q_p} |x_p - t_{_f}|^2 |f(x_p)|^2\mathrm{d}x_p \Big/ \int_{Q_p} |f(x_p)|^2\mathrm{d}x_p \right)^{1/2} \tag{4.2}$$

if the integral (4.2) exists.

Theorem 4.3: *Let* $\mathbf{P}(t_f^{(R)} - a_R) = \mathbf{P}(t_f^{(R)}) - \mathbf{P}(a_R)$, $a_R = \mathbf{P}^{-1}(a), a \in \mathbf{Q}_p \backslash \mathbf{M}_p$.

(1) If f is increasing, then $t_{T_a f} = t_f - a$

(2) Suppose $\mathrm{supp} f \subset B_r(0)$. *For* $a = p^{-\beta}$, *if* $\beta > r$, *then* $t_{T_a f} = t_f - a$

(3) For $a = p^{-\beta}$, $\beta \in Z$, *then*

$$t_{s_a f} = a t_f,$$

where $T_a f(x_p) = f(x_p + a)$, $S_a f(x_p) = f(\frac{x_p}{a})$.

Proof: (1) Under the condition of (1) in this theorem, using

$$\mathbf{P}(x_R + a_R) \geq \mathbf{P}(x_R) + \mathbf{P}(a_R)$$

we have

$$(f \circ \mathbf{P})(x_R + a_R) \geq f(x + a)$$

where $x_R = \mathbf{P}^{-1}(x_p)$, $x_p \in \mathbf{Q}_p \setminus \mathbf{M}_p$. Hence

$$\begin{aligned}
t_{T_a f}^{(R)} &= \int_{Q_p \backslash M_p} \mathbf{P}^{-1}(x_p) |T_a f(x_p)|^2 \mathrm{d}x_p \Big/ \int_{Q_p} |T_a f(x_p)|^2 \mathrm{d}x_p \\
&= \int_{Q_p \backslash M_p} \mathbf{P}^{-1}(x_p) |f(x + a)|^2 \mathrm{d}x_p \Big/ \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
&\leq \int_{R^+} x_R |(f \circ \mathbf{P})(x_R + a_R)|^2 \mathrm{d}x_R \Big/ \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
&= \int_{R^+} (x_R - a_R) |(f \circ \mathbf{P})(x_R)|^2 \mathrm{d}x_R \Big/ \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
&= -a_R + \int_{R^+} x_R |(f \circ \mathbf{P})(x_R)|^2 \mathrm{d}x_R \Big/ \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
&= -a_R + \int_{Q_p \backslash M_p} \mathbf{P}^{-1}(x_p) |f(x_p)|^2 \mathrm{d}x_p \Big/ \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
&= -a_R + t_f^{(R)} \qquad (4.3)
\end{aligned}$$

where we used $\mu(\mathbf{M}_p) = 0$, and for $x_p, a \in B_r(0) \cap (\mathbf{Q}_p \backslash \mathbf{M}_p), x_p + a \in B_r(0)$. On the other hand, using inequation $\mathbf{P}^{-1}(x - a) \geq \mathbf{P}^{-1}(x) - \mathbf{P}^{-1}(a)$, we can easily obtain

$$t_{T_a f}^{(R)} \geq t_f^{(R)} - a_R \qquad (4.4)$$

From (4.3) and (4.4), we have

$$t_{T_a f}^{(R)} = t_f^{(R)} - a_R.$$

Finally, from the condition of (1) in theorem, we have

$$t_{_{T_af}} = \mathbf{P}(t_{_f}^{(R)} - a_{_R}) = \mathbf{P}(t_{_f}^{(R)}) - \mathbf{P}(a_{_R}) = t_{_f} - a$$

.

