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ABSTRACT Kazhdan and Lusztig have shown how to partition a Coxeter group into cells. In this paper, we use the theory of Vogan classes to obtain a first characterisation of the left cells of type B_n with respect to a certain choice of weight function.

1. INTRODUCTION

Lusztig has described how to partition a Coxeter group into left, right and two-sided cells with respect to a weight function [20]. This is done via certain equivalence relations that are calculated in the corresponding Iwahori–Hecke algebra, and the resulting cells afford representations of both the group and the algebra. Algebraic techniques have been developed to reduce the determination of cells to combinatorial calculations at the level of the group.

These ideas, along with their connections to other areas of mathematics, were first outlined by Kazhdan and Lusztig [18]. Their paper contains a quintessential result in the theory of cells – the classification of cells in type A_n. In this setting, the left cells afford a complete list of irreducible representations of the corresponding Iwahori–Hecke algebra, a pair of left cells afford isomorphic representations if and only if they are contained in the same two-sided cell, and two elements of the group are in the same left cell if and only if they have the same recording tableaux under the Robinson–Schensted correspondence.

There is ample motivation then, to have an interest in the theory of cells. One of the aims of this theory is to classify the cells of finite Coxeter groups.

Only three types of finite irreducible Coxeter groups may have associated Iwahori–Hecke algebras with unequal parameters; these are type F_4, type B_n, and type $I_2(m)$, where m is even. The theory here is more complex than the equal parameter case; the conditions for determining whether two weight functions are cell-equivalent, that is, they give rise to the same partition of the Coxeter group into cells, are far from obvious.

Lusztig comprehensively discussed the cells of type $I_2(m)$ in [21], while Geck used a combination of theoretical considerations and explicit computer calculations to resolve the case of type F_4 in [13]. This leaves the case of type B_n to be considered.

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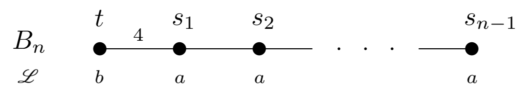
Denote by W_n the Coxeter group of type B_n. The weighted Coxeter system $(W_n, S, \mathscr{L})$ may be concisely described by its Coxeter diagram:



When $b/a = 1$, the cells of W_n are known following a series of papers of Garfinkle [9, 10, 11]. In this case, the left cells may be combinatorially described using domino tableaux and associated operations. A description of the left cells when $b/a \in \{\frac{1}{2}, \frac{3}{2}\}$ is due to Lusztig [20]. The other known case is detailed in papers of Bonnafé and Iancu [8] and Bonnafé [2]; any weight function such that $b/a > n-1$ corresponds to an "asymptotic" choice of parameters, and the resulting cells are governed by a generalised Robinson–Schensted correspondence. Further, the left cells afford a complete list of irreducible representations of the Iwahori–Hecke algebra, and two left cells afford isomorphic representations if and only if they are in the same two-sided cell. Thus their results are analogous to the aforementioned results of Kazhdan and Lusztig on cells in type A_{n-1}.

An important development in the study of the cells of W_n came in the form of a number of conjectures by Bonnafé, Geck, Iancu and Lam [7]. These conjectures state conditions for two weight functions on W_n to be cell-equivalent, as well as a unified combinatorial description of the left, right and two-sided cells for each of these cases. Although there are results in this direction due to Bonnafé [4, 5], a proof of these conjectures remains elusive.

In this paper, our focus is on the left cells of W_n when $b/a > n-2$. The conjectures of [7] suggests that the corresponding weight functions belong to one of the following three classes of cell-equivalence:

- when $b/a > n-1$; this is the *asymptotic case*,
- when $b/a = n-1$; which we refer to as the *intermediate case*,
- when $b/a \in (n-2, n-1)$; referred to as the *sub-asymptotic case*.

In the following two sections we recall some necessary background material. In Section 4 we generalise the notion of the *enhanced right descent set* of Bonnafé and Geck [6] to obtain an invariant $\overline{\mathcal{R}}^{\mathscr{L}}$ of left cells for finite weighted Coxeter systems. Section 5 is a collection of technical results for later use. In Section 6, we recall the concept of Vogan classes from [6], and establish a particular set Ξ of KL-admissible pairs for use in type B_n, defined exactly when $b/a > n-2$.

These KL-admissible pairs describe maps that can be used to determine cellular information that is common to the three cases mentioned above; Section 7 is dedicated to understanding this. In the final section, we determine the left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes, which leads to the following result.

Theorem. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a \geqslant n-1$. Then two elements of W_n are in the same left cell if and only if they lie in the same left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-class.*

As such, the left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes offer a new characterisation of the asymptotic left cells of W_n, and the first characterisation of the left cells when $b/a = n-1$.

2. Kazhdan–Lusztig cells with unequal parameters

Let (W, S) be a Coxeter system, let $y, w \in W$, $s, t \in S$, and denote by $\ell : W \to \mathbb{Z}_{\geqslant 0}$ the standard length function. A *weight function* for W is any map $\mathscr{L} : W \to \mathbb{Z}$ such that $\mathscr{L}(yw) = \mathscr{L}(y) + \mathscr{L}(w)$ whenever $\ell(yw) = \ell(y) + \ell(w)$. A weight function is uniquely determined by its values on S; conversely, any function $\mathscr{L} : S \to \mathbb{Z}$ such that $\mathscr{L}(s) = \mathscr{L}(t)$ whenever st has odd order extends uniquely to a weight function on W.

We will assume throughout that $\mathscr{L}(s) > 0$ for all $s \in S$; see Lusztig [20] and Bonnafé [3] for the original, more general framework.

We will denote by $\leqslant$ the Bruhat–Chevalley order on W, and write $y < w$ if $y \leqslant w$ with $y \neq w$. If y is a suffix of w, we write $y \leqslant_e w$; see 2.1 of Geck–Pfeiffer [16] for details.

Let $I \subseteq S$ be non-empty. The parabolic subgroup $W_I \coloneqq \langle I \rangle$ has a corresponding set X_I of distinguished left coset representatives. For all $w \in W$, there exist unique elements $x \in X_I$, $u \in W_I$ such that $w = xu$; moreover, $\ell(w) = \ell(x) + \ell(u)$. In this context, we denote:

$$\mathrm{rep}_I(w) \coloneqq x, \quad \mathrm{pr}_I(w) \coloneqq u.$$

There exists a bijection:

$$\begin{aligned} W &\longleftrightarrow X_I \times W_I, \\ w &\longleftrightarrow (\mathrm{rep}_I(w), \mathrm{pr}_I(w)). \end{aligned}$$

The Iwahori–Hecke algebra $\mathcal{H} \coloneqq \mathcal{H}(W, S, \mathscr{L})$ is a deformation of the group algebra of W over $\mathcal{A} \coloneqq \mathbb{Z}[v, v^{-1}]$, the ring of Laurent polynomials with indeterminate v.

The Iwahori–Hecke algebra has an $\mathcal{A}$-basis $\{T_w : w \in W\}$; multiplication between basis elements may be described by the formula:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ T_{sw} + (v^{\mathscr{L}(s)} - v^{-\mathscr{L}(s)})\, T_w & \text{if } sw < w. \end{cases}$$

We refer to the elements of the set $\{v^{\mathscr{L}(s)} : s \in S\}$ as the *parameters* of the Iwahori–Hecke algebra. If there exist $s, t \in S$ such that $v^{\mathscr{L}(s)} \neq v^{\mathscr{L}(t)}$ then we say that $\mathcal{H}$ has *unequal parameters*. Otherwise, we say that we are in the *equal parameter* case.

Multiplication in the Kazhdan–Lusztig basis $\{C_w : w \in W\}$ of $\mathcal{H}$ (see Kazhdan–Lusztig [18] and Lusztig [20]) determines the left cells of W as follows. The relation defined by:

$$y \leqslant'_L w \quad \text{if} \quad \begin{cases} \text{there exists } s \in S \text{ such that} \\ C_y \text{ occurs with non-zero coefficient in } C_s C_w \end{cases}$$

can be extended to its reflexive, transitive closure – a preorder $\leqslant_L$ called the *Kazhdan–Lusztig preorder*. The associated equivalence relation on W, denoted $\sim_L$, is defined by:

$$y \sim_L w \iff y \leqslant_L w \text{ and } w \leqslant_L y.$$

The resulting equivalence classes are called *left cells*. Similar definitions exist for the preorder $\leqslant_R$ and right cells, as well as the preorder $\leqslant_{LR}$ and two-sided cells; see Lusztig [21, 22] for details.

The Coxeter group W_I has its own left, right and two-sided cells, arising from relations denoted $\sim_{L,I}$, $\sim_{R,I}$ and $\sim_{LR,I}$ respectively.

3. The Coxeter group of type B_n

We retain the notation for the Coxeter group of type B_n from § 1. It is often useful to identify W_n with the group of signed permutations. The map given by $t \mapsto (1, -1)$, $s_i \mapsto (i, i+1)(-i, -i-1)$ defines an isomorphism between these groups. Thus we can write $w \in W_n$ as a sequence $w(1), \ldots, w(n)$, where for $1 \leqslant i \leqslant n$ we have $w(i) = \varepsilon_i p_i$, with $\varepsilon_i \in \{\pm 1\}$, and $p_1, \ldots, p_n$ forming a permutation of n.

One benefit of this identification is the use of the following classical result. Let $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$. Set $t_j \coloneqq s_{j-1} \cdots s_1 t s_1 \cdots s_{j-1} = (j, -j)$.

Lemma 3.1. *Let $w \in W_n$. Then:*

(i) $\ell(ws_i) < \ell(w) \iff w(i+1) < w(i)$,
(ii) $\ell(wt_j) < \ell(w) \iff w(j) < 0$.

There exists a generalised Robinson–Schensted correspondence from the elements of W_n to pairs of standard bitableaux of size n with the same shape; see Bonnafé–Iancu [8] for details. We denote this correspondence

$$w \longmapsto \big(A_n(w),\, B_n(w)\big),$$

and set $\mathrm{sh}(w)$ to be the shape of $A_n(w)$ which is equal to the shape of $B_n(w)$. The bitableau $A_n(w)$ is called the *insertion bitableau of* w, and $B_n(w)$ is called the *recording bitableau of* w.

In [8, § 1], it is noted that the left cells of $(W_n, S, \mathscr{L})$ are independent of the exact value of b/a, provided it is sufficiently large (with respect to n). Such a weight function is termed an *asymptotic weight function*. Whenever W_n is equipped with an asymptotic weight function, we say that we are in the *asymptotic case*.

Theorem 3.2 (Bonnafé–Iancu, [8, Theorem 7.7]; Bonnafé, [2, Remark 3.7 and Proposition 5.5]). *A weight function on W_n is asymptotic if and only if $b/a > n-1$. Moreover, let $y, w \in W_n$ and suppose that we are in the asymptotic case. Then:*

(i) $y \sim_L w \iff B_n(y) = B_n(w)$,
(ii) $y \sim_R w \iff A_n(y) = A_n(w)$,
(iii) $y \sim_{LR} w \iff \mathrm{sh}(y) = \mathrm{sh}(w)$.

We conclude with some additional notation. Denote by ℓ_t the function that counts the number of occurences of the generator t in a reduced expression for $w \in W_n$. Let e be the identity element, and let w_0 be the longest word of W_n.

4. A new descent set for finite Coxeter groups

Let $(W, S, \mathscr{L})$ be a weighted Coxeter system. The *right descent set of* $w \in W$ is given by

$$\mathcal{R}(w) \coloneqq \{s \in S : \ell(ws) < \ell(w)\}.$$

The right descent set is an invariant of the left cells of W (see [20]), so if $y \sim_L w$ then $\mathcal{R}(y) = \mathcal{R}(w)$. This concept may be refined as in Bonnafé–Geck [6, Example 7.4]. Let

$$S^{\mathscr{L}} \coloneqq S \cup \{sts : s, t \in S, \text{ such that } \mathscr{L}(t) > \mathscr{L}(s)\}.$$

Then the *enhanced right descent set* of $w \in W$ is

$$\mathcal{R}^{\mathscr{L}}(w) \coloneqq \{\sigma \in S^{\mathscr{L}} : \ell(w\sigma) < \ell(w)\}.$$

This is again an invariant of left cells. Although this is only a slightly finer invariant than the right descent set, its strength lies in being fine enough to determine left cells of dihedral groups with respect to any of the three cell-equivalence classes of weight functions. It is therefore a useful tool when considering parabolic subgroups of W of rank 2. In this section, we further refine this concept to obtain an invariant of left cells of finite weighted Coxeter systems that is sensitive to the choice of weight function.

Definition 4.1. *Let $(W, S, \mathscr{L})$ be a weighted Coxeter system, and let*

$$\overline{S}^{\mathscr{L}} \coloneqq S \cup \left\{ s_k \cdots s_1 t s_1 \cdots s_k \ : \begin{array}{l} \mathscr{L}(t) > k \cdot \mathscr{L}(s_i) \quad \textit{and} \\ \mathrm{order}(s_i s_{i+1}) = 3 \textit{ for } 1 \leqslant i \leqslant k-1 \end{array} \right\}.$$

For $w \in W$, let

$$\overline{\mathcal{R}}^{\mathscr{L}}(w) \coloneqq \{\sigma \in \overline{S}^{\mathscr{L}} : \ell(w\sigma) < \ell(w)\}.$$

Note that $\overline{\mathcal{R}}^{\mathscr{L}}(w) = \mathcal{R}(w)$ when $\mathscr{L}$ is constant on S.

