



ALGEBRAIC COMBINATORICS


Gil Kalai & Eran Nevo

Turán, involution and shifting

Volume 2, issue 3 (2019), p. 367-378.

http://alco.centre-mersenne.org/item/ALCO_2019__2_3_367_0

© The journal and the authors, 2019.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>

Access to articles published by the journal *Algebraic Combinatorics* on
the website <http://alco.centre-mersenne.org/> implies agreement with the
Terms of Use (<http://alco.centre-mersenne.org/legal/>).



Algebraic Combinatorics is member of the
Centre Mersenne for Open Scientific Publishing
www.centre-mersenne.org



Turán, involution and shifting

Gil Kalai & Eran Nevo

ABSTRACT We propose a strengthening of the conclusion in Turán’s (3,4)-conjecture in terms of algebraic shifting, and show that its analogue for graphs does hold. In another direction, we generalize the Mantel–Turán theorem by weakening its assumption: for any graph G on n vertices and any involution on its vertex set, if for any 3-set S of the vertices, the number of edges in G spanned by S , plus the number of edges in G spanned by the image of S under the involution, is at least 2, then the number of edges in G is at least the Mantel–Turán bound, namely the number achieved by two disjoint cliques of sizes $\frac{n}{2}$ rounded up and down.

1. INTRODUCTION

Let $T(n)$ be a graph on the vertex set $[n] = \{1, 2, \dots, n\}$ which is the disjoint union of two cliques of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. Our starting point is the following influential Mantel–Turán theorem [18, 24].

THEOREM 1.1 (Mantel, Turán). *Let G be a graph on $[n]$ where every 3-subset of $[n]$ contains an edge of G . Then the edge sets satisfy*

$$|E(G)| \geq |E(T(n))|.$$

Let $B(n)$ be the set of edges on $[n]$

$$B(n) = \{ab : 1 \leq a < b \leq n, a + b \leq n\}.$$

Note that $|B(n)| = |E(T(n))|$, and that $B(n)$ is a *shifted* graph, a.k.a. threshold graph. (By a slight abuse of notation we refer to $B(n)$ also as the graph with the vertex set $[n]$ and edge set $B(n)$.) However, not every 3-subset of $[n]$ contains an edge of $B(n)$. We relate $B(n)$ to the Mantel–Turán theorem in two ways: first, by weakening the condition in Theorem 1.1 so that its conclusion applies to $B(n)$ as well; second, by strengthening the conclusion in Theorem 1.1 to an algebraic one, namely that any graph G as in the theorem *dominates* $B(n)$. The later, as we shall see, serves as an inspiration for an algebraic approach to Turán’s (3, 4)-problem.

Manuscript received 5th March 2018, accepted 26th June 2018.

KEYWORDS. Turán’s (3, 4)-conjecture, shifting, threshold graphs.

ACKNOWLEDGEMENTS. Research of Kalai is partially supported by ERC advanced grant 320924, BSF grant 2006066, and NSF grant DMS-1300120, and of Nevo by Israel Science Foundation grant ISF-1695/15, by grant 2528/16 of the ISF-NRF Singapore joint research program, and by ISF-BSF joint grant 2016288.

THEOREM 1.2 (Turán with involution). *Let G be a graph on $[n]$. Assume there exists an involution τ on $[n]$ such that any 3-set $S \subseteq [n]$ satisfies*

$$(1) \quad |E(G[S])| + |E(G[\tau(S)])| \geq 2.$$

Then $|E(G)| \geq |E(T(n))|$.

Here $G[A]$ denotes the induced subgraph of G on the subset A of the vertex set. First, note that the condition in Theorem 1.1 (“any triple supports an edge”) implies the condition in the above theorem (“any triple and its τ -image support two edges”, counted with repetitions). In fact, if $\langle \tau \rangle = \mathbb{Z}_2$ acts trivially on $[n]$, then the conditions in both theorems coincide.

Second, if τ acts on $[n]$ with at most one fixed point, then by applying a suitable permutation of the vertices we can assume $\tau(i) = n + 1 - i$ for any $i \in [n]$. For this τ , $(B(n), \tau)$ satisfies the assumptions of Theorem 1.2.

We now turn to an algebraic statement that implies Theorem 1.1. First we define the notion of domination.

Let $X = (x_{i,j})$ be an $n \times n$ matrix of variables, and $C_k(X) = (c_{S,T})$ its k -th compound matrix, namely the $\binom{n}{k} \times \binom{n}{k}$ matrix, where for k -subsets S, T of $[n]$, $c_{S,T}$ equals the determinant of the (S, T) -minor of X , computed in the field extension $\mathbb{Q}(x_{i,j})$ over the rationals say. Let F_1 and F_2 be two families of k -subsets of $[n]$. Then F_1 dominates F_2 if the submatrix of $C_k(X)$ with rows indexed by F_2 and columns indexed by F_1 has rank $|F_2|$. This implies, of course, that $|F_2| \leq |F_1|$.

THEOREM 1.3 (Turán with domination). *Let G be a graph on $[n]$ where every 3-subset of $[n]$ contains an edge of G . Then $E(G)$ dominates $B(n)$.*

We prove this statement via exterior algebraic shifting, introduced by Kalai [11], and via relations between algebraic and combinatorial shifting established by Hibi and Murai [19, 20]. (We can apply algebraic shifting w.r.t. any fixed term order \prec_t that satisfies $ab \prec_t a'b'$ whenever $a + b < a' + b'$.)

We do not know if there is a common generalization of Theorems 1.2 and 1.3.

