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Standard Subalgebras of Semisimple Lie Algebras and Computer-Aided for Enumeration

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Abstract

The aim of this work is to enumerate the standard subalgebras of a semisimple Lie algebra. The computations are based on the approach developed by Yu. Khakimdjanov in 1974. In this paper, we give a general formula for the number of standard subalgebras not necessarly nilpotent of a semisimple Lie algebra of type A_p and the exceptional semisimple Lie algebras. With computer aided, we enumerate this number for the other types of small rank. Therefore, We deduce the number in the nilpotent case and describe a family of complete nilpotent standard subalgebras, these algebras are the nilradical of their normalizer.

1 Introduction

The motivation for this study can be found in the theory of complex homogeneous spaces : Let M be a compact homogeneous space $M = G/H$, G being a complex Lie group and H a closed subgroup. Tits has established the following result : if $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras corresponding to G and H, the normalizer of $\mathfrak{t}$ in $\mathfrak{g}$ is parabolic, such a Lie algebra $\mathfrak{t}$ is called *standard* . Then, one may translate the study of homogeneous complex spaces into study of standard subalgebras. The computations of standard Lie algebra, in the nilpotent case, were done by different authors (G. Favre and L. Santharoubane [2], P. Cellini and P. Papi [1], L. Orsina [5]).

This paper is organized as follows. In section 2, we summarize the basic facts about semisimple Lie algebras and standard subalgebras. In section 3, we characterize the complete nilpotent standard subalgebras and prove that if the semisimple Lie algebra has rank p then their number is 2^p. The section 4 is devoted to the semisimple Lie algebras A_p, we give a recursive formula of the number of standard subalgebras (not necessarily nilpotent). We prove that for A_p this number is

$$ST(p) = \frac{1}{p}\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix} + \frac{1}{p+2}\begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix} + \sum_{1\leq j\leq p-1} \frac{2}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} + \sum_{1\leq j\leq p-2} \frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} \cdot \left(\sum_{1\leq i\leq p-j-1} ST(i)\right)$$

In the last section, we enumerate for the exceptional semisimple Lie algebras and for A_p, B_p, C_p, D_p of small rank the number of nilpotent standard subalgebras, the sequence of nilpotent standard subalgebras for each value of nilindex, the number of complete nilpotent standard subalgebras and the number of standard subalgebras. The computations uses Mathematica package available in : http://www.math.uha.fr/publi2002.html.

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2 Standard Lie algebras

2.1 Parabolic and Borel subalgebras of semisimple Lie algebra

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra of rank p. A Borel subalgebra of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$, and a parabolic subalgebra of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra of $\mathfrak{g}$.

We fix the following notations : $\hbar$ is a Cartan subalgebra of $\mathfrak{g}$, Δ is the set of roots corresponding to $\hbar$, S is a basis of Δ (the simple roots), Δ_+ (respectively, Δ_-) is the set of positive (respectively, negative) roots. Let $\alpha \in \Delta$:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [X,H] = \alpha(H)X \quad \text{for all } H \in \hbar\}$$

This space $\mathfrak{g}_\alpha$ has dimension one. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is conjugate, up to inner automorphism, to subalgebra of the following type :

$$\mathfrak{b}' = \hbar \oplus \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$$

The parabolic subalgebra may be characterized, up to inner automorphism, by a subset T of S. Let Ω_1 be the set of roots whose decomposition on S contains only elements of $S\backslash T$. We set $\Omega_2 = \Delta\backslash\Omega_1$, $\Omega_2^+ = \Omega_2 \cap \Delta_+$. The Lie algebra

$$\rho = \hbar \oplus \sum_{\alpha\in\Omega_2^+\cup\Omega_1} \mathfrak{g}_\alpha \tag{2.1}$$

is a parabolic subalgebra and every parabolic subalgebra is conjugate to this Lie algebra.

We note that the reductive Levi subalgebra of ρ is

$$r = \hbar \oplus \sum_{\alpha\in\Omega_1} \mathfrak{g}_\alpha \tag{2.2}$$

and its nilradical part is

$$\mathfrak{n} = \sum_{\alpha\in\Omega_2^+} \mathfrak{g}_\alpha \tag{2.3}$$

2.2 Standard subalgebras

Definition: A subalgebra of a semisimple Lie algebra is called *standard* if its normalizer is a parabolic subalgebra.