Conclusion of (1) in theorem is proved. (2) and (3) can be proved similarly. □

Theorem 4.4: *(1) If $f(x)$ and a, t_f satisfy the condition of theorem 3.3, then*

$$\triangle_{T_af} = \triangle_f$$

(2) $\triangle_{S_af} = |a|_p \triangle_f$

Proof: For (1), we have

$$\triangle_{T_af} = \left(\int_{Q_p} |x_p - t_{_{T_af}}|_p^2 |T_af|^2(x_p)\mathrm{d}x_p \Big/ \int_{Q_p} |T_af|^2(x_p)\mathrm{d}x_p\right)^{1/2}$$

$$= \left(\int_{Q_p} |x_p - (t_{_f} - a)|_p^2 |f(x_p + a)|^2\mathrm{d}x_p \Big/ \int_{Q_p} |f(x_p + a)|^2\mathrm{d}x_p\right)^{1/2}$$

$$= \left(\int_{Q_p} |t_p - t_{_f}|_p^2 |f(t_p)|^2\mathrm{d}t_p \Big/ \int_{Q_p} |f(t_p)|^2\mathrm{d}t_p\right)^{1/2}$$

$$= \triangle_f$$

(2) can be proved similarly. □

After doing the preparation of section 1-4, we will give a theorem on harmonic analysis which is about the relation of the width of complex function in $\mathbf{Q}_p$ and the width of its Fourier transform. This theorem is similar to the Heisenberg uncertainty relation in quantum mechanics.

5 Main theorem

Lemma 5.1: *Let $x_p \in \mathbf{Q}_p$, then $\mu([0, x_p]) \leq |x_p|_p$*

Proof: For $x_p \in \mathbf{P}_p \setminus \mathbf{M}_p$

$$x_p = p^{-r} \sum_{k=0}^{\infty} x_k p^k \in \mathbf{Q}_p, \quad x_0 \neq 0, 0 \leq x_k \leq p-1$$

and therefore, we have

$$\mathbf{P}^{-1}(x_p) = p^{r-1} \sum_{k=0}^{\infty} x_k p^{-k} \leq p^{r-1}(p-1) \sum_{k=0}^{\infty} p^{-k} = |x_p|_p \tag{5.1}$$

By definition of measure μ_p, we have

$$\frac{1}{p}\mathbf{P}^{-1}(x_p) = \mu([0, x_p])$$

which leads to

$$\overline{\mu}(x_p) \stackrel{\text{def}}{=} \mu([0, x_p]) \leq |x_p|_p / p \tag{5.2}$$

□

Theorem 5.2: *Let f be complex-valued function of p-adic variable. If $f \in L^2(\mathbf{Q}_p), f' \in L^2(\mathbf{Q}_p)$ and*

$$\lim_{|b_p|_p \to \infty} |b_p|_p |f(b_p)|^2 = 0, \; f(0) = 0 \tag{5.3}$$

then the following inequality is valid:

$$\frac{1}{4\pi} \leq \triangle_f \triangle_{\widehat{f}} \tag{5.4}$$

where $\widehat{f}$ is the transform of f,

$$\widehat{f}(\xi_p) = \int_{Q_p} f(x_p) \exp(2\pi i \{\xi_p x_p\}) \mathrm{d}x_p$$

and by means of representation (1.1), $\{x_p\}$ is defined as

$$\{x_p\} = \begin{cases} 0 & \textit{if } \; r(x_p) \geq 0 \;\; \textit{or} \;\; x_p = 0 \\ p^r(x_0 + x_1 p + \cdots + x_{|r|-1} p^{|r|-1}) & \textit{if } \; r(x_p) < 0 \end{cases}$$

Inequality (5.4) is called the Heisenberg uncertainty relation in harmonic analysis on p-adic numbers field.

