ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 10, n°2 (2003), p. 181-194.

<http://ambp.cedram.org/item?id=AMBP_2003__10_2_181_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

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Annales Mathematiques Blaise Pascal **10**, 181-194 (2003)

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Abstract

We study in this paper a Lipschitz control problem associated to a semilinear second order ordinary differential equation with pointwise state constraints. The control acts as a coefficient of the state equation. The nonlinear part of the equation is governed by a Nemytskij operator defined by a Lipschitzian but possibly nonsmooth function. We prove the existence of optimal controls and obtain a necessary optimality conditions looking somehow to the Pontryagin's maximum principle. These conditions utilize the notion of Clarke's generalized directional derivative. We point out that this work provides complements to our previous paper [2], where a similar problem was studied but with tools only from classical analysis.

Key words and phrases. Semilinear second order ordinary differential equation. Optimality conditions. Nemytskij operator. Clarke's generalized directional derivative.

2000 Mathematics Subject Classification. 49K15, 49J15.

Résumé: L'objet de cet article est d'étudier un problème de contrôle optimal gouverné par une équation différentielle ordinaire du second ordre sous conditions aux limites et contraintes sur l'état. Les contrôles sont Lipschitziens et agissent comme des coefficients pour cette équation. La partie non linéaire de cette équation est donnée par l'action d'un opérateur de composition (de Nemytskij) défini par une fonction Lipschitzienne mais non nécessairement régulière. Nous établissons l'existence des contrôles optimaux et trouvons des conditions nécessaires d'optimalité qui ressemblent au principe du maximum de Pontriaguine. Ces conditions utilisent des notions d'analyse non régulière telles que les notions de sous-différentiel et la derivée directionnelle généralisée de Clarke. Ainsi, ce travail complète notre article [2] qui traite le même problème mais dans le cas régulier avec des outils d'analyse classique. A la fin de ce travail, nous donnons un exemple d'applications.

1 Introduction and position of the problem

The purpose of this paper is to generalize and complete the results of the paper [2]. In a precise manner, for a given function h in the Banach space $C([0,1])$ (of continuous functions on $[0,1]$), we are interested by finding

$$\inf \mathcal{J}(y), \qquad \mathcal{J}(y) = \int_0^1 (y(x) - h(x))^2 \, dx, \tag{1.1}$$

where the state y verifies the nonlinear boundary value problem:

$$\frac{d}{dx}\left(\frac{y'(x)}{u(x)}\right) + q(x)y(x) + \theta(y(x)) = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{1.2}$$

under the constraint

$$0 \leq y(x) \leq a \quad \forall x \in [0,1], \tag{1.3}$$

the controls u are Lipschitz continuous functions belonging to some compact subset $\mathcal{U}_{ad}$ of the Banach space $C([0,1])$ which is contained in $C^1([0,1])$. q is a fixed continuous function on $[0,1]$, and θ is a fixed Lipschitz continuous function on $[0,a]$, which is possibly not belonging to $C^1([0,a])$. We set $Q := \sup\{\mid q(x) \mid : x \in [0,1]\}$ and $b := \sup\{\mid \theta(x) \mid : x \in [0,a]\}$. Throughout this paper, we suppose that the following assumption holds:

$$\theta(y) + q(x)y \geq 0, \quad \forall x \in [0,1], \ \forall y \in [0,a]. \tag{1.4}$$

Since θ is Lipschitzian, we can find a positive constant l such that

$$\mid \theta(x_1) - \theta(x_2) \mid \leq l \mid x_1 - x_2 \mid, \forall x_1, x_2 \in [0,a].$$

In all this paper, $\mathcal{U}_{ad}$ will be the closure, in the Banach space $C([0,1])$, of the following set:

$$\mathcal{S}_{\phi,k,r} := \Big\{ u \in C^2([0,1]) : \phi(x) \leq u(x) \leq \Omega \quad \forall x \in [0,1], \\ \mid u'(x) \mid \leq k, \quad \text{and} \quad \mid u''(x) \mid \leq r, \ \forall x \in [0,1] \Big\} \tag{1.5}$$

where k, r, Ω are fixed numbers in $]0, \infty[$, and ϕ is a fixed continuous function on $[0, 1]$ such that $\phi(x) > 0$ for all $x \in [0, 1]$.