PROBLEM 1.4. *Let G be a graph on $[n]$. Assume there exists an involution τ on $[n]$ such that any 3-set $S \subseteq [n]$ satisfies relation (1). Must $E(G)$ dominate $B(n)$?*

1.1. TURÁN’S (3,4)-CONJECTURE. Theorem 1.3 is motivated by an algebraic approach to a famous conjecture by Turán [24, 25]: Turán asked for the minimum number of edges in a 3-uniform hypergraph on $[n]$ such that every four vertices span at least one edge. (Equivalently he asked for the maximum number of edges if every four vertices span at most three edges.) Turán conjectured in 1940 that the minimum is attained by the following hypergraph, now called the *Turán’s (3,4)-hypergraph*:

Divide $[n]$ to three equal-as-possible sets A_1, A_2, A_3 and consider all 3-sets which are either contained in A_i or contain two elements from A_i and one element from $A_{i-1 \pmod 3}$, $i = 1, 2, 3$. Thus Turán’s conjecture is:

CONJECTURE 1.5 (Turán’s (3,4)-conjecture). *Let H be a collection of 3-sets on $[n]$ such that any 4-subset on $[n]$ supports a 3-set in H . Then $|H| \geq h(n)$, where*

$$h(n) := \binom{s_1}{3} + \binom{s_2}{3} + \binom{s_3}{3} + s_1 \binom{s_2}{2} + s_2 \binom{s_3}{2} + s_3 \binom{s_1}{2}, \quad n = s_1 + s_2 + s_3,$$

and $|s_i - s_j| \leq 1$ for any $1 \leq i, j \leq 3$.

For more background on this problem see [4, 5, 6, 8, 10, 15, 17, 22, 23].

Consider, now, the collection of triples

$$C(n) := \{abc : a + c \leq n, 2a + b \leq n, 1 \leq a < b < c \leq n\}.$$

Then $C(n)$ is a shifted family, and we expect it to play a similar role for Turán's (3,4)-conjecture as the role of $B(n)$ for the Mantel–Turán's theorem. Indeed, we note first that

$$(2) \quad |C(n)| = h(n) = \max_s s \cdot \binom{n-s-1}{2} = \left\lfloor \frac{n}{3} \right\rfloor \left(n - \left\lfloor \frac{n}{3} \right\rfloor - 1 \right).$$

The following conjecture implies Turán's (3,4)-conjecture. It also strengthens Kalai's conjecture from [12, (3)].

CONJECTURE 1.6. *Let H be a 3-uniform hypergraph on $[n]$ such that any four vertices span an edge. Then H dominates $C(n)$.*

Conjecture 1.6 was verified, by computer, for all H arising from Kostochka's construction [17] with $n = 3k \leq 18$ vertices ($|H| = h(n)$).

OUTLINE. Section 2 contains background on dominance and shifting. We prove Theorem 1.2 in Section 4, and Theorem 1.3 in Section 3. In Section 5 we discuss Turán's (3,4)-conjecture.

2. DOMINANCE AND SHIFTING

2.1. DOMINANCE. Let $X = (x_{ij})_{1 \leq i, j \leq n}$ be a matrix of n^2 variables. Recall that the k -th compound matrix $C_k(X)$ is the matrix of k by k minors, namely,

$$C_k(X) = (c_{S,T})_{S, T \in \binom{[n]}{k}},$$

where

$$c_{ST} = \det(x_{ij})_{i \in S, j \in T}.$$

(The order of rows and columns is not important in this subsection; it will be in the next).

DEFINITION 2.1. *Given two k -uniform hypergraphs on the vertex set $[n]$ E_1 and E_2 ,*

- (1) E_1 dominates E_2 if the matrix $C_{E_1, E_2}(X)$, whose rows and columns correspond to sets in E_2 and E_1 respectively, has rank $|E_2|$.
- (2) E_1 and E_2 are weakly isomorphic if each dominates the other.

Of course, if E_2 dominates E_1 then $|E_2| \geq |E_1|$, and if E_1 and E_2 are weakly isomorphic then $|E_1| = |E_2|$.

One observes that if E_1 and E_2 are combinatorially isomorphic then they are weakly isomorphic. This relies on the condition that the matrix X is generic; compare with the Permutation Lemma of [3].

EXAMPLE 2.2. A graph G on $[n]$ dominates a star on $[n]$ iff G is connected, and G is dominated by a star on $[n]$ iff G contains no cycle. In particular, G is weakly isomorphic to a star on $[n]$ iff G is a tree.

This example is the case $k = 2$ of Proposition 2.5 below.

However, as we shall see, weak isomorphism is not a transitive relation.

EXAMPLE 2.3. Let C_5 be a cycle of length five. We will see later that C_5 is weakly isomorphic to the graphs G_1 and G_2 which are themselves not weakly isomorphic, where $G_1 = \{12, 13, 14, 15, 23\}$ and $G_2 = \{12, 13, 14, 23, 24\}$.

The following problem naturally arises:

PROBLEM 2.4. *What is the transitive closure of weak isomorphism for k -uniform hypergraph on n vertices?*

2.2. CONNECTION WITH HOMOLOGY. Given a k -uniform hypergraph G let $K(G)$ be the $(k - 1)$ -dimensional simplicial complex whose $(k - 1)$ -faces are the edges in G , with complete $(k - 2)$ -dimensional skeleton.

The i -th reduced rational homology group of a simplicial complex K is denoted by $\tilde{H}_i(K, \mathbb{Q})$.

A k -uniform hypergraph G whose edges are all the k -subsets of $[n]$ containing some fixed element $v \in [n]$ is called a (spanning) k -star on $[n]$; a 2-star is simply a star, in the graph theoretical sense.

PROPOSITION 2.5.

- (1) A k -uniform hypergraph G on $[n]$ dominates a (spanning) k -star on $[n]$ if and only if $\tilde{H}_{k-2}(K(G), \mathbb{Q}) = 0$.
- (2) A k -star on $[n]$ dominates a k -uniform hypergraph G if and only if $\tilde{H}_{k-1}(K(G), \mathbb{Q}) = 0$.

For a proof one can either argue directly, or, as we shall do, use known properties of algebraic shifting, discussed next. The proof is postponed then to the next subsection.

2.3. ALGEBRAIC SHIFTING.

DEFINITION 2.6.

- (1) The partial order $<$ on $\mathcal{F} \subseteq \binom{[n]}{k}$ is defined as follows: If $S = \{s_1, s_2, \dots, s_k\}$, $s_1 < s_2 < \dots < s_k$ and $T = \{t_1, t_2, \dots, t_k\}$, $t_1 < t_2 < \dots < t_k$, then $S < T$ if $s_1 \leq t_1, s_2 \leq t_2, \dots, s_k \leq t_k$.
- (2) A term order $<_t$ on $\binom{[n]}{k}$ is a linear extension of the partial order $<$.

