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Comparing formulas for type GL_n Macdonald polynomials – Supplement

Weiying Guo & Arun Ram

Dedicated to Hélène Barcelo

Abstract This paper is a supplement to [5], containing examples, remarks and additional material that could be useful to researchers working with Type GL_n Macdonald polynomials. In the course of our comparison of the alcove walk formula and the nonattacking fillings formulas for type GL_n Macdonald polynomials we did many examples and significant analysis of the literature. In the preparation of [5] it seemed sensible to produce a document with focus and this material was removed. This is paper resurrects and organizes that material, in hopes that others may also find it useful.

0. Introduction

This paper is a supplement to [5], containing examples, remarks and additional material that could be useful to researchers working with Type GL_n Macdonald polynomials. In the course of our comparison of the alcove walk formula and the nonattacking fillings formulas for type GL_n Macdonald polynomials we did many examples and significant analysis of the literature. In the preparation of [5] it seemed sensible to produce a document with focus and this material was removed. This is paper resurrects and organizes that material, in hopes that others may also find it useful.

1. The material in Section 1: Several colleagues have asked us questions about permuted basement Macdonald polynomials and KZ-families (the permuted basement Macdonald polynomials are called relative Macdonald polynomials in this paper). These questions are helpfully considered in the context of the results of the two paragraphs following equation (6.6) in Macdonald's Séminaire Bourbaki article [11] and Sections 5.4 and 5.5 of Macdonald's followup book [12] treating the fully general case. In hopes of making these results more accessible, in Section 1 we have recast these completely in the type GL_n and included their proofs (which are not difficult). These results are the H-decomposition in Section 1.1, symmetrization statement in Proposition 1.1, and the KZ-family characterization in Proposition 1.2. We hope that these type GL_n specific expositions of these results can be helpful to the community.
2. The material in Section 2: This section has a focus on counting the number of alcove walks and the number of nonattacking fillings, in order to compare the number of terms that appear in alcove walks formula and the nonattacking fillings formula for Macdonald polynomials. Some explicit formulas for these counts, which may not have been widely noticed before, are included.

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Keywords. Macdonald polynomials, affine Hecke algebras, tableaux.

3. The material in Section 3: This section explains how to recast the alcove walks and nonattacking fillings into path form and pipe dream form. Pictures are provided.

4,5,6. The material in Sections 4–6: These sections provide explicit examples of the main results of [5]: the inversions and the box-greedy reduced word for u_μ proved in [5, Proposition 2.2], the step-by-step and box-by-box recursions for computing Macdonald polynomials in [5, Proposition 4.1 and 4.3] and some specific examples to help support the exposition of the type GL_n double affine Hecke algebra (DAHA) given in [5, Section 5].

7. The material in Section 7: In this final section we provide additional explicit expansions of Macdonald polynomials for special cases: $n=2$, $n=3$, a single column, partitions with 3 boxes, and explicit nonattacking fillings and their weights for E_μ where μ has less than 3 boxes.

8. Section 8 contains some brief remarks about the queue tableaux and multiline queues which appear in [4, Section 1.2 and Definition A.2].

A small warning: Even though they all have a Type A root system, type SL_n Macdonald polynomials, type PGL_n Macdonald polynomals and type GL_n Macdonald polynomials are all *different* (though the relationship is well known and not difficult). We should stress that this paper is specific to the GL_n-case and some results of this paper do not hold for Type SL_n or type PGL_n unless properly modified.

1. Symmetrization, H decomposition of $\mathbb{C}[X]$ and KZ-families

Let $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$. Following the notation of [10, Ch. VI (3.1)], let T_{q^{-1},x_1} be the operator on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ given by

$$T_{q^{-1},x_n} h(x_1, \dots, x_n) = h(x_1, \dots, x_{n-1}, q^{-1}x_n).$$

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by permuting the the variables $x_1, \dots, x_n$. Define operators $T_1, \dots, T_{n-1}$, g and $g^\vee$ on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$T_i = t^{-\frac{1}{2}}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1-s_i)\big), \tag{1}$$

$$g = s_1 s_2 \cdots s_{n-1} T_{q^{-1},x_n}, \qquad g^\vee = x_1 T_1 \cdots T_{n-1},$$

where $s_1, \dots, s_{n-1}$ are the simple transpositions in S_n. The *Cherednik-Dunkl operators* are

$$Y_1 = gT_{n-1}\cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad \dots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}. \tag{2}$$

For $\mu \in \mathbb{Z}^n$ the *nonsymmetric Macdonald polynomial* E_μ is the (unique) element $E_\mu \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that the coefficient of $x_1^{\mu_1} \cdots x_n^{\mu_n}$ in E_μ is 1 and

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu, \tag{3}$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing. Let $\mu = (\mu_1, \dots, \mu_n)$ and let $z \in S_n$.