Our control problem will be denoted by $\mathbf{P}_\theta$. We see that the controls are acting in this setting as coefficients for the state equation associated to $\mathbf{P}_\theta$. We recall that when q is zero and when $\theta \in C^1([0, 1])$, this problem was studied in [2] by using tools from classical and regular analysis. In this paper, we are interested by the more general case where q is not zero and θ is not necessary in $C^1([0, a])$. In this case, we are led to use the notions of non smooth analysis. Hence, we consider that this paper provides a generalization and gives complements to our previous paper [2]. Similar problems were studied in [7], [14], and [12], however, using completely different methods. General remarks concerning coefficient control problems in both ordinary and partial differential equations can be found in [13]. We point out that this paper makes a sequel of the papers [1], [9] and [10], where investigations were made for smooth and nonsmooth optimal Lipschitz control for problems governed by semilinear second order differential equations in which the nonlinear part is given by the action of a Nemytskij operator. For other related subjects, one can see the papers [11], [3], [4], [5], [8].

This paper is organized as follows. In the second section, we establish existence of the states and optimal controls. The main result of this section is Theorem 2.1. In the section three, we establish necessary optimality conditions using tools and notions from nonsmooth analysis. The principal result of this section is Theorem 3.1. We end this paper by providing, in section four, an illustrative example where our results are applied.

2 Existence of the states and optimal controls

In this section, we provide some sufficient conditions ensuring existence for solutions to our problem. These conditions are expressed by some inequalities between the fixed parameters Ω, a, b, l, Q involved in the problem $\mathbf{P}_\theta$.

2.1 Preliminaries

2.1.1 Let $u \in \mathcal{U}_{ad}$, and let $G_u = G_u(x,\xi)$ be the uniquely determined Green's function to the next boundary problem

$$\frac{d}{dx}\left[\frac{y'(x)}{u(x)}\right] = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{2.1}$$

An easy computation shows that G_u is given by

$$\begin{cases} G_u(x,\xi) = -\displaystyle\int_0^\xi u(s)\,ds & \text{for } 0 \le \xi \le x \le 1, \quad \text{and} \\ G_u(x,\xi) = -\displaystyle\int_0^x u(s)\,ds & \text{for } 0 \le x \le \xi \le 1. \end{cases}$$

G_u is continuous and symmetric on $[0,1]\times[0,1]$ and the following estimation holds

$$0 \le -\int_0^1 G_u(x,\xi)\,d\xi \le \frac{\Omega}{2} \quad \forall x \in [0,1].$$

2.1.2 In all this paper, we suppose that

$$\Omega < \min\left\{\frac{2a}{b+aQ}, \frac{2}{l+Q}\right\}. \tag{2.2}$$

The Banach space $C([0,1])$ will be equipped with its usual norm denoted by $\|.\|_{C([0,1])}$. We consider the subset $B_+(a)$ of $C([0,1])$ defined by

$$B_+(a) := \{y \in C([0,1]) : 0 \le y(x) \le a \quad \forall x \in [0,1]\}.$$

For any arbitrary control $u \in \mathcal{U}_{ad}$, an element $y \in C^2([0,1]) \cap B_+(a)$ is a solution to the nonlinear boundary value problem (2) if, and only if, $y \in B_+(a)$ and y is a solution to the Hammerstein integral equation

$$y(x) = -\int_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi - \int_0^1 G_u(x,\xi)q(\xi)y(\xi)\,d\xi \quad \forall x \in [0,1]. \tag{2.3}$$

2.1.3 To each control $u \in \Sigma_{ad}$ we associate the unique solution $S(u) = y_u$ to the problem (1.2) (under condition (1.3)). One can see that S is a Lipschitz continuous map from the compact convex subset Σ_{ad} of $C([0,1])$ to the Banach space $C([0,1])$. Indeed, for all controls $u, v \in \mathcal{U}_{ad}$, an easy computation will give the following estimation

$$\|y_u - y_v\|_{C([0,1])} \le \frac{b+aQ}{2-(l+Q)\Omega}\|u-v\|_{C([0,1])}. \tag{2.4}$$

With these preliminaries, we are in position to establish the existence of the solutions to our control problem.