DEFINITION 2.7. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is shifted if it is closed down under the partial order, namely, for any $F \in \mathcal{F}$ such that $i < j \in F$ and $i \notin F$, the set $S = (F \setminus \{j\}) \cup \{i\}$ is in \mathcal{F} .

For example, G_1 and G_2 of Example 2.3 are shifted. Shifted graphs are called threshold graphs.

DEFINITION 2.8. Let A be an invertible $n \times n$ matrix over some field, and let $<_t$ be a term order on $\binom{[n]}{k}$.

For $\mathcal{F} \subseteq \binom{[n]}{k}$, the $(<_t, A)$ -exterior shifting $\mathcal{F}^{<_t, A}$ of \mathcal{F} is the $<_t$ -smallest family $\mathcal{G} \subseteq \binom{[n]}{k}$ of size $|\mathcal{F}|$ such that \mathcal{G} is weakly isomorphic to \mathcal{F} w.r.t. $C_k(A)$. (For two same size families of k -subsets of $[n]$, Y and Z , say Y is $<_t$ -smaller than Z iff the least element of the symmetric difference $Y \Delta Z$ w.r.t. $<_t$ belongs to Y .)

Clearly, such \mathcal{G} exists, as \mathcal{F} is weakly isomorphic to itself.

Equivalently, start with \mathcal{G} empty, and add elements to \mathcal{G} one-by-one “greedily” according to the order $<_t$ if the rank of the corresponding matrix $C_{\mathcal{F}, \mathcal{G}}(A)$ increases by appending the row. For the matrix $A = X$ over the field $\mathbb{Q}(x_{i,j})$ (the degree n^2 transcendental extension of the field of rationales), denote $\mathcal{F}^{<_t, X}$ by $\mathcal{F}^{<_t}$, called the exterior shifting of \mathcal{F} w.r.t. $<_t$.

Kalai [11] proved that $\mathcal{F}^{<_t}$ is shifted, and showed that for K a simplicial complex and K_k its family of k -faces, the union $K^{<_t, A} := \cup_k (K_k)^{<_t, A}$ is a simplicial complex, provided that the order $S <_t T$ depends only on the symmetric difference $S \Delta T$.

Of special importance is exterior shifting with respect to the lexicographic order $<_L$, where $S <_L T$ if $\min S \Delta T \in S$. Another case, that was less studied, is exterior shifting w.r.t. the reverse lexicographic order $<_{RL}$ defined by $S <_{RL} T$ if $\max S \Delta T \in T$. (In [2] shifting w.r.t. $<_{RL}$ over the symmetric algebra, rather than exterior algebra, was studied.) For the graph C_5 its exterior shifting w.r.t. the lexicographic order gives

G_1 and its exterior shifting w.r.t. the reverse lexicographic order gives G_2 . It follows that C_5 is weakly isomorphic to both G_1 and G_2 . On the other hand Kalai [13, Proposition 4.2] proved that shifted families are weakly isomorphic if and only if they are equal. This explains Example 2.3.

The following relation between exterior shifting w.r.t. $<_L$ and reduced homology [3, 11] is important.

LEMMA 2.9. *For any simplicial complex Δ ,*

$$\dim \tilde{H}_i(\Delta; \mathbb{Q}) = |\{F \in \Delta^{<_L} : |F| = i + 1, \{1\} \cup F \notin \Delta\}|.$$

Proof of Proposition 2.5. From Lemma 2.9 it follows that a k -uniform hypergraph G on $[n]$ has $\tilde{H}_{k-2}(K(G), \mathbb{Q}) = 0$ iff $G^{<_L}$ contains the k -star on $[n]$ with apex 1, denoted by S , and $\tilde{H}_{k-1}(K(G), \mathbb{Q}) = 0$ iff $G^{<_L} \subseteq S$. As domination is preserved under a permutation of $[n]$, we can assume the k -star in the proposition to be S . Both parts (1) and (2) then follow, as an *initial* segment $I \subseteq \binom{[n]}{k}$ w.r.t. $<_L$ dominates (resp. is dominated by) G iff $G^{<_L} \subseteq I$ (resp. $I \subseteq G^{<_L}$). \square

2.4. COMBINATORIAL SHIFTING.

DEFINITION 2.10. *The combinatorial shifting \mathcal{F}^c of \mathcal{F} refers to any family \mathcal{G} of k -sets that can be obtained from \mathcal{F} by the following procedure:*

- (1) *Pick some $1 \leq i < j \leq n$. For all $F \in \mathcal{F}$ with $j \in F$, $i \notin F$ and $S = (F \setminus j) \cup i$ not in \mathcal{F} , replace F by S , and leave the other elements of \mathcal{F} intact. Denote this operation by Shift_{ij} .*
- (2) *Repeat (1) until a shifted family is achieved.*

Combinatorial shifting was introduced in seminal papers by Erdős, Ko and Rado [7] and Kleitman [16], see also [9]. As before, for a simplicial complex K , repeating this procedure for all faces of K (rather than of \mathcal{F}), the union $K^c := \cup_k (K_k)^c$ is a simplicial complex, with same face numbers as K .

3. TURÁN WITH SHIFTING

Let $<_t$ be a term order that satisfies $ab <_t a'b'$ whenever $a + b < a' + b'$. Note that $B(n)$ is an initial segment w.r.t. $<_t$. Recall we denote by $G^{<_t}$ the exterior shifting of G w.r.t. $<_t$, and by G^c a combinatorial shifting of G .

By the definition of exterior shifting (via a greedy choice of a new basis w.r.t. $<_t$), and as the algebraic shifting w.r.t. any term order does not change under a permutation of $[n]$, see [3, Permutation Lemma]⁽¹⁾, Theorem 1.3 is thus equivalent to the following statement.

THEOREM 3.1 (Turán with shifting). *Let G be a graph on $[n]$ where every 3-subset of $[n]$ contains an edge of G . Then $G^{<_t}$ contains $B(n)$.*

The following relation between combinatorial and algebraic shifting essentially appears in the combination of Murai [19] and Murai–Hibi [20].