Proof: By using (3.4) and theorem 3.3, we have

$$\left(\bar{\mu}(x_p - t_{_f})|f(x_p)|^2\right)' = \left(\bar{\mu}(x_p - t_{_f})f(x_p)\chi_p(t_{_{\hat{f}}}x_p)\overline{f(x_p)}\chi_p(t_{_{\hat{f}}}x_p)\right)'$$

$$= |f(x_p)|^2 + \bar{\mu}(x_p - t_{_f})\ \left(f(x_p)\chi_p(t_{_{\hat{f}}}x_p)\right)'\overline{f(x_p)\chi_p(t_{_{\hat{f}}}x_p)}$$

$$+\bar{\mu}(x_p - t_{_f})f(x)\chi_p(t_{_{\hat{f}}}x_p)\overline{[f(x_p)\chi_p(t_{_{\hat{f}}}x_p)]'} \tag{5.5}$$

Therefore, from (3.7) we have

$$\int_0^{b_p} |f(x_p)|^2 \mathrm{d}x = \bar{\mu}(x_p - t_{_f})|f(x_p)|^2|_0^{b_p}$$

$$-\int_0^{b_p} \bar{\mu}(x_p - t_{_f})\overline{f(x_p)\chi_p(t_{_{\hat{f}}}x_p)}\left(f(x_p)\chi_p(t_{_{\hat{f}}}x_p)\right)' \mathrm{d}x_p$$

$$-\int_0^{b_p} \bar{\mu}(x_p - t_{_f})f(x_p)\chi_p(t_{_{\hat{f}}}x_p)\overline{[f(x_p)\chi_p(t_{_{\hat{f}}}x_p)]'}\mathrm{d}x_p \tag{5.6}$$

where the function $\chi_p(t_{_{\hat{f}}}x_p) = \exp(2\pi i\{t_{_{\hat{f}}}x_p\})$

By taking the limit of (5.5) as $|b|_p \to \infty$ and using (5.2),(5.3), we obtain

$$\begin{aligned}
&\int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p \\
\leq\ & 2\left(\int_{Q_p} (\bar{\mu}(x_p - t_{_f}))^2|f(x_p)|^2\mathrm{d}x_p\right)^{1/2}\left(\int_{Q_p} |[f(x_p)\chi_p(t_{_{\hat{f}}}x_p)]'|^2\mathrm{d}x_p\right)^{1/2} \\
=\ & \left(\int_{Q_p} (\bar{\mu}(x_p - t_{_f}))^2|f(x_p)|^2\mathrm{d}x_p\right)^{1/2}\left(\int_{Q_p} |[f(\cdot)\chi_p(t_{_{\hat{f}}}\cdot)]'^{\wedge}(\xi)|^2\mathrm{d}\xi\right)^{1/2} \\
=\ & 2\left(\int_{Q_p} \left(\bar{\mu}(x_p - t_{_f})\right)^2|f(x_p)|^2\mathrm{d}x_p\right)^{1/2}\left(\int_{Q_p} 4\pi^2|\xi|^2|\widehat{f}(\xi + t_{_{\hat{f}}})|^2\mathrm{d}\xi\right)^{1/2}
\end{aligned} \tag{5.7}$$

where we used the Hölder inequality for the integral and $f'(\cdot)^{\wedge}(\xi) = -2\pi i\xi\hat{f}(\xi)$, $(f, f)_{L^2(Q_p)} = (\widehat{f}, \widehat{f})_{L^2(Q_p)}$, $(f(\cdot)\chi_p(a\cdot))^{\wedge}(\xi) = \widehat{f}(\xi + a)$ From (5.2), we have

$$\frac{1}{4\pi} \leq \left(\int_{Q_p} |x_p - t_{_f}|_p^2|f(x_p)|^2\mathrm{d}x_p / \int_{Q_p} |f(x_p)|^2\mathrm{d}x_p\right)^{1/2}.$$

$$\left(\int_{Q_p}|\xi_p-t_{\widehat{f}}|_p^2|\widehat{f}(\xi_p)|^2\mathrm{d}\xi_p/\int_{Q_p}|\widehat{f}(\xi_p)|^2\mathrm{d}\xi_p\right)^{1/2}$$

$$=\triangle_f\triangle_{\widehat{f}} \tag{5.8}$$

Hence we have completed our proof. □

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The Heisenberg uncertainty relation

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