(4) The *relative Macdonald polynomial* E_μ^z is $\qquad E_\mu^z = t^{-\frac{1}{2}(\ell(zv_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu.$

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n) \in \mathbb{Z}^n$.

(5) The *symmetric Macdonald polynomial* P_λ is $\qquad P_\lambda = \sum_{\nu \in S_n\lambda} t^{\frac{1}{2}\ell(z_\nu)} T_{z_\nu} E_\lambda,$

where the sum is over rearrangements ν of λ and $z_\nu \in S_n$ is minimal length such that $\nu = z_\nu \lambda$.

1.1. The H-modules $\mathbb{C}[X]^\lambda$. Let H be the algebra generated by the operators $T_1, \ldots, T_{n-1}$ and $Y_1, \ldots, Y_n$ (so that H is an affine Hecke algebra) and let

$$\tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} \quad \text{for } i \in \{1, \ldots, n-1\}.$$

As H-modules

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \bigoplus_\lambda \mathbb{C}[X]^\lambda \qquad \text{where} \qquad \mathbb{C}[X]^\lambda = \operatorname{span}\{E_\mu \mid \mu \in S_n\lambda\},$$

and the direct sum is over decreasing $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n) \in \mathbb{Z}^n$. A description of the action of H on $\mathbb{C}[X]^\lambda$ is given by the following. Let $\mu \in \mathbb{Z}^n$ and, with notations as in (3), let

$$\begin{aligned} a_\mu &= q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}, \\ a_{s_i\mu} &= q^{\mu_{i+1} - \mu_i} t^{v_\mu(i+1) - v_\mu(i)}, \end{aligned} \qquad \text{and} \qquad D_\mu = \frac{(1-ta_\mu)(1-ta_{s_i\mu})}{(1-a_\mu)(1-a_{s_i\mu})}.$$

Assume that $\mu_i > \mu_{i+1}$. By using the identity $E_{s_i\mu} = t^{\frac{1}{2}}\tau_i^\vee E_\mu$ from [5, (3.5)], the eigenvalue from (3) and [5, Proposition 5.5 (5.23)], it is straightforward to compute that

$$\begin{aligned} Y_i^{-1}Y_{i+1}E_\mu &= a_\mu E_\mu, & t^{\frac{1}{2}}\tau_i^\vee E_\mu &= E_{s_i\mu}, \\ Y_i^{-1}Y_{i+1}E_{s_i\mu} &= a_{s_i\mu}E_{s_i\mu}, & t^{\frac{1}{2}}\tau_i^\vee E_{s_i\mu} &= D_\mu E_\mu, \end{aligned} \tag{6}$$

$$t^{\frac{1}{2}}T_iE_\mu = -\frac{1-t}{1-a_\mu}E_\mu + E_{s_i\mu}, \qquad \text{and} \qquad t^{\frac{1}{2}}T_iE_{s_i\mu} = D_\mu E_\mu + \frac{1-t}{1-a_{s_i\mu}}E_{s_i\mu}.$$

Now assume that $\mu_i = \mu_{i+1}$. Then $v_\mu(i+1) = v_\mu(i) + 1$ and $a_\mu = t^{-1}$ so that

$$Y_i^{-1}Y_{i+1}E_\mu = t^{-1}E_\mu, \qquad (t^{\frac{1}{2}}\tau_i^\vee)E_\mu = 0, \qquad \text{and} \qquad (t^{\frac{1}{2}}T_i)E_\mu = tE_\mu. \tag{7}$$

These formulas make explicit the action of H on $\mathbb{C}[X]^\lambda$ in the basis $\{E_\mu \mid \mu \in S_n\lambda\}$. The formulas in (6) are the type GL_n special cases of [12, (5.4.3),(5.6.6)].