For two simplicial complexes on $[n]$, Z and Y , and a term order \prec , let $Z \prec Y$ denote that the minimal element in the symmetric difference $Z_k \Delta Y_k$ w.r.t. \prec belongs to Z , for any k .

LEMMA 3.2. *For any simplicial complex Y , any term order \prec , and any combinatorial shifting, $Y^\prec \prec Y^c$.*

⁽¹⁾The proof of Björner and Kalai refers to the lex order, however it extends to any term order with no difficulty.

Proof. There is a composition $Y^c = \text{Shift}_{i_1 j_1} \circ \text{Shift}_{i_2 j_2} \circ \dots \circ \text{Shift}_{i_l j_l} Y$, with $i_s < j_s$ for all $1 \leq s \leq l$.

We show that for any simplicial complex Z on $[n]$, any $1 \leq i < j \leq n$, and any term order \prec , the following holds:

$$(3) \quad Z^\prec \prec (\text{Shift}_{ij} Z)^\prec.$$

Then, iterating (3) we obtain the desired:

$$Y^\prec \prec (\text{Shift}_{i_l j_l} Y)^\prec \prec (\text{Shift}_{i_{l-1} j_{l-1}} (\text{Shift}_{i_l j_l} Y)^\prec)^\prec \prec \dots \prec (Y^c)^\prec = Y^c,$$

where the last equality follows from the fact that $Z^\prec = Z$ for Z shifted.

Now, (3) follows from a result of Murai, rephrased here in terms of algebraic shifting, rather than of GINs: for $S \subseteq [n]$, $K \subseteq 2^{[n]}$ and term order $<$, denote $m_S^<(K) := |\{T \in K : |T| = |S|, T \leq S\}|$.

LEMMA 3.3 ([19, Proposition 2.4]). *For any simplicial complex K , any term order $<'$, any invertible $n \times n$ matrix ϕ and any $S \subseteq [n]$,*

$$m_S^{<'}(K^{<'}) \geq m_S^{<'}((K^{<', \phi})^{<'}).$$

Thus, all that is left to verify is that there exists ϕ s.t. $\text{Shift}_{ij} K = K^{<', \phi}$. Let ϕ be the linear map defined by $\phi(e_k) = e_k$ for all $k \neq j$ and $\phi(e_j) = e_i + e_j$. One can check directly that this ϕ works, as mentioned in [1, Lemma 1.10], or apply [20, Lemma 2.5(b)] to this ϕ . \square

LEMMA 3.4. *For any graph G on n vertices where every 3-subset of the vertices contains an edge of G , there exists a labeling of the vertex set by $[n] = \{1 < 2 < \dots < n\}$ and a combinatorial shifting such that G^c contains $B(n)$.*

Proof. We argue by induction on $n = |V(G)|$; the cases $n = 2, 3$ clearly hold. Let $n \geq 4$. First we define a labeling of the vertices: if G is not complete let v_n and v_{n-1} be nonadjacent, else label v_1, \dots, v_n arbitrarily (and the assertion is trivial). Repeat this rule for labeling the induced graph on $V(G) \setminus \{v_n, v_{n-1}\}$.

By induction, there is a combinatorial shifting $(G[V \setminus \{v_n, v_{n-1}\}])^c$ that contains $B(n-2)$, and let C be the result of applying this combinatorial shifting to G .

Note that any vertex v_i , $i < n-1$, is connected to at least one of v_n and v_{n-1} in G . Thus, also in C there are at least $n-2 = |B(n)| - |B(n-2)|$ edges between one of v_n, v_{n-1} and the other v_i 's, $i < n-1$. Hence, $G^c = \text{Shift}_{n-2, n}(\text{Shift}_{n-3, n}(\dots \text{Shift}_{1, n}(\text{Shift}_{n-2, n-1}(\dots \text{Shift}_{1, n-1}(C)) \dots))$ contains $B(n)$. \square

Proof of Theorem 3.1. As exterior shifting is stable under permutation of the vertices, we can relabel the vertices as needed in Lemma 3.4 without effecting the resulted shifted graph $G^{<t}$. Lemmas 3.2 and 3.4 imply that $G^{<t} <_t G^c$ for some G^c that contains $B(n)$. However, as $B(n)$ is an initial segment w.r.t. $<t$, and $|E(G^c)| = |E(G)| = |E(G^{<t})|$, also $G^{<t}$ contains $B(n)$. \square

3.1. A GENERALIZATION. Turán's theorem says that if any r vertices of a graph G span at least one edge then $|E(G)| \geq |E(T_r(n))|$ where $T_r(n)$ is the unique graph on n vertices composed as the disjoint union of r complete graphs, each of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$. Let $B_r(n) := \{ab : (r-1)a + b \leq n, a < b\}$. Let $<_r$ be the term order defined by $ab <_r a'b'$ (where $a < b$ and $a' < b'$) iff either $(r-1)a + b < (r-1)a' + b'$ or $(r-1)a + b = (r-1)a' + b'$ and $a < a'$. Then $B_r(n)$ is initial w.r.t. $<_r$.

Theorems 1.3 and 3.1 generalize from the case $r = 2$ to general r , with essentially the same proof.

THEOREM 3.5. *If G has independence number at most r (i.e., any $r + 1$ vertices span at least one edge) then $E(G^{<r})$ contains $B_r(n)$. In particular, $E(G)$ dominates $B_r(n)$.*

Proof sketch. By removing edges, and by permuting vertices, we may assume $I = \{n, n - 1, \dots, n - r + 1\}$ is an independent set of G . By induction, there exists a sequence of combinatorial shifts $\text{Shift}_{y,x}$, $y < x \leq n - r$ resulting in a graph G' such that the induced subgraph $G'[n - r]$ contains $B_r(n - r)$, and by assumption, also in G' there are at least $n - r$ edges from $[n - r]$ to I . Now perform $\text{Shift}_{1,n-r+1}$, then $\text{Shift}_{2,n-r+1} \dots$ then $\text{Shift}_{n-r,n-r+1}$, then $\text{Shift}_{1,n-r+2}$ etc. to obtain a graph G'' containing $B_r(n)$. Conclude as before. \square

REMARK 3.6. By iteratively applying [21, Theorem .1.2], on the exterior shifting of a disjoint union w.r.t. the lexicographic order $<_{lex}$, one gets

$$(T_r(n))^{<_{lex}} = B_r(n).$$

REMARK 3.7. Combining Theorem 3.5 with [14, Subsection 3.5.6] on the relation between algebraic shifting and complement, gives that if G is a graph with no complete subgraph on $r + 1$ vertices, then G is dominated by $\bar{B}_r(n) := \{ab : 1 \leq a < b \leq n, a + (r - 1)b \leq (r - 1)(n + 1)\}$.

