



ALGEBRAIC COMBINATORICS


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Volume 1, issue 4 (2018), p. 425-440.

http://alco.centre-mersenne.org/item/ALCO_2018__1_4_425_0

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ABSTRACT Let G be a finite group, V a complex permutation module for G over a finite G -set \mathcal{X} , and $f: V \times V \rightarrow \mathbb{C}$ a G -invariant positive semidefinite hermitian form on V . In this paper we show how to compute the radical V^\perp of f , by extending to nontransitive actions the classical combinatorial methods from the theory of association schemes. We apply this machinery to obtain a result for standard Majorana representations of the symmetric groups.

1. INTRODUCTION

A major difficulty in studying linear representations of certain finite groups, such as the large sporadic simple groups, arises when the degrees of these representations become so large that applying the general methods from linear algebra gets hard, if not practically impossible, even by machine computation. In this paper we cope with a frequent problem when dealing with the usual representation of the Monster (and many of its simple subgroups) on the Norton–Conway–Griess algebra, or, more generally, with Majorana representations of finite groups (see [7]), and can be stated as follows: given a finite group G , a complex permutation module V on a finite G -set \mathcal{X} , and a G -invariant positive semidefinite hermitian form f , determine the radical V^\perp of f from the Gram matrix Γ associated to f with respect to \mathcal{X} . In this context, the G -invariance of the form f implies strong restrictions on the Gram matrix Γ that can be exploited, via the theory of association schemes, to get a significantly more manageable situation. In fact, by [5, p. 11] or [15, §2.6 and §2.7], Γ is equivalent to a block diagonal matrix Γ' , whose blocks have sizes corresponding to the multiplicities of the irreducible $\mathbb{C}[G]$ -submodules of V , so that the decomposition of V^\perp into irreducible $\mathbb{C}[G]$ -submodules can be recovered from the ranks of the diagonal blocks of Γ' . The key step to compute the diagonal blocks of Γ' is to determine a generalised first eigenmatrix (see [3]) of the association scheme related to the action of G on \mathcal{X} . If this action is multiplicity-free (or, better, if the graph associated to this action is distance transitive), there are well established combinatorial methods (see [1] or [4]) to compute this matrix. On the other hand, if the action is not multiplicity-free, this strategy becomes much more awkward, though still possible in some cases: in [3], for example, this machinery has been extended to the case where at most one irreducible $\mathbb{C}[G]$ -submodule of the complex permutation module on \mathcal{X} has multiplicity 2 and all the others have multiplicity 1 (a more detailed description on how to deal with this

Manuscript received 3rd August 2017, revised 4th June 2018, accepted 10th June 2018.

KEYWORDS. Hermitian form, Symmetric group, Majorana representation, Monster group, Association scheme, Specht module.

$(x, y)^{S_n}$	$f(x, y)$	
$((1, 2)(3, 4), (1, 2)(3, 4))^{S_n}$	1	$n \geq 4$
$((1, 2)(3, 4), (1, 3)(2, 4))^{S_n}$	1/8	$n \geq 4$
$((1, 2)(3, 4), (1, 2)(3, 5))^{S_n}$	13/2 ⁸	$n \geq 5$
$((1, 2)(3, 4), (1, 3)(2, 5))^{S_n}$	3/2 ⁷	$n \geq 5$
$((1, 2)(3, 4), (1, 2)(5, 6))^{S_n}$	1/8	$n \geq 6$
$((1, 2)(3, 4), (1, 3)(5, 6))^{S_n}$	1/2 ⁶	$n \geq 6$
$((1, 2)(3, 4), (1, 5)(2, 6))^{S_n}$	1/2 ⁶	$n \geq 6$
$((1, 2)(3, 4), (1, 5)(3, 6))^{S_n}$	13/2 ⁸	$n \geq 6$
$((1, 2)(3, 4), (1, 5)(6, 7))^{S_n}$	5/2 ⁸	$n \geq 7$
$((1, 2)(3, 4), (5, 6)(7, 8))^{S_n}$	0	$n \geq 8$
$(\{1, 2, 3\}, \{1, 2, 3\})^{S_n}$	2 ³ /5	$n \geq 3$
$(\{1, 2, 3\}, \{1, 2, 4\})^{S_n}$	2 ³ · 17/(3 ⁴ · 5)	$n \geq 4$
$(\{1, 2, 3\}, \{1, 4, 5\})^{S_n}$	2 ⁴ /(3 ⁴ · 5)	$n \geq 5$
$(\{1, 2, 3\}, \{4, 5, 6\})^{S_n}$	0	$n \geq 6$
$(\{1, 2, 3\}, (1, 2)(3, 4))^{S_n}$	1/3 ²	$n \geq 4$
$(\{1, 2, 5\}, (1, 2)(3, 4))^{S_n}$	1/4	$n \geq 5$
$(\{1, 3, 5\}, (1, 2)(3, 4))^{S_n}$	1/(2 · 3 ²)	$n \geq 5$
$(\{1, 5, 6\}, (1, 2)(3, 4))^{S_n}$	1/(2 ² · 3 ²)	$n \geq 6$
$(\{5, 6, 7\}, (1, 2)(3, 4))^{S_n}$	0	$n \geq 7$

TABLE 1. The relevant values of the form f

case will be given in Section 3). We'll show in the sequel how, in the case of nontransitive actions (which are definitely not multiplicity-free), V^\perp can be determined from the generalised first eigenmatrices of the association schemes related to the actions induced by G on the G -orbits of \mathcal{X} . As an application, we prove the following result:

THEOREM 1.1. *Let n be a positive integer with $4 \leq n$, S_n the symmetric group on $\{1, \dots, n\}$, \mathcal{T} the set of the permutations of S_n of type $(2, 2)$, on which S_n acts via conjugation, \mathcal{U} the set of 3-subsets of $\{1, \dots, n\}$, on which S_n acts in the natural way, \mathcal{X} the union (as S_n -sets) of \mathcal{T} with \mathcal{U} , V the complex permutation module of S_n on \mathcal{X} , and f the S_n -invariant hermitian form on V defined as in Table 1. Then, denoting by S^λ the Specht module associated to a partition λ of $\{1, \dots, n\}$, we have*

- (1) f is positive semidefinite if and only if $n \leq 12$;
- (2) if $n = 12$, then $V^\perp \cong 1 \oplus S^{(11,1)} \oplus 2S^{(10,2)} \oplus S^{(9,3)} \oplus S^{(9,2,1)}$;
- (3) if $n = 11$, then $V^\perp \cong S^{(2,9)}$;
- (4) if $10 \geq n \geq 8$, then $V^\perp = \{0\}$.

We remark that under the hypothesis of Theorem 1.1, $S^{(n-2,2)}$ has multiplicity 3 in V and the homogeneous component $V(S^{(n-2,2)})$ splits into the orthogonal direct sum of its intersections with the linear spans T and U of \mathcal{T} and \mathcal{U} , respectively. In U the module $S^{(n-2,2)}$ has multiplicity 1 and, as we shall see, $V(S^{(n-2,2)}) \cap T$ splits as the direct sum of an irreducible submodule, canonically associated to the Johnson scheme $J(n, 4)$, and its orthogonal complement with respect to a natural S_n -invariant hermitian form κ defined in Section 3. The choices of \mathcal{X} and f in Theorem 1.1 arise from the theory of Majorana representations; in particular, the form f is the one induced on V by a standard Majorana representation of S_n and Theorem 1.1 is

needed to determine the subalgebra generated by the Majorana axes associated to this representation. When $n < 8$, this subalgebra has been determined by Ivanov, Seress, Pasechnik, and Shpectorov in [8], [9], [10]. Since the Specht modules are defined over \mathbb{Q} and they are absolutely irreducible [11, Theorem 4.12], restricting the scalars to \mathbb{R} , we obtain immediately from Theorem 1.1 the following result about Majorana representations of the symmetric groups:

THEOREM 1.2. *Let n be an integer greater than 7 and S_n the symmetric group on $\{1, \dots, n\}$. Let $(S_n, \mathcal{T}, W, \phi, \psi)$ be a standard Majorana representation of S_n , Y and Z be the subspaces of W generated by the axial vectors associated to the bitranspositions and the 3-cycles of S_n respectively. Then*