Lemma 4.2. *Let W be of type F_4, and let $\mathscr{L} : W \to \mathbb{Z}$ be a weight function. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ is an invariant of the left cells of W with respect to $\mathscr{L}$.*

Proof. The cell-equivalence classes of weight functions on W are known following Corollary 4.8 of Geck [13]. The result can then be verified for a representative of each class with some elementary computer code in conjunction with the Python module PyCox; see Geck [15]. □

Remark 4.3. For $W = W_n$ and $b/a \geqslant 1$, we have

$$\overline{S}^{\mathscr{L}} \subseteq \{s_1, s_2, \dots, s_{n-1}, t_1, t_2, \dots, t_n\},$$

and for $w \in W_n$, we have

$$\overline{\mathcal{R}}^{\mathscr{L}}(w) = \mathcal{R}(w) \cup \left\{ t_k \ : \ \frac{b}{a} > k-1 \ \text{ and } \ \ell(wt_k) < \ell(w) \right\}.$$

We set $K \coloneqq \{t, s_1, \dots, s_{n-2}\}$ so that $W_K = W_{n-1}$. We may describe the set X_K as the set of all suffixes of the Coxeter word $t_n = s_{n-1} \cdots s_1 t s_1 \cdots s_{n-1}$.

From now until the end of this section only, we adopt the setup of Remark 4.3. A similar setting to this was considered in Pietraho [23, Definition 3.4].

Remark 4.4. Let $x \in X_K$ and $u \in W_K$. Then $\overline{\mathcal{R}}^{\mathscr{L}}(xu) = \overline{\mathcal{R}}^{\mathscr{L}}(u) \cup A$, for some $A \subseteq \{s_{n-1}, t_n\}$.

Proposition 4.5. *Let $y, w \in W_n$. If $y \sim_L w$, then $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$.*

Proof. Fix a weight function $\mathscr{L}$ as in Remark 4.3, let $y, w \in W_n$, and suppose that $y \sim_L w$. Recall that if $y \sim_L w$ then $\mathcal{R}(y) = \mathcal{R}(w)$, and so we only need to evaluate the membership of t_j in $\overline{\mathcal{R}}^{\mathscr{L}}(y)$ and $\overline{\mathcal{R}}^{\mathscr{L}}(w)$, where j satisfies both $2 \leqslant j \leqslant n$ and $b/a > j-1$. Proceed by induction on n.

Suppose $n = 2$. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ coincides with $\mathcal{R}^{\mathscr{L}}$, and W_2 coincides with $W(I_2(4))$. The statement is then true by [6, Example 7.4]. Now assume the statement is true for all $r < n$. We show that it is true for $r = n$.

Let $2 \leqslant j \leqslant n-1$ and suppose $b/a > j-1$. Then by Remark 4.4, we have

$$t_j \in \overline{\mathcal{R}}^{\mathscr{L}}(w) \iff t_j \in \overline{\mathcal{R}}^{\mathscr{L}}(\mathrm{pr}_K(w)). \tag{1}$$

By Geck [12, Theorem 1], $y \sim_L w$ implies that $\mathrm{pr}_K(y) \sim_{L,K} \mathrm{pr}_K(w)$. By our induction hypothesis, $\mathrm{pr}_K(y) \sim_{L,K} \mathrm{pr}_K(w)$ implies that $\overline{\mathcal{R}}^{\mathscr{L}}(\mathrm{pr}_K(y)) = \overline{\mathcal{R}}^{\mathscr{L}}(\mathrm{pr}_K(w))$. So if $b/a \leqslant n-1$, then we are done by (1). If not, then it remains to determine the membership of t_n in $\overline{\mathcal{R}}^{\mathscr{L}}(y)$ and $\overline{\mathcal{R}}^{\mathscr{L}}(w)$.

So, suppose now that $b/a > n-1$. Then by [8, Corollary 6.7], we know that $\ell_t(y) = \ell_t(w)$. By [12, Theorem 1], we have $\ell_t(\mathrm{pr}_K(y)) = \ell_t(\mathrm{pr}_K(w))$. It follows that $\ell_t(\mathrm{rep}_K(y)) = \ell_t(\mathrm{rep}_K(w))$. It remains to observe that $t_n \in \overline{\mathcal{R}}^{\mathscr{L}}(w)$ if and only if $\ell_t(\mathrm{rep}_K(w)) = 1$. □

Corollary 4.6. *Let $(W, S, \mathscr{L})$ be a finite weighted Coxeter system. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ is an invariant of its left cells.*

Proof. We use the classification of finite irreducible Coxeter groups to reduce to the case where W is of type B_n, type F_4, or type $I_2(m)$ with m even. The case of type F_4 was looked at in Lemma 4.2. The case of type $I_2(m)$ has been considered in [6, Example 7.4]. For type B_n, if $b/a \in (0,1)$, then $\overline{\mathcal{R}}^{\mathscr{L}} = \mathcal{R}^{\mathscr{L}}$, and we are in the case of [6, Example 7.4] again. Finally, we appeal to Proposition 4.5 to conclude the proof. □

From the definition of $\overline{\mathcal{R}}^{\mathscr{L}}$, we see that we can write

$$W_n = \bigsqcup_{I \subseteq \overline{S}^{\mathscr{L}}} \{w \in W_n : \overline{\mathcal{R}}^{\mathscr{L}}(w) = I\},$$

and so $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions W_n into up to 2^{2n-1} subsets. However, unlike with $\mathcal{R}$, some of these subsets will be empty. For instance, if W_2 is equipped with an asymptotic weight function, then $\overline{\mathcal{R}}^{\mathscr{L}}$ distinguishes all six of its left cells. On the other hand, there is no element $w \in W_2$ such that $\overline{\mathcal{R}}^{\mathscr{L}}(w)$ is equal to either $\{t, s_1\}$ or $\{t_2\}$. These observations motivate us to work out just how fine an invariant $\overline{\mathcal{R}}^{\mathscr{L}}$ is for the left cells of W_n.

Lemma 4.7. *Suppose W_n is equipped with an asymptotic weight function. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions W_n into exactly $2 \cdot 3^{n-1}$ non-empty subsets.*

Proof. We proceed by induction on n. For the case $n = 2$, see [6, Example 7.4].

Let $x \in X_K$ and $u \in W_K$. Then by Remark 4.4, $\overline{\mathcal{R}}^{\mathscr{L}}(xu)$ is equal to one of the following:

(i) $\overline{\mathcal{R}}^{\mathscr{L}}(u)$, (ii) $\overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{s_{n-1}\}$,
(iii) $\overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{t_n\}$, (iv) $\overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{s_{n-1}, t_n\}$.

Let $I \subseteq \overline{S}^{\mathscr{L}}$, and consider the sets $C_I \coloneqq \{u \in W_K : \overline{\mathcal{R}}^{\mathscr{L}}(u) = I\}$ and $D_I \coloneqq \{xu : x \in X_K,\ u \in C_I\}$. By our inductive hypothesis, $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions W_K into exactly $2 \cdot 3^{n-2}$ non-empty subsets. It suffices to show that if $C_I \neq \varnothing$, then $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions D_I into exactly three non-empty subsets. So, fix some $I \subseteq \overline{S}^{\mathscr{L}}$ such that $C_I \neq \varnothing$, and let $u \in C_I$.

First note that $e \in X_K$ is such that $s_{n-1}, t_n \notin \overline{\mathcal{R}}^{\mathscr{L}}(eu)$ for all $u \in W_K$, while $t_n \in X_K$ commutes with all $u \in W_K$ to give $s_{n-1}, t_n \in \overline{\mathcal{R}}^{\mathscr{L}}(t_n u)$. We now distinguish two cases.

Case A. Suppose that $t_{n-1} \notin I$. By Lemma 3.1, we have $u(n-1) > 0$. Consider $x = s_1 \cdots s_{n-1} \in X_K$. We have $xu(n-1) > 0$ and $xu(n) = 1$, and it follows that $\overline{\mathcal{R}}^{\mathscr{L}}(xu) = \overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{s_{n-1}\}$.

So now suppose there is some $x' \in X_K$ such that $s_{n-1} \notin \overline{\mathcal{R}}^{\mathscr{L}}(x'u)$. Using Lemma 3.1 and Remark 4.4, we have that $x'u(n) > x'u(n-1) > 0$, and so $t_n \notin \overline{\mathcal{R}}^{\mathscr{L}}(x'u)$. Thus if $t_{n-1} \notin I$, then $\overline{\mathcal{R}}^{\mathscr{L}}(x'u)$ is not equal to $\overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{t_n\}$.

Case B. Suppose that $t_{n-1} \in I$, and let $x = ts_1 \cdots s_{n-1} \in X_K$. Via similar considerations to those in Case A, we see that $\overline{\mathcal{R}}^{\mathscr{L}}(xu) = \overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{t_n\}$ while $\overline{\mathcal{R}}^{\mathscr{L}}(x'u) \neq \overline{\mathcal{R}}^{\mathscr{L}}(u) \cup \{s_{n-1}\}$ for all $x' \in X_K$. ☐

Corollary 4.8. *Suppose $b/a \in (k, k+1] \subseteq (1, n]$ for some $k \in \mathbb{Z}$. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions W_n into exactly $2^{n-k} \cdot 3^k$ non-empty subsets.*

5. The set Z_n

In this section, we collect a number of technical results for later use.

5.1. Shape. To any $w \in W_n$ we associate the bipartition $\mathrm{sh}(w)$ as in § 3. For any bipartition $\lambda \Vdash n$, we set $\Omega_\lambda \coloneqq \{w \in W_n : \mathrm{sh}(w) = \lambda\}$, and for $0 \leqslant q \leqslant n$ we set $\zeta_q \coloneqq (1^{n-q} \mid 1^q) \Vdash n$. If we are in the asymptotic case, then Ω_λ is equal to a two-sided cell, as in Theorem 3.2 (iii).

Set $J \coloneqq \{s_1, \ldots, s_{n-1}\}$, and denote by $w_J \in W_n$ the longest word of the parabolic subgroup W_J. Then $\mathrm{sh}(w_J) = \zeta_0$ and $\mathrm{sh}(w_J w_0) = \zeta_n$. We note that if $w \in W_n$ is such that either $A_n(w) = A_n(w_J)$ or $B_n(w) = B_n(w_J)$, then $w = w_J$. By Theorem 3.2, the element w_J lies in an asymptotic left, right and two-sided cell of cardinality one. Analogous statements hold for the element $w_J w_0$.

Definition 5.1.

$$Z_n \coloneqq \bigsqcup_{q=0}^{n} \Omega_{\zeta_q},$$
$$\widetilde{Z}_n \coloneqq Z_n \smallsetminus \{w_J,\, w_J w_0\}.$$

We have $Z_n = w_0 Z_n$ and $Z_n = \{w^{-1} : w \in Z_n\}$, as well as analogous statements for $\widetilde{Z}_n$.

The cardinality of $\{B_n(w) : w \in \Omega_{\zeta_q}\}$ (and of $\{A_n(w) : w \in \Omega_{\zeta_q}\}$) is $\binom{n}{q}$. If we are in the asymptotic case, this is equivalent to saying that Ω_{ζ_q} contains $\binom{n}{q}$ left (equivalently, right) cells, each of size $\binom{n}{q}$. For later use, we may therefore state:

$$\big|\{B_n(w) : w \in Z_n\}\big| = \sum_{q=0}^{n} \binom{n}{q} = 2^n. \tag{2}$$

5.2. Signed permutations. The condition $w \in Z_n$ is quite a strict one, and in turn places conditions on the row form of w. Indeed, let us consider $w = w(1), \dots, w(n) \in Z_n$ as a signed permutation. Then the subsequence $x_1, \dots, x_q$ of negative integers must be such that $|x_1| > |x_2| > \cdots > |x_q|$. Similarly, the subsequence $y_1, \dots, y_{n-q}$ of positive integers must be such that $y_1 > y_2 > \cdots > y_{n-q}$. Further, if $y, w \in Z_n$ then $B_n(y) = B_n(w)$ if and only if $y(i)$ and $w(i)$ have the same sign for all $1 \leqslant i \leqslant n$.