4. TURÁN WITH INVOLUTION

Proof of Theorem 1.2. First we prove in detail the even case $n = 2m$ where τ acts with no fixed points, by induction on m . Then we indicate the modification for all n and all actions of τ .

Case where τ acts freely. As τ acts with no fixed points, w.l.o.g. assume $\tau(i) = n + 1 - i$ for all $i \in [n]$, and $n = 2m$. For the base case, one verifies the cases $n \leq 4$ by inspection.

Assume $m > 2$.

First Step. Assume there exists $i \in [n]$ such that the edge $\{i, \tau(i)\} \notin E(G)$. For any triple $T = \{i, \tau(i), j\}$, T and $\tau(T) = \{i, \tau(i), \tau(j)\}$ support at least two edges of G , thus there are at least $n - 2$ edges between $V \setminus \{i, \tau(i)\}$ and $\{i, \tau(i)\}$. By induction,

$$|E(G)| \geq n - 2 + |E(G[V \setminus \{i, \tau(i)\}])| \geq n - 2 + |E(T(n - 2))| = |E(T(n))|.$$

Thus, assume $\{i, \tau(i)\} \in E(G)$ for every $i \in [n]$.

Second Step. Consider the case where G contains an *induced* matching with two edges $M = (V_M, E_M)$, $E_M = \{i\tau(i), j\tau(j)\}$. For any vertex $k \notin V_M$ the two triples ijk and $\tau(i)\tau(j)\tau(k)$ support at least 2 edges, and the two triples $ij\tau(k), \tau(i)\tau(j)k$ support at least 2 edges, so there are at least 4 edges between V_M and $\{k, \tau(k)\}$. All together, there are at least $2(n - 4)$ edges in the cut $(V_M, V \setminus V_M)$. By induction, and as the edges $i\tau(i)$ and $j\tau(j)$ exist,

$$\begin{aligned} |E(G)| &\geq 2 + 2(n - 4) + |E(G[V \setminus V_M])| \\ &\geq (n - 2) + (n - 4) + |E(T(n - 4))| = |E(T(n))|. \end{aligned}$$

Thus, assume further that

(*) for any $j \neq i, \tau(i)$, there is a crossing edge from $\{i, \tau(i)\}$ to $\{j, \tau(j)\}$.

Third Step. We define 3 auxiliary graphs, $W = ([m], E_W)$, $Z = ([m], E_Z)$ and $R = ([m], E_R)$ as follows: $ij \in E_W/E_Z/E_R$ iff there exists, resp., exactly one/ exactly two/ at least 3 edges in G crossing from $\{i, \tau(i)\}$ to $\{j, \tau(j)\}$. Let $N_X(x)$ denote the set of neighbors of vertex x in a graph X .

LEMMA 4.1. *If $j, k \in N_W(i)$, $j \neq k$, then there exist at least 3 edges in G crossing from $\{k, \tau(k)\}$ to $\{j, \tau(j)\}$. In particular, W is triangle free.*

Proof of Lemma 4.1. By interchanging the labeling of v and $\tau(v)$ if needed, we can assume that either

- (a) $ij, \tau(i)\tau(k) \in E(G)$, or
- (b) $ij, ik \in E(G)$.

In case (a): the two triples $i\tau(j)\tau(k)$ and $\tau(i)jk$ support at least 2 edges, so $jk, \tau(j)\tau(k) \in E(G)$. Likewise, the two triples $ij\tau(k)$ and $\tau(i)\tau(j)k$ support at least 2 edges, one of them is ij , so at least one of $j\tau(k), \tau(j)k$ is in G ; in total we found 3 crossing edges from $\{k, \tau(k)\}$ to $\{j, \tau(j)\}$.

In case (b): similarly, considering the two triples $i\tau(j)\tau(k), \tau(i)jk$ implies $jk, \tau(j)\tau(k) \in E(G)$, and considering the two triples $ij\tau(k), \tau(i)\tau(j)k$ implies that at least one of $j\tau(k), \tau(j)k$ is in $E(G)$; in total we found 3 crossing edges from $\{k, \tau(k)\}$ to $\{j, \tau(j)\}$ in this case as well. \square

By definition, clearly the edge sets E_W, E_R, E_Z are pairwise disjoint, so $|E(G)| \geq m + |E(W)| + 3|E(R)| + 2|E(Z)|$. By Lemma 4.1, the graph R contains the union of cliques $\cup_{v \in [m]} K_{N_W(v)}$. Thus, the following claim, of independent interest, finishes the proof of the even case $n = 2m$ (details follow):

CLAIM 4.2. *Let X be a triangle free graph on m vertices, and define the graph $Y = Y(X) = \cup_{v \in X} K_{N_X(v)}$. Then*

$$|E(Y)| + \lfloor \frac{m}{2} \rfloor \geq |E(X)|.$$

REMARK 4.3. The Mantel–Turán theorem (phrased for the complementary graph w.r.t. Theorem 1.1) easily follows from this claim, combined with the trivial inequality $|E(X)| + |E(Y)| \leq \binom{m}{2}$. Indeed, we get $2|E(X)| - \lfloor \frac{m}{2} \rfloor \leq \binom{m}{2}$, equivalently $|E(X)| \leq \lfloor \frac{m^2}{4} \rfloor$.

While in the Mantel–Turán theorem equality holds only for the complete bipartite graph $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$, in Claim 4.2 equality holds for more graphs, e.g. for perfect matchings.