- (1) $n \leq 12$;
- (2) if $n = 12$, then $Z \leq Y$, and Y decomposes into irreducible $\mathbb{R}[S_n]$ -submodules as follows

$$1 \oplus S^{(11,1)} \oplus S^{(10,2)} \oplus S^{(9,3)} \oplus S^{(8,4)} \oplus S^{(8,2,2)};$$

- (3) if $n = 11$, then $Y \cap Z \cong S^{(9,2)}$, and $Y + Z$ decomposes into irreducible $\mathbb{R}[S_n]$ -submodules as follows

$$1 \oplus 1 \oplus 2S^{(10,1)} \oplus 2S^{(9,2)} \oplus 2S^{(8,3)} \oplus S^{(7,4)} \oplus S^{(8,2,1)} \oplus S^{(7,2,2)};$$

- (4) if $n \in \{8, 9, 10\}$, $Y \cap Z = \{0\}$ and $Y + Z$ decomposes into irreducible $\mathbb{R}[S_n]$ -submodules as follows

$$1 \oplus 1 \oplus 2S^{(n-1,1)} \oplus 3S^{(n-2,2)} \oplus 2S^{(n-3,3)} \oplus S^{(n-4,4)} \oplus S^{(n-3,2,1)} \oplus S^{(n-4,2,2)}.$$

Note that, for $n = 12$, the inclusion $Z \leq Y$ in Theorem 1.2 was proved, with different methods, by Castillo-Ramirez and Ivanov in [2].

2. STRATEGY

Let G be a finite group acting on a finite set $\mathcal{X} := \{x_1, \dots, x_m\}$, V the complex permutation module of G on \mathcal{X} , and f a G -invariant hermitian form on V . Let Γ be the Gram matrix associated to f with respect to the basis (x_1, \dots, x_m) . As stated in the introduction, for each irreducible $\mathbb{C}[G]$ -submodule of V , we want to determine its multiplicity in V^\perp in terms of the matrix Γ . For the remainder of this paper all modules are $\mathbb{C}[G]$ -modules. For an irreducible submodule S of a module N , denote by $N(S)$ the S -homogeneous component of N (i.e. the submodule of N generated by all submodules of N isomorphic to S) and by $m_N(S)$ the multiplicity of S in N . By Maschke's Theorem, V is a completely reducible module, so that V is the direct sum of its homogeneous components.

LEMMA 2.1. *Each two distinct homogeneous components of V are orthogonal to each other.*

Proof. Since f is positive semidefinite and G -invariant, each submodule of V has a G -invariant orthogonal complement. So V decomposes as an orthogonal direct sum of irreducible submodules and the result follows by Schur's Lemma. □

COROLLARY 2.2. *Let S be an irreducible submodule of V and assume $V(S)$ is contained in the linear span W of a G -orbit \mathcal{O} of G in \mathcal{X} . Then the multiplicity of S in V^\perp is equal to the multiplicity of S in $W \cap W^\perp$.*

Proof. By Lemma 2.1, $V(S)$ is orthogonal to the linear span of every G -orbit different from \mathcal{O} , whence $V(S) \cap V^\perp = V(S) \cap W^\perp$. □

LEMMA 2.3. *Let N and S be finite dimensional modules with S irreducible. Then, for every subgroup H of G , we have*

$$\dim_{\mathbb{C}}(C_{N(S)}(H)) = m_N(S) \cdot \dim_{\mathbb{C}}(C_S(H))$$

(where, as usual, $C_{N(S)}(H)$ denotes the centraliser of H in $N(S)$, i.e. the set of the vectors in $N(S)$ fixed by H).

Proof. This is an immediate consequence of the complete reducibility of the involved modules. □

COROLLARY 2.4. *Assume f is positive semidefinite. Let S be an irreducible $\mathbb{C}[G]$ -submodule of V , let H be a subgroup of G , let C be the centraliser of H in $V(S)$, and let $\Gamma_{(S,H)}$ be the Gram matrix associated to $f|_{C \times C}$ with respect to a basis of C . Then,*

$$m_{V^\perp}(S) \cdot \dim_{\mathbb{C}}(C_S(H)) = \text{corank}(\Gamma_{(S,H)}).$$

Proof. The corank of $\Gamma_{(S,H)}$ is equal to the dimension of $(C \cap C^\perp)$ and, since f is positive semidefinite, $(C \cap V^\perp) = (C \cap C^\perp)$, so the result follows from Lemma 2.3, with $N = V^\perp$. □

The idea is now to choose H in such a way that $\dim_{\mathbb{C}}(C_S(H))$ is as small as possible. In particular, if, as in the case we are interested in, $\dim_{\mathbb{C}}(C_S(H)) = 1$ then, for every decomposition

$$V(S) = \bigoplus_{i=1}^{m_V(S)} S_i$$

into irreducible submodules S_i , each one isomorphic with S , we can get a basis

$$\mathcal{C}_S := (s_1, \dots, s_{m_V(S)})$$

for C by choosing, for every $i \in \{1, \dots, m_V(S)\}$, a nontrivial vector s_i in $C_{S_i}(H)$. A way to obtain the vectors s_i is to take the images $t_i^{\pi_i}$ of suitable H -invariant vectors t_i of V via the projection π_i of V onto S_i associated to a decomposition of V into irreducible submodules that involves S_i . The expression of the s_i 's as linear combinations of the vectors in \mathcal{X} , can be obtained from any generalised first eigenmatrix of the association scheme related to the action of G on \mathcal{X} (see [5, § 3]). As already mentioned, if the action of G on \mathcal{X} is not multiplicity-free, there are no standard methods to compute a generalised first eigenmatrix. If the action is not transitive, one might hope to get to a simpler situation by restricting the action to each orbit and decomposing $V(S)$ into the direct sum of its intersections with the linear spans of the G -orbits in \mathcal{X} . Let

$$\mathcal{X}_1, \dots, \mathcal{X}_r$$

be the distinct G -orbits of \mathcal{X} and let R_1, \dots, R_p be representatives of the isomorphism classes of the irreducible submodules of V . For $j \in \{1, \dots, r\}$ let V_j be the linear span of \mathcal{X}_j and, for $S \in \{R_1, \dots, R_p\}$, let $V_j(S) := V(S) \cap V_j$, so that V decomposes as follows:

$$(1) \quad \begin{array}{ccccccc} V & = & V_1(R_1) \oplus & V_2(R_1) \oplus & \cdots & \oplus & V_r(R_1) \oplus \\ & & V_1(R_2) \oplus & V_2(R_2) \oplus & \cdots & \oplus & V_r(R_2) \oplus \\ & & \vdots & \vdots & \ddots & & \vdots \\ & & V_1(R_p) \oplus & V_2(R_p) \oplus & \cdots & \oplus & V_r(R_p) \end{array}$$

Clearly, for $h \in \{1, \dots, p\}$, the row sums are the homogeneous components $V(R_h)$ and, for $j \in \{1, \dots, r\}$, the column sums are the submodules V_j 's. For $j \in \{1, \dots, r\}$ let $\mathcal{O}_{j1}, \dots, \mathcal{O}_{js_j}$ be the orbitals of G on \mathcal{X}_j and let $P^{\mathcal{X}_j}$ be a generalised first eigenmatrix associated to the action of G on \mathcal{X}_j (see [3, §2]). Recall that the rows (resp. the columns) of $P^{\mathcal{X}_j}$ are in one to one correspondence with isomorphism classes of the

irreducible submodules of V_j (resp. the orbitals of G on \mathcal{X}_j) and the entries of $P^{\mathcal{X}_j}$ are square matrices. If P_{hk}^j is the hk -entry of $P^{\mathcal{X}_j}$ corresponding to the irreducible submodule R_k and the orbital \mathcal{O}_{jh} , the il -entry of P_{hk}^j will be denoted by $(P_{hk}^j)_{il}$. For each such entry define

$$q_{kh}^{il}(\mathcal{X}_j) := \frac{\overline{(P_{hk}^j)_{il}}}{|\mathcal{O}_{jh}|} \dim_{\mathbb{C}}(R_k),$$

where $\overline{P_{hk}^j}$ is the complex conjugate of the matrix P_{hk}^j . Let Q_{kh}^j be the matrix whose il -entry is $q_{kh}^{il}(\mathcal{X}_j)$ and let $Q^{\mathcal{X}_j}$ be the matrix whose kh -entry is Q_{kh}^j . We shall call $Q^{\mathcal{X}_j}$ a *generalised second eigenmatrix* associated to the action of G on \mathcal{X}_j .