Example 5.2. We have $y, w \in Z_7$, where:

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -7 & -5 & 6 & 4 & 3 & -2 & 1 \end{pmatrix},\ A_7(y) = \begin{array}[t]{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 6\\\hline\end{array}\ \begin{array}[t]{|c|}\hline 2\\\hline 5\\\hline 7\\\hline\end{array},\ B_7(y) = \begin{array}[t]{|c|}\hline 3\\\hline 4\\\hline 5\\\hline 7\\\hline\end{array}\ \begin{array}[t]{|c|}\hline 1\\\hline 2\\\hline 6\\\hline\end{array},$$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -4 & -2 & 7 & 6 & 5 & -1 & 2 \end{pmatrix},\ A_7(w) = \begin{array}[t]{|c|}\hline 3\\\hline 5\\\hline 6\\\hline 7\\\hline\end{array}\ \begin{array}[t]{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array},\ B_7(w) = \begin{array}[t]{|c|}\hline 3\\\hline 4\\\hline 5\\\hline 7\\\hline\end{array}\ \begin{array}[t]{|c|}\hline 1\\\hline 2\\\hline 6\\\hline\end{array}.$$

5.3. Reduced expressions. The forthcoming notation $\sigma_{n,q}$ and $p_{n,q}$ will be crucial in discussions regarding Z_n. First set $a_0 \coloneqq e$, b_{n-1}, $b_n \coloneqq e$, and $p_{n,0}$, $p_{n,n} \coloneqq e$, and then:

- for $1 \leqslant q \leqslant n$, set $a_q \coloneqq (t)(s_1 t) \cdots (s_{q-1} \cdots s_1 t)$,
- for $0 \leqslant q \leqslant n-2$, set $b_q \coloneqq (s_{q+1})(s_{q+2} s_{q+1}) \cdots (s_{n-1} \cdots s_{q+1})$,
- for $0 \leqslant q \leqslant n$, set $\sigma_{n,q} \coloneqq a_q \cdot b_q$,
- for $1 \leqslant q \leqslant n-1$, set:
$$\begin{aligned} p_{n,q} &\coloneqq (s_{n-q} s_{n-q-1} \cdots s_1)(s_{n-q+1} s_{n-q} \cdots s_2) \cdots (s_{n-1} s_{n-2} \cdots s_q) \\ &= (s_{n-q} s_{n-q+1} \cdots s_{n-1})(s_{n-q-1} s_{n-q} \cdots s_{n-2}) \cdots (s_1 \cdots s_q). \end{aligned}$$

Note that a_q and b_q commute in W_n, with $\ell(\sigma_{n,q}) = \ell(a_q) + \ell(b_q)$.

Proposition 5.3. *Consider the asymptotic two-sided cell $\Omega_{\zeta_q} \subseteq Z_n$. It contains the asymptotic left cell*

$$\Gamma_q \coloneqq \{\pi \sigma_{n,q} : \pi \leqslant_e p_{n,q}\},$$

and the following is a complete list of asymptotic left cells contained in Ω_{ζ_q}:

$$\{\Gamma_q \tau^{-1} : \tau \leqslant_e p_{n,q}\}.$$

Thus,

$$\Omega_{\zeta_q} = \{\pi \sigma_{n,q} \tau^{-1} : \pi, \tau \leqslant_e p_{n,q}\}.$$

Further, for any $0 \leqslant q \leqslant n$ and $\pi, \tau \leqslant_e p_{n,q}$, the expression $\pi \sigma_{n,q} \tau^{-1}$ is reduced.

Together with § 5.1, this result implies that $p_{n,q}$ has $\binom{n}{q}$ suffixes.

5.4. Cell decomposition. We now give a pair of basic lemmas with useful corollaries.

Lemma 5.4. *Let $w \in Z_n$. Then:*

(i) *If $wt > w$ and $wt \in Z_n$, then $s_1 \in \mathcal{R}(w)$ and $\mathcal{R}(wt) = \mathcal{R}(w) \cup \{t\} \setminus \{s_1\}$.*
(ii) *For $1 \leqslant i \leqslant n-2$, if $ws_i > w$ and $ws_i \in Z_n$, then $s_{i+1} \in \mathcal{R}(w)$. We have $\mathcal{R}(ws_1) = \mathcal{R}(w) \cup \{s_1\} \setminus \{t, s_2\}$, and for $2 \leqslant i \leqslant n-2$ we have $\mathcal{R}(ws_i) = \mathcal{R}(w) \cup \{s_i\} \setminus \{s_{i+1}\}$.*
(iii) *If $ws_{n-1} > w$ and $ws_{n-1} \in Z_n$, then $\mathcal{R}(ws_{n-1}) = \mathcal{R}(w) \cup \{s_{n-1}\}$.*
(iv) *Left-handed versions of the above statements also hold.*

Proof. Following the discussion in § 5.2, this may be verified with judicious application of Lemma 3.1. □

Corollary 5.5. *Let $y, w \in Z_n$ with $\mathcal{R}(y) = \mathcal{R}(w)$, and let $p \in W_n$. Suppose that*

$$y\tau^{-1},\ w\tau^{-1} \in Z_n \qquad \forall\, \tau \leqslant_e p.$$

Then we have $\mathcal{R}(y\tau^{-1}) = \mathcal{R}(w\tau^{-1})$ for all $\tau \leqslant_e p$.

Lemma 5.6. *Let $w \in Z_n$, and $1 \leqslant i \leqslant n-2$. Suppose that $s_i w \in Z_n$ with $w < s_i w$. Then $w \sim_L s_i w$ with respect to any choice of parameters.*

Proof. Application of Lemma 5.4 (iv) shows that $M^{s_{i+1}}_{w, s_i w} = 1$, and thus $w \sim_L s_i w$, with respect to any choice of parameters[(1)]. □

Denote by S_w the set of all $s \in S$ such that s occurs in a reduced expression for w (as in [21, § 9.2]).

Corollary 5.7. *Let $\Gamma \subseteq Z_n$ be an asymptotic left cell. Then the cell can be decomposed as*

$$\Gamma = \gamma_1 \sqcup \gamma_2$$

where γ_1 and γ_2 are each contained within a left cell for all choices of parameters. Denote by σ the element in Γ of minimal length, and set $q = \ell_t(\sigma)$. Then we have

$$\begin{aligned}
\gamma_1 &= \{\pi\sigma : \pi \leqslant_e p_{n,q} \text{ and } s_{n-1} \notin S_\pi\} = \{\pi\sigma : \pi \leqslant_e p_{n-1,q}\},\\
\gamma_2 &= \{\pi\sigma : \pi \leqslant_e p_{n,q} \text{ and } s_{n-1} \in S_\pi\} = \{\pi\chi_q\sigma : \pi \leqslant_e p_{n-1,q-1}\},
\end{aligned}$$

where $\chi_q \coloneqq s_{n-1} \cdots s_q$.

Proof. We first check that the two descriptions of γ_1 and γ_2 are consistent. By Proposition 5.3, we may describe the cell Γ as

$$\Gamma = \{\pi\sigma : \pi \leqslant_e p_{n,q}\}.$$

Recall the following reduced expressions for $p_{n,q}$:

$$\begin{aligned}
p_{n,q} &= (s_{n-q} \cdots s_{n-1})(s_{n-q-1} \cdots s_{n-2}) \cdots (s_1 \cdots s_q)\\
&= (s_{n-q} \cdots s_1)(s_{n-q+1} \cdots s_2) \cdots (s_{n-1} \cdots s_q).
\end{aligned}$$

In the first expression, we have moved the lone s_{n-1} term as far to the left as possible; in the second, as far to the right as possible. We see that $s_{n-1} \notin S_\pi$ if and only if $\pi \leqslant_e p_{n-1,q}$. Similarly, $s_{n-1} \in S_\pi$ if and only if there exists a reduced expression for π ending in the block sequence $\chi_q = s_{n-1} \cdots s_q$. Now note that

$$p_{n,q} = p_{n-1,q-1}\chi_q \quad \text{with} \quad \ell(p_{n,q}) = \ell(p_{n-1,q-1}) + \ell(\chi_q).$$

---

(1) For the definition of M-polynomials, the reader may consult [21, Chapter 6], where they are denoted by μ.

Following this, we may apply Lemma 5.6 to see that for $i \in \{1,2\}$, if $y, w \in \gamma_i$, then $y \sim_L w$ for any choice of parameters. In this result, we note that $\gamma_2 = \varnothing$ if $|\Gamma| = 1$. □

5.5. The partition of Z_n with respect to $\overline{\mathcal{R}}^{\mathscr{L}}$.

Lemma 5.8. *Let $w \in Z_n$, and let $1 \leqslant i \leqslant n-1$. Then:*

$$\ell(ws_i) < \ell(w) \iff \ell(wt_i) > \ell(w).$$

Proof. The statement follows from Lemma 3.1 and § 5.2. □

Corollary 5.9. *Consider $(W_n, S, \mathscr{L})$ and let $y, w \in Z_n$.*

(i) *If $b/a > n-2$, then*

$$\overline{\mathcal{R}}^{\mathscr{L}}(y) \cap \{t_1, \ldots, t_n\} = \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\} \iff \overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w).$$

(ii) *If $b/a \in [1, n-1]$, then*

$$\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w) \iff \mathcal{R}(y) = \mathcal{R}(w).$$

Lemma 5.10. *Consider $(W_n, S, \mathscr{L})$.*

(i) *If $b/a > n-1$, then Z_n is partitioned by $\overline{\mathcal{R}}^{\mathscr{L}}$ into 2^n non-empty subsets; these subsets are exactly the asymptotic left cells.*

(ii) *Suppose $b/a > n-1$, and let $T \subseteq \{t_1, \ldots, t_n\}$ be arbitrary. Then there exists a unique left cell $\Gamma \subseteq Z_n$ such that for any $w \in \Gamma$, we have $\overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\} = T$.*

(iii) *If $b/a \in [1, n-1]$, then Z_n is partitioned by $\overline{\mathcal{R}}^{\mathscr{L}}$ (and $\mathcal{R}$) into 2^{n-1} non-empty subsets, each of which is the union of exactly two asymptotic left cells.*

Proof. We first look at part (i). By Corollary 5.9 (i), we know that $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$ if and only if $\overline{\mathcal{R}}^{\mathscr{L}}(y) \cap \{t_1, \ldots, t_n\} = \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\}$. By Lemma 3.1, this is in turn equivalent to the statement that $y(i)$ and $w(i)$ have the same sign for all $1 \leqslant i \leqslant n$. As was noted in § 5.2, this is if and only if $B_n(y) = B_n(w)$, which is true if and only if y and w are in the same asymptotic left cell. Recalling (2) from § 5.1 finishes the proof.

Part (ii) follows from part (i), Corollary 5.9 (i), and the fact that the power set of $\{t_1, \ldots, t_n\}$ contains 2^n elements.

Turning to part (iii), we first prove the statement for $b/a \in (n-2, n-1]$. So, let $T' \subseteq \{t_1, \ldots, t_{n-1}\}$ be an arbitrary subset. Then

$$\begin{aligned}
&\{w \in Z_n : \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_{n-1}\} = T'\} \text{ (with } b/a \in (n-2, n-1]) \\
&\quad = \{w \in Z_n : \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\} = T'\} \sqcup \ \{w \in Z_n : \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\} \\
&\quad = T' \sqcup \{t_n\}\} \text{ (with } b/a > n-1).
\end{aligned}$$

The parts of this disjoint union are non-empty by part (ii). So when $b/a \in (n-2, n-1]$, $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions Z_n into half as many non-empty subsets compared to when $b/a > n-1$. Now apply part (i). Corollary 5.9 (ii) then extends the scope of the result to all $b/a \in [1, n-1]$. □

Later, it will benefit us to have a description of how $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions Z_n. Lemma 5.10 indicates that we can split this into two cases. If $b/a > n-1$, then descriptions of this partition are given in both Theorem 3.2 (i) and Proposition 5.3. Otherwise, it suffices to look at how $\mathcal{R}$ partitions Z_n.

So let $I \subseteq S$ be such that $\{w \in Z_n : \mathcal{R}(w) = I\}$ is non-empty. This set comprises exactly two asymptotic left cells. In order to describe all such sets in this form, we need to show how the left cells "pair up" under $\mathcal{R}$. This has three steps; first finding representative pairings, then taking advantage of Corollary 5.5 to "translate" the resulting sets (pairs of cells) via right multiplication, and finally showing that this method encompases all elements in Z_n. Some notation:

- denote by $\Gamma(w)$ the asymptotic left cell containing $w \in W_n$,
- denote by $\Gamma'(w)$ the asymptotic right cell containing $w \in W_n$,
- set $\Upsilon(w) \coloneqq \{z \in Z_n : \mathcal{R}(z) = \mathcal{R}(w)\}$ for $w \in Z_n$,
- and recall that $\chi_q \coloneqq s_{n-1} \cdots s_q$.

Let us begin by using Lemma 3.1 to see that $\mathcal{R}(\sigma_{n,q}) = \{t, s_{q+1}, \ldots, s_{n-1}\}$. As $\chi_{q+1} \leqslant_e p_{n,q+1}$, Proposition 5.3 indicates that $\sigma_{n,q+1}\chi_{q+1}^{-1} \in Z_n$, and by applying Lemma 3.1 again we see that $\mathcal{R}(\sigma_{n,q}) = \mathcal{R}(\sigma_{n,q+1}\chi_{q+1}^{-1})$.