Proof of Claim 4.2. We give an easy proof by induction on m . The assertion is clear for $E(X) = \emptyset$, and for the case $m \leq 2$; assume $m \geq 3$ and $uw \in E(X)$. As X is triangle free, $N_X(u) \cap N_X(w) = \emptyset$. By the definition of Y , for any $u \neq w' \in N_X(w)$, and for any $w \neq u' \in N_X(u)$, we have $uw', u'w \in E(Y)$. Thus,

$$\begin{aligned} |E(Y)| &\geq |E(Y(X - \{u, w\}))| + |N_{X-u}(w)| + |N_{X-w}(u)| \\ &\geq |E(X - \{u, w\})| - \lfloor \frac{m-2}{2} \rfloor + |E_X(\{u, w\}, V_X - \{u, w\})| \\ &= |E(X)| - 1 - \left(\lfloor \frac{m}{2} \rfloor - 1 \right), \end{aligned}$$

as desired, where the second inequality is by the induction hypothesis, and the first “ -1 ” stands for the edge uw . $E_X(A, V - A)$ stands for edges of X in the cut from A to its complement. \square

Back to the proof of the free action case of Theorem 1.2, we get

$$|E(G)| \geq m + |E(W)| + 3|E(R)| + 2|E(Z)| \geq m + 2 \binom{m}{2} - \lfloor \frac{m}{2} \rfloor > |E(T(2m))|,$$

as desired, where we used Claim 4.2 for the second inequality.

General case. The proof is similar to the free action case; we indicate the differences, and keep the notations from the proof of the previous case. W.l.o.g. $n = 2m + l$, $\tau(i) = 2m + 1 - i$ for $i \in [2m]$, and $\tau(z) = z$ for $2m + 1 \leq z \leq n$.

First Step. Note that for any $i \in [2m]$, $z \in [2m + 1, n]$ and $T = \{i, \tau(i), z\}$, $\tau(T) = T$, so T must support an edge of G , and as in the free action case we conclude there are at least $n - 2$ edges crossing from $\{i, \tau(i)\}$ to $V(G) - \{i, \tau(i)\}$. So assume all edges $i\tau(i)$ exist in G . By the same reasoning, also assume that any two fixed points form an edge in G .

Second Step. The triples $T = ijz$ and $\tau(T) = \tau(i)\tau(j)z$, for z a fixed point, support at least 2 edges of G , both contain the vertex z , and again we conclude there are at least $2(n - 4)$ edges in G crossing from $\{i, j, \tau(i), \tau(j)\}$ to the complementary set of vertices. So assume there is at least one crossing edge from $\{i, \tau(i)\}$ to $\{j, \tau(j)\}$. By similar reasoning, we can assume there is at least one edge crossing from $\{i, \tau(i)\}$ to any pair of fixed points $\{z, z'\}$.

Third Step. Define the graphs W, R, Z on the vertex set $[m]$ as before, and define bipartite graphs W'', Z'', R'' with edges crossing from $[m]$ to $[2m + 1, n]$ as follows: for $z \in [2m + 1, n]$ and $i \in [m]$ $iz \in E(W'')/E(Z'')/E(R'')$ iff, resp., in G there are exactly 0/1/2 crossing edges from z to $\{i, \tau(i)\}$. Let $W' = W \cup W''$, and similarly define the graphs Z' and R' . For a graph H denote $e(H) := |E(H)|$. Thus,

$$e(W') + e(R') + e(Z') + \binom{l}{2} = \binom{m+l}{2}.$$

In order to lower bound the number of edges in G , we need the following analog of Lemma 4.1: for any $z \in [2m + 1, n]$ and $i, j \in [m]$, $i \neq j$, we have

- (a) if $i, j \in N_{W'}(z)$ then there exist all 4 crossing edges from $\{i, \tau(i)\}$ to $\{j, \tau(j)\}$ in G , and
- (b) if $i, z \in N_{W'}(j)$ then there exist all 2 crossing edges from $\{i, \tau(i)\}$ to z in G .

The verification of (a) and (b) is similar to the proof of Lemma 4.1.

We estimate $|E(G)|$:

$$\begin{aligned} e(G) &\geq \left(m + \binom{l}{2}\right) + (3e(R) + 2e(Z) + e(W)) + (2e(R'') + e(Z'')) \\ &\geq m + \binom{l}{2} + \left(e(W') - \left\lfloor \frac{m+l}{2} \right\rfloor\right) + 2e(R) + 2e(Z) + e(W) + e(R'') + e(Z'') \\ &= m + \binom{m+l}{2} - \left\lfloor \frac{m+l}{2} \right\rfloor + e(R) + e(Z) + e(W) \\ &= m - \left\lfloor \frac{m+l}{2} \right\rfloor + \binom{m+l}{2} + \binom{m}{2}, \end{aligned}$$

where in the first inequality the first summand accounts for edges from the first step and the other summands are by definition of W', R', Z' ; the second inequality follows from combining the second and third steps with Claim 4.2, yielding $e(R) + e(R'') \geq e(W') - \lfloor \frac{m+l}{2} \rfloor$; and the equalities are by the definition of W', R', Z' (and of W, R, Z).

Now, one verifies by a direct computation that the RHS is indeed $> e(T(2m + l))$. This completes the proof of Theorem 1.2. \square

REMARK 4.4. Unlike Theorem 1.1, where the extremal example is unique, in Theorem 1.2, for $\tau(i) = n + 1 - i$ say, there are multiple extremal examples. In fact, following the first and second steps in the proof gives a recursive way to construct

all of them, as labeled graphs. It may be of interest to characterize / count these extremal examples up to graph isomorphism.

REMARK 4.5. In support of Problem 1.4 we note that if G is a shifted graph on $[n]$ satisfying (1) w.r.t. the involution $\tau(i) = n + 1 - i$, then $E(G) \supseteq B(n)$, so $E(G)$ dominates $B(n)$. Indeed, assume by contradiction that $ab \in B(n) \setminus E(G)$, $a < b$, and consider the triple $T = \{a, b, b + 1\}$. Then T contains no edge of G , as G is shifted. As $ab \in B(n)$ $b < \tau(a)$, so $a \geq \tau(b + 1)$, thus $\tau(b + 1)\tau(a)$ is not in $E(G)$ as G is shifted. We conclude that $\tau(T)$ contains at most one edge of G , violating condition (1).