2.2 Existence results

Theorem 2.1: (i) *For each control* $u \in \mathcal{U}_{ad}$, *the boundary value problem* (1.2) *has a unique solution* y_u. *Moreover this solution belongs to* $C^2([0,1]) \cap B_+(a)$.
(ii) *The optimal control problem* $\mathbf{P}_\theta$ *has (at least) an optimal solution* $u_0 \in \Sigma_{ad}$.

Proof: (i) Let $u \in \Sigma_{ad}$ be fixed and associate to it the map T_u defined from $B_+(a)$ to $C([0,1])$ by

$$T_u(y)(x) := -\int_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi - \int_0^1 G_u(x,\xi)q(\xi)y(\xi)\,d\xi \quad \forall x \in [0,1]. \tag{2.5}$$

An easy computation will show that for all $y, z \in B_+(a)$, we have

$$\|T_u(y) - T_u(z)\|_{C([0,1])} \leq \frac{(l+Q)\Omega}{2}\|y-z\|_{C([0,1])}. \tag{2.6}$$

By using assumption (2.2), we see that $T_u(B_+(a)) \subset B_+(a)$, and that T_u must be a contraction from $B_+(a)$ to itself. Since the set $B_+(a)$ is a closed (convex) subset of the Banach space $C([0,1])$, we deduce by using the Banach fixed point theorem that T_u has a unique fixed point $y_u \in B_+(a)$. This proves (i). It remains to prove (ii).

(ii) For each control $u \in \Sigma_{ad}$ we set $J(u) := \mathcal{J}(y_u)$. We obtain by easy computation the following estimation

$$\mid J(u) - J(v) \mid \leq \frac{2(b+aQ)(a+\|h\|_{C([0,1]})}{2-(l+Q)\Omega}\|u-v\|_{C([0,1])}. \tag{2.7}$$

This inequality says that the map J is Lipschitz continuous from the compact subset $\mathcal{U}_{ad}$ of the Banach space $C([0,1])$ to the set of real numbers. Hence, using the classical Weierstrass theorem, we deduce that there exists at least one optimal control to our problem $(\mathbf{P}_\theta)$. □

3 Necessary optimality conditions

3.1 Preliminaries and recalls

3.1.1 Let $u_0 \in \mathcal{U}_{ad}$ be an optimal control to the problem $\mathbf{P}_\theta$ and $u \in \mathcal{U}_{ad}$ another admissible control. The respective states are denoted by $y_0 = S(u_0)$

and $y_u = S(u)$. For any $\lambda \in [0,1]$ we set $u_\lambda := u_0 + \lambda(u - u_0) \in \Sigma_{ad}$, and $y_\lambda := S(u_\lambda)$. From the inequality (2.4) we obtain

$$\|y_\lambda - y_0\|_{C([0,1])} \leq \frac{\lambda(b + aQ)}{2 - (l + Q)\Omega} \|u - u_0\|_{C([0,1])}, \quad \forall \lambda \in [0,1]. \tag{3.1}$$

3.1.2 As in the paper [2], we can prove that if θ belongs to $C^1([0,a])$, then the quotient $\phi_\lambda^u := \frac{y_\lambda - y_0}{\lambda}$ ($\lambda \in]0,1]$) converges in the Banach space $C([0,1])$, when $\lambda \longrightarrow 0^+$, to the unique fixed point $\tilde{y}(u)$ of the selfmapping Υ of $C([0,1])$ defined for all $z \in C([0,1])$, by

$$\Upsilon(z)(x) := -y_0(x) - m(x) - \int_0^1 G_{u_0}(x,\xi)\theta'(y_0(\xi))z(\xi)\, d\xi \quad \forall x \in [0,1],$$

where m is the function defined for all $x \in [0,1]$, by

$$m(x) := \int_0^1 G_u(x,\xi)\theta(y_0(\xi))\, d\xi + \int_0^1 G_u(x,\xi)q(\xi)y_0(\xi)\, d\xi.$$

Indeed, it is easy to see that $\Upsilon(z)$ verifies the following inequality:

$$\|\Upsilon(z_1) - \Upsilon(z_2)\|_{C([0,1])} \leq \frac{l\Omega}{2} \|z_1 - z_2\|_{C([0,1])}. \tag{3.2}$$

Therefore, the inequality (2.2) and the Banach fixed point theorem ensure the existence and uniqueness of a unique fixed point which we have denoted here by $\tilde{y}(u)$.