Thus for $0 \leqslant q \leqslant n-1$, we have:

$$\Upsilon(\sigma_{n,q}) = \Gamma(\sigma_{n,q}) \sqcup \Gamma(\sigma_{n,q+1}\chi_{q+1}^{-1}).$$

By Proposition 5.3, we know that $\Gamma'(\sigma_{n,q}) = \{\sigma_{n,q}\pi^{-1} : \pi \leqslant_e p_{n,q}\}$. As $p_{n-1,q} \leqslant_e p_{n,q}$ and $\Gamma'(\sigma_{n,q}) \subseteq Z_n$, we observe that:

$$\{\sigma_{n,q}\tau^{-1} : \tau \leqslant_e p_{n-1,q}\} \subseteq Z_n.$$

Similarly, we have $\Gamma'(\sigma_{n,q+1}) = \{\sigma_{n,q+1}\pi^{-1} : \pi \leqslant_e p_{n,q+1}\}$. Noting that $p_{n,q+1} = p_{n-1,q}\chi_{q+1}$ with $\ell(p_{n,q+1}) = \ell(p_{n-1,q}) + \ell(\chi_{q+1})$, we also observe that:

$$\{\sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1} : \tau \leqslant_e p_{n-1,q}\} \subseteq Z_n.$$

These two observations allow us to apply Corollary 5.5 (setting $y = \sigma_{n,q}$, $w = \sigma_{n,q+1}\chi_{q+1}^{-1}$ and $p = p_{n-1,q}$) to see that for all $0 \leqslant q \leqslant n-1$ and all $\tau \leqslant_e p_{n-1,q}$ we have:

$$\Upsilon(\sigma_{n,q}\tau^{-1}) = \Gamma(\sigma_{n,q}\tau^{-1}) \sqcup \Gamma(\sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1}).$$

Sets procured in this way are non-empty, mutually distinct and contained in Z_n. Recalling from § 5.4 that the number of suffixes of $p_{n,q}$ is $\binom{n}{q}$, we see that the number of sets that we have obtained is 2^{n-1}, which by Lemma 5.10 (ii) is the number of non-empty sets that $\mathcal{R}$ partitions Z_n into. We therefore obtain Proposition 5.11 parts (i) and (ii) below.

There is a more straightforward description of these sets. Using the braid relations for W_n, we can verify that for $0 \leqslant q \leqslant n-1$, we have

$$s_{n-q-1} \cdots s_1 \cdot t \cdot p_{n,q} = p_{n,q+1} \cdot t \cdot s_1 \cdots s_q.$$

Denote this element by $\Pi_{n,q}$, and note that both of these expressions are reduced.

Proposition 5.11. *Let $w \in Z_n$.*

(i) *There exists $0 \leqslant q \leqslant n-1$ and $\tau \leqslant_e p_{n-1,q}$ such that:*
$\Upsilon(w) = \Upsilon(\sigma_{n,q}\tau^{-1}) = \Gamma(\sigma_{n,q}\tau^{-1}) \sqcup \Gamma(\sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1})$,

(ii) $Z_n = \displaystyle\bigsqcup_{q=0}^{n-1} \bigsqcup_{\tau \leqslant_e p_{n-1,q}} \Upsilon(\sigma_{n,q}\tau^{-1})$.

For $0 \leqslant q \leqslant n-1$ and $\tau \leqslant_e p_{n-1,q}$, we have:

(iii) $\Upsilon(\sigma_{n,q}\tau^{-1}) = \{\pi\sigma_{n,q}\tau^{-1} : \pi \leqslant_e \Pi_{n,q}\}$,

(iv) $|\Upsilon(\sigma_{n,q}\tau^{-1})| = \dbinom{n}{q} + \dbinom{n}{q+1}$.

Proof. Verifying that the descriptions of $\Upsilon(\sigma_{n,q}\tau^{-1})$ in parts (i) and (iii) are equivalent is done in an entirely similar way to verifying the asymptotic left cell decomposition in the proof of Corollary 5.7. For part (iv), consider the decomposition of $\Upsilon(\sigma_{n,q}\tau^{-1})$ in part (i) into asymptotic cells. □

6. An extension of the generalised τ-invariant in type B_n

A classical pair of results in the theory of Kazhdan–Lusztig cells with equal parameters are as follows: if two elements of a Coxeter group lie in the same left cell then they have the same generalised τ-invariant, and two such elements remain equivalent under $\sim_L$ after a $*$-operation has been applied to them; see Vogan [25, § 3] or [18, § 4]. The theory of $*$-operations and the generalised τ-invariant has been substantially generalised in [6], providing compatibility with unequal parameters as well as the framework for maps richer than the traditional $*$-operations. In this section, we recall some definitions and results from [6], and introduce new ones.

6.1. Vogan classes. Let $(W, S, \mathscr{L})$ be a weighted Coxeter system. We denote by $[\Gamma]$ the $\mathcal{H}$-module afforded by a left cell $\Gamma \subseteq W$; see [20, § 6] for details.

Definition 6.1 (Bonnafé–Geck, [6, Definition 6.1]). *A pair (I, δ) consisting of a non-empty subset $I \subseteq S$ and a left cellular map $\delta : W_I \to W_I$ is called* KL-admissible. *We recall that this means that the following conditions are satisfied for every left cell $\Gamma \subseteq W_I$ (with respect to $\mathscr{L}|_{W_I}$):*

(A1) *$\delta(\Gamma)$ also is a left cell.*
(A2) *The map δ induces an $\mathcal{H}_I$-module isomorphism $[\Gamma] \cong [\delta(\Gamma)]$.*

We say that (I, δ) is strongly KL-admissible *if, in addition to* (A1) *and* (A2)*, the following condition is satisfied:*

(A3) *We have $u \sim_{R,I} \delta(u)$ for all $u \in W_I$.*

If $I \subseteq S$ and if $\delta : W_I \to W_I$ is a map, we obtain a map $\delta^L : W \to W$ by

$$\delta^L(xu) := x\delta(u) \qquad \text{for all } x \in X_I \text{ and } u \in W_I.$$

The map δ^L is called the *left extension of δ to W*. However, by abuse of notation, we will often use δ to refer to δ^L where the meaning is clear.

Theorem 6.2 (Bonnafé–Geck, [6, Theorem 6.2]). *Let (I, δ) be a (strongly) KL-admissible pair. Then (S, δ^L) is (strongly) KL-admissible.*

This theorem brings us to consider strongly KL-admissible pairs that give us as much information about the cells of W as possible; to this end, we introduce an additional condition.

(B) If $u, v \in W_I$ are such that $u \sim_{R,I} v$, then there exists some $k \in \mathbb{Z}_{\geqslant 0}$ such that $u = \delta^k(v)$.

Definition 6.3. *We say that a pair (I, δ) is* maximally KL-admissible *if conditions* (A1)*,* (A2)*,* (A3) *and* (B) *hold.*

Example 6.4. Consider W_n equipped with any weight function, and let $J := \{s_1, \ldots, s_{n-1}\}$, so that $W_J \cong \mathfrak{S}_n$. The cells of $\mathfrak{S}_n$ are described by the Robinson–Schensted correspondence; see [18, § 5] or Ariki [1].

Let $\lambda \vdash n$ be a partition, and Ω_λ be the two-sided cell of W_J such that $\mathrm{sh}(w) = \lambda$ for any $w \in \Omega_\lambda$. Fix an arbitrary total order on the left cells contained in Ω_λ; say $\Gamma_1 < \Gamma_2 < \cdots < \Gamma_k$.

We now define a map $\varepsilon_\lambda : \Omega_\lambda \to \Omega_\lambda$. Let $1 \leqslant i \leqslant k-1$. For any $w \in \Gamma_i$, set $\varepsilon_\lambda(w)$ to be equal to the unique element $w' \in \Gamma_{i+1}$ such that $w \sim_{R,J} w'$. For $w \in \Gamma_k$, set $\varepsilon_\lambda(w)$ to be equal to the unique element $w' \in \Gamma_1$ such that $w \sim_{R,J} w'$, as in Figure 1.

This map can be then extended to the rest of the group by setting:

$$\varepsilon'_\lambda(w) = \begin{cases} \varepsilon_\lambda(w) & \text{if } w \in \Omega_\lambda, \\ w & \text{otherwise.} \end{cases}$$

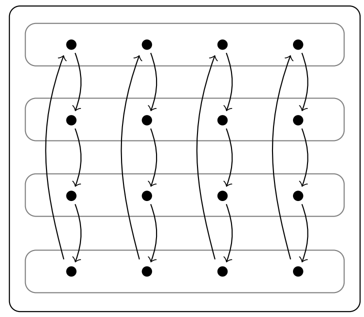


Figure 1. A two-sided cell $\Omega_\lambda \subseteq \mathfrak{S}_n$ with left cells given by rows, and right cells by columns. The map ε_λ is indicated by the arrows.

Finally, we can define a map $\varepsilon : W_J \to W_J$ by taking the composition of all maps $\varepsilon'_\lambda : W_J \to W_J$ over the indexing set $\{\lambda \ : \ \lambda \vdash n\}$.

We know that ε satisfies condition (A2) by [14, Example 2.6]. From the construction, we can see that (J, ε) is maximally KL-admissible, and well-defined up to the choice of total ordering on left cells in each two-sided cell.

Example 6.5. Consider $(W_n, S, \mathscr{L})$ and set $K \coloneqq \{t, s_1, \ldots, s_{n-2}\}$ so that $W_K = W_{n-1}$. Suppose that the restriction of $\mathscr{L}$ to W_K is an asymptotic weight function for W_K. Then the left, right and two-sided cells of W_K are determined by the generalised Robinson–Schensted correspondence, as in Theorem 3.2. Now we can proceed analogously to Example 6.4 to obtain a map $\psi : W_K \to W_K$. Both [14, Theorem 6.3] and [6, Example 6.8] tell us that ψ satsifies condition (A2). The resulting pair (K, ψ) is maximally KL-admissible.

Recall from Theorem 3.2 that $b/a > n-1$ is the optimal lower bound for a weight function on W_n to result in asymptotic cells. Thus, the maximally KL-admissible pair (K, ψ) is well-defined (up to a choice of total ordering on left cells in two-sided cells) with respect to any $\mathscr{L} : W_n \to \mathbb{Z}_{\geqslant 0}$ such that $b/a > n-2$.

Conversely, if $b/a \leqslant n-2$, then $\mathscr{L}|_{W_K}$ is not an asymptotic weight function for W_K. Without a suitable analogue of [14, Theorem 6.3], we cannot readily produce a non-trivial map $\psi' : K \to K$ such that (K, ψ') is a KL-admissible pair with respect to $\mathscr{L}'$.

From now until the end of this section we fix an arbitrary weighted Coxeter system $(W, S, \mathscr{L})$, as well as a map $\rho : W \to E$ (where E is a fixed set) such that the fibres of ρ are (possibly empty)(2) unions of left cells.

Definition 6.6 (Bonnafé–Geck, [6, § 7]). *Let Δ be a collection of KL-admissible pairs with respect to $(W, S, \mathscr{L})$. We define by induction on n a family of equivalence relations $\approx_n^{\Delta,\rho}$ on W as follows. Let $y, w \in W$.*

- *For $n = 0$, we write $y \approx_0^{\Delta,\rho} w$ if $\rho(y) = \rho(w)$.*
- *For $n \geqslant 1$, we write $y \approx_n^{\Delta,\rho} w$ if $y \approx_{n-1}^{\Delta,\rho} w$ and $\delta^L(y) \approx_{n-1}^{\Delta,\rho} \delta^L(w)$ for all $(I, \delta) \in \Delta$.*

We write $y \approx^{\Delta,\rho} w$ if $y \approx_n^{\Delta,\rho} w$ for all $n \geqslant 0$. The equivalence classes under this relation are called the left Vogan *(Δ, ρ)*-classes.

Theorem 6.7 (Bonnafé–Geck, [6, Theorem 7.2]). *Let $y, w \in W$. Then*

$$y \sim_L w \ \Rightarrow \ y \approx^{\Delta,\rho} w.$$

(2) In [6, § 7], the condition that ρ is surjective is not necessary.

6.2. Orbits. In the case of equal parameters, the $*$-orbit of an element w is the set consisting of w and all elements that can be reached from w by a sequence of $*$-operations. As shown in [18], the $*$-orbit of w is contained in the right cell containing w. Theorem 6.2 offers an analogue of this result in the more general framework of strongly KL-admissible pairs.

Let Δ be a collection of strongly KL-admissible pairs, and $y, w \in W$. If y can be reached from w via a sequence of applications of maps δ^L such that $(I,\delta) \in \Delta$, then we write $y \xleftrightarrow{\Delta} w$, and we denote by $\mathrm{Orb}^R_\Delta(w)$ the set of all elements related to w in this way. We refer to this set as the Δ-*orbit of* w, and it is a subset of the right cell containing w. By Theorem 6.2 and Theorem 6.7 respectively, we have:

$$y^{-1} \xleftrightarrow{\Delta} w^{-1} \ \Rightarrow\ y \sim_L w \ \Rightarrow\ y \approx^{\Delta,\rho} w.$$

Hence left Vogan (Δ,ρ)-classes are unions of left cells, which are in turn unions of inverses of Δ-orbits.

Since inverses of Δ-orbits play an important role later, we also define:

$$\mathrm{Orb}^L_\Delta(w) \coloneqq \{y \in W : y^{-1} \xleftrightarrow{\Delta} w^{-1}\} = \{y^{-1} \in W : y \in \mathrm{Orb}^R_\Delta(w^{-1})\}.$$

The superscripts are chosen so that $\mathrm{Orb}^R_\Delta(w)$ is contained in the *right* cell containing w, and $\mathrm{Orb}^L_\Delta(w)$ is contained in the *left* cell containing w.

Suppose that every $(I,\delta) \in \Delta$ is such that the map δ has finite order. It follows that each δ is a bijection and that the corresponding left extension δ^L is a bijection of finite order. So

$$\mathcal{V}_\Delta \coloneqq \langle\, \delta^L : (I,\delta) \in \Delta \,\rangle$$

is a group of permutations of the elements of W, and the orbit of $w \in W$ with respect to $\mathcal{V}_\Delta$ is precisely the Δ-orbit of w.