LEMMA 2.5. *With the above notation, there exists a decomposition of $V_j(R_k)$ as a direct sum of irreducible submodules S_1, \dots, S_l (isomorphic to R_k) such that, for every $x \in \mathcal{X}_j$, the projection map $\pi_i : V \rightarrow S_i$ maps*

$$(2) \quad x \mapsto \sum_{h=1}^{s_j} q_{kh}^{ii}(\mathcal{X}_j) \sum_{y \in \Delta_{jh}(x)} y,$$

where $\Delta_{jh}(x) := \{y \in \mathcal{X}_j \mid (x, y) \in \mathcal{O}_{jh}\}$.

Proof. Since the action of G on \mathcal{X}_j is transitive, the result follows from [3, Equation (13) and Lemma 1 (i)]. \square

Note that in case of a multiplicity-free action $P^{\mathcal{X}_j}$ and $Q^{\mathcal{X}_j}$ are, respectively, the usual first and second eigenmatrices (see [1, p. 60]) and are uniquely determined for fixed orders of the orbitals and of the isomorphism classes of the irreducible submodules. On the other hand, if the action is not multiplicity free, the generalised first eigenmatrix $P^{\mathcal{X}_j}$ depends on the chosen decomposition into irreducibles of $V_j(R_i)$, for each $i \in \{1, \dots, p\}$.

3. TRANSITIVE NON MULTIPLICITY-FREE ACTIONS

In this section, we briefly describe how the relevant information on a generalised first eigenmatrix $P^{\mathcal{X}_j}$ (i.e. the diagonal entries $(P_{hk}^j)_{ii}$ of each block P_{hk}^j) can be obtained in the case where G acts transitively on \mathcal{X}_j and all irreducible submodules of the linear span V_j of \mathcal{X}_j have multiplicity less or equal 2 and only one, which we can assume to be R_p , has multiplicity 2. The entries of P relative to an irreducible submodule of multiplicity 1 can be computed inside the submodule itself using Lemma 3 in [3]. To compute the diagonal entries of the 2×2 blocks corresponding to R_p , we proceed as follows. Let

$$\kappa : V_j \times V_j \rightarrow \mathbb{C}$$

be the unique nondegenerate hermitian form on V_j such that the elements of \mathcal{X}_j are mutually orthogonal vectors of norm 1. Find a G -set \mathcal{Y} and a surjective homomorphism of G -sets

$$\theta : \mathcal{X}_j \rightarrow \mathcal{Y},$$

such that, if M is the complex permutation module for G on \mathcal{Y} and $\bar{\theta} : V_j \rightarrow M$ is the $\mathbb{C}[G]$ -homomorphism induced by θ , the following conditions hold:

- (1) M is multiplicity-free;
- (2) the first eigenmatrix associated to the action of G on \mathcal{Y} is known;
- (3) M contains a submodule isomorphic to R_p

(e.g. in [3], as in Section 4 of this paper, \mathcal{X}_j is the set \mathcal{T} , \mathcal{Y} is the set of 4-subsets of $\{1, \dots, n\}$, θ is the map that sends a permutation of type $(2, 2)$ to its support, and M is the Young module $M^{(n-4,4)}$, corresponding to the Johnson scheme $J(n, 4)$). Let I be the orthogonal complement to $\ker(\theta)$ in V_j with respect to the form κ . Since V_j^θ contains a submodule isomorphic to R_p , and M is multiplicity-free, both I and $\ker(\theta)$ contain a copy of R_p . Denote by R_I and R_K the copies of R_p contained in I and $\ker(\theta)$ respectively. Then $V_j(R_p)$ decomposes as the orthogonal (with respect to κ) direct sum of the submodules R_I and R_K and we get one of the diagonal entries (which we may assume to be the $(1, 1)$ entry), given by the following formula (which is an obvious generalisation of Lemma 7 in [3] and is proved in the same way):

$$(P_{kp}^j)_{11} = c_{lp} \frac{|\mathcal{O}_{jk}^\theta||\mathcal{Y}|}{|\overline{\mathcal{O}}_l||\mathcal{X}_j|}$$

where $\overline{\mathcal{O}}_l$ is the orbital of G on \mathcal{Y} containing \mathcal{O}_{jk}^θ and c_{lp} is the entry of the first eigenmatrix of G on \mathcal{Y} corresponding to the irreducible module R_p and to the orbital $\overline{\mathcal{O}}_l$. Finally, for each column of P , the last missing “diagonal” entry $(P_{kp}^j)_{22}$ can be computed using the Second Generalised Orthogonality Relation [3, Lemma 1 (iii)].

4. PROOF OF THEOREM 1.1

From now on let $n, G, \mathcal{T}, \mathcal{U}, \mathcal{X}, V$, and f be as in Theorem 1.1. Let T be the linear span of \mathcal{T} and U the linear span of \mathcal{U} , so that $V = T \oplus U$.

LEMMA 4.1. *With the above notation, f is positive semidefinite if and only if $n \leq 12$.*

Proof. By [3, Table 13], T has an irreducible submodule isomorphic with $S^{(n-3,2,2)}$ that contains a nonzero vector v such that

$$f(v, v) = \frac{15}{128}(12 - n),$$

which is clearly negative for $n > 12$. Conversely, if $n \leq 12$, by [13], we may assume S_n to be a subgroup of the Monster such that the bitranspositions of S_n are involutions of type $2A$ in the Monster. Moreover, by [14], there is an S_n -invariant subset \mathcal{Y} of the (complex) Conway–Norton–Griess algebra \mathcal{G} , such that \mathcal{Y} is S_n -isomorphic to \mathcal{X} and the Gram matrix of the hermitian form $(\cdot, \cdot)_{\mathcal{G}}$ of \mathcal{G} with respect to \mathcal{Y} is the same as Γ . Since $(\cdot, \cdot)_{\mathcal{G}}$ is positive definite, it follows that f is positive semidefinite on V . \square

4.1. THE ACTIONS OF S_n ON \mathcal{T} AND \mathcal{U} . Denote the 10 orbitals

$$\mathcal{O}_1^{\mathcal{T}}, \dots, \mathcal{O}_{10}^{\mathcal{T}}$$

of S_n on \mathcal{T} and the 4 orbitals

$$\mathcal{O}_1^{\mathcal{U}}, \dots, \mathcal{O}_4^{\mathcal{U}}$$

of S_n on \mathcal{U} as in Table 2. For $\mathcal{R} \in \{\mathcal{T}, \mathcal{U}\}$ and $x \in \mathcal{X} \cap \mathcal{R}$, let

$$\Delta_k^{\mathcal{R}}(x) := \{y \in \mathcal{X} \cap \mathcal{R} \mid (x, y) \in \mathcal{O}_k^{\mathcal{R}}\},$$

where k ranges from 1 to 10, if $\mathcal{R} = \mathcal{T}$, and from 1 to 4, if $\mathcal{R} = \mathcal{U}$.

LEMMA 4.2. *Let T be as above, then*

(1) *T decomposes into irreducible submodules as follows:*

$$T = T_{1,1} \oplus T_{2,1} \oplus T_{3,1} \oplus T_{4,1} \oplus T_{4,2} \oplus T_{5,1} \oplus T_{6,1} \oplus T_{7,1}$$

where $T_{1,1}$ is the trivial module, $T_{2,1} \cong S^{(n-1,1)}$, $T_{3,1} \cong S^{(n-3,3)}$, $T_{4,1} \cong T_{4,2} \cong S^{(n-2,2)}$, $T_{5,1} \cong S^{(n-4,4)}$, $T_{6,1} \cong S^{(n-3,2,1)}$, and $T_{7,1} \cong S^{(n-4,2,2)}$.