In our situation, we have no information about the convergence of the quotient ϕ_λ^u in the Banach space $C([0,1])$. Instead of this, we shall see next that we can always find at least a subsequence $(\lambda_n)_n$ of elements in $]0,1]$ converging to zero for which the quotient $((y_{\lambda_n} - y_0){\lambda_n}^{-1})_n$ converges in $C([0,1])$.

3.1.3. Recalls. For the generalized gradient and all related topics used below, one can see [[6], chapter two]. For the sake of completeness, let us make a brief recall on Clarke's subdifferential and directional derivative. Let X be a Banach space. Let $x \in X$ and let f be a real valued function defined and Lipschitz near x. Then, the generalized directional derivative of f at x in the direction v (in X) is given by

$$f^0(x;v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}. \tag{3.3}$$

Let X^* be the dual space of X. For all $\xi \in X^*$ and $v \in X$, we denote $\xi(v) =< \xi, v >$. The generalized gradient of f at x is denoted by $\partial f(x)$, it is the subset of X^* given by

$$\partial f(x) := \{\xi \in X^* : f^0(x; v) \geq < \xi, v > \ \forall v \in X\}. \tag{3.4}$$

By Proposition 2.1.2 of [[6], p. 27], we know that

$$f^0(x; v) = \max\{< \xi, v >: \xi \in \partial f(x)\}. \tag{3.5}$$

Let C be a nonempty subset of X, and consider its distance function, that is

$$d_C(x) := \inf\{\|x - c\| : c \in C\}.$$

Suppose now that $x \in C$ and $v \in X$. The vector v is said to be tangent to C at x provided $d^0_C(x; v) = 0$. The set of all tangents to C at x is denoted $T_C(x)$. The normal cone to C at x is defined by polarity with $T_C(x)$

$$N_C(x) := \{\xi \in X^* :< \xi, v >\leq 0 \ \forall\, v \in T_C(x)\}. \tag{3.6}$$

3.1.4 As it was done in the papers [1], [9] and [10], we shall use tools from nonsmooth analysis to derive our optimal conditions. We start by pointing out that since the map $J : \mathcal{U}_{ad} \to \mathbb{R}$ is Lipschitz continuous (but surely not convex), we have the optimality condition

$$0 \in \partial J(u_0) + N_{\mathcal{U}_{ad}}(u_0). \tag{3.7}$$

where J is now a Lipschitz continuous extension of the map $J : \mathcal{U}_{ad} \to \mathbb{R}$ to the whole Banach space $C([0, 1])$. As usual, the notation $\partial J(u_0)$ designates Clarke's subdifferential of J at u_0 and $N_{\mathcal{U}_{ad}}(u_0)$ is the normal cone to $\mathcal{U}_{ad}$ at u_0 (see [6], p. 52). For each admissible control u we introduce (as in [1], [9], [10]) the set $\Phi(u)$ given by

$$\Phi(u) := \left\{\phi_\lambda \in C([0, 1]) : \phi_\lambda = \frac{y_\lambda - y_0}{\lambda}, \quad \lambda \in]0, 1]\right\},$$

where $y_\lambda = y_{u_\lambda}$ is the same as before, and define two new functions ϕ_- and ϕ_+ by setting

$$\phi_-(x) := \liminf_{\lambda\to 0+} \phi_\lambda(x), \quad \phi_+(x) := \limsup_{\lambda\to 0+} \phi_\lambda(x), \quad \forall x \in [0, 1].$$

By using the inequality (3.1), we see that $\Phi(u)$ is a bounded subset of $C([0,1])$. Next, we will show that the set $\Phi(u)$ is equicontinuous. Indeed, since every ϕ_λ is differentiable, then it is sufficient to show that the set of derivatives $\{\phi'_\lambda : \lambda \in]0,1]\}$ is uniformly bounded. For all $x \in [0,1]$ and all $\lambda \in]0,1]$, we have

$$y'_\lambda(x) = u_\lambda(x) \int_x^1 \theta(y_\lambda(\xi))\, d\xi + u_\lambda(x) \int_x^1 q(\xi) y_\lambda(\xi)\, d\xi, \tag{3.8}$$

and

$$y'_0(x) = u_0(x) \int_x^1 \theta(y_0(\xi))\, d\xi + u_0(x) \int_x^1 q(\xi) y_0(\xi)\, d\xi. \tag{3.9}$$