6.3. KL-admissible pairs. The following lemma lists some properties of KL-admissible pairs.

Lemma 6.8. *All sets of KL-admissible pairs mentioned will be with respect to* $(W, S, \mathscr{L})$.

(i) *If* (I,δ) *is a strongly KL-admissible pair, then* $\Delta \coloneqq \{(I,\delta)\}$ *may be replaced by a collection* Δ' *of strongly KL-admissible pairs, where each* $(I',\delta') \in \Delta'$ *is such that* $W_{I'}$ *is an irreducible Coxeter group, and for all* $y, w \in W$ *we have:*
- $y \xleftrightarrow{\Delta} w \ \Rightarrow\ y \xleftrightarrow{\Delta'} w$
- $y \approx^{\Delta',\rho} w \ \Rightarrow\ y \approx^{\Delta,\rho} w$.

(ii) *Suppose that* (I,δ) *is strongly KL-admissible and* (I',δ') *is maximally KL-admissible, with* $I \subseteq I'$. *Then for all* $y, w \in W$, *we have:*
- $y \xleftrightarrow{\{(I,\delta)\}} w \ \Rightarrow\ y \xleftrightarrow{\{(I',\delta')\}} w$
- $y \approx^{\{(I',\delta')\},\rho} w \ \Rightarrow\ y \approx^{\{(I,\delta)\},\rho} w$.

(iii) *Suppose that* W *has a complete list* $W_{I_1}, \ldots, W_{I_d}$ *of distinct irreducible parabolic subgroups of rank* $|S|-1$, *and for each* W_{I_i} *there is a corresponding maximally KL-admissible pair* (I_i,δ_i). *Set* $\Xi \coloneqq \{(I_i,\delta_i) : 1 \leqslant i \leqslant d\}$, *and let* Δ *be any other set of strongly KL-admissible pairs (such that*[(3)] *for any* $(I,\delta) \in \Delta$ *we have* $I \neq S$). *Then for all* $y, w \in W$, *we have:*
- $y \xleftrightarrow{\Delta} w \ \Rightarrow\ y \xleftrightarrow{\Xi} w$
- $y \approx^{\Xi,\rho} w \ \Rightarrow\ y \approx^{\Delta,\rho} w$.

(3) Vogan classes are a tool for inductively obtaining cellular information from parabolic subgroups. It makes sense, therefore, to restrict ourselves with the condition $I \neq S$.

Proof. We begin by noting that in all three parts, the first claim implies the second by Definition 6.6. We prove only part (iii).

Assume towards a contradiction that we have two elements $y, w \in W$ in the same Δ-orbit but not in the same Ξ-orbit. So there is some $(I, \delta) \in \Delta$ such that we have $y \in \mathrm{Orb}^R_{\{(I,\delta)\}}(w)$ but not $y \in \mathrm{Orb}^R_\Xi(w)$. We have $y = (\delta^L)^k(w)$ for some $k \in \mathbb{Z}_{\geqslant 0}$; by the definition of the left extension of a map we must have $\mathrm{rep}_I(y) = \mathrm{rep}_I(w)$. Using part (i), it will suffice if we assume that W_I is irreducible, and thus lies in some parabolic subgroup of rank $|S| - 1$. So consider some $(I', \delta') \in \Xi$ with $I \subseteq I'$, and apply part (ii) to obtain the contradiction. □

Conventions. For the rest of this paper, we shall only work with weight functions $\mathscr{L} : W_n \to \mathbb{Z}$ such that $b/a > n - 2$. We also establish the following notation:

- let (J, ε) be as in Example 6.4,
- let (K, ψ) be as in Example 6.5,
- and set $\Xi := \{(J, \varepsilon), (K, \psi)\}$.

Further, we will take $\rho = \overline{\mathcal{R}}^{\mathscr{L}}$ from now on.

The pair (J, ε) does not depend on our choice of weight function (the Iwahori–Hecke algebra of symmetric group does not admit unequal parameters), while we have mentioned in Example 6.5 that the pair (K, ψ) is well-defined (up to a choice of ordering on the left cells) for any $\mathscr{L}$ such that $b/a > n - 2$.

A corollary to Lemma 6.8 (ii) is that the Ξ-orbits (and thus, left Vogan (Ξ, ρ)-classes) are independent of the choice of total ordering placed on the left cells in Examples 6.4 and 6.5.

Noting that Ξ satisfies the conditions in Lemma 6.8 (iii) motivates this particular choice of set of KL-admissible pairs.

As the maps ε and ψ have finite order, $\mathcal{V}_\Xi$ is a group of permutations of the elements of W_n.

7. Determining Ξ-orbits

This section will be used to describe the decomposition of W_n into Ξ-orbits. The key to this is relating the group $\mathcal{V}_\Xi$ to (generalised) Knuth relations, and thus to (bi)tableaux and cells. For $w \in \mathfrak{S}_n$, denote by $(P(w), Q(w))$ the application of the Robinson–Schensted algorithm.

7.1. Generalised Knuth relations. Let $w = w(1), \dots, w(n) \in W_n$ be a signed permutation. We define the following relations, being modified versions of the ones found in 5.1.4 of Knuth [19] and [8, § 3].

For $1 \leqslant i \leqslant k-2$: if $w(i+1) < w(i) < w(i+2)$ or $w(i+2) < w(i) < w(i+1)$, then set $w' := ws_{i+1}$ and say that w and w' differ by a relation of type I_k.

For $1 \leqslant i \leqslant k-2$: if $w(i+1) < w(i+2) < w(i)$ or $w(i) < w(i+2) < w(i+1)$, then set $w' := ws_i$ and say that w and w' differ by a relation of type II_k.

For $1 \leqslant i \leqslant k-1$: if the sign of $w(i)$ and $w(i+1)$ differ, then set $w' := ws_i$ and say that w and w' differ by a relation of type III_k.

If two (signed) permutations y, w are linked by a series of Knuth relations of, say, types I_k and II_k, then we denote this by $y \sim_{\mathrm{I}_k, \mathrm{II}_k} w$. We may now state two important results in this context.

Theorem 7.1 (Knuth, [19, Theorem 6]). *Let $u, v \in W(A_{n-1})$. Then $u \sim_{\mathrm{I}_n, \mathrm{II}_n} v$ if and only if $P(u) = P(v)$.*

Proposition 7.2 (Bonnafé–Iancu, [8, Proposition 3.8]). *Let $y, w \in W_n$. Then $y \sim_{\mathrm{I}_n, \mathrm{II}_n, \mathrm{III}_n} w$ if and only if $A_n(y) = A_n(w)$.*

We will determine to what degree we can recover the Knuth relations using only the maps ε^L and ψ^L.

Lemma 7.3. *Let $w \in W_n$, and denote $u \coloneqq \mathrm{pr}_J(w)$. Then $w(i) > w(j)$ if and only if $u(i) > u(j)$, for $1 \leqslant i, j \leqslant n$.*

Proof. We identify $x \coloneqq \mathrm{rep}_J(w)$. Let $p_1, p_2, \dots, p_q$ be the increasing sequence comprising the $q = \ell_t(w)$ negative entries in the row form of w, and similarly $m_1, m_2, \dots, m_{n-q}$ the increasing sequence comprising the positive entries. Set $r_1 \coloneqq t$ and recursively define $r_{i+1} \coloneqq s_i r_i = s_i \cdots s_1 t$. Then from § 4.1 of [8], we have

$$x = r_{|p_q|} \cdots r_{|p_1|} = \begin{pmatrix} 1 & 2 & \cdots & q & q+1 & \cdots & n-1 & n \\ p_1 & p_2 & \cdots & p_q & m_1 & \cdots & m_{n-q-1} & m_{n-q} \end{pmatrix}.$$

We can read off from this that the element x has the property of preserving the ordering of the row form of u under left multiplication; that is to say $u(i) > u(j)$ if and only if $xu(i) > xu(j)$. ☐

Lemma 7.4. *Let $w \in W_n$ and denote $u \coloneqq \mathrm{pr}_K(w)$. Then*

(i) *$w(i) > w(j)$ if and only if $u(i) > u(j)$, for $1 \leqslant i, j \leqslant n-1$,*
(ii) *$w(j) > 0$ if and only if $u(j) > 0$, for $1 \leqslant j \leqslant n-1$.*

Proof. Let $k = w(n)$. There exists a unique $x \in X_K$ such that $x(n) = k$, and so $x = \mathrm{rep}_K(w)$. Denote by $\mathrm{sgn}(z)$ the sign of $z \in \mathbb{Z}$. Then we have

$$x = \begin{pmatrix} 1 & 2 & \cdots & |k|-1 & |k| & \cdots & n-1 & n \\ 1 & 2 & \cdots & |k|-1 & |k|+1 & \cdots & n & k \end{pmatrix},$$

and in turn,

$$w(i) = \begin{cases} u(i) & \text{if } 1 \leqslant |u(i)| \leqslant |k|-1, \\ u(i) + \mathrm{sgn}(u(i)) & \text{if } |k| \leqslant |u(i)| \leqslant n-1, \\ k & \text{if } i = n. \end{cases}.$$

Parts (i) and (ii) follow from this formula and Lemma 3.1. ☐

Lemma 7.5. *Let $y, w \in W_n$.*

(i) *If $y \sim_{\mathrm{I}_n, \mathrm{II}_n} w$ then $\mathrm{rep}_J(y) = \mathrm{rep}_J(w)$.*
(ii) *If $y \sim_{\mathrm{I}_{n-1}, \mathrm{II}_{n-1}, \mathrm{III}_{n-1}} w$, then $\mathrm{rep}_K(y) = \mathrm{rep}_K(w)$.*

Corollary 7.6. *Let $y, w \in W_n$. Then*

(i) *$w = \varepsilon^k(y)$ for some $k \in \mathbb{Z}$ if and only if $y \sim_{\mathrm{I}_n, \mathrm{II}_n} w$,*
(ii) *$w = \psi^k(y)$ for some $k \in \mathbb{Z}$ if and only if $y \sim_{\mathrm{I}_{n-1}, \mathrm{II}_{n-1}, \mathrm{III}_{n-1}} w$.*

Proof. We prove only part (i). Suppose $w = \varepsilon^k(y)$. By the definition of the left extension of a map (see § 6.1), we have $\mathrm{rep}_J(y) = \mathrm{rep}_J(w)$. Now set $u \coloneqq \mathrm{pr}_J(y)$, $v \coloneqq \mathrm{pr}_J(w)$, and $x \coloneqq \mathrm{rep}_J(y) = \mathrm{rep}_J(w)$. Then

$$\begin{aligned} \exists\, k \in \mathbb{Z} : v = \varepsilon^k(u) &\iff u \sim_{R,J} v && \text{by property (B)} \\ &\iff P(u) = P(v) && \text{[18, § 5]} \\ &\iff u \sim_{\mathrm{I}_n, \mathrm{II}_n} v && \text{Theorem 7.1} \\ &\iff xu \sim_{\mathrm{I}_n, \mathrm{II}_n} xv. \end{aligned}$$

For the last equivalence, the forward implication is given by Lemma 7.3, while the converse uses Lemma 7.5 (i) as well. ☐

The following result is due to Welsh through private communication.

Lemma 7.7 (Welsh). *Let $w = w(1), \dots, w(n) \in W_n \smallsetminus Z_n$ be a signed permutation such that $w(n-1)$ and $w(n)$ have opposing signs. Then the element $w' = ws_{n-1}$ is connected to w by a sequence of Knuth relations of types I_n, II_n and III_{n-1}.*

Proof. Denote by $w_{n-1} \coloneqq w(n-1)$, $w_n \coloneqq w(n)$, by $x_1, \dots, x_f$ the subsequence of negative entries in $w(1), \dots, w(n-2)$, and by $y_1, \dots, y_g$ the subsequence of positive entries. Suppose that $w_{n-1} < 0 < w_n$, so that we have

$$w \ \sim_{\mathrm{III}_{n-1}} \ x_1, \dots, x_f, w_{n-1}, y_1, \dots, y_g, w_n.$$

If we had instead $w_n < 0 < w_{n-1}$, then we would repeatedly use relations of type III_{n-1} to move all of the positive entries to the left, and then proceed with an analogous argument to the following.

Suppose that we do not have $y_1 > y_2 > \dots > y_g > w_n$. Then there exists a maximal index k such that $y_k < y_{k+1}$, with the convention $y_{g+1} \coloneqq w_n$. If $y_k < w_n$, then it is straightforward to prove the lemma.