$\mathcal{O}_1^{\mathcal{T}} := ((1, 2)(3, 4), (1, 2)(3, 4))^{S_n}$	$\mathcal{O}_1^{\mathcal{U}} := (\{1, 2, 3\}, \{1, 2, 3\})^{S_n}$
$\mathcal{O}_2^{\mathcal{T}} := ((1, 2)(3, 4), (1, 3)(2, 4))^{S_n}$	
$\mathcal{O}_3^{\mathcal{T}} := ((1, 2)(3, 4), (1, 2)(3, 5))^{S_n}$	$\mathcal{O}_2^{\mathcal{U}} := (\{1, 2, 3\}, \{1, 2, 4\})^{S_n}$
$\mathcal{O}_4^{\mathcal{T}} := ((1, 2)(3, 4), (1, 3)(2, 5))^{S_n}$	
$\mathcal{O}_5^{\mathcal{T}} := ((1, 2)(3, 4), (1, 2)(5, 6))^{S_n}$	$\mathcal{O}_3^{\mathcal{U}} := (\{1, 2, 3\}, \{1, 4, 5\})^{S_n}$
$\mathcal{O}_6^{\mathcal{T}} := ((1, 2)(3, 4), (1, 3)(5, 6))^{S_n}$	
$\mathcal{O}_7^{\mathcal{T}} := ((1, 2)(3, 4), (1, 5)(2, 6))^{S_n}$	$\mathcal{O}_4^{\mathcal{U}} := (\{1, 2, 3\}, \{4, 5, 6\})^{S_n}$
$\mathcal{O}_8^{\mathcal{T}} := ((1, 2)(3, 4), (1, 5)(3, 6))^{S_n}$	
$\mathcal{O}_9^{\mathcal{T}} := ((1, 2)(3, 4), (1, 5)(6, 7))^{S_n}$	
$\mathcal{O}_{10}^{\mathcal{T}} := ((1, 2)(3, 4), (5, 6)(7, 8))^{S_n}$	

TABLE 2. Orbitals of S_n on \mathcal{T} and on \mathcal{U}

- (2) We can choose the $T_{h,i}$'s in such a way that the images of the vectors of the basis \mathcal{X} under the projection maps $\pi_{hi}^{\mathcal{T}} : T \rightarrow T_{h,i}$ are given by the following formula:

$$x^{\pi_{hi}^{\mathcal{T}}} = \sum_{k=1}^{10} q_{hk}^{ii}(\mathcal{T}) \sum_{y \in \Delta_k^{\mathcal{T}}(x)} y.$$

- (3) For $h \in \{1, \dots, 4\}$, $q_{hk}^{ii}(\mathcal{T})$ is the entry of Table 3 corresponding to the pair $(\mathcal{O}_k^{\mathcal{T}}, T_{h,i})$.

Proof. The decomposition into irreducible submodules follows from [3, Lemma 6] and the remaining assertions follow from Lemma 2.5 and [3, Tables 8 and 9]. \square

LEMMA 4.3. The first eigenmatrix associated to the action of S_n on the set \mathcal{U} is displayed in Table 4.

Proof. This follows by routine computation using, e.g., the formulas in [1, Corollary to Theorem 2.9, pp. 219-220]. \square

LEMMA 4.4. Let U be as above, then

- (1) U decomposes into irreducible submodules as follows:

$$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

where U_1 is the trivial module, $U_2 \cong S^{(n-1,1)}$, $U_3 \cong S^{(n-3,3)}$, and $U_4 \cong S^{(n-2,2)}$.

- (2) We can choose the U_i 's in such a way that the images of the vectors of the basis \mathcal{X} under the projection maps $\pi_i^{\mathcal{U}} : U \rightarrow U_i$ are given by the following formula:

(3)
$$x^{\pi_i^{\mathcal{U}}} = \sum_{k=1}^4 q_{ik}^{11}(\mathcal{U}) \sum_{y \in \Delta_k^{\mathcal{U}}(x)} y.$$

- (3) For $i \in \{1, \dots, 4\}$, $q_{ik}^{11}(\mathcal{U})$ is the entry of Table 5 corresponding to the pair $(\mathcal{O}_k^{\mathcal{U}}, U_i)$.

Proof. The decomposition into irreducible submodules follows by a standard argument in representation theory of the symmetric group (see [11]). The remaining assertions follow by Lemma 2.5 and Lemma 4.3. \square

	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{4,1}$	$T_{4,2}$
\mathcal{O}_1^T	1	$n - 1$	$\frac{n(n-1)(n-5)}{6}$	$\frac{n(n-3)}{2}$	$\frac{n(n-3)}{2}$
\mathcal{O}_2^T	1	$n - 1$	$\frac{n(n-1)(n-5)}{6}$	$\frac{-n(n-3)}{4}$	$\frac{n(n-3)}{2}$
\mathcal{O}_3^T	1	$\frac{(3n-16)(n-1)}{4(n-4)}$	$\frac{n(n-1)(n-5)(n-10)}{24(n-4)}$	$\frac{n(n-3)}{4}$	$\frac{n(n-3)(n-7)}{4(n-4)}$
\mathcal{O}_4^T	1	$\frac{(3n-16)(n-1)}{4(n-4)}$	$\frac{n(n-1)(n-5)(n-10)}{24(n-4)}$	$\frac{-n(n-3)}{8}$	$\frac{n(n-3)(n-7)}{4(n-4)}$
\mathcal{O}_5^T	1	$\frac{(n-1)(n-8)}{2(n-4)}$	$\frac{-n(n-1)(n-8)}{6(n-4)}$	$\frac{n(n-3)}{6}$	$\frac{n(n-3)(n^2-21n+92)}{12(n-4)(n-5)}$
\mathcal{O}_6^T	1	$\frac{(n-1)(n-8)}{2(n-4)}$	$\frac{-n(n-1)(n-8)}{6(n-4)}$	$\frac{-n(n-3)}{12}$	$\frac{n(n-3)(n^2-21n+92)}{12(n-4)(n-5)}$
\mathcal{O}_7^T	1	$\frac{(n-1)(n-8)}{2(n-4)}$	$\frac{-n(n-1)(n-8)}{6(n-4)}$	$\frac{-n(n-3)}{12}$	$\frac{n(n-3)(n^2-21n+92)}{12(n-4)(n-5)}$
\mathcal{O}_8^T	1	$\frac{(n-1)(n-8)}{2(n-4)}$	$\frac{-n(n-1)(n-8)}{6(n-4)}$	$\frac{n(n-3)}{24}$	$\frac{n(n-3)(n^2-21n+92)}{12(n-4)(n-5)}$
\mathcal{O}_9^T	1	$\frac{(n-1)(n-16)}{4(n-4)}$	$\frac{n(n-1)(3n-22)}{4(n-4)(n-6)}$	0	$\frac{-3n(n-3)(n-9)}{4(n-4)(n-5)}$
\mathcal{O}_{10}^T	1	$\frac{-4(n-1)}{(n-4)}$	$\frac{-4n(n-1)}{(n-4)(n-6)}$	0	$\frac{6n(n-3)}{(n-4)(n-5)}$

TABLE 3. The coefficients $q_{hk}^{ii}(\mathcal{T})$

1	$3(n-3)$	$\frac{3}{2}(n-3)(n-4)$	$\frac{(n-3)(n-4)(n-5)}{6}$
1	$2n-9$	$\frac{(n-4)(n-9)}{2}$	$-\frac{1}{2}(n-4)(n-5)$
1	$n-7$	$-2n+11$	$n-5$
1	-3	3	-1

