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ABSTRACT We study the action of a differential operator on Schubert polynomials. Using this action, we first give a short new proof of an identity of I. Macdonald (1991). We then prove a determinant conjecture of R. Stanley (2017). This conjecture implies the (strong) Sperner property for the weak order on the symmetric group, a property recently established by C. Gaetz and Y. Gao (2019).

1. INTRODUCTION

This paper is motivated by a conjecture of R. Stanley [10, Conjecture 2.2]. Let \mathcal{S}_n be the symmetric group⁽¹⁾ with its standard generating set $S = \{s_1, s_2, \dots, s_{n-1}\}$, and let $\mathcal{S}_n(\ell)$ denote the subset of those permutations of (Coxeter) length ℓ . For $1 \leq \ell \leq \binom{n}{2}$, let M_ℓ be the matrix with rows indexed by $\mathcal{S}_n(\ell-1)$ and columns indexed by $\mathcal{S}_n(\ell)$, where the entry in position (u, v) is

$$M_\ell[u, v] = \begin{cases} k, & \text{if } v = us_k \text{ and} \\ 0, & \text{if } u^{-1}v \notin S. \end{cases}$$

For $\ell \leq \binom{n}{2} - \ell$, the product $\widetilde{M}^{(\ell)} = M_{\ell+1}M_{\ell+2} \cdots M_{\binom{n}{2}-\ell}$ is a square matrix with rows indexed by $\mathcal{S}_n(\ell)$ and columns indexed by $\mathcal{S}_n(\binom{n}{2} - \ell)$. Stanley conjectures an explicit formula for $\det \widetilde{M}^{(\ell)}$, which implies that $\widetilde{M}^{(\ell)}$ is invertible. A motivation for this conjecture is that $M_{\ell(u)+1}M_{\ell(u)+2} \cdots M_{\ell(v)}[u, v]$ is nonzero if and only if $u \leq v$ in weak order. Hence, by standard linear-algebraic arguments (cf. [9, 10]), showing that $\det \widetilde{M}^{(\ell)}$ is nonzero implies that the weak order on the symmetric group has the (strong) Sperner property.

Recently, C. Gaetz and Y. Gao [5] proved the invertibility of Stanley's matrix by constructing an action of the Lie algebra \mathfrak{sl}_2 . We give a new proof of invertibility by proving Stanley's determinant conjecture. Our proof also involves an \mathfrak{sl}_2 -representation, but relies on a new identity for derivatives of Schubert polynomials.

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KEYWORDS. Sperner property, weak order, Schubert polynomial, Macdonald identity.

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⁽¹⁾Interpreting \mathcal{S}_n as the group of bijections from $\{1, 2, \dots, n\}$ to itself, we define multiplication by $wv := v \circ w$.

The generator s_i acts on $\text{Poly}_n = \mathbb{C}[x_1, \dots, x_n]$ by

$$s_i \cdot f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i+1}, x_i, \dots, x_n).$$

We define the *Newton divided difference operators* on Poly_n by

$$N_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}.$$

(We avoid the more standard notation ∂_i because of potential confusion with partial derivatives.) For $w \in \mathcal{S}_n$, the *Schubert polynomials* \mathfrak{S}_w are defined by the recurrence

$$\mathfrak{S}_{s_k w} = N_k \mathfrak{S}_w \text{ for } \ell(s_k w) < \ell(w)$$

with $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. For background on Schubert polynomials, we refer the reader to [6, 7].

We consider the differential operator $\nabla = \sum_{i=1}^n \frac{\partial}{\partial x_i}$. Our key result is the following:

PROPOSITION 1.1. *For $w \in \mathcal{S}_n$, we have*

$$\nabla(\mathfrak{S}_w) = \sum_{\ell(ws_k) < \ell(w)} k \mathfrak{S}_{ws_k}.$$

Thus, Stanley’s M -matrices are the matrices of the operator ∇ in the basis of Schubert polynomials.