So assume that $y_k > w_n$. Let $j \in \{k+1, \dots, g\}$ be the unique index such that $y_j > y_k > y_{j+1}$. If two elements differ only by the block sequences $y_k, \dots, y_g, w_n$ and $y_{k+1}, \dots, y_{j-1}, y_k, y_{j+1}, \dots, y_g, w_n, y_j$, then they are linked by a sequence of relations of types I_n and II_n. We use this in the second and fifth statements of the following. We have:

$$\begin{aligned}
w &\sim_{\mathrm{III}_{n-1}} x_1 \cdots x_f w_{n-1} y_1 \cdots y_{k-1} y_k y_{k+1} \cdots y_{j-1} y_j y_{j+1} \cdots y_g w_n \\
&\sim_{\mathrm{I}_n, \mathrm{II}_n} x_1 \cdots x_f w_{n-1} y_1 \cdots y_{k-1} y_{k+1} \cdots y_{j-1} y_k y_{j+1} \cdots y_g w_n y_j \\
&\sim_{\mathrm{III}_{n-1}} x_1 \cdots x_f y_1 \cdots y_{k-1} y_{k+1} \cdots y_{j-1} y_k y_{j+1} \cdots y_g w_n w_{n-1} y_j \\
&\sim_{\mathrm{I}_n} \quad x_1 \cdots x_f y_1 \cdots y_{k-1} y_{k+1} \cdots y_{j-1} y_k y_{j+1} \cdots y_g w_n y_j w_{n-1} \\
&\sim_{\mathrm{I}_n, \mathrm{II}_n} x_1 \cdots x_f y_1 \cdots y_{k-1} y_k y_{k+1} \cdots y_{j-1} y_j y_{j+1} \cdots y_g w_n w_{n-1} \\
&\sim_{\mathrm{III}_{n-1}} w_1 \cdots w_{n-2} w_n w_{n-1} = w'.
\end{aligned}$$

Now suppose instead that $y_1 > y_2 > \dots > y_g > w_n$. As $w \notin Z_n$, by § 5.2 we cannot have $x_1 < x_2 < \dots < x_f < w_{n-1}$. With similar arguments to the above, the result holds. $\square$

Recall from Corollary 7.6 that if $y \sim_{\mathrm{I}_n, \mathrm{II}_n} w$, then we have $w = \varepsilon^{k_1}(y)$ for some $k_1 \in \mathbb{Z}$, and if $y \sim_{\mathrm{III}_{n-1}} w$, then we have $w = \psi^{k_2}(y)$ for some $k_2 \in \mathbb{Z}$. So under the hypothesis of Lemma 7.7, we know that w can be transformed into $w' = ws_{n-1}$ by applying a combination of the maps ε and ψ suitably. In other words, if $w \notin Z_n$ and $w \sim_{\mathrm{III}_n} ws_{n-1}$, then $ws_{n-1} \in \mathrm{Orb}^R_\Xi(w)$.

7.2. Ξ-orbits. We begin by showing how $W_n \smallsetminus \widetilde{Z}_n$ is partitioned into Ξ-orbits. Let $y, w \in W_n$. Then:

$$y \overset{\Xi}{\longleftrightarrow} w \ \Rightarrow \ A_n(y) = A_n(w). \tag{3}$$

Indeed, Ξ-orbits are subsets of asymptotic right cells by Theorem 6.2, and so (3) follows by the characterisation of these cells in Theorem 3.2 (ii).

A converse to (3) exists if we assume that $y, w \in W_n \smallsetminus \widetilde{Z}_n$. Combining our discussion of the elements w_J and $w_J w_0$ in § 5.1 with Proposition 7.2, it suffices to show that for $y, w \in W_n \smallsetminus Z_n$, if $y \sim_{\mathrm{I}_n, \mathrm{II}_n, \mathrm{III}_n} w$ then y and w lie in the same Ξ-orbit. However, this follows from Corollary 7.6 and Lemma 7.7, yielding the following result.

Proposition 7.8. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$, and let $y, w \in W_n \smallsetminus \widetilde{Z}_n$. Then*

$$y \overset{\Xi}{\longleftrightarrow} w \iff A_n(y) = A_n(w).$$

We now consider how $\widetilde{Z}_n$ is partitioned into Ξ-orbits.

Lemma 7.9. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$, and let $y, w \in \widetilde{Z}_n$. Then we may have $A_n(y) = A_n(w)$ without $w \in \mathrm{Orb}^R_\Xi(y)$. Indeed, let $1 \leqslant q \leqslant n-1$, and let T_q be any standard bitableau of shape $(1^{n-q} \mid 1^q)$. Then the set $\Gamma'(T_q) := \{w \in W_n : A_n(w) = T_q\}$ is equal to the union of exactly two Ξ-orbits.*

Proof. Consider $w = w(1), \dots, w(n) \in \Gamma'(T_q)$ as a signed permutation, and note that $q = \ell_t(w)$. As in § 5.2, we write $x_1, \dots, x_q$ for the (increasing) subsequence of negative integers in the row form of w, and $y_1, \dots, y_{n-q}$ for the (decreasing) subsequence of positive integers. Schensted [24, Theorems 1 and 2] motivates the following observations.

- Suppose $w(n) > 0$; that is, $w(n) = y_{n-q}$. Then by Lemma 7.3, $\mathrm{pr}_J(w)$ has a longest increasing subsequence of length $q+1$, and a longest decreasing subsequence of length $n-q$. By Lemma 7.4 (ii), we note that $\ell_t(w) = \ell_t(\mathrm{pr}_K(w))$.
- Suppose $w(n) < 0$; that is, $w(n) = x_q$. Then by Lemma 7.3, $\mathrm{pr}_J(w)$ has a longest increasing subsequence of length q, and a longest decreasing subsequence of length $n-q+1$. By Lemma 7.4 (ii), we note that $\ell_t(w) = \ell_t(\mathrm{pr}_K(w)) + 1$.

We denote $r := n-q$ and $\overline{x}_i := |x_i|$ for reasons of typesetting. Applying the generalised Robinson–Schensted insertion algorithm, we see that:

$$A_n(w) = \begin{array}{|c|}\hline y_r \\ \hline y_{r-1} \\ \hline \vdots \\ \hline y_1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline \overline{x}_q \\ \hline \overline{x}_{q-1} \\ \hline \vdots \\ \hline \overline{x}_1 \\ \hline \end{array} \, .$$

Now we use our observations to deduce the contents of Figure 2.

We have a disjoint union of non-empty sets $\Gamma'(T_q) = \gamma'_1 \sqcup \gamma'_2$, where

- $\gamma'_1 := \{w \in W_n : A_n(w) = T_q \text{ and } w(n) > 0\}$,
- $\gamma'_2 := \{w \in W_n : A_n(w) = T_q \text{ and } w(n) < 0\}$.

By noting that $\Gamma'(T_q)$ coincides with an asymptotic right cell, we see that this decomposition may be identified with that in (a right-handed version of) Corollary 5.7. Let $y, w \in \Gamma'(T_q)$. Then using Figure 2, we deduce that the following four statements are true if and only if $y, w \in \gamma'_i$ for some $i \in \{1, 2\}$:

$$\begin{array}{ll} \text{(i)}\ A_{n-1}(\mathrm{pr}_K(y)) = A_{n-1}(\mathrm{pr}_K(w)), & \text{(ii)}\ y \overset{\{(K,\psi)\}}{\longleftrightarrow} w, \\ \text{(iii)}\ P(\mathrm{pr}_J(y)) = P(\mathrm{pr}_J(w)), & \text{(iv)}\ y \overset{\{(J,\varepsilon)\}}{\longleftrightarrow} w. \end{array}$$

This proves the lemma, as well as Proposition 7.10 (iii). □

Proposition 7.10. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$.*

(i) *The number of Ξ-orbits that comprise W_n is* $\mathrm{YBT} + 2^n - 2$*, where* YBT *is the number of standard bitableaux of size n.*

(ii) *Let $w \in W_n$.*
- *If $w \in W_n \setminus \widetilde{Z}_n$ then*
$$\mathrm{Orb}^R_\Xi(w) = \{w' \in W_n : A_n(w) = A_n(w')\}.$$
- *If $w \in \widetilde{Z}_n$ and $w(n) > 0$, then*
$$\mathrm{Orb}^R_\Xi(w) = \{w' \in W_n : A_n(w) = A_n(w') \text{ and } w'(n) > 0\}.$$

$$A_{n-1}(\mathrm{pr}_K(w)) \;=\; \begin{array}{|c|}\hline y'_{r-1} \\ \hline y'_{r-2} \\ \hline \vdots \\ \hline y'_1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline \overline{x}'_q \\ \hline \overline{x}'_{q-1} \\ \hline \vdots \\ \hline \overline{x}'_1 \\ \hline \end{array}\,, \qquad \begin{array}{ll} \text{where if } w(i) = x_j & (\text{resp. } y_j), \\ \text{then } x'_j \coloneqq \mathrm{pr}_K(w)(i) & (\text{resp. } y'_j), \end{array}$$

$$\iff w(n) > 0$$

$$\iff P(\mathrm{pr}_J(w)) \;=\; \begin{array}{|c|ccccc}\hline x''_1 & \multicolumn{1}{|c|}{x''_2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{x''_q} & \multicolumn{1}{c|}{y''_r} \\ \hline y''_{r-1} \\ \cline{1-1} y''_{r-2} \\ \cline{1-1} \vdots \\ \cline{1-1} y''_1 \\ \cline{1-1} \end{array}\,, \qquad \begin{array}{ll} \text{where if } w(i) = x_j & (\text{resp. } y_j), \\ \text{then } x''_j \coloneqq \mathrm{pr}_J(w)(i) & (\text{resp. } y''_j). \end{array}$$

We also have:

$$A_{n-1}(\mathrm{pr}_K(w)) \;=\; \begin{array}{|c|}\hline y'_{r} \\ \hline y'_{r-1} \\ \hline \vdots \\ \hline y'_1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline \overline{x}'_{q-1} \\ \hline \overline{x}'_{q-2} \\ \hline \vdots \\ \hline \overline{x}'_1 \\ \hline \end{array}\,, \qquad \begin{array}{ll} \text{where if } w(i) = x_j & (\text{resp. } y_j), \\ \text{then } x'_j \coloneqq \mathrm{pr}_K(w)(i) & (\text{resp. } y'_j), \end{array}$$

$$\iff w(n) < 0$$

$$\iff P(\mathrm{pr}_J(w)) \;=\; \begin{array}{|c|cccc}\hline x''_1 & \multicolumn{1}{|c|}{x''_2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{x''_q} \\ \hline y''_{r} \\ \cline{1-1} y''_{r-1} \\ \cline{1-1} \vdots \\ \cline{1-1} y''_1 \\ \cline{1-1} \end{array}\,, \qquad \begin{array}{ll} \text{where if } w(i) = x_j & (\text{resp. } y_j), \\ \text{then } x''_j \coloneqq \mathrm{pr}_J(w)(i) & (\text{resp. } y''_j). \end{array}$$

Figure 2. Two collections of equivalent statements pertaining to the proof of Lemma 7.9.

- *If* $w \in \widetilde{Z}_n$ *and* $w(n) < 0$, *then*
$$\mathrm{Orb}^R_\Xi(w) = \{w' \in W_n : A_n(w) = A_n(w') \text{ and } w'(n) < 0\}.$$

(iii) *If* $w \in Z_n$, *then*
$$\mathrm{Orb}^R_\Xi(w) = \mathrm{Orb}^R_{\{(J,\varepsilon)\}}(w) = \mathrm{Orb}^R_{\{(K,\psi)\}}(w).$$

Proof. Part (ii) follows from Proposition 7.8 and the proof of Lemma 7.9, while part (i) holds by part (ii) and application of (2) from § 5.1. □

An immediate consequence of this is that $w \in W_n \setminus \widetilde{Z}_n$ if and only if the Ξ-orbit containing w coincides with the asymptotic right cell containing w.

As noted in § 6.3, Ξ satisfies the conditions in Lemma 6.8; together with this result we confirm the conjecture mentioned at the end of [6, Example 6.8].

A table comparing the number of Ξ-orbits to the number of asymptotic and intermediate right cells of W_n is given in Figure 3.

	cells		Ξ-*orbits*
	$\frac{b}{a} = n-1$	$\frac{b}{a} > n-1$	$\frac{b}{a} > n-2$
$n = 2$	4	6	8
3	16	20	26
4	68	76	90
5	296	312	342
6	1352	1384	1446
7	6448	6512	6638

Figure 3. On the left, the number of left/right cells of W_n with respect to the given weights. On the right, the number of Ξ-orbits that comprise W_n.

7.3. The property $(\star)$. We require a final discussion regarding bipartitions and Ξ-orbits before moving on to Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes.

If π is a partition of n with conjugate denoted π', and $\lambda = (\lambda^+ \mid \lambda^-) = (\lambda^+_1, \ldots, \lambda^+_f \mid \lambda^-_1, \ldots, \lambda^-_g) \Vdash n$ is a bipartition of n, then $\lambda' := ((\lambda^-)' \mid (\lambda^+)')$ is the conjugate bipartition of λ. If we write $\lambda = (\lambda_1, \ldots, \lambda_{f+g})$ where $\lambda_i = \lambda^+_i$ for $1 \leqslant i \leqslant f$ and $\lambda_{f+j} = \lambda^-_j$ for $1 \leqslant j \leqslant g$, then we can define:
$$I(\lambda) := S \setminus \{t,\, s_{\lambda_1},\, s_{\lambda_1+\lambda_2},\, \ldots,\, s_{\lambda_1+\lambda_2+\cdots+\lambda_{f+g-1}}\}.$$
Set $q := \lambda_1 + \cdots + \lambda_f$, denote by $w_{I(\lambda)}$ the longest word of $W_{I(\lambda)}$, and by w_q the longest word of W_q. Then we set
$$w_\lambda := w_{I(\lambda)} w_q.$$
Thus $\mathcal{R}(w_{I(\lambda)}) = I(\lambda)$, and if $b/a > q-1$, then $\overline{\mathcal{R}}^{\mathscr{L}}(w_\lambda) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_q\}$. With the notation of § 5, we have $w_{(q|n-q)} = \sigma_{n,q} \in \Omega_{\zeta_q}$.