TABLE 4. The first eigenmatrix $P^{\mathcal{U}}$

	U_1	U_2	U_3	U_4
$\mathcal{O}_1^{\mathcal{U}}$	1	$n - 1$	$\frac{n(n-3)}{2}$	$\frac{n(n-1)(n-5)}{6}$
$\mathcal{O}_2^{\mathcal{U}}$	1	$\frac{(2n-9)(n-1)}{3(n-3)}$	$\frac{n(n-7)}{6}$	$\frac{-n(n-1)(n-5)}{6(n-3)}$
$\mathcal{O}_3^{\mathcal{U}}$	1	$\frac{(n-1)(n-9)}{3(n-3)}$	$\frac{n(-2n+11)}{3(n-4)}$	$\frac{n(n-1)(n-5)}{3(n-3)(n-4)}$
$\mathcal{O}_4^{\mathcal{U}}$	1	$\frac{-3(n-1)}{n-3}$	$\frac{3n}{n-4}$	$\frac{-n(n-1)}{(n-3)(n-4)}$

TABLE 5. The second eigenmatrix $Q^{\mathcal{U}}$

COROLLARY 4.5. A set of representatives for the irreducible $\mathbb{C}[S_n]$ -submodules of V is given by

$$\mathcal{S} := \{1, S^{(n-1,1)}, S^{(n-2,2)}, S^{(n-3,3)}, S^{(n-4,4)}, S^{(n-3,2,1)}, S^{(n-4,2,2)}\}.$$

Moreover

- (1) $S^{(n-2,2)}$ has multiplicity 3 in V ,
- (2) the trivial module, $S^{(n-1,1)}$, and $S^{(n-3,3)}$ have multiplicity 2,
- (3) $S^{(n-4,4)}, S^{(n-3,2,1)}, S^{(n-4,2,2)}$ have multiplicity 1 and appear only as submodules of T .

Proof. This follows from Lemma 4.2(1) and Lemma 4.4(1). □

4.2. SUBMODULES OF MULTIPLICITY 3. In this subsection we assume $S = S^{(n-2,2)}$. We apply the method described in Section 2: let

$$H_1 \text{ be the stabiliser in } S_n \text{ of the set } \{1, 2, 3, 4\},$$

and denote the H_1 -orbits of the bitranspositions as follows:

$$(4) \quad \begin{aligned} \mathcal{P}_1 &:= ((1, 2)(3, 4))^{H_1}, & \mathcal{P}_2 &:= ((1, 2)(3, 5))^{H_1}, & \mathcal{P}_3 &:= ((1, 2)(5, 6))^{H_1}, \\ \mathcal{P}_4 &:= ((1, 5)(2, 6))^{H_1}, & \mathcal{P}_5 &:= ((1, 5)(6, 7))^{H_1}, & \mathcal{P}_6 &:= ((5, 6)(7, 8))^{H_1}. \end{aligned}$$

LEMMA 4.6. *The complex permutation module Z of S_n on the set of its 4-subsets decomposes into irreducible submodules as follows*

$$Z = 1 \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-3,3)} \oplus S^{(n-4,4)}.$$

Proof. This follows by standard representation theory of the symmetric groups (see [11]). \square

LEMMA 4.7. $C_S(H_1)$ has dimension 1 over \mathbb{C} .

Proof. The result follows from Lemma 4.6, since, by Frobenius Reciprocity [6, Lemma 5.2], $\dim_{\mathbb{C}} C_S(H_1)$ is equal to the multiplicity of S in the permutation module of S_n on the set of 4-subsets of $\{1, \dots, n\}$. \square

Let

$$u := \{1, 2, 3\} + \{1, 2, 4\} + \{1, 3, 4\} + \{2, 3, 4\}$$

and

$$s := \sum_{v \in \mathcal{P}_3} v.$$

Clearly both u and s are H_1 -invariant. We determine the projections $s^{\pi_{41}^T}$, $s^{\pi_{42}^T}$, and $u^{\pi_4^U}$ on $T_{4,1}$, $T_{4,2}$, and U_4 , respectively, and show that

$$(s^{\pi_{41}^T}, s^{\pi_{42}^T}, u^{\pi_4^U})$$

is the basis \mathcal{C}_S for $C_{V(S^{(n-2,2)})(H_1)}$ as in Section 2.

LEMMA 4.8. *For $(h, i) \in \{(1, 1), (2, 1), (3, 1), (4, 1), (4, 2)\}$, we have $s^{\pi_{hi}^T} \in C_{T_{h,i}}(H_1)$, and*

$$(5) \quad s^{\pi_{hi}^T} = \sum_{l=1}^6 s_{hi}^l \sum_{v \in \mathcal{P}_l} v$$

where

$$\begin{aligned} s_{hi}^1 &:= (n-4)(n-5)(q_{h5}^{ii}(\mathcal{T}) + 2q_{h7}^{ii}(\mathcal{T})), \\ s_{hi}^2 &:= \frac{1}{2}(n-5) [2q_{h3}^{ii}(\mathcal{T}) + 4q_{h4}^{ii}(\mathcal{T}) + (n-6)q_{h5}^{ii}(\mathcal{T}) \\ &\quad + 2q_{h6}^{ii}(\mathcal{T}) + 2(n-6)q_{h7}^{ii}(\mathcal{T}) + 4q_{h8}^{ii}(\mathcal{T}) + 3(n-6)q_{h9}^{ii}(\mathcal{T})], \\ s_{hi}^3 &:= q_{1h}^{ii}(\mathcal{T}) + 2(n-4)q_{h3}^{ii}(\mathcal{T}) + \frac{(n-6)(n-7) + 2}{2}q_{h5}^{ii}(\mathcal{T}) + 8(n-6)q_{h8}^{ii}(\mathcal{T}) \\ &\quad + 2(n-6)^2q_{h9}^{ii}(\mathcal{T}) + \frac{(n-6)(n-7)}{2}q_{h10}^{ii}(\mathcal{T}), \\ s_{hi}^4 &:= q_{h2}^{ii}(\mathcal{T}) + 2(n-4)q_{h4}^{ii}(\mathcal{T}) + 4(n-6)q_{h6}^{ii}(\mathcal{T}) + \frac{(n-6)(n-7) + 2}{2}q_{h7}^{ii}(\mathcal{T}) \\ &\quad + 4(n-6)q_{h8}^{ii}(\mathcal{T}) + 2(n-6)^2q_{h9}^{ii}(\mathcal{T}) + \frac{(n-6)(n-7)}{2}q_{h10}^{ii}(\mathcal{T}), \\ s_{hi}^5 &:= 3 [q_{h3}^{ii}(\mathcal{T}) + 2q_{h4}^{ii}(\mathcal{T}) + q_{h5}^{ii}(\mathcal{T}) + (n-7)q_{h6}^{ii}(\mathcal{T}) + 2q_{h7}^{ii}(\mathcal{T}) + 2(n-7)q_{h8}^{ii}(\mathcal{T})] \\ &\quad + 3 \left[\frac{(n-7)(n-2)}{2}q_{h9}^{ii}(\mathcal{T}) + \frac{(n-7)(n-8)}{2}q_{h10}^{ii}(\mathcal{T}) \right], \end{aligned}$$

$$s_{hi}^6 := 6 \left[2q_{h5}^{ii}(\mathcal{T}) + 4q_{h7}^{ii}(\mathcal{T}) + 4(n-8)q_{h9}^{ii}(\mathcal{T}) + \frac{(n-8)(n-9)}{2}q_{h10}^{ii}(\mathcal{T}) \right].$$

Proof. Since $s \in C_T(H_1)$ and π_{hi}^T is a homomorphism of $\mathbb{C}[S_n]$ -modules, $s^{\pi_{hi}^T} \in C_{T_k}(H_1)$. The formula follows from Lemma 4.4, since, for $(x, z) \in \mathcal{O}_k^T$, the set $\Delta_k^T(x)$ is the orbit of z under the action of the stabiliser in S_n of x . \square

Denote the H_1 -orbits of \mathcal{U} as follows:

$$(6) \quad \mathcal{Q}_1 := \{1, 2, 3\}^{H_1}, \mathcal{Q}_2 := \{1, 2, 4\}^{H_1}, \mathcal{Q}_3 := \{1, 5, 6\}^{H_1}, \mathcal{Q}_4 := \{5, 6, 7\}^{H_1}.$$