We first apply Proposition 1.1 to give a short new proof of a theorem of Macdonald [6, (6.11)]. We then use Proposition 1.1 to prove Stanley’s conjecture.

THEOREM 1.2 (Conjectured by Stanley [10, Conjecture 2.2]). *For $\ell \leq \binom{n}{2} - \ell$,*

$$\det \widetilde{M}^{(\ell)} = \pm \prod_{k=0}^{\ell} \left((\ell - k + 1)(\ell - k + 2) \cdots \left(\binom{n}{2} - \ell - k \right) \right)^{|\mathcal{S}_n(k)| - |\mathcal{S}_n(k-1)|}.$$

We write \pm because we have not specified an order on the rows and columns of each M_i . In more recent work, C. Gaetz and Y. Gao [4] build on the results here to obtain, among other consequences, an explicit description of the Smith normal form of $\widetilde{M}^{(\ell)}$.

2. PROOF OF PROPOSITION 1.1 AND A MACDONALD IDENTITY

We start with a straightforward lemma.

LEMMA 2.1. *The differential operator ∇ commutes with N_i for all i . That is, for any $f \in \text{Poly}_n$, we have*

$$\nabla(N_i(f)) = N_i(\nabla(f)).$$

Proof. If $j \notin \{i, i + 1\}$, then

$$\frac{\partial}{\partial x_j} N_i(f) = \frac{1}{x_i - x_{i+1}} \left(\frac{\partial}{\partial x_j} (f) - \frac{\partial}{\partial x_j} (s_i \cdot f) \right).$$

If $j = i$, then

$$\frac{\partial}{\partial x_i} N_i(f) = \frac{1}{x_i - x_{i+1}} \left(\frac{\partial}{\partial x_i} (f) - \frac{\partial}{\partial x_i} (s_i \cdot f) \right) - \frac{f - s_i \cdot f}{(x_i - x_{i+1})^2}.$$

Similarly, if $j = i + 1$, then

$$\frac{\partial}{\partial x_{i+1}} N_i(f) = \frac{1}{x_i - x_{i+1}} \left(\frac{\partial}{\partial x_{i+1}} (f) - \frac{\partial}{\partial x_{i+1}} (s_i \cdot f) \right) + \frac{f - s_i \cdot f}{(x_i - x_{i+1})^2}.$$

Therefore,

$$\begin{aligned} \nabla(N_i(f)) &= \frac{f - s_i \cdot f}{(x_i - x_{i+1})^2} - \frac{f - s_i \cdot f}{(x_i - x_{i+1})^2} + \sum_{j=1}^n \frac{1}{x_i - x_{i+1}} \left(\frac{\partial}{\partial x_j}(f) - \frac{\partial}{\partial x_j}(s_i \cdot f) \right) \\ &= \frac{\sum_{j=1}^n \frac{\partial}{\partial x_j}(f) - \sum_{j=1}^n \frac{\partial}{\partial x_j}(s_i \cdot f)}{x_i - x_{i+1}} = \frac{\nabla(f) - \nabla(s_i \cdot f)}{x_i - x_{i+1}}. \end{aligned}$$

Since ∇ is \mathcal{S}_n -invariant, it commutes with s_i , so we obtain

$$\frac{\nabla(f) - \nabla(s_i \cdot f)}{x_i - x_{i+1}} = \frac{\nabla(f) - s_i \cdot \nabla(f)}{x_i - x_{i+1}} = N_i(\nabla(f)).$$

Thus, $\nabla(N_i(f)) = N_i(\nabla(f))$, as desired. □

Proof of Proposition 1.1. We first verify Proposition 1.1 in the case that $w = w_0$. Since $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, we have

$$\begin{aligned} \nabla(\mathfrak{S}_{w_0}) &= \sum_{j=1}^{n-1} (n-j)x_1^{n-1} x_2^{n-2} \cdots x_{j-1}^{n-j+1} x_j^{n-j-1} x_{j+1}^{n-j-1} \cdots x_{n-1} \\ &= \sum_{k=1}^{n-1} kx_1^{n-1} x_2^{n-2} \cdots x_{n-k-1}^{k+1} x_{n-k}^{k-1} x_{n-k+1}^{k-1} \cdots x_{n-1}. \end{aligned}$$