4.1. A GENERAL PROBLEM. Let Γ be a finite group acting on $[n]$, G a graph on $[n]$, and consider the condition

$$(**) |\Gamma| \leq \sum_{g \in \Gamma} e(G[g(T)]) \text{ for all } T \subseteq [n] \text{ with } |T| = 3.$$

PROBLEM 4.6. For which group actions $\Gamma \curvearrowright [n]$ does condition $(**)$ imply $e(G) \geq e(T(n))$?

Clearly this is the case for the trivial action of Γ , where $(**)$ is equivalent to the condition of Mantel–Turán theorem. Theorem 1.2 says this is the case for any action of \mathbb{Z}_2 . This is not the case for the symmetric group acting on $[n]$ by permutations, where $(**)$ is equivalent to the weaker bound $e(G) \geq \frac{n(n-1)}{6}$.

We can ask similar questions for other Turán graphs and hypergraphs problems. Some of those may be interesting on their own right and may shed light on the original problem.

5. TURÁN’S (3,4)-PROBLEM

Call a 3-uniform hypergraph on $[n]$ such that any 4 vertices span an edge a *Turán hypergraph on $[n]$* . First we recall Conjectures 1.5 and 1.6:

CONJECTURE 5.1 (Turán’s (3, 4)-conjecture). Let H be a Turán hypergraph on $[n]$. Then $|H| \geq h(n) := \lfloor \frac{n}{3} \rfloor \binom{n - \lfloor \frac{n}{3} \rfloor - 1}{2}$.

CONJECTURE 5.2. Any Turán hypergraph on $[n]$ dominates $C(n) := \{abc : a + c \leq n, 2a + b \leq n, 1 \leq a < b < c \leq n\}$.

This last conjecture can be rephrased in terms of algebraic shifting: let $<_c$ be any term order on $\binom{[n]}{3}$ in which $C(n)$ forms an initial segment. (Such $<_c$ exists, e.g. $\{a < b < c\} <_c \{a' < b' < c'\}$ if and only if either (i) $a + c < a' + c'$, or (ii) $a + c = a' + c'$ and $2a + b < 2a' + b'$, or (iii) $a + c = a' + c'$, $2a + b = 2a' + b'$ and $a < a'$.)

CONJECTURE 5.3. If H is a Turán hypergraph on $[n]$ then $C(n) \subseteq H^{<_c}$.

We now relate Turán’s Conjecture 1.5 to exterior shifting w.r.t. the lexicographic order $<_L$.

CONJECTURE 5.4. If H is a Turán hypergraph on $[n]$ then for any r , $|\{F \in H^{<_L} : F \cap [r] \neq \emptyset\}| \geq r \binom{n-1-r}{2}$.

This conjecture is equivalent to Kalai’s [12, (3)], where algebraic shifting is not mentioned. It is implied by Conjecture 1.6, by combining the definition of exterior shifting w.r.t. $<_L$ and a direct computation that shows, for any $r \in [n]$,

$$|\{F \in C(n) : F \cap [r] \neq \emptyset\}| = r \binom{n-r-1}{2}.$$

Here is another equivalent formulation of Conjecture 5.4, in terms of the compound matrix.

Let $H(k, r, n) = \{F \in \binom{[n]}{k} : F \cap [r] \neq \emptyset\}$, and $H(r, n) = H(3, r, n)$ for short. For a k -uniform hypergraph H on $[n]$ let $\text{rank}_r(H)$ be the rank of the submatrix $C_{H, H(k, r, n)}$ of the compound matrix.

CONJECTURE 5.5. *If H is a Turán hypergraph on $[n]$ then for any r , $\text{rank}_r(H) \geq r \binom{n-1-r}{2}$.*

The case $r = \lfloor \frac{n}{3} \rfloor$, where the right hand side of the inequality is maximized, implies the original Conjecture 1.5 by Turán.

The case $r = 1$ of Conjecture 5.4 holds, as mentioned in [12]. Here we provide a proof.

Proof for $r = 1$. Let $\Delta = K(H)$, namely the 2-dimensional simplicial complex consisting of H and the complete graph on $[n]$, and let $\partial_i : C_i(\Delta) \rightarrow C_{i-1}(\Delta)$ be the usual boundary map from i -chains to $(i - 1)$ -chains on Δ with \mathbb{Q} -coefficients.

By Lemma 2.9, $\dim H_1(\Delta) := \dim(\frac{\ker \partial_1}{\partial_2(C_2(\Delta))}) = |\{ab \in \Delta^{<L} : 1ab \notin \Delta^{<L}\}|$. As $\dim \ker \partial_1 = \binom{n-1}{2}$, we need to show

$$\dim H_1(\Delta) \leq \binom{n-1}{2} - \binom{n-2}{2} = n - 2.$$

As the 1-cycles $\{c(ab) := 1a + ab - 1b : 1 < a < b \leq n\}$ form a basis of the space of 1-cycles in Δ , we can choose a subset B of them that forms a basis of $H_1(\Delta; \mathbb{Q})$, and let G be the graph on $\{2, 3, \dots, n\}$ spanned by the edges ab such that $c(ab) \in B$. In order to show that $|B| \leq n - 2$ we show that

CLAIM 5.6. *G is a forest. In particular G has at most $n - 2$ edges.*

Indeed, suppose by contradiction that for some basis B , G contains a cycle, and let C be such cycle with minimal number of vertices among all choices of B . Note that for any $1 < i < j < k \leq n$,

- (a) if $ijk \in \Delta$ then $c(ij) + c(jk) - c(ik) = 0$ in $H_1(\Delta; \mathbb{Q})$, and
- (b) if $1ij \in \Delta$ then $c(ij) = 0$ in $H_1(\Delta; \mathbb{Q})$.

Thus C cannot have 3 vertices i, j, k as the subset $1ijk$ supports a triangle in Δ .