LEMMA 4.9. For $k \in \{1, \dots, 4\}$, $u^{\pi_k^U} \in C_{U_k}(H_1)$ and

$$u^{\pi_k^U} = (q_{k1}^{11}(\mathcal{U}) + 3q_{k2}^{11}(\mathcal{U})) \sum_{v \in \mathcal{Q}_1} v + (2q_{k2}^{11}(\mathcal{U}) + 2q_{k3}^{11}(\mathcal{U})) \sum_{v \in \mathcal{Q}_2} v + (3q_{k3}^{11}(\mathcal{U}) + q_{k4}^{11}(\mathcal{U})) \sum_{v \in \mathcal{Q}_3} v + 4q_{k4}^{11}(\mathcal{U}) \sum_{v \in \mathcal{Q}_4} v.$$

Proof. The proof is the same as in the previous lemma. \square

COROLLARY 4.10. The multiplicity of $S^{(n-2,2)}$ in V^\perp is equal to the corank of the matrix

$$\Gamma_4 := \begin{pmatrix} f(s^{\pi_{41}^T}, s^{\pi_{41}^T}) & f(s^{\pi_{41}^T}, s^{\pi_{42}^T}) & f(s^{\pi_{41}^T}, u^{\pi_4^U}) \\ f(s^{\pi_{42}^T}, s^{\pi_{41}^T}) & f(s^{\pi_{42}^T}, s^{\pi_{42}^T}) & f(s^{\pi_{42}^T}, u^{\pi_4^U}) \\ f(u^{\pi_4^U}, s^{\pi_{41}^T}) & f(u^{\pi_4^U}, s^{\pi_{42}^T}) & f(u^{\pi_4^U}, u^{\pi_4^U}) \end{pmatrix}$$

Proof. An easy, though tedious, computation shows that none of $s^{\pi_{41}^T}$, $s^{\pi_{42}^T}$, and $u^{\pi_4^U}$ is the zero vector. Therefore, since they belong to distinct irreducible components of V , $(s^{\pi_{41}^T}, s^{\pi_{42}^T}, u^{\pi_4^U})$ is a basis of $C_{V(S^{(n-2,2)})(H_1)}$. The result then follows by Corollary 2.4 and Lemma 4.7. \square

4.3. SUBMODULES OF MULTIPLICITY 2. In this subsection assume

$$S \in \{1, S^{(n-1,1)}, S^{(n-3,3)}\}.$$

In this case, we still have $\dim_{\mathbb{C}} C_S(H_1) = 1$, but the elements $s^{\pi_{k1}^T}$ are equal to 0 if $k \in \{1, 2, 3\}$ and $n = 8$. Not to treat separately these cases we replace the subgroup H_1 and the element s by more suitable ones. Let

$$t := (1, 2)(3, 4), \text{ and } H_2 \text{ be the centraliser in } S_n \text{ of } t.$$

Obviously t is H_2 -invariant and, in the same way as in Lemma 4.7 using [3, Lemma 6] in place of Lemma 4.6, one proves that

$$\dim_{\mathbb{C}} C_S(H_2) = 1.$$

Denote the H_2 -orbits of \mathcal{T} as follows:

$$(7) \quad \begin{aligned} \mathcal{R}_1 &:= ((1, 2)(3, 4))^{H_2}, & \mathcal{R}_2 &:= ((1, 3)(2, 4))^{H_2}, & \mathcal{R}_3 &:= ((1, 2)(3, 5))^{H_2}, \\ \mathcal{R}_4 &:= ((1, 3)(2, 5))^{H_2}, & \mathcal{R}_5 &:= ((1, 2)(5, 6))^{H_2}, & \mathcal{R}_6 &:= ((1, 3)(5, 6))^{H_2}, \\ \mathcal{R}_7 &:= ((1, 5)(2, 6))^{H_2}, & \mathcal{R}_8 &:= ((1, 5)(3, 6))^{H_2}, & \mathcal{R}_9 &:= ((1, 5)(6, 7))^{H_2}, \\ \mathcal{R}_{10} &:= ((5, 6)(7, 8))^{H_2}. \end{aligned}$$

LEMMA 4.11. For $k \in \{1, \dots, 3\}$, $t^{\pi_{k1}^T} \in C_{T_{k,1}}(H_2)$, and

$$(8) \quad t^{\pi_{k,1}^T} = \sum_{h=1}^{10} q_{kh}^{11}(\mathcal{T}) \sum_{v \in \mathcal{R}_h} v$$

Proof. Clearly $t \in C_T(H_2)$ and so $t^{\pi_{k1}^T} \in C_{T_{k,1}}(H_2)$, since π_{k1}^T is a homomorphism of $\mathbb{C}[S_n]$ -modules. The formula follows from Lemma 4.2, since, for $(x, z) \in \mathcal{O}_k^T$, $\Delta_k^T(x) = z^{C_{S_n}(x)}$. \square

Let u be as in Subsection 4.2. Since $H_2 \leq H_1$, u and its projections $u^{\pi_1^U}$, $u^{\pi_2^U}$, and $u^{\pi_3^U}$ are H_2 -invariant. As above, for $k \in \{1, 2, 3\}$, $(t^{\pi_{k1}^T}, u^{\pi_k^U})$ is a basis of $C_{V(U_k)}(H_2)$, whence the following result holds.

COROLLARY 4.12. *For $S \in \{1, S^{(n-1,1)}, S^{(n-3,3)}\}$, the multiplicity of S in V^\perp is equal to the corank of the matrix*

$$\Gamma_k := \begin{pmatrix} f(t^{\pi_{k1}^T}, t^{\pi_{k1}^T}) & f(t^{\pi_{k1}^T}, u^{\pi_k^U}) \\ f(t^{\pi_{k1}^T}, u^{\pi_k^U}) & f(u^{\pi_k^U}, u^{\pi_k^U}) \end{pmatrix}$$

for $k = 1, 2, 3$ respectively.

4.4. SUBMODULES OF MULTIPLICITY 1. Finally, assume

$$S \in \{S^{(n-4,4)}, S^{(n-3,2,1)}, S^{(n-4,2,2)}\}.$$

By Corollary 4.5 and Corollary 2.2, we may restrict to the action of S_n on \mathcal{T} and apply the results in [3].

LEMMA 4.13.

- (1) For $8 \leq n < 12$, V^\perp has no $\mathbb{C}[S_n]$ -submodules isomorphic to $S^{(n-4,4)}$, $S^{(n-3,2,1)}$, or $S^{(n-4,2,2)}$,
- (2) for $n = 12$, V^\perp has no $\mathbb{C}[S_n]$ -submodules isomorphic to $S^{(n-4,4)}$ or $S^{(n-4,2,2)}$, and has exactly one $\mathbb{C}[S_n]$ -submodule isomorphic to $S^{(n-3,2,1)}$.

Proof. Let $S \in \{S^{(n-4,4)}, S^{(n-3,2,1)}, S^{(n-4,2,2)}\}$. By Corollary 2.2, the multiplicity of S in V^\perp is equal to the multiplicity of S in $T \cap T^\perp$. The result then follows from [3, Table 13]. \square

4.5. THE MATRICES Γ_k . In this subsection we show how to compute the entries of the matrices $\Gamma_1, \dots, \Gamma_4$. Note first that the values $f(t^{\pi_{ki}^T}, t^{\pi_{ki}^T})$ and $f(s^{\pi_{ki}^T}, s^{\pi_{ki}^T})$ (resp. $f(u^{\pi_k^U}, u^{\pi_k^U})$) can be computed inside the module T , (resp. U), so we can avoid a direct computation by applying the results from [3] (resp. Table 6).