Therefore, by using (3.8) with (3.9) and after easy manipulations, we obtain

$$\begin{aligned} \phi'_\lambda(x) &= (u(x) - u_0(x)) \int_x^1 \theta(y_\lambda(\xi))\, d\xi \\ &-u_0(x) \int_x^1 q(\xi) \frac{[y_\lambda(\xi) - y_0(\xi)]}{\lambda}\, d\xi \\ &+ (u(x) - u_0(x)) \int_x^1 q(\xi) y_\lambda(\xi)\, d\xi \\ &:= A + B + C. \end{aligned} \tag{3.10}$$

By using inequality (3.1) and the assumptions, we get the following estimates:

$$\begin{cases} \mid A \mid \leq 2b\Omega, \\ \mid B \mid \leq \dfrac{2(b + aQ)Q\Omega^2}{2 - (l + Q)\Omega}, \\ \mid C \mid \leq 2aQ\Omega. \end{cases}$$

Whence, the equicontinuity of the set $\Phi(u)$ is proved. Therefore, by Ascoli's theorem, this set is relatively compact in the Banach space $C([0,1])$. As a consequence, we deduce that the functions ϕ_- and ϕ_+ are continuous on $[0,1]$. Another consequence is that, if we introduce the new set $\Phi_0(u)$ given by

$$\Phi_0(u) := \Big\{ \phi \in C([0,1]) : \phi = \lim_{n\to+\infty} \phi_{\lambda_n} \quad \text{in} \quad C([0,1]), \quad \text{where} \\ (\phi_{\lambda_n})_n \subset \Phi(u) \quad \text{and} \quad (\lambda_n)_n \subset [0,1] \quad \text{with} \quad \lambda_n \to 0+ \Big\},$$

then $\Phi_0(u)$ is nonempty and bounded. Now, let us denote by θ any given Lipschitz continuous extension of the function $\theta : [0,1] \to \mathbb{R}$ to the whole $\mathbb{R}$, and let $\theta^0(x;\xi)$ designates Clarke's directional derivative of θ at x in the direction ξ for all $x, \xi \in \mathbb{R}$. Then with these notations and preliminaries, we can state our result providing the optimality conditions.

3.2 Optimality conditions

Theorem 3.1: *Let $u_0 \in \mathcal{U}_{ad}$ be an optimal control to P_θ and $y_0 \in C([0,1])$ the related optimal state. Then for all $u \in \mathcal{U}_{ad}$ and all $\phi \in \Phi_0(u)$ it holds*

(i) $\phi(x) \in [\phi_-(x), \phi_+(x)] \quad \forall x \in [0,1]$.

(ii) $\phi(x) \leq -\int_0^1 G_{u_0}(x,\xi)\theta^0(y_0(\xi);\phi(\xi))\,d\xi - \int_0^1 \Big(G_u(x,\xi) - G_{u_0}(x,\xi)\Big)\theta(\phi(\xi))\,d\xi$

$\qquad - \int_0^1 G_{u_0}(x,\xi)q(\xi)\phi(\xi)\,d\xi, \qquad$ *for all* $x \in [0,1]$.

(iii) $0 \leq \int_0^1 \phi(x)(y_0(x) - h(x))\,dx$.

Proof: Let $u \in \mathcal{U}_{ad}$. We verify first that the assertions (i), (ii) and (iii) hold if $u = u_0$. Indeed, in this case, we have $\Phi(u_0) = \{0\}$, and therefore $\Phi_0(u_0) = \{0\}$, and $\phi_- = \phi_+ = 0$. Thus we may suppose that $u \neq u_0$. Consider an element $\phi \in \Phi_0(u)$. Then by the definition of the set $\Phi_0(u)$, one can find a zero sequence $(\lambda_n)_n \subset]0,1]$ and a sequence $(\phi_{\lambda_n})_n \subset \Phi(u)$ converging to ϕ in the Banach space $C([0,1])$. Assertion (i) is obvious. Next, we prove assertion (ii).