Suppose C has more than 3 vertices, and that i, j, k, l are consecutive vertices on C . Consider the subset $1ijk$. Then $1ij, 1jk \notin \Delta$ by (b), and $ijk \notin \Delta$ by (a), as otherwise we can shorten C by replacing the two edges ij, jk by the chord ik , contradicting the minimality of C . Thus $1ik \in \Delta$. Similarly, considering the subset $1jkl$ gives $1jl \in \Delta$, so $c(ik) = 0 = c(jl)$ in $H_1(\Delta; \mathbb{Q})$.

Consider the subset $ijkl$, it spans a triangle in Δ , and we show that any such triangle yields a contradiction. If $ijk \in \Delta$ then, as $c(ik) = 0$, $c(ij)$ and $c(jk)$ are homologues, a contradiction; similarly $jkl \in \Delta$ yields to contradiction. If $ikl \in \Delta$ then, as $c(ik) = 0$, $c(il)$ and $c(kl)$ are homologues, and thus replace $c(kl)$ by $c(il)$ in B to get a new basis B' with graph G' . If C has only 4 vertices then B' is dependent, a contradiction, and if C has more than 4 vertices then G' contains a shorter cycle obtained from C by deleting the edges ij, jk, kl and adding the chord il , contradicting the minimality of C . Similarly, $ijl \in \Delta$ yields to contradiction. \square

Acknowledgements. We are grateful to Sonia Balagopalan for conducting computer experiments to test Conjecture 1.6 and related conjectures. We thank Satoshi Murai and the referees for helpful comments.

REFERENCES

- [1] Annetta Aramova, Jürgen Herzog, and Takayuki Hibi, *Shifting operations and graded Betti numbers*, J. Algebr. Comb. **12** (2000), no. 3, 207–222.

- [2] Eric Babson, Isabella Novik, and Rekha Thomas, *Reverse lexicographic and lexicographic shifting*, J. Algebr. Comb. **23** (2006), no. 2, 107–123.
- [3] Anders Björner and Gil Kalai, *An extended Euler-Poincaré theorem*, Acta Math. **161** (1988), no. 3-4, 279–303.
- [4] William G. Brown, *On an open problem of Paul Turán concerning 3-graphs*, in Studies in pure mathematics, Birkhäuser, 1983, pp. 91–93.
- [5] Fan Chung and Linyuan Lu, *An upper bound for the Turán number $t_3(n, 4)$* , J. Comb. Theory, Ser. A **87** (1999), no. 2, 381–389.
- [6] D. de Caen, *The current status of Turán’s problem on hypergraphs*, in Extremal problems for finite sets (Visegrád, 1991), Bolyai Society Mathematical Studies, vol. 3, János Bolyai Mathematical Society, 1994, pp. 187–197.
- [7] Pál Erdős, Chao Ko, and Richard Rado, *Intersection theorems for systems of finite sets*, Q. J. Math., Oxf. II. Ser. **12** (1961), 313–320.
- [8] Dmitriĭ G. Fon-Der-Flaass, *A method for constructing $(3, 4)$ -graphs*, Mat. Zametki **44** (1988), no. 4, 546–550.
- [9] Peter Frankl, *The shifting technique in extremal set theory*, in Surveys in combinatorics 1987 (New Cross, 1987), London Mathematical Society Lecture Note Series, vol. 123, Cambridge University Press, 1987, pp. 81–110.
- [10] Andrew Frohmader, *More constructions for Turán’s $(3, 4)$ -conjecture*, Electron. J. Comb. **15** (2008), no. 1, article no. R137 (23 pages).
- [11] Gil Kalai, *Characterization of f -vectors of families of convex sets in \mathbf{R}^d . I. Necessity of Eckhoff’s conditions*, Isr. J. Math. **48** (1984), no. 2-3, 175–195.
- [12] ———, *A new approach to Turán’s conjecture*, Graphs Comb. **1** (1985), 107–109.
- [13] ———, *Symmetric matroids*, J. Comb. Theory, Ser. B **50** (1990), no. 1, 54–64.
- [14] ———, *Algebraic shifting*, in Computational commutative algebra and combinatorics (Osaka, 1999), Advanced Studies in Pure Mathematics, vol. 33, Mathematical Society of Japan, 2002, pp. 121–163.
- [15] Peter Keevash, *Hypergraph Turán problems*, in Surveys in combinatorics 2011, London Mathematical Society Lecture Note Series, vol. 392, Cambridge University Press, 2011, pp. 83–139.
- [16] Daniel J. Kleitman, *Maximal number of subsets of a finite set no k of which are pairwise disjoint*, J. Comb. Theory **5** (1968), 157–163.
- [17] Aleksandr V. Kostochka, *A class of constructions for Turán’s $(3, 4)$ -problem*, Combinatorica **2** (1982), no. 2, 187–192.
- [18] W. Mantel, *Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff)*, Wiskundige Opgaven **10** (1907), 60–61.
- [19] Satoshi Murai, *Generic initial ideals and exterior algebraic shifting of the join of simplicial complexes*, Ark. Mat. **45** (2007), no. 2, 327–336.
- [20] Satoshi Murai and Takayuki Hibi, *Algebraic shifting and graded Betti numbers*, Trans. Am. Math. Soc. **361** (2009), no. 4, 1853–1865.
- [21] Eran Nevo, *Algebraic shifting and basic constructions on simplicial complexes*, J. Algebr. Comb. **22** (2005), no. 4, 411–433.
- [22] Oleg Pikhurko, *The minimum size of 3-graphs without a 4-set spanning no or exactly three edges*, Eur. J. Comb. **32** (2011), no. 7, 1142–1155.
- [23] Alexander A. Razborov, *On 3-hypergraphs with forbidden 4-vertex configurations*, SIAM J. Discrete Math. **24** (2010), no. 3, 946–963.
- [24] Paul Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452.
- [25] ———, *Research problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 417–423.

GIL KALAI, Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem (Israel)
E-mail : kalai@math.huji.ac.il

ERAN NEVO, Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem (Israel)
E-mail : nevo@math.huji.ac.il