LEMMA 4.14. *Let $\kappa: V \times V \rightarrow \mathbb{C}$ be the hermitian form on V with respect to which \mathcal{X} is an orthonormal basis. Then, for every $v \in T_{k,i}$ and $w \in U_l$, where $(k, i) \in \{(1, 1), (2, 1), (3, 1), (4, 1), (4, 2)\}$ and $l \in \{1, \dots, 4\}$, we have*

$$f(v, v) = \kappa(v, v)\delta_{ki} \text{ and } f(w, w) = \kappa(w, w)\epsilon_l,$$

where the δ_{ki} 's and the ϵ_l 's are listed in Table 6.

k	δ_{k1}	δ_{k2}	ϵ_l
1	$\frac{5}{128}n^3 - \frac{25}{128}n^2 - \frac{15}{64}n + \frac{45}{16}$		$\frac{8}{135}n^2 + \frac{16}{27}n - \frac{32}{45}$
2	$\frac{5}{512}n^3 - \frac{35}{512}n^2 - \frac{15}{64}n + \frac{45}{16}$		$\frac{8}{405}n^2 + \frac{56}{135}n - \frac{32}{45}$
3	$\frac{15}{128}(18 - n)$		$\frac{752}{675}$
4	$\frac{5}{768}(n - 32)(n - 13)$	$\frac{37}{768}n^2 - \frac{97}{256}n + \frac{311}{192}$	$\frac{8}{2025}(31n + 158)$

TABLE 6. Values of δ_{ki} and ϵ_l

Proof. Since $T_{k,i}$ (resp U_k) is an irreducible $\mathbb{C}[S_n]$ -module, by [12, p. 534], any two S_n -invariant hermitian forms on T_k (resp. U_k) differ only by a scalar. The result for T_k then follows from [3, Table 13], of which Table 6 is an extract. The same argument gives the values ϵ_k from the first eigenmatrix associated to the action of S_n on U (Table 4). \square

The remaining entries of the matrices Γ_i have been computed, using the formulae in Section A.

k	$f(u^{\pi_k^U}, u^{\pi_k^U})$
1	$\frac{64}{405}n(n-1)(n-2)(n^2+10n-12)$
2	$\frac{16}{405} \frac{n(n-1)^2(n-2)(n-4)(n^2+21-36)}{(n-3)}$
3	$\frac{64}{405} \frac{n^2(n-1)^2(n-2)(n-5)(n-6)}{(n-3)}$
4	$\frac{208}{1215}n^2(n-1)(n-2)(n-5)(n-\frac{16}{13})$

TABLE 7. Values of $f(u^{\pi_k^U}, u^{\pi_k^U})$

i	$\det(\Gamma_i)$
1	$-\frac{1}{108}n^2(n-1)^2(n-2)^2(n-3)(n-12)(n^2-2n+6)$
2	$-\frac{1}{288}n^2(n-1)^4(n-2)^2(n-4)(n-12)(n^2-3n+12)$
3	$-\frac{1}{2592}n^4(n-1)^4(n-2)^2(n-5)^2(n-6)(n-12)$
4	$-\frac{1}{221 \cdot 3^2}n^6(n-1)^4(n-2)^4(n-3)^4(n-4)(n-5)^2(n-7)^2(n-11)(n-12)^2(n-14)^2(n^2-3n+12)$

TABLE 8. Determinants of matrices Γ_i

4.6. PROOF OF THEOREM 1.1. The first assertion follows from Lemma 4.1. Assume

$$S \in \{S^{(n-4,4)}, S^{(n-4,2,2)}, S^{(n-3,2,1)}\},$$

then the multiplicities of S in V^\perp are given in Lemma 4.13. Assume

$$S \in \{1, S^{(n-1,1)}, S^{(n-3,3)}\}.$$

By Tables 7 and 8, for $i \in \{1, 2, 3\}$, the corank of Γ_i is 0, if $n \neq 12$, and 1, if $n = 12$. The result then follows from Corollary 4.12. Finally assume

$$S = S^{(n-2,2)}.$$

If $n \leq 10$, by Table 8, $\det(\Gamma_4) \neq 0$, so the result follows from Corollary 4.10. If $n = 11$, then $\det(\Gamma_4) = 0$ and, since, by Lemma 4.1, $T \cap T^\perp = \{0\}$, it follows that Γ_4 has rank 2. The result then follows from Corollary 4.10. If $n = 12$, $\det(\Gamma_4) = 0$, hence, Γ_4 has rank at most 2. Let β, δ be the submatrices of Γ_4 defined as follows

$$\beta := \begin{pmatrix} f(s^{\pi_5^V}, s^{\pi_5^V}) & f(s^{\pi_5^V}, u^{\pi_4^U}) \\ f(u^{\pi_4^U}, s^{\pi_5^V}) & f(u^{\pi_4^U}, u^{\pi_4^U}) \end{pmatrix}, \quad \delta := \begin{pmatrix} f(s^{\pi_4^V}, s^{\pi_4^V}) & f(s^{\pi_4^V}, s^{\pi_5^V}) \\ f(s^{\pi_5^V}, s^{\pi_4^V}) & f(s^{\pi_5^V}, s^{\pi_5^V}) \end{pmatrix}$$

Using the formulae in Section A, we get that the determinant of β is equal to

$$-\frac{1}{29 \cdot 3^4 \cdot 5}n^4(n-1)^3(n-2)^3(n-3)^2(n-4)(n-5)^2(n-12)(n^2+117n-148)$$

that vanishes when $n = 12$. By Lemma 4.14 and [3, Table 13 and Equation 25], the determinant of δ is a multiple of

$$\frac{15}{2^{16}}(n-12)(n^3 - 56n^2 + 411n - 1596)$$

that also vanishes when $n = 12$. Since the maximum rank of the 2×2 submatrices of Γ_4 is equal to the maximum rank of the two matrices β and δ , we get that Γ_4 has rank 1, so, again, the claim follows from Corollary 4.10.

APPENDIX A.

We give here some formulae we used to compute the entries of matrices Γ_i . They have been obtained by straightforward computation using Lemma 1 and the following easy observation.

LEMMA A.1. *Let $L \leq S_n$ and let \mathcal{O}_1 and \mathcal{O}_2 be two L -orbits on \mathcal{X} . Then, for every $x \in \mathcal{O}_1$, we have*

$$f\left(\sum_{v \in \mathcal{O}_1} v, \sum_{w \in \mathcal{O}_2} w\right) = |\mathcal{O}_1| \sum_{y \in \mathcal{O}_2} f(x, y).$$

Let $a, b \in C_T(H_2)$, $d \in C_T(H_1)$, and $c \in C_U(H_2)$ and suppose that

$$a = \sum_{h=1}^6 a_h \sum_{v \in \mathcal{P}_h} v, \quad b = \sum_{h=1}^6 b_h \sum_{v \in \mathcal{P}_h} v, \quad c = \sum_{h=1}^4 c_h \sum_{v \in \mathcal{Q}_h} v, \quad d = \sum_{h=1}^{10} d_h \sum_{v \in \mathcal{R}_h} v$$

with $a_i, b_i, c_i, d_i \in \mathbb{C}$. Then

$$\begin{aligned} f(a, b) &= a_1 b_1 \frac{15}{4} + (a_1 b_2 + a_2 b_1) \frac{75}{64} (n-4) + (a_1 b_3 + a_3 b_1) \frac{15}{32} (n-4)(n-5) \\ &\quad + (a_1 b_4 + a_4 b_1) \frac{45}{64} (n-4)(n-5) + (a_1 b_5 + a_5 b_1) \frac{15}{128} (n-4)(n-5)(n-6) \\ &\quad + (a_1 b_6 + a_6 b_1) 0 + a_2 b_2 \frac{15}{64} (n-4)(25n-46) \\ &\quad + (a_2 b_3 + a_3 b_2) 3 \frac{55}{128} (n-4)^2 (n-5) \\ &\quad + (a_2 b_4 + a_4 b_2) \frac{45}{64} (n-4)(n-5)(3n-10) \\ &\quad + (a_2 b_5 + a_5 b_2) \frac{15}{128} (n-4)(n-5)(n-6)(3n+2) \\ &\quad + (a_2 b_6 + a_6 b_2) \frac{15}{128} (n-4)(n-5)(n-6)(n-7) \\ &\quad + 3a_3 b_3 (n-4)(n-5) \left(\frac{13}{128} n^2 - \frac{99}{128} n + \frac{37}{16} \right) \\ &\quad + (a_3 b_4 + a_4 b_3) \frac{9}{64} (n-4)(n-5)(2n^2 - 11n + 4) \\ &\quad + (a_3 b_5 + a_5 b_3) \frac{15}{256} (n-4)(n-5)(n-6)(n^2 + 3n - 44) \\ &\quad + (a_3 b_6 + a_6 b_3) \frac{15}{256} (n-4)^2 (n-5)(n-6)(n-7) \\ &\quad + a_4 b_4 \frac{9}{128} (n-4)(n-5)(11n^2 - 53n + 52) \\ &\quad + (a_4 b_5 + a_5 b_4) \frac{15}{128} (n-4)(n-5)(n-6)(n^2 + 5n - 62) \\ &\quad + (a_4 b_6 + a_6 b_4) \frac{15}{128} (n-4)(n-5)^2 (n-6)(n-7) \end{aligned}$$