According to the definition of ϕ_{λ_n}, for all $x \in [0,1]$ we have

$$\begin{aligned}
\phi_{\lambda_n}(x) &= \frac{y_{\lambda_n}(x) - y_0(x)}{\lambda_n} \\
&= -\int_0^1 G_{u_0}(x,\xi)\frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\,d\xi \\
&\quad - \int_0^1 [G_u(x,\xi) - G_{u_0}(x,\xi)]\theta(y_{\lambda_n}(x))\,d\xi \\
&\quad - \int_0^1 G_{u_0}(x,\xi)q(\xi)\phi_{\lambda_n}(\xi)\,d\xi \\
&\quad - \int_0^1 [G_u(x,\xi) - G_{u_0}(x,\xi)]q(\xi)y_{\lambda_n}(\xi)\,d\xi. \qquad (3.11)
\end{aligned}$$

Because of (3.1) and the fact that θ is $l-$Lipschitz continuous, we have

$$\lim_{n\to+\infty} y_{\lambda_n} = y_0, \quad \text{and} \quad \lim_{n\to+\infty} \theta(y_{\lambda_n}) = \theta(y_0) \quad \text{in} \quad C([0,1]).$$

Because of (3.1) and the fact that θ is $l-$Lipschitz continuous, for all $\xi \in [0,1]$, we have the following inequality

$$\frac{\mid \theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi)) \mid}{\lambda_n} \le \frac{l}{\lambda_n} \mid y_{\lambda_n}(\xi) - y_0(\xi) \mid \le \frac{b+aQ}{2-(l+Q)\Omega} \|u-u_0\|_{C([0,1])}.$$

By taking in (3.11) the limit superior and applying Fatou's lemma we obtain

$$\begin{aligned}\phi(x) \le &- \int_0^1 G_{u_0}(x,\xi) \limsup_{n\to+\infty} \frac{\theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi))}{\lambda_n}\, d\xi \\ &- \int_0^1 [G_u(x,\xi) - G_{u_0}(x,\xi)]\theta(y_0(x))\, d\xi \\ &- \int_0^1 G_{u_0}(x,\xi) q(\xi)\phi(\xi)\, d\xi \\ &- \int_0^1 [G_u(x,\xi) - G_{u_0}(x,\xi)] q(\xi) y_0(\xi)\, d\xi \quad \forall x \in [0,1]. \qquad (3.12)\end{aligned}$$

Since $\theta : [0,a] \to \mathbb{R}$ is $l-$Lipschitz continuous it can be extended to the whole $\mathbb{R}$ in a Lipschitz continuous function denoted again by θ having the same Lipschitz constant l. For this extension and for all $x, \xi \in \mathbb{R}$ we will denote Clarke's directional derivative of θ at x in the direction ξ by $\theta^0(x;\xi)$, and $\partial\theta(x)$ will be the corresponding subdifferential (see for example [6], [15]). By using the mean-value theorem for subdifferentials (see for example [15], p. 40) we get the following property: For any natural number $n \in \mathbb{N}$ and any $x \in [0,1]$ there are two real numbers $\zeta_n(x)$, $z_n(x) \in \mathbb{R}$ satisfying the following conditions

$$\zeta_n(x) \in \partial\theta(z_n(x)) \;\; \forall n \in \mathbb{N} \quad \lim_{n\to+\infty} (z_n(x)) = y_0(x) \quad (\text{in}\, \mathbb{R}),$$

$$\frac{\mid \theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi)) \mid}{\lambda_n} = \zeta_n(x)\phi_{\lambda_n}(x) \le \theta^0(z_n(x); \phi_{\lambda_n}(x)).$$

According to the upper semicontinuity of Clarke's directional derivative we obtain

$$\limsup_{n\to+\infty} \frac{\mid \theta(y_{\lambda_n}(\xi)) - \theta(y_0(\xi)) \mid}{\lambda_n} \le \theta^0(y_0(x); \phi(x)) \quad \forall x \in [0,1]. \qquad (3.13)$$