1.2. Symmetrization of E_μ for $\mu \in \mathbb{Z}^n$. If $z \in S_n$ and

$$z = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word,} \qquad \text{let} \qquad T_z = T_{i_1} \cdots T_{i_\ell}.$$

Let w_0 be the longest element of S_n so that

$$w_0(i) = n - i + 1, \text{ for } i \in \{1, \ldots, n\}, \qquad \text{and} \qquad \ell(w_0) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Following [12, (5.5.7), (5.5.16), (5.5.17)], let

$$\mathbf{1_0} = t^{-\frac{1}{2}\ell(w_0)} \sum_{z \in S_n} t^{\frac{1}{2}\ell(z)} T_z, \tag{8}$$

so that $T_i\mathbf{1}_0 = \mathbf{1}_0 T_i = t^{\frac{1}{2}}\mathbf{1}_0$ for $i \in \{1, \ldots, n-1\}$, and

$$\mathbf{1}_0^2 = W_0(t)\mathbf{1}_0, \qquad \text{where} \quad W_0(t) = \sum_{z \in S_n} t^{\ell(z)} \tag{9}$$

is the *Poincaré polynomial for S_n*.

For $\mu \in \mathbb{Z}^n$, the *symmetrization of E_μ* is (see [12, (5.7.1)] and [11, Remarks after (6.8)])

$$F_\mu = \mathbf{1}_0 E_\mu = t^{-\frac{1}{2}\ell(w_0)} \sum_{z \in S_n} t^{\frac{1}{2}(\ell(z) - \ell(zv_\mu^{-1}) + \ell(v_\mu^{-1}))} E_\mu^z, \tag{10}$$

so that F_μ is a (weighted) sum of the relative Macdonald polynomials E_μ^z defined in (4)). The following Proposition shows that F_μ is always, up to an explicit constant factor, equal to the symmetric Macdonald polynomial P_λ (defined in (5)). Proposition 1.1 is the specialization of [11, remarks after (6.8)] and [12, (5.7.2)] to our setting.

Proposition 1.1. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the weakly decreasing rearrangement of μ and let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu\lambda$. Let*

$$S_\lambda = \{y \in S_n \mid y\lambda = \lambda\} \qquad \text{and} \qquad W_\lambda(t) = \sum_{y\in S_\lambda} t^{\ell(y)}.$$

Then

$$P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\Big(\prod_{(i,j)\in \mathrm{Inv}(z_\mu)} \frac{1-q^{\lambda_i-\lambda_j}t^{j-i}}{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}\Big)F_\mu.$$

Proof. The proof is by induction on $\ell(z_\mu)$. The base case $z_\mu = 1$ has $\mu = \lambda$ and $v_\lambda = w_0 z_\lambda$ so that

$$\begin{aligned} F_\lambda = \mathbf{1}_0 E_\lambda &= t^{-\frac{1}{2}\ell(w_0)}\Big(\sum_{u\in S_n/S_\lambda}\sum_{v\in S_\lambda} t^{\frac{1}{2}\ell(x)+\ell(y)}T_xT_y\Big)E_\lambda \\ &= t^{-\frac{1}{2}\ell(w_0)}\Big(\sum_{u\in S_n/S_\lambda} t^{\frac{1}{2}\ell(x)}T_x\Big)W_\lambda(t)E_\lambda = t^{-\frac{1}{2}\ell(w_0)}W_\lambda(t)P_\lambda, \end{aligned}$$

where $T_yE_\lambda = t^{\frac{1}{2}\ell(y)}E_y$ is a consequence of (7) and the last equality is (5). For the induction step, assume that μ is not weakly decreasing and let $i \in \{1, \dots, n-1\}$ be such that $\mu_i < \mu_{i+1}$. Then $z_{s_i\mu} = s_iz_\mu$ and $\ell(z_{s_i\mu}) = \ell(z_\mu) - 1$. Using $E_\mu = t^{\frac{1}{2}}\tau_i^\vee E_{s_i\mu}$ and $\mathbf{1}_0T_i = \mathbf{1}_0t^{\frac{1}{2}}$ from (6) and (7) gives

$$\begin{aligned} F_\mu = \mathbf{1}_0E_\mu = \mathbf{1}_0t^{\frac{1}{2}}\tau_{i_1}E_{s_i\mu} &= \mathbf{1}_0\Big(t^{\frac{1}{2}}T_i + \frac{1-t}{1-Y_i^{-1}Y_{i+1}}\Big)E_{s_i\mu} \\ &= \mathbf{1}_0\Big(t + \frac{1-t}{1-Y_i^{-1}Y_{i+1}}\Big)E_{s_i\mu} = \mathbf{1}_0\frac{1-tY_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}E_{s_i\mu} \\ &= \mathbf{1}_0\frac{1-tq^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}E_{s_i\mu} = \frac{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)+1}}{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}F_{s_i\mu} \end{aligned}$$

and the result follows by induction (see Section 1.3.3 for an example). ☐

1.3. The KZ-family basis of $\mathbb{C}[X]^\lambda$. For $\mu \in \mathbb{Z}^n$, let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ be the decreasing rearrangement of μ and let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu\lambda$. Define

$$f_\mu = E_\lambda^{z_\mu} = t^{\frac{1}{2}\ell(z_\mu)}T_{z_\mu}E_\lambda. \tag{11}$$