In order to simplify terminology, we shall call standard algebra every standard subalgebra of a semisimple Lie algebra. These algebras have been studied, in the first time, by G.B. Gurevich [3], in the case where the simple Lie algebra is of type A_p.

In the following, we give the characterization of these algebras, using the roots, due to Yu. Khakimdjanov [4].

We consider a partial order relation on the dual $\hbar^*$ of the Cartan subalgebra $\hbar$: $\omega_1 \leq \omega_2$ if and only if $\omega_2 - \omega_1$ is linear combination of simple roots with non-negative coefficients.

Proposition 2.1: *Let $\mathfrak{t}$ be a standard algebra such that its normalizer can be written $\rho(\mathfrak{t}) = \hbar \oplus \sum_{\alpha\in\Omega_1\cup\Omega_2^+} \mathfrak{g}_\alpha$. Suppose that γ and β are positive roots with $\gamma \leq \beta$. If the subspace $\mathfrak{g}_\gamma$ is included in $\mathfrak{t}$, then $\mathfrak{g}_\beta$ is also in $\mathfrak{t}$.*

This proposition is useful for the study of nilpotent standard algebras.

2.3 Nilpotent standard algebras

Let $\mathfrak{R}$ be a subset of Δ_+ whose elements are pairwise noncomparable (for the previous ordering on $\hbar^*$). We put :

$$\mathfrak{R}_1 = \{\alpha \in \Delta_+ : \beta \leq \alpha \text{ for some } \beta \in \mathfrak{R}\} \tag{2.4}$$

The subspace $\mathfrak{m} = \sum_{\alpha \in \mathfrak{R}_1} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of $\mathfrak{g}$.

The normalizer of $\mathfrak{m}$ contains a Borel subalgebra. Thus, $\mathfrak{m}$ is a nilpotent standard algebra of $\mathfrak{g}$. We say that $\mathfrak{m}$ is the nilpotent standard algebra associated to $\mathfrak{R}$. This process permits to construct more easily such subalgebra. The following theorem due to Yu. Khakimdjanov [4] shows that every nilpotent standard algebra is of this type.

Theorem 2.2: *Let* $\mathfrak{m}$ *be a nilpotent standard algebra whose normalizer has the form* $\rho(m) = \hbar \oplus \sum_{\alpha \in \Omega_1 \cup \Omega_2^+} \mathfrak{g}_\alpha$. *Then, there is a subset* $\mathfrak{R} \subset \Delta_+$ *of pairwise noncomparable roots such that* $\mathfrak{m}$ *is the nilpotent standard algebra associated to* $\mathfrak{R}$.

Corollary 2.3: *Every nilpotent standard algebra is conjugate to a nilpotent standard algebra associated to a set* $\mathfrak{R} \subset \Delta_+$ *of pairwise noncomparable roots.*

Remark: Let $\mathfrak{g}^+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ be a nilpotent subalgebra of $\mathfrak{g}$. If $\mathfrak{m}$ is nilpotent standard algebra then the quotient $\mathfrak{g}^+/\mathfrak{m}$ is also nilpotent. These quotients contain all nilpotent Lie algebras of maximal rank studied by G. Favre and L. Santharoubane [2].

Remark: In their paper, P. Cellini and P. Papi [1] called these nilpotent standard algebras, ad-nilpotent ideals of a Borel subalgebra and gave the number for each type of semisimple Lie algebra. In another paper, L. Orsina and P. Papi [5] gave for A_p the number of nilpotent standard algebras for each nilindex. We compute, in section 5, these numbers for exceptional semisimple Lie algebras and other type of semisimple Lie algebras of small rank p.

2.4 Structure and construction of standard algebras

In this section, we generalize the study to standard algebra not necessarily nilpotent.

Definition: The root $\alpha \in S$ is called *extremal* for $\beta \in \Delta_+$ if it satisfies $\alpha = \beta$ or $\beta - \alpha \in \Delta$.

In the following proposition, we characterize the normalizer of a nilpotent standard algebra using the extremal roots [4]. Let $\mathfrak{R} \subset \Delta_+$ be a subsystem of pairwise noncomparable roots, $\mathfrak{m}$ be the nilpotent standard algebra defined by this subsystem and S^β be the set of all extremal roots for $\beta \in \Delta_+$.

Proposition 2.4: *The normalizer $\rho(\mathfrak{m})$ of $\mathfrak{m}$ is defined by the subsystem $S_2 = \bigcup_{\beta\in\mathfrak{R}} S^\beta \subset S$.*

We construct a standard algebra whose nilpotent part $\mathfrak{m}$ is given by a subsystem $\mathfrak{R} \subset \Delta_+$.