But also

$$\mathfrak{S}_{w_0 s_k} = \mathfrak{S}_{s_{n-k} w_0} = N_{n-k}(\mathfrak{S}_{w_0}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-k-1}^{k+1} x_{n-k}^{k-1} x_{n-k+1}^{k-1} \cdots x_{n-1}.$$

Comparing these equations gives

$$\nabla(\mathfrak{S}_{w_0}) = \sum_{k=1}^{n-1} k \mathfrak{S}_{w_0 s_k}.$$

Consider an arbitrary permutation w . Let $r = \binom{n}{2} - \ell(w)$ and write $w = s_{i_1} s_{i_2} \cdots s_{i_r} w_0$. By Lemma 2.1, we have

$$(\star) \quad \nabla(\mathfrak{S}_w) = \nabla N_{i_1} \cdots N_{i_r}(\mathfrak{S}_{w_0}) = N_{i_1} \cdots N_{i_r} \nabla(\mathfrak{S}_{w_0}) = N_{i_1} \cdots N_{i_r} \sum_{k=1}^{n-1} k \mathfrak{S}_{w_0 s_k}.$$

Hence,

$$N_{i_1} N_{i_2} \cdots N_{i_r}(\mathfrak{S}_{w_0 s_k}) = \begin{cases} \mathfrak{S}_{s_{i_1} \cdots s_{i_r} w_0 s_k}, & \text{if } \ell(ws_k) = \binom{n}{2} - r - 1 = \ell(w) - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathfrak{S}_{s_{i_1} \cdots s_{i_r} w_0 s_k} = \mathfrak{S}_{ws_k}$, Equation (\star) then becomes

$$\nabla(\mathfrak{S}_w) = \sum_{\ell(ws_k) = \ell(w) - 1} k \mathfrak{S}_{ws_k},$$

as desired. □

Proposition 1.1 yields a short proof of an identity of Macdonald [6, (6.11)]. Another proof of this result was given by S. Fomin and R. Stanley in terms of nilCoxeter algebras [3], while a bijective proof was given by S. Billey, A. Holroyd and B. Young [2]. A reduced word for $w \in \mathcal{S}_n$ is a tuple $a = (a_1, \dots, a_{\ell(w)})$ such that $w = s_{a_1} s_{a_2} \cdots s_{a_{\ell(w)}}$. We write $R(w)$ for the set of reduced words of w .

THEOREM 2.2 (Macdonald [6, (6.11)]). *Let $w \in \mathcal{S}_n$ with $\ell(w) = k$. Then*

$$\frac{1}{k!} \sum_{a \in R(w)} a_1 a_2 \cdots a_k = \mathfrak{S}_w(1, 1, \dots, 1).$$

Proof. For any monomial μ of degree k , we have $\nabla^k(\mu) = k!$. Since \mathfrak{S}_w is homogeneous of degree k , we then see $\nabla^k(\mathfrak{S}_w) = k! \mathfrak{S}_w(1, 1, \dots, 1)$. On the other hand, by Proposition 1.1,

$$\nabla^k(\mathfrak{S}_w) = \sum_{a \in R(w)} a_1 a_2 \cdots a_k. \quad \square$$

3. A VECTOR SPACE OF POLYNOMIALS

Let $W \subset \text{Poly}_n$ be the span of the monomials of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $0 \leq a_j \leq n - j$. (In particular, the exponent of x_n is required to be 0, so the variable x_n does not occur in any polynomial in W .) Let W_ℓ be the subspace of W spanned by monomials of degree ℓ . We will need the following lemma.