Now, by using the following estimate

$$\theta^0(y_0(x); \phi(x)) \leq l \mid \phi(x) \mid \quad \forall x \in [0,1],$$

we deduce that the function $\theta^0(y_0(.); \phi(.))$ is bounded. Furtheremore, since it is upper semicontinuous on $[0,1]$ it is measurable and, hence it is integrable on $[0,1]$. The property (ii) follows then from (3.12) and (3.13). It remains to prove (iii). To this end, we start by noticing that for all $\lambda \in]0,1]$, we have the following inequality:

$$0 \leq \frac{j(u_\lambda) - j(u_0)}{\lambda} = 2\int_0^1 \frac{y_\lambda - y_0}{\lambda}(y_0 - h)\,dx + \frac{1}{\lambda}\int_0^1 (y_\lambda - y_0)^2\,dx. \quad (3.14)$$

Now, we use the inequalities (3.1), (3.14) and obtain (iii) by applying Lebesgue's theorem of dominated convergence to the sequence ϕ_{λ_n} which is converging to ϕ in the Banach space $C([0,1])$. □

This theorem may be considered as a generalization of our theorem 3.1.3 stated in the paper [2] in a particular case where θ was supposed to be continously differentiable.

4 Illustrative example

4.1. We want to determine $\inf_{u\in\mathcal{U}_{ad}} J(u)$, where

$$J(u) = \int_0^1 [y_u(x) - 1]^2\,dx, \quad (4.1)$$

$\mathcal{U}_{ad} = [\omega, \Omega]$, with $0 < \omega < \Omega < \infty$, and y_u is the unique solution of the following boundary problem

$$y'' - uy + u = 0, \quad y(0) = 0 = y'(1), \quad (4.2)$$

with the constraint on the state $0 \leq y \leq 1$.

The solution of this problem is given for all $0 \leq x \leq 1$ by

$$y_u(x) = 1 - \frac{\cosh((1-x)\sqrt{u})}{\cosh(\sqrt{u})}. \quad (4.3)$$

After some computations, we get

$$J(u) = \frac{1}{2}\left(1 - \tanh^2(\sqrt{u}) + \frac{\tanh(\sqrt{u})}{\sqrt{u}}\right). \tag{4.4}$$

4.2. Let u_0 be an optimal control and let u be any arbitrary control in $[\omega, \Omega]$. We apply (iii) of Theorem 3.1 to find u_0. By easy computations, we obtain $\Phi_0(u_0) = \{z_{(u_0,u)}\}$, where

$$z_{(u_0,u)}(x) := \frac{(u-u_0)\cosh((x-1)\sqrt{u_0})\left[\tanh(\sqrt{u_0}) + (x-1)\tanh((1-x)\sqrt{u_0})\right]}{2\sqrt{u_0}\cosh(\sqrt{u_0})}. \tag{4.5}$$

By (iii) we have

$$0 \leq \int_0^1 z_{(u_0,u)}(x)[y_0(x) - 1]\, dx, \tag{4.6}$$

which implies

$$0 \leq (u-u_0)\int_0^1 \frac{\cosh^2((x-1)\sqrt{u_0})\left[(1-x)\tanh((1-x)\sqrt{u_0}) - \tanh(\sqrt{u_0})\right]}{2\sqrt{u_0}\cosh^2(\sqrt{u_0})}\, dx. \tag{4.7}$$

It is easy to see that

$$\int_0^1 \cosh^2((x-1)\sqrt{u_0})\,[(1-x)\tanh((1-x)\sqrt{u_0}) - \tanh(\sqrt{u_0})]\; dx < 0. \tag{4.8}$$

From (4.7) and (4.8) we deduce that

$$u \leq u_0, \quad \forall u \in [\omega, \Omega], \tag{4.9}$$

From (4.9), we deduce that $u_0 = \Omega$. Thus,

$$\inf_{u\in[\omega,\Omega]} J(u) = J(\Omega) = \frac{1}{2}\left(1 - \tanh^2(\sqrt{\Omega}) + \frac{\tanh(\sqrt{\Omega})}{\sqrt{\Omega}}\right). \tag{4.10}$$

Acknowledgements

The authors thank very much the anonymous referee for his (or her) helpful comments and suggestions.

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