Consider the subsystems $S_1 = \mathfrak{R} \cap S$ and $S_2 = \bigcup_{\beta\in\mathfrak{R}} S^\beta$ of the system S. These subsystems define respectively the parabolic subalgebras ρ_1 and ρ_2 of the form (2.1). Let r_i denote the reductive Levi subalgebra of the form (2.2) and $\mathfrak{n}_i$ denote the nilradical of Lie algebra ρ_i of the form (2.3), $i = 1, 2$.

Let r_0 be an ideal in r_1 contained in r_2.

Lemma 2.5: *The semidirect sum $\mathfrak{t} = r_0 + \mathfrak{m}$ is a standard algebra.*

Theorem 2.6: *Given any ideal r_0 of the algebra r_1 lying in r_2, the subalgebra $\mathfrak{t} = r_0 + \mathfrak{m}$ is standard algebra, while $\rho(\mathfrak{t}) = \rho(\mathfrak{m})$. Conversely, any standard algebra is conjugate to subalgebra $\mathfrak{t}$ for some subsystem $\mathfrak{R} \subset \Delta_+$ of pairwise noncomparable roots and for some ideal r_0.*

3 Complete nilpotent standard Lie algebras

In this section, we characterize the complete nilpotent standard algebras and count their number.

Definition: A nilpotent standard algebra $\mathfrak{m}$ is called *complete* if it is the nilradical of its normalizer.

Proposition 3.1: *Let $\mathfrak{m}$ be the nilpotent standard algebra defined by a subsystem $\mathfrak{R} \subset \Delta_+$ of pairwise noncomparable roots.*

Then, $\mathfrak{m}$ is complete if and only if $\mathfrak{R}$ is formed by simple roots.

The proposition leads to the following result.

Corollary 3.2: *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank p, then it has exactly 2^p complete nilpotent standard algebras.*

Remark: The number 2^p is independent from the type of $\mathfrak{g}$.

4 The general case for $\mathbf{A}_p$

In this section, we establish a formula which gives the number of standard algebras (not necessarily nilpotent) of A_p. Let $\mathfrak{g}$ be a semisimple complex Lie algebra of type A_p.

Let $S = \{\alpha_1, \cdots, \alpha_p\}$ be a basis of Δ (the simple roots), then the set of positive roots is

$$\Delta_+ = \{ \sum_{i \leq k \leq j} \alpha_k,\ 1 \leq i \leq j \leq p\}$$

Let $\alpha \in \Delta$:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [X, H] = \alpha(H)X \quad \text{for all } H \in \hbar\}.$$

This space $\mathfrak{g}_\alpha$ has dimension one. Let e_α be a non-null vector in $\mathfrak{g}_\alpha$.

Let $\mathfrak{R} \subset \Delta_+$ be a subsystem of pairwise noncomparable roots and $\mathfrak{m}$ be a nilpotent standard algebra defined by this subsystem. We fix the following notations : $S_1 = \mathfrak{R} \cap S$, $S_2 = \underset{\beta\in\mathfrak{R}}{\cup} S^\beta$ and Ω_1(respectively, Ω_2) the set of roots whose decomposition on S contains only elements of $S\backslash S_1$ (respectively, $S\backslash S_2$).

From formula 2.2, we have two reductive Levi subalgebras r_1 and r_2 defined by the subsystems S_1 and S_2 :

$$r_1 = \hbar + \sum_{\alpha\in\Omega_1} \mathfrak{g}_\alpha$$

and

$$r_2 = \hbar + \sum_{\alpha\in\Omega_2} \mathfrak{g}_\alpha$$

Let r_0 be an ideal of r_1 contained in r_2. Since $r_2 = \sum_{\alpha\in\Delta_+\backslash\Omega_2} \mathbb{C}[e_\alpha, e_{-\alpha}] + \sum_{\alpha\in\Omega_2} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ where $\mathbb{C}[e_\alpha, e_{-\alpha}]$ denotes the vector space generated by

$[e_\alpha, e_{-\alpha}]$, then the ideal r_0 has the form $h_0 + \sum\limits_{\alpha\in\Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ where $h_0 \subset \hbar$, and $\Omega_0 \subset \Omega_2$.

In the following, we consider an ideal r_0 of the form $\sum\limits_{\alpha\in\Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ where $\Omega_0 \subset \Omega_2$ (the other cases would be obtained by adding a subalgebra h_0 of $\hbar$ such that $h_0 + r_0$ is an ideal of r_1 contained in r_2).