Note that $\mathrm{sh}(w_\lambda) = \lambda'$, and so $w_{\lambda'} \in \Omega_\lambda$. We are interested in whether or not an element $z \in W_n$ has the property:
$$(\star) \qquad \exists\, \nu \in \mathcal{V}_\Xi : \nu(z) \in \mathrm{Orb}^L_\Xi(w_{\mathrm{sh}(z)'}).$$
For such an element z, it follows from Proposition 7.10 (ii) that there exists some $\hat{z} \in \mathrm{Orb}^L_\Xi(z)$ such that $\nu(\hat{z}) = w_{\mathrm{sh}(z)'}$; this $\hat{z}$ is the unique element in the intersection of $\mathrm{Orb}^L_\Xi(z)$ with $\mathrm{Orb}^R_\Xi(w_{\mathrm{sh}(z)'})$.

Remark 7.11. From Proposition 7.10, we can state that $w \in W_n$ satisfies ($\star$) if and only if $w \in W_n \smallsetminus \widetilde{Z}_n$ or $w \in \widetilde{Z}_n$, $w(n) > 0$ and $w^{-1}(n) > 0$.

Remark 7.12. Let $y, w \in W_n \smallsetminus \widetilde{Z}_n$. By Propositions 7.8 and 7.2 respectively, we have:

$$y \overset{\Xi}{\longleftrightarrow} w \iff A_n(y) = A_n(w) \iff y \sim_{\mathrm{I}_n, \mathrm{II}_n, \mathrm{III}_n} w.$$

8. Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes when $b/a > n-2$

In this section, we prove the main result of the paper. To do this, we begin by stating two results of Bonnafé from [4]. Recall that $\chi_q \coloneqq s_{n-1} \cdots s_q$.

Proposition 8.1 (Bonnafé, [4, Proposition 4.1]). *Let $0 \leqslant q \leqslant n$, let $\alpha, \beta \leqslant_e p_{n,q}$ and let $\sigma, \sigma' \in W_{S \smallsetminus \{t, s_q\}}$ be such that $\sigma \sim_L \sigma'$. Assume that $b/a \geqslant n-1$. Then*

$$\alpha a_q \sigma \beta^{-1} \sim_L a_q \sigma' \beta^{-1}.$$

Proposition 8.2 (Bonnafé, [4, Proposition 6.1]). *Let $0 \leqslant q \leqslant n-1$ and assume that $b/a \leqslant n-1$. Then*

$$s_1 \cdots s_q \cdot \sigma_{n,q} \sim_L \sigma_{n,q+1} \chi_{q+1}^{-1}.$$

Note that these two results of Bonnafé's are valid simultaneously if and only if $b/a = n-1$.

Lemma 8.3. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a \geqslant n-1$ and $y, w \in Z_n$. Then the following are equivalent:*

(i) $y \approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}} w$,
(ii) $y \sim_L w$,
(iii) $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$.

Proof. We have (ii) implies (i) by Theorem 6.7, while Definition 6.6 shows that (i) implies (iii). If $b/a > n-1$, then (iii) implies (ii) by Lemma 5.10 (i). If $b/a = n-1$, then we have $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$ if and only if $\mathcal{R}(y) = \mathcal{R}(w)$ by Corollary 5.9 (ii). So, it remains to prove that when $b/a = n-1$, $\mathcal{R}(y) = \mathcal{R}(w)$ implies that $y \sim_L w$.

We have seen in Lemma 5.10 (iii) and the subsequent discussion in § 5.5 that $\Upsilon(w) \coloneqq \{z \in Z_n : \mathcal{R}(z) = \mathcal{R}(w)\}$ comprises exactly two asymptotic left cells, a decomposition which we denote by $\Upsilon(w) = \Gamma_1 \sqcup \Gamma_2$. Our first step will be to show that

$$y \sim_L w \text{ when } \frac{b}{a} > n-1 \;\Rightarrow\; y \sim_L w \text{ when } \frac{b}{a} = n-1. \tag{4}$$

Following our discussion of the elements w_J and $w_J w_0$ in § 5.1, we can assume $y, w \in \widetilde{Z}_n$ when proving (4).

Suppose $y, w \in \Gamma \subseteq \widetilde{Z}_n$, where Γ is an asymptotic left cell. We can decompose Γ as $\Gamma = \gamma_1 \sqcup \gamma_2$, where the disjoint union is as in Corollary 5.7. Let $q = \ell_t(y) = \ell_t(w)$. By Corollary 5.7, the sets γ_1 and γ_2 are subsets of left cells with respect to any choice of parameters; in particular, they must be subsets of left cells when $b/a = n-1$. Thus to prove (4), it suffices to show that there exist representatives $y' \in \gamma_1$ and $w' \in \gamma_2$ such that $y' \sim_L w'$ when $b/a = n-1$. So set $y' \coloneqq \sigma_{n,q}\theta^{-1}$ where $\theta \leqslant_e p_{n,q}$ is such that $y' \in \gamma_1$, and set $w' \coloneqq \chi_q y' \in \gamma_2$. Let b_q be as in § 5.3.

We now apply Proposition 8.1 with $\alpha = \chi_q$, $\beta = \theta$ and $\sigma = \sigma' = b_q$ to see that $y' \sim_L w'$ when $b/a = n-1$; (4) is now proved.

By Proposition 5.11 (i), we know that the sets Γ_1 and Γ_2 are of the form $\Gamma(\sigma_{n,q}\tau^{-1})$ and $\Gamma(\sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1})$ respectively for some $0 \leqslant q \leqslant n-1$ and $\tau \leqslant_e p_{n-1,q}$, which we fix for the remainder of this proof. Thus it remains to show that $\sigma_{n,q}\tau^{-1} \sim_L \sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1}$ when $b/a = n-1$. Indeed, the following holds for $b/a \in (n-2, n-1]$, resulting in Corollary 8.4.

By Corollary 5.7, we have $\sigma_{n,q} \sim_L s_1 \cdots s_q \sigma_{n,q}$ for all choices of parameters. Then by Proposition 8.2, we have $\sigma_{n,q} \sim_L \sigma_{n,q+1}\chi_{q+1}^{-1}$. Recall that if $y \sim_L w$, then $\nu(y) \sim_L \nu(w)$ for all $\nu \in \mathcal{V}_\Xi$.

Consider $\Gamma'(\sigma_{n,q})$, and the corresponding subset γ_1 as in (a right-handed version of) Corollary 5.7. Compare γ_1 with the description of Ξ-orbits in Proposition 7.10 (ii) to see that

$$\sigma_{n,q} \overset{\Xi}{\longleftrightarrow} \sigma_{n,q}\tau^{-1} \text{ if and only if } \tau \leqslant_e p_{n-1,q}.$$

Now we use Proposition 7.10 (iii) to reduce this statement to

$$\sigma_{n,q} \overset{\{(I,\delta)\}}{\longleftrightarrow} \sigma_{n,q}\tau^{-1} \text{ if and only if } \tau \leqslant_e p_{n-1,q},$$

where $(I,\delta) \in \Xi$. By the same results, we have

$$\sigma_{n,q+1}\chi_{q+1}^{-1} \overset{\{(I,\delta)\}}{\longleftrightarrow} \sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1} \text{ if and only if } \tau \leqslant_e p_{n-1,q}.$$

It may be verified that $\mathrm{pr}_I(\sigma_{n,q}) = \mathrm{pr}_I(\sigma_{n,q+1}\chi_{q+1}^{-1})$ for $(I,\delta) \in \Xi$. It now follows that $\nu(\sigma_{n,q}) = \sigma_{n,q}\tau^{-1}$ if and only if $\nu(\sigma_{n,q+1}\chi_{q+1}^{-1}) = \sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1}$. Thus $\sigma_{n,q}\tau^{-1} \sim_L \sigma_{n,q+1}\chi_{q+1}^{-1}\tau^{-1}$, as needed. □

Corollary 8.4. *No asymptotic left cell contained in Z_n is a left cell with respect to $b/a \in (n-2, n-1]$.*

We will write $\approx$ for $\approx^{\Xi,\overline{\mathcal{R}}^{\mathscr{L}}}$ in proofs for cleaner notation. Remark 7.11 describes the scope of the following result.

Lemma 8.5. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$. Let $y, w \in W_n$ be elements satisfying $(\star)$ with $\ell_t(y) = \ell_t(w)$. Then*

$$y \approx^{\Xi,\overline{\mathcal{R}}^{\mathscr{L}}} w \;\Rightarrow\; y \in \mathrm{Orb}_\Xi^L(w).$$

We note that the following proof is adapted from part of Ariki's proof of Theorem A in [1, § 3.4], who in turn remarks to have adapted an argument from the proof of Jantzen [17, Satz 5.25].

Proof. Denote $q := \ell_t(y) = \ell_t(w)$. Let $\lambda, \mu \Vdash n$ be such that $\mathrm{sh}(y) = \lambda'$ and $\mathrm{sh}(w) = \mu'$. Let $\alpha, \beta \in \mathcal{V}_\Xi$ be such that $\alpha(y) \in \mathrm{Orb}_\Xi^L(w_\lambda)$ and $\beta(w) \in \mathrm{Orb}_\Xi^L(w_\mu)$. Let $\hat{y}, \hat{w} \in W_n$ be the unique elements in the intersection of $\mathrm{Orb}_\Xi^L(y)$ with $\mathrm{Orb}_\Xi^R(w_\lambda)$ and $\mathrm{Orb}_\Xi^L(w)$ with $\mathrm{Orb}_\Xi^R(w_\mu)$ respectively. Note that $\alpha(\hat{y}) = w_\lambda$ and $\beta(\hat{w}) = w_\mu$.

As $\hat{y} \in \mathrm{Orb}_\Xi^L(y)$ and $\hat{w} \in \mathrm{Orb}_\Xi^L(w)$, we have $y \sim_L \hat{y}$ and $w \sim_L \hat{w}$. Applying Theorem 6.7, we see that

$$\hat{y} \approx y \approx w \approx \hat{w}.$$

We apply the generalised Robinson–Schensted algorithm to the first q entries of the row forms of $\alpha(\hat{w})$ and $\beta(\hat{y})$; that is, their negative entries. Our focus will first be on the recording bitableaux of $\alpha(\hat{w})$, with that of $\beta(\hat{y})$ being obtained in a similar fashion.

As $\hat{y} \approx \hat{w}$ and $\alpha(\hat{y}) = w_\lambda$, we have $\overline{\mathcal{R}}^{\mathscr{L}}(\alpha(\hat{w})) = \overline{\mathcal{R}}^{\mathscr{L}}(\alpha(\hat{y})) = \overline{\mathcal{R}}^{\mathscr{L}}(w_\lambda)$. So by Lemma 3.1, the first λ_1^+ entries in the row form of $\alpha(\hat{w})$ form an increasing sequence (of negative integers), the next λ_2^+ entries form an increasing sequence, and so on. So the first column of the second tableau of $B_n(\alpha(\hat{w}))$ is a column of length at least λ_1^+. As $\mathrm{sh}(\alpha(\hat{w})) = \mathrm{sh}(w) = \mu'$, the length of this first column equals μ_1^+.

Similarly, the length of the first column of the second tableau of $B_n(\beta(\hat{y}))$ is a column of length at least μ_1^+, and equal to λ_1^+. Combining these observations indicates that $\lambda_1^+ = \mu_1^+$.

We continue in this way, noting that bumping between the columns during the algorithm produces a contradiction to the previous statements. We thus determine that $\lambda^+ = \mu^+$. By similarly working with the remaining $n-q$ entries in the row forms of $\alpha(\hat{w})$ and $\beta(\hat{y})$ – that is, the positive entries – we determine that $\lambda^- = \mu^-$. Thus $\lambda = \mu$, and we have:

$$B_n(\alpha(\hat{w})) = B_n(w_\lambda) = B_n(w_\mu) = B_n(\beta(\hat{y})).$$

As $\beta(\hat{y}) \in \mathrm{Orb}^R_\Xi(w_\lambda)$, we have $A_n(\beta(\hat{y})) = A_n(w_\lambda)$ by Proposition 7.10 (ii). Similarly, $\alpha(\hat{w}) \in \mathrm{Orb}^R_\Xi(w_\mu)$, and so $A_n(\alpha(\hat{w})) = A_n(w_\mu)$. Now that we know the recording and insertion bitableaux of $\alpha(\hat{w})$ and $\beta(\hat{y})$, we can determine these elements; we have:

$$\alpha(\hat{w}) = \beta(\hat{y}) = w_\lambda = w_\mu = \alpha(\hat{y}) = \beta(\hat{w}).$$

Thus $\hat{y} = \hat{w}$, and so $y \in \mathrm{Orb}^L_\Xi(w)$ as needed. □

Lemma 8.6. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$. Let $y, w \in W_n \smallsetminus Z_n$ with $\ell_t(y) \neq \ell_t(w)$. Then y and w do not lie in the same left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-class.*

Proof. Suppose, towards a contradiction, that $y \approx w$. As $\ell_t(y) \neq \ell_t(w)$ and $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$, we must have $b/a \in (n-2, n-1]$ and $\ell_t(y) = \ell_t(w) \pm 1$. Without loss of generality, we will assume that $\ell_t(y) = q$ and $\ell_t(w) = q+1$ for some $0 \leqslant q \leqslant n-1$.