LEMMA 3.1 ([1, Proof of Corollary 3.9]). *For $w \in \mathcal{S}_n$, the Schubert polynomial \mathfrak{S}_w lies in W . If we choose a term order with $x_n > x_{n-1} > \cdots > x_1$, then the leading term of \mathfrak{S}_w is*

$$\prod_{j=1}^n x_j^{\#\{k : k > j, w(k) < w(j)\}}. \quad \square$$

The list of numbers $\#\{k : k > j, w(k) < w(j)\}$ is the (Lehmer) code of w ; taking the code is a bijection between \mathcal{S}_n and $\{(a_1, \dots, a_n) \in \mathbb{Z}^n : 0 \leq a_j \leq n - j\}$ (see, e.g. [7, Proposition 2.1.2]). Thus, Lemma 3.1 implies that the Schubert polynomials have distinct leading terms and we deduce:

COROLLARY 3.2. *The Schubert polynomials \mathfrak{S}_w for $w \in \mathcal{S}_n$ are a basis for W . The change of basis matrix between $\{\mathfrak{S}_w : \ell(w) = k\}$ and $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : 0 \leq a_j \leq n - j, \sum a_j = k\}$ has determinant ± 1 .* \square

Proposition 1.1 shows that $\nabla : W_\ell \rightarrow W_{\ell-1}$, in the Schubert basis, is represented by the matrix M_ℓ . Therefore, to prove Theorem 1.2, we must compute

$$\det \left(W_{\binom{n}{2}-\ell} \xrightarrow{\nabla^{\binom{n}{2}-2\ell}} W_\ell \right)$$

in the Schubert basis. By Corollary 3.2, we may compute this determinant instead in the monomial basis. For the remainder of this note, Schubert polynomials disappear and our goal is to compute the determinant of ∇^j acting with respect to the monomial basis.

We would prefer to have a map from a vector space to itself, so that we could speak of its determinant without any reference to bases. There is a simple bijection between the monomial bases of W_k and $W_{\binom{n}{2}-k}$, taking $\prod x_j^{a_j}$ to $\prod x_j^{n-j-a_j}$. For reasons that will become clear in Section 4, we prefer to twist this map by $(-1)^k$, so we define J to be the linear endomorphism of W with

$$J \left(\prod x_j^{a_j} \right) = (-1)^{\sum a_j} \prod x_j^{n-j-a_j}.$$

Note that J has determinant ± 1 in the monomial basis and that $\nabla^{\binom{n}{2}-2\ell} \circ J$ maps W_ℓ to itself. To finish our proof of Theorem 1.2, it remains to establish

$$\det \tau = \pm \prod_{k=0}^{\ell} \left((\ell - k + 1)(\ell - k + 2) \cdots \left(\binom{n}{2} - \ell - k \right) \right)^{|\mathcal{S}_n(k)| - |\mathcal{S}_n(k-1)|},$$

where τ is the linear transformation $W_\ell \xrightarrow{\nabla^{\binom{n}{2}-2\ell} \circ J} W_\ell$. We now turn to this task.

4. SL_2 -REPRESENTATIONS AND A PROOF OF THEOREM 1.2

In this section, we discuss some representations of the Lie group SL_2 and its Lie algebra \mathfrak{sl}_2 . Our approach to proving Theorem 1.2 via these representations mirrors the general framework of R. Proctor from [8]; however, we provide a self-contained argument. We will denote group actions by variants of the letter ρ and the corresponding Lie algebra actions by variants of σ . We write the standard basis of \mathfrak{sl}_2 as

$$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and define the element

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2.$$

Let V_k be the $(k + 1)$ -dimensional irreducible representation of SL_2 and \mathfrak{sl}_2 ; we write ρ_k and σ_k for the action maps $\rho_k : SL_2 \rightarrow GL(V_k)$ and $\sigma_k : \mathfrak{sl}_2 \rightarrow \text{End}(V_k)$.