Let Π_1 be a subsystem of S.

Definition: The subsystem Π_1 is called *connected* if its diagram in the Dynkin diagram of A_p is connected.

Notation: Let Φ be a subsystem of S. The set of roots expressed only with elements of Φ is denoted by $< \Phi >$.

Let Π_1 be a connected subsystem of $S\backslash S_1$ and $S\backslash S_2$ and Γ_1 be a set of roots expressed only with elements of Π_1. We set $I_1 = \sum\limits_{\alpha\in\Gamma_1} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$.

Lemma 4.1: *The subspace I_1 is an ideal of r_1 contained in r_2.*

Proof: Since $\Pi_1 \subset S\backslash S_2 \subset S\backslash S_1$, then $\Gamma_1 \subset \Omega_2 \subset \Omega_1$ and $I_1 \subset r_2 \subset r_1$.

We can write $S\backslash S_1 = \bigcup\limits_{1\leq j\leq m} C_j$ with $m \in \mathbb{N}^*$, $C_1 = \Pi_1$ and $\{C_j\}_{2\leq j\leq m}$ is the family of connected subsystems of $S\backslash S_1$ such that C_j are pairwise disjoint. We have $r_1 = \hbar + \sum\limits_{\alpha\in\Omega_1} \mathfrak{g}_\alpha$, then $[I_1, r_1] = \sum\limits_{\alpha\in\Gamma_1} \mathfrak{g}_\alpha + \sum\limits_{(\alpha,\beta)\in\Gamma_1\times\Omega_1} [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$.

Now, let $j \in [\![2, m]\!]$ and $\beta \in< C_j > \cap\Delta_+$. For all $\alpha \in \Gamma_1 \cap \Delta_+$, we have $\alpha - \beta \notin \Delta$ and $\alpha + \beta \notin \Delta$. Therefore, for $(\alpha, \beta) \in \Gamma_1 \times \Omega_1$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ equal to 0 or $\mathbb{C}[e_\alpha, e_{-\alpha}]$ or $\mathfrak{g}_{\alpha+\beta}$ with $\alpha + \beta \in \Gamma_1$.

It follows $[I_1, r_1] \subset I_1$. $\square$

Let $\Pi = \{\Pi_i\}_{1\leq i\leq k}$ be a family of connected subsystems of $S\backslash S_1$ and $S\backslash S_2$ such that Π_i are pairwise disjoint. Let I_i be the ideal associated to Π_i, $1 \leq i \leq k$. We set $I = \sum\limits_{1\leq i\leq k} I_i$ and it is called ideal associated to family Π.

Lemma 4.2: *The semidirect sum $I = \sum\limits_{1\leq i\leq k} I_i$ is an ideal of r_1 contained in r_2.*

Proof: Let $i \in [\![1, k]\!]$. We have $I_i \subset r_2$ and $[I_i, r_1] \subset I_i$. Then $I \subset r_2$ and $[I, r_1] = \sum\limits_{1\leq i\leq k} [I_i, r_1] \subset \sum\limits_{1\leq i\leq k} I_i = I$. $\square$

Proposition 4.3: *Let $r_0 = \sum\limits_{\alpha\in\Omega_0} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ be an ideal of r_1 contained in r_2, then there exists a family $\Pi = \{\Pi_i\}_{1\leq i\leq k}$ of connected subsystems* *of* $S\backslash S_1$ *and* $S\backslash S_2$ *such that* Π_i *are pairwise disjoint,* $I = \sum_{1\leq i\leq k} I_i$ *the ideal associated to* Π *and* $r_0 = I$.

Proof: First we show that $\Omega_0 \cap S \neq \emptyset$. Let $\beta \in \Omega_0$, $\beta = \beta_1 + \beta_2 + \cdots + \beta_m$ with $m \in \mathbb{N}^*$, $\beta_i \in S$, for $1 \leq i \leq m$. The partial sum $\beta_1 + \cdots + \beta_h$ is a root, for $1 \leq h \leq m$. Since $\Omega_0 \subset \Omega_1$, therefore $\beta' = \beta_1 + \cdots + \beta_{m-1}$ is a root in Ω_1. Then $\beta_m = \beta - \beta' \in \Omega_0$ because $[r_0, r_1] \subset r_0$.