As y satisfies ($\star$) by Remark 7.11, there exists some $\alpha \in \mathcal{V}_\Xi$ such that $\alpha(y) \in \mathrm{Orb}^L_\Xi(w_{\mathrm{sh}(y)'})$. Thus $\overline{\mathcal{R}}^{\mathscr{L}}(\alpha(y)) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_q\}$, and so $\alpha(y)(i) < 0$ if and only if $i \in \{1, \ldots, q\}$. By Remark 7.12, we have

$$\alpha(y) \xleftrightarrow{\ \Xi\ } \alpha(y) \cdot s_q \xleftrightarrow{\ \Xi\ } \cdots \xleftrightarrow{\ \Xi\ } \alpha(y) \cdot s_q s_{q+1} \cdots s_{n-1}.$$

Choose $\gamma \in \mathcal{V}_\Xi$ such that $\gamma(y) = \alpha(y) \cdot s_q s_{q+1} \cdots s_{n-1}$. Then $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(y)) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_{q-1}\}$. On the other hand, the set $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(w)) \cap \{t_1, \ldots, t_n\}$ must contain at least q elements. As $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(y)) \neq \overline{\mathcal{R}}^{\mathscr{L}}(\gamma(w))$ implies that $y \not\approx w$, a contradiction is reached. □

Finally, we need to show that Z_n is closed under $\approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}}$.

Lemma 8.7. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a > n-2$. Let $y \in Z_n$ and $w \in W_n$. Then*

$$y \approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}} w \;\Rightarrow\; w \in Z_n.$$

Proof. If w does not satisfy ($\star$), then $w \in \widetilde{Z}_n$ by Remark 7.11, and we are done. So we may assume from now on that w satisfies ($\star$).

Case A. Suppose that y satisfies ($\star$); there exists some $\alpha \in \mathcal{V}_\Xi$ such that $\alpha(y) \in \mathrm{Orb}^L_\Xi(w_{\mathrm{sh}(y)'})$. Following Lemma 8.5, we may assume that $\ell_t(y) \neq \ell_t(w)$. Thus $b/a \in (n-2, n-1]$ and $\ell_t(y) = \ell_t(w) \pm 1$.

Subcase I. Suppose that $\ell_t(y) = q$ for some $0 \leqslant q \leqslant n-1$ and that $\ell_t(w) = q+1$.

If $q = n-1$, there are sufficient conditions on y for us to identify it; we have $y = \sigma_{n,n-1}$. Thus $\overline{\mathcal{R}}^{\mathscr{L}}(w) = \overline{\mathcal{R}}^{\mathscr{L}}(\sigma_{n,n-1})$. We know that $\ell_t(w) = n$, so we have enough information to apply Lemma 3.1 and determine that $w = \sigma_{n,n} \in Z_n$.

If $0 \leqslant q \leqslant n-2$, then we have $\overline{\mathcal{R}}^{\mathscr{L}}(\alpha(w)) \cap \{t_1, \ldots, t_n\} = \{t_1 \ldots, t_q\}$ and $\alpha(w)(n) < 0 < \alpha(w)(n-1)$. Suppose, towards a contradiction, that $w \notin Z_n$. By Remark 7.12, we may choose some $\gamma \in \mathcal{V}_\Xi$ such that $\gamma(w) = \alpha(w) \cdot s_{n-1}$. The set $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(w)) \cap \{t_1, \ldots, t_n\}$ contains $q+1$ elements, while $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(y)) \cap \{t_1, \ldots, t_n\}$ contains at most q elements. Thus $y \not\approx w$.

Subcase II. Suppose that $\ell_t(y) = q$ for some $1 \leqslant q \leqslant n$ and that $\ell_t(w) = q-1$.

We note that if $\ell_t(y) > \ell_t(w)$ and $\overline{\mathcal{R}}^{\mathscr{L}}(\alpha(y)) = \overline{\mathcal{R}}^{\mathscr{L}}(\alpha(w))$, then we must have $q = n$, and so $y = \sigma_{n,n}$. Now apply Lemma 3.1 to see that if $\ell_t(w) = n-1$ and $\overline{\mathcal{R}}^{\mathscr{L}}(\alpha(w)) = \overline{\mathcal{R}}^{\mathscr{L}}(\sigma_{n,n})$ then we must have $w = \sigma_{n,n-1} \in Z_n$.

Case B. Suppose that y does not satisfy $(\star)$. By Remark 7.11, this assumption precludes the possibility of either $\ell_t(y) = 0$ or $\ell_t(y) = n$.

Subcase I. Suppose that $\ell_t(y) = \ell_t(w)$; denote this value by q.

As w satisfies $(\star)$, there exists some $\beta \in \mathcal{V}_\Xi$ such that $\beta(w) \in \mathrm{Orb}^L_\Xi(w_{\mathrm{sh}(w)'})$. As $y \approx w$ and $\beta(w)(n) > 0$, it must be the case that $\beta(y)(n) > 0$. By Proposition 7.10 (ii), we have:

$$\nu(y)(n) > 0 \qquad \text{for all } \ \nu \in \mathcal{V}_\Xi. \tag{5}$$

Suppose now, towards a contradiction, that $w \notin Z_n$. By Remark 7.12, we may apply a particular sequence of generalised Knuth relations of type III_n to $\beta(w)$ while remaining in $\mathrm{Orb}^R_\Xi(w)$; let $\gamma \in \mathcal{V}_\Xi$ be such that $\gamma(w) = \beta(w) \cdot s_q s_{q+1} \cdots s_{n-1}$. Then we have $\beta(w)(j) < 0$ if and only if $j \in \{1, \ldots, q\}$, and $\gamma(w)(j) < 0$ if and only if $j \in \{1, \ldots, q-1, n\}$. As $y \approx w$ and $q \leqslant n-1$, we must also have $\gamma(y)(j) < 0$ if and only if $j \in \{1, \ldots, q-1, n\}$. However, this contradicts (5).

Subcase II. Suppose that $\ell_t(y) \neq \ell_t(w)$. Let $q \coloneqq \ell_t(y)$, so that $y \in \Omega_{\zeta_q}$.

From Proposition 5.3, we observe that Ω_{ζ_q} contains a unique shortest word and a unique longest word; they are $\sigma_{n,q}$ and $\lambda_{n,q} \coloneqq p_{n,q}\sigma_{n,q}p_{n,q}^{-1}$ respectively. From Proposition 7.10, we determine that:

$$\exists\, \nu \in \mathcal{V}_\Xi \ : \ B_n(\nu(y)) = B_n(\sigma_{n,q}) \iff y(n) > 0, \tag{6}$$

$$\exists\, \nu \in \mathcal{V}_\Xi \ : \ B_n(\nu(y)) = B_n(\lambda_{n,q}) \iff y(n) < 0. \tag{7}$$

Subsubcase (i). Suppose that we are in the setting of (6).

Further suppose that $q = n-1$. As $\overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$, we have $\ell_t(w) = n$. Note that $\overline{\mathcal{R}}^{\mathscr{L}}(y) \cap \{t_1, \ldots, t_n\} = \overline{\mathcal{R}}^{\mathscr{L}}(w) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_{n-1}\}$, and $\overline{\mathcal{R}}^{\mathscr{L}}(\sigma_{n,n-1}) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_{n-1}\}$. As $y, \sigma_{n,n-1} \in Z_n$, by Corollary 5.9 (i) we have $\overline{\mathcal{R}}^{\mathscr{L}}(w) = \overline{\mathcal{R}}^{\mathscr{L}}(\sigma_{n,n-1})$. We note that $\ell_t(w) = n$, and so we have enough information to apply Lemma 3.1 and determine that $w = \sigma_{n,n} \in Z_n$.

We may now suppose that $1 \leqslant q \leqslant n-2$. As $\overline{\mathcal{R}}^{\mathscr{L}}(\nu(y)) = \overline{\mathcal{R}}^{\mathscr{L}}(\sigma_{n,q})$, we see that $\overline{\mathcal{R}}^{\mathscr{L}}(\nu(y)) \cap \{t_1, \ldots, t_n\} = \overline{\mathcal{R}}^{\mathscr{L}}(\nu(w)) \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_q\}$. This forces $\ell_t(w) = q+1$ and $\nu(w)(n) < 0$. Assume, towards a contradiction, that $w \notin Z_n$. By Remark 7.12, we may choose some $\gamma \in \mathcal{V}_\Xi$ such that $\gamma(w) = \nu(w) \cdot s_{n-1}$. The set $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(w)) \cap \{t_1, \ldots, t_n\}$ contains $q+1$ elements, while $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(y)) \cap \{t_1, \ldots, t_n\}$ contains at most q elements. Thus $y \not\approx w$.

Subsubcase (ii). Suppose that we are in the setting of (7).

Then $\overline{\mathcal{R}}^{\mathscr{L}}(\nu(y)) \cap \{t_1, \ldots, t_n\} = \overline{\mathcal{R}}^{\mathscr{L}}(\lambda_{n,q}) \cap \{t_1, \ldots, t_n\} = \{t_{n-q+1}, \ldots, t_{n-1}\}$ and $\nu(y)(n) < 0$. If $\ell_t(w) = q+1$ then $\overline{\mathcal{R}}^{\mathscr{L}}(\nu(w)) \cap \{t_1, \ldots, t_n\}$ contains at least q elements, in contradiction to the previous statement. So we may assume that $\ell_t(w) = q-1$.

Suppose that $q = 1$. Then the conditions on y are sufficient to determine that $B_n(y) = B_n(\lambda_{n,1})$. Thus $\overline{\mathcal{R}}^{\mathscr{L}}(w) = \overline{\mathcal{R}}^{\mathscr{L}}(\lambda_{n,1}) = \{s_1, \ldots, s_{n-1}\}$. As $\ell_t(w) = 0$, application of Lemma 3.1 shows that $w = \sigma_{n,0} \in Z_n$.

So we may now assume that $2 \leqslant q \leqslant n-1$. We have $\overline{\mathcal{R}}^{\mathscr{L}}(\nu(w)) \cap \{t_1, \ldots, t_n\} = \{t_{n-q+1}, \ldots, t_{n-1}\}$ and $\nu(w)(n) > 0$. Assume, towards a contradiction, that $w \notin Z_n$. Then by Remark 7.12, we may choose some map $\gamma \in \mathcal{V}_\Xi$ such that $\gamma(w) = \nu(w) \cdot s_{n-1}$. It remains to note that $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(w)) \cap \{t_1, \ldots, t_n\}$ contains $q-2$ elements, while $\overline{\mathcal{R}}^{\mathscr{L}}(\gamma(y)) \cap \{t_1, \ldots, t_n\}$ contains at least $q-1$ elements. □

Theorem 8.8. *Suppose $(W_n, S, \mathscr{L})$ is such that $b/a \geqslant n-1$. Then*

$$y \sim_L w \iff y \approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}} w.$$

Proof. This follows from Lemmas 8.3, 8.5, 8.6 and 8.7. □

Using the results of this section, we may now give a combinatorial description of the left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes for $b/a > n-2$. The classes differ depending on whether $b/a > n-1$ or $b/a \in (n-2, n-1]$, and they do so precisely on Z_n.

If $b/a > n-1$, then we obtain a description of Z_n using Proposition 5.3. If $b/a \in (n-2, n-1]$, it is sufficient to determine the classes for a representative in this interval. So take $b/a = n-1$, and look to the decomposition of Z_n given in Proposition 5.11 (ii).

Corollary 8.9. *If $y, w \in W_n \smallsetminus Z_n$ and $b/a > n-2$, then*

$$y^{-1} \overset{\Xi}{\longleftrightarrow} w^{-1} \iff y \sim_L w \iff B_n(y) = B_n(w) \iff y \approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}} w.$$

If $b/a \in (n-2, n-1]$ we have:

$$Z_n = \bigsqcup_{q=0}^{n-1} \bigsqcup_{\tau \leqslant_e p_{n-1,q}} \{\pi \cdot \sigma_{n,q} \cdot \tau^{-1} : \pi \leqslant_e \Pi_{n,q} = s_{n-q-1} \cdots s_1 \cdot t \cdot p_{n,q}\}.$$

Each term of the disjoint union is a left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-class, and a left cell if $b/a = n-1$.

If $b/a > n-1$ we have:

$$Z_n = \bigsqcup_{q=0}^{n} \bigsqcup_{\tau \leqslant_e p_{n,q}} \{\pi \cdot \sigma_{n,q} \cdot \tau^{-1} : \pi \leqslant_e p_{n,q}\}.$$

Each term of the disjoint union is both a left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-class and a left cell.

Further, for $b/a > n-2$, Z_n is closed under the relation $\approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}}$.

From this, we may infer some information about the intermediate two-sided cells of W_n.

Corollary 8.10. *Let $\Omega \subseteq W_n$ be an asymptotic two-sided cell. Then Ω is contained within an intermediate two-sided cell. Further, Z_n is contained within an intermediate two-sided cell of W_n.*

We may also characterise the asymptotic left cells that change when passing to the intermediate or sub-asymptotic case.

Corollary 8.11. *Let $\Gamma \subseteq W_n$ be an asymptotic left cell. Then Γ is also a left cell with respect to $b/a \in (n-2, n-1]$ if and only if $\Gamma \subseteq W_n \smallsetminus Z_n$.*

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Edmund Howse, Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076

E-mail : `edmund.howse@nus.edu.sg`