One usually describes V_k as the natural action on degree k polynomials in two variables. For our purposes, it is more convenient to describe V_k as an action on polynomials of degree $\leq k$ in one variable x . We have

$$\begin{aligned} \sigma_k(F)(x^j) &= jx^{j-1}, \\ \sigma_k(H)(x^j) &= (2j - k)x^j, \\ \sigma_k(E)(x^j) &= (k - j)x^{j+1}, \\ \rho_k(J)(x^j) &= (-1)^j x^{k-j}. \end{aligned} \tag{†}$$

In particular, for any polynomial f , we have $\sigma_k(F)(f) = \frac{df}{dx}$.

Identify the vector space W from Section 3 with $V_{n-1} \otimes V_{n-2} \otimes \cdots \otimes V_0$ by identifying $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $x^{a_1} \otimes x^{a_2} \otimes \cdots \otimes x^{a_n}$. We let SL_2 and \mathfrak{sl}_2 act on this tensor product in the standard way, and denote these actions by σ_W and ρ_W . We note that W_ℓ is the $2\ell - \binom{n}{2}$ weight space, i.e. the $2\ell - \binom{n}{2}$ eigenspace of $\sigma_W(H)$.

We have

$$\sigma_W(F) = \sum_{k=1}^n \text{Id} \otimes \text{Id} \otimes \cdots \otimes \sigma_{n-k}(F) \otimes \cdots \otimes \text{Id}$$

where $\sigma_{n-k}(F)$ occurs in the k -th position. Therefore,

$$\sigma_W(F) \cdot f = \sum_{k=1}^n \frac{\partial}{\partial x_k} f = \nabla f.$$

Similarly,

$$\rho_W(J) = \rho_{n-1}(J) \otimes \rho_{n-2}(J) \otimes \cdots \otimes \rho_0(J),$$

so

$$\rho_W(J) \cdot \left(\prod_{j=1}^n x_j^{a_j} \right) = \prod_{j=1}^n (-1)^{a_j} x_j^{n-j-a_j} = (-1)^{\sum a_j} \prod_{j=1}^n x_j^{n-j-a_j} = J \left(\prod_{j=1}^n x_j^{a_j} \right).$$

Thus, our goal of computing $\det \left(\nabla \binom{n}{2}^{-2\ell} \circ J \right)$ on W_ℓ is the same as computing the determinant of $\sigma_W(F) \binom{n}{2}^{-2\ell} \rho_W(J)$ as a map from the $2\ell - \binom{n}{2}$ weight space of $V_{n-1} \otimes V_{n-2} \otimes \cdots \otimes V_0$ to itself.

This is a standard computation. By comparing dimensions of weight spaces,

$$V_{n-1} \otimes V_{n-2} \otimes \cdots \otimes V_0 \cong \bigoplus_{0 \leq k \leq \binom{n}{2} - k} V_{\binom{n}{2} - 2k}^{\oplus |\mathcal{S}_n(k)| - |\mathcal{S}_n(k-1)|}.$$

Thus, Theorem 1.2 comes down to showing that $\sigma_W(F) \binom{n}{2}^{-2\ell} \rho_W(J)$ as a map from the $2\ell - \binom{n}{2}$ weight space of $V_{\binom{n}{2} - 2k}$ to itself is $\pm(\ell - k + 1)(\ell - k + 2) \cdots \left(\binom{n}{2} - \ell - k \right)$. Consulting the formulas from Equation (†), one sees that

$$\rho_{\binom{n}{2} - 2k}(J) x^{\ell - k} = (-1)^{\ell - k} x^{\binom{n}{2} - 2k - (\ell - k)} = (-1)^{\ell - k} x^{\binom{n}{2} - \ell - k}$$

and

$$\begin{aligned} \sigma_{\binom{n}{2} - 2k}(F) \binom{n}{2}^{-2\ell} x^{\binom{n}{2} - \ell - k} &= \left(\frac{d}{dx} \right)^{\binom{n}{2} - 2\ell} x^{\binom{n}{2} - \ell - k} \\ &= (\ell - k + 1)(\ell - k + 2) \cdots \left(\binom{n}{2} - \ell - k \right) x^{\ell - k}. \end{aligned}$$

Theorem 1.2 follows. □

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Derivatives of Schubert polynomials

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