Now, we construct the family Π. We set $S_0 = \Omega_0 \cap S$, S_0 may be written $S_0 = \bigcup_{1\leq i\leq k} \Pi_i$ where $\Pi = \{\Pi_i\}_{1\leq i\leq k}$ is a family of connected subsystems of S_0 such that Π_i are pairwise disjoint. It remains to prove that Π is a family of connected subsystems of $S\backslash S_1$ and $S\backslash S_2$.

Let $i \in [\![1, k]\!]$ and $\gamma \in (S\backslash S_1)\backslash S_0$. We suppose that $\Pi_i \cup \{\gamma\}$ is a connected subsystem of $S\backslash S_1$. Then, there exists a family $\{\gamma_j\}_{1\leq j\leq s}$ of simple roots of Π_i where :

1. $\gamma_1 + \cdots + \gamma_s + \gamma$ is a root and $\gamma' = \gamma_1 + \cdots + \gamma_s$ is a root of Ω_0. Since $[r_0, r_1] \subset r_0$ then $[\mathfrak{g}_{\gamma'+\gamma}, \mathfrak{g}_{-\gamma'}] = \mathfrak{g}_\gamma \subset r_0$ i.e $\gamma \in S_0$, contradiction.

Or

2. $\gamma + \gamma_1 + \cdots + \gamma_s$ is a root. We have $[[\mathfrak{g}_{\gamma+\gamma_1+\cdots+\gamma_m}, \mathfrak{g}_{-\gamma_s}], \cdots, \mathfrak{g}_{-\gamma_1}] = \mathfrak{g}_\gamma \subset r_0$ i.e $\gamma \in S_0$, contradiction.

Finally, for each $i \in [\![1, k]\!]$, Π_i is a connected subsystem of $S\backslash S_1$. Since $S\backslash S_2 \subset S\backslash S_1$, then Π_i is also a connected subsystem of $S\backslash S_2$. Therefore, the ideal $I = \sum_{1\leq i\leq k} \sum_{\alpha\in<\Pi_i>} (\mathbb{C}[e_\alpha, e_{-\alpha}] + \mathfrak{g}_\alpha)$ associated to the family Π is equal to r_0, because $\sum_{1\leq i\leq k} < \Pi >= \Omega_0$. □

Let $\Pi = \{\Pi_i\}_{1\leq i\leq k}$ be a family of connected subsystems of S such that Π_i are pairwise disjoint and $I = \sum_{1\leq i\leq k} I_i$ the ideal associated to Π. We introduce the following property : *Let* m *be a nilpotent standard algebra such that* I *is an ideal of* r_1 *contained in* r_2. Let $\mathfrak{F}$ denote the set of nilpotent standard algebras satisfing the above property. Let $C(\Pi)$ be the cardinal of $\mathfrak{F}$. Then, the number of standard algebras is a sum of the number $C(\Pi)$ for different family Π.

Let $ST(j)$ be the number of standard algebras of semisimple Lie algebra of type A_j, for $j = 1, \cdots, p$.

If $k = 1$, there is 3 cases :

1. $\Pi = \Pi_1 = \emptyset$ then $C(\Pi) = \frac{1}{p+2}\begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix}$ (It's the number of nilpotent

standard algebras).

2. $\Pi = \Pi_1 = \{\alpha_j, \alpha_{j+1}, \cdots, \alpha_{j+t}\}$ for t in $\{p-j-1, p-j\}$ then $C(\Pi) = \frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix}$ for $j = 1, \cdots, p-1$.

3. $\Pi_1 = \{\alpha_p\}$ then $C(\Pi) = \frac{1}{p}\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix}$.

If $k > 1$, there is one case :

1. $\Pi_1 = \{\alpha_j, \alpha_{j+1}, \cdots, \alpha_{j+t}\}$ for $t = 0, \cdots, p-j-2$ then $C(\Pi) = \frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} ST(p-j-t-1)$ for $j = 1, \cdots, p-2$.