$$\begin{aligned}
 &+ a_5 b_5 \frac{1}{256} (n-4)(n-5)(n-6)(5n^3 + 115n^2 - 1820n + 6180) \\
 &+ (a_5 b_6 + a_6 b_5) \frac{5}{256} (n-4)(n-5)(n-6)(n-7)(3n^2 - 31n + 66) \\
 &+ a_6 b_6 \frac{(n-4)(n-5)(n-6)(n-7)}{8} \left(\frac{5}{128} n^3 - \frac{85}{128} n^2 + \frac{205}{64} n - \frac{15}{8} \right),
 \end{aligned}$$

$$\begin{aligned}
 f(a, c) = & (c_1 + 3c_2) \left[a_1 \frac{4}{3} + a_2 (n-4) \frac{17}{3} + a_3 (n-4)(n-5) \frac{5}{3} \right. \\
 & \left. + a_4 (n-4)(n-5) + a_5 (n-4)(n-5)(n-6) \frac{1}{6} \right] \\
 & + 2(c_2 + c_3) \left[a_1 (n-4) \frac{13}{6} + a_2 (n-4) \frac{(16n-55)}{3} \right. \\
 & \left. + a_3 (n-4)^2 (n-5) \frac{13}{12} + a_4 (n-4)(n-5)(n+3) \right. \\
 & \left. + a_5 (n-4)(n-5)(n-6)(n+9) \frac{1}{6} \right. \\
 & \left. + a_6 (n-4)(n-5)(n-6)(n-7) \frac{1}{12} \right] \\
 & + (3c_3 + c_4) \left[a_1 (n-4)(n-5) \frac{1}{6} + a_2 (n-4)(n-5)(3n+10) \frac{1}{6} \right. \\
 & \left. + a_3 (n-4)(n-5) \left(\frac{13}{6} + (n-6) + (n-6)(n-7) \frac{1}{12} \right) \right. \\
 & \left. + a_4 (n-4)(n-5) \left(2 + (n-6) \frac{13}{3} + (n-6)(n-7) \frac{1}{6} \right) \right. \\
 & \left. + a_5 (n-4)(n-5)(n-6) \left(\frac{17}{6} + \frac{11}{9}(n-7) \right. \right. \\
 & \left. \left. + \frac{1}{36}(n-7)(n-8) \right) \right. \\
 & \left. + a_6 (n-4)(n-5)(n-6)(n-7) \left(\frac{13}{36} + \frac{1}{18}(n-8) \right) \right] \\
 & + 4c_4 \left[a_2 (n-4)(n-5)(n-6) \frac{1}{6} \right. \\
 & \left. + a_3 (n-4)(n-5)(n-6) \left(\frac{1}{12}(n-7) + \frac{3}{4} \right) \right. \\
 & \left. + a_4 (n-4)(n-5)(n-6) \left(\frac{1}{6}(n-7) + \frac{1}{3} \right) \right. \\
 & \left. + a_5 (n-4)(n-5)(n-6) \left(\frac{2}{9} + \frac{13}{18}(n-7) \right. \right. \\
 & \left. \left. + \frac{1}{12}(n-8)(n-7) \right) \right. \\
 & \left. + a_6 \frac{1}{8} (n-4)(n-5)(n-6)(n-7) \left(\frac{4}{9} + \frac{13}{18}(n-8) \right. \right. \\
 & \left. \left. + \frac{1}{18}(n-8)(n-9) \right) \right],
 \end{aligned}$$

and

$$\begin{aligned}
 f(d, c) = & (c_1 + 3c_2) \left[\frac{4}{9}d_1 + \frac{8}{9}d_2 + 4(n-4)\frac{17}{36}d_3 + 4(n-4)\frac{17}{18}d_4 \right. \\
 & + 2(n-4)(n-5)\frac{5}{18}d_5 + 4(n-4)(n-5)\frac{5}{18}d_6 \\
 & + \frac{(n-4)(n-5)}{3}d_7 + 2\frac{(n-4)(n-5)}{3}d_8 \\
 & \left. + \frac{(n-4)(n-5)(n-6)}{6}d_9 \right] \\
 & + 2(c_2 + c_3) \left[\frac{13}{18}(d_1 + 2d_2)(n-4) + d_3(n-4) \left(\frac{16}{9}n - \frac{55}{9} \right) \right. \\
 & + 2d_4(n-4) \left(\frac{16}{9}n - \frac{55}{9} \right) + d_5(n-4)(n-5)\frac{13}{36}(n-4) \\
 & + d_6(n-4)\frac{13}{18}(n-5)(n-4) \\
 & + d_7(n-4)(n-5)(n+3)\frac{1}{3} + d_8(n-4)(n-5)(n+3)\frac{2}{3} \\
 & + d_9(n-4)(n-5)(n-6)(n+9)\frac{1}{6} \\
 & \left. + d_{10}(n-4)(n-5)(n-6)(n-7)\frac{1}{12} \right] \\
 & + (3c_3 + c_4) \left\{ d_1(n-4)(n-5)\frac{1}{18} + d_2(n-4)(n-5)\frac{1}{9} \right. \\
 & + d_3(n-4)(n-5)(3n+10)\frac{1}{18} \\
 & + d_4(n-4)(n-5)(3n+10)\frac{1}{9} \\
 & + d_5(n-4)(n-5) \left[\frac{13}{18} + (n-6)\frac{1}{3} + (n-6)(n-7)\frac{1}{36} \right] \\
 & + d_6(n-4)(n-5) \left[\frac{13}{9} + \frac{2}{3}(n-6) + (n-6)(n-7)\frac{1}{18} \right] \\
 & + d_7(n-4)(n-5) \left[\frac{2}{3} + \frac{13}{9}(n-6) + (n-6)(n-7)\frac{1}{18} \right] \\
 & + 2d_8(n-4)(n-5) \left[\frac{2}{3} + \frac{13}{9}(n-6) + (n-6)(n-7)\frac{1}{18} \right] \\
 & + d_9(n-4)(n-5)(n-6) \left[\frac{17}{6} + \frac{11}{9}(n-7) \right. \\
 & \quad \left. + \frac{1}{36}(n-7)(n-8) \right] \\
 & \left. + d_{10}(n-4)(n-5)(n-6)(n-7) \left[\frac{13}{36} + (n-8)\frac{1}{18} \right] \right\} \\
 & 4c_4 \left\{ d_3(n-4)(n-5)(n-6)\frac{1}{18} + d_4(n-4)(n-5)(n-6)\frac{1}{9} \right. \\
 & \quad \left. + d_5(n-4)(n-5)(n-6) \left[\frac{1}{36}(n-7) + \frac{1}{4} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
& + d_6(n-4)(n-5)(n-6) \left[\frac{1}{18}(n-7) + \frac{1}{2} \right] \\
& + d_7(n-4)(n-5)(n-6) \left[\frac{1}{18}(n-7) + \frac{1}{9} \right] \\
& + d_8(n-4)(n-5)(n-6) \left[\frac{1}{9}(n-7) + \frac{2}{9} \right] \\
& + d_9(n-4)(n-5)(n-6) \left[\frac{2}{9} + \frac{13}{18}(n-7) + \frac{1}{12}(n-8)(n-7) \right] \\
& + d_{10} \frac{(n-4)(n-5)(n-6)(n-7)}{8} \left[\frac{4}{9} + \frac{13}{18}(n-8) \right. \\
& \quad \left. + \frac{1}{18}(n-8)(n-9) \right] \Big\}.
\end{aligned}$$

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