Then, the number of standard algebras is :

$$ST(p) = \frac{1}{p+2}\begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix} + \sum_{1\le j\le p-1}\left(\sum_{p-j-1\le t\le p-j}\frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix}\right) + \frac{1}{p}\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix} + \sum_{1\le j\le p-2}\left(\sum_{0\le t\le p-j-2}\frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} ST(p-j-t-1)\right).$$

After developing this formula, we obtain :

Theorem 4.4: *The number of standard algebras (not necessarily nilpotent) of semisimple Lie algebra $\mathfrak{g}$ of type A_p is given by the following recursive function :*

$$ST(p) = \frac{1}{p}\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix} + \frac{1}{p+2}\begin{pmatrix} 2p+2 \\ p+1 \end{pmatrix} + \sum_{1\le j\le p-1}\frac{2}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix} + \sum_{1\le j\le p-2}\frac{1}{j}\begin{pmatrix} 2j-2 \\ j-1 \end{pmatrix}\left(\sum_{1\le i\le p-j-1} ST(i)\right)$$

5 Computer aided for enumeration of standard algebras.

In this section, we compute the number of standard algebras and nilpotent standard algebras of a semisimple Lie algebra. We give also for each value of the nilindex the number of corresponding standard subalgebras. These results are obtained using a Mathematica package available in

http://www.math.uha.fr/publi2002.html.

5.1 Exceptional semisimple Lie algebras

Let NS denotes the number of nilpotent standard algebras and the sequence IND_p denotes the sequence of the number of nilpotent standard algebras for each value of the nilindex, the number in IND_p at position j corresponds to the number of nilpotent standard algebras of nilindex j. The number CNS denotes the number of complete nilpotent standard algebras as defined in section 3 and the number ST denotes the number of standard algebras.

Algebra	NS	IND	CNS	ST
E_6	833	{1,63,210,217,150,92,51,28,12,6,2,1}	64	1092
E_7	4160	{1,127,662,894,766,576,403,279, 175,115,68,44,23,14,7,4,1,1}	128	5048
E_8	25080	{1,255,2200,3804,3872,3372,2752,2182, 1656,1277,955,737,536,412,300,227, 157,123,81,61,40,30,18,14,7,5,3,2,0,1}	256	28355
F_4	105	{1,15,28,21,14,12,5,4,2,2,0,1}	16	132
G_2	8	{1,3,2,1,0,1}	4	11

5.2 Algebras A_p, B_p, C_p, D_p for $p \leq 7$

Let NS_p be the number of nilpotent standard algebras. The sequence IND_p denotes the sequence of the number of nilpotent standard algebras for each value of the nilindex. Let CNS_p be the number of complete nilpotent standard algebras and ST_p be the number of standard algebras.

5.2.1 Algebra A_p

p	NS_p	IND_p	CNS_p	ST_p
2	5	{1,3,1}	4	8
3	14	{1,7,5,1}	8	23
4	42	{1,15,18,7,1}	16	69
5	132	{1,31,57,33,9,1}	32	215
6	429	{1,63,169,132,52,11,1}	64	691
7	1430	{1,127,482,484,247,75,13,1}	128	2278

5.2.2 Algebras B_p

p	NS_p	IND_p	CNS_p	ST_p
2	6	{1,3,1,1}	4	9
3	20	{1,7,6,4,1,1}	8	29
4	70	{1,15,23,16,7,6,1,1}	16	98
5	252	{1,31,75,62,36,28,9,8,1,1}	32	343
6	924	{1,63,226,229,162,121,54,45,11,10,1,1}	64	1231
7	3432	{1,127,651,811,674,504, 274,220,77,66,13,12,1,1}	128	4499

5.2.3 Algebras C_p

p	NS_p	IND_p	CNS_p	ST_p
2	6	{1,3,1,1}	4	9
3	20	{1,7,5,5,1,1}	8	29
4	70	{1,15,18,20,7,7,1,1}	16	98
5	252	{1,31,57,73,35,35,9,9,1,1}	32	343
6	924	{1,63,169,253,152,154,54,54,11,11,1,1}	64	1231
7	3432	{1,127,482,848,611,635,273, 273,77,77,13,13,1,1}	128	4499

5.2.4 Algebra D_p

p	NS_p	IND_p	CNS_p	ST_p
2	4	{1,3}	4	9
3	14	{1,7,5,1}	8	23
4	50	{1,15,20,10,3,1}	16	77
5	182	{1,31,65,48,23,10,3,1}	32	264
6	672	{1,63,195,190,118,62,27,12,3,1}	64	937
7	2508	{1,127,560,691,516,313,164,85,33,14,3,1}	128	3401

Remark: The values of NS_p for any semisimple Lie algebra and the IND_p for A_p correspond to the formula given in [1] and [5]. The values CNS_p for any semisimple Lie algebra and ST_p in A_p's case correspond to the formula given in section 3 and section 4 of this paper.

Remark: In forthcoming paper we will give a formulas for ST_p for the other semisimple Lie algebras.

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