It follows from the identities in the last column of (6) that

$$\{f_\mu \mid \mu \in S_n\lambda\} \quad \text{is another basis of } \mathbb{C}[X]^\lambda.$$

The following Proposition says that the $\{f_\mu \mid \mu \in \mathbb{Z}^n\}$ form a KZ-family, in the terminology of [8, Def. 3.3] (see also [4, Def. 1.13], [2, (17), (18), (19)], [3, Def. 2]).

Proposition 1.2. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$. Let $i \in \{1, \dots, n-1\}$ and let T_i and g be as defined in* (1). *Then*

$$t^{\frac{1}{2}}T_if_\mu = \begin{cases} f_{s_i\mu}, & \text{if } \mu_i > \mu_{i+1}, \\ tf_\mu, & \text{if } \mu_i = \mu_{i+1}, \end{cases} \qquad \text{and} \qquad gf_\mu = q^{-\mu_n}f_{(\mu_n,\mu_1,\dots,\mu_{n-1})}.$$

Proof. Assume $\mu_i > \mu_{i+1}$. Then $z_{s_i\mu} = s_iz_\mu$ and $\ell(z_{s_i\mu}) = \ell(z_\mu) + 1$ so that

$$t^{\frac{1}{2}}T_if_\mu = t^{\frac{1}{2}}T_it^{\frac{1}{2}\ell(z_\mu)}T_{z_\mu}E_\lambda = t^{\frac{1}{2}\ell(z_{s_i\mu})}T_{z_{s_i\mu}}E_\lambda = f_{s_i\mu}.$$

Assume $\mu_i = \mu_{i+1}$. Then there exists $j \in \{1, \dots, n-1\}$ such that $s_j\lambda = \lambda$ and $s_iz_\mu = z_\mu s_j$ (so that $s_i\mu = s_iz_\mu\lambda = z_\mu s_j\lambda$). Then

$$t^{\frac{1}{2}}T_if_\mu = t^{\frac{1}{2}}T_it^{\frac{1}{2}\ell(z_\mu)}T_{z_\mu}E_\lambda = t^{\frac{1}{2}\ell(z_\mu)}T_{z_\mu}t^{\frac{1}{2}}T_jE_\lambda = t^{\frac{1}{2}\ell(z_\mu)}T_{z_\mu}tE_\lambda = tf_\mu.$$

(c) Let $\mu = (\mu_1, \dots, \mu_n)$ and let i and j be such that λ_i is the first part of λ equal to μ_n and λ_j is the last part of λ equal to μ_n. Thus $\mu_n = \lambda_i = \lambda_{i+1} = \cdots = \lambda_j$. Write $z_\mu = z s_{n-1} \cdots s_j$ with $z \in S_{n-1}$ and let $c_n = s_1 \cdots s_{n-1}$. Then, using $v_\lambda(j) = 1 + (j-i) + n - j = n - i + 1$ from [5, Proposition 2.1(a)],

$$\begin{aligned}
g f_\mu &= g t^{\frac{1}{2}\ell(z_\mu)} T_{z_\mu} E_\lambda = g t^{\frac{1}{2}\ell(z)} T_z t^{\frac{1}{2}(n-j)} T_{n-1} \cdots T_j E_\lambda \\
&= t^{\frac{1}{2}(n-j)} g t^{\frac{1}{2}\ell(z)} T_z g^{-1} g T_{n-1} \cdots T_j E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (g t^{\frac{1}{2}\ell(z)} T_z g^{-1}) T_1 \cdots T_{j-1} (T_{j-1}^{-1} \cdots T_1^{-1} g T_{n-1} \cdots T_j) E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{j-1} Y_j E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{j-1} q^{-\lambda_j} t^{-(v_\lambda(j)-1)+\frac{1}{2}(n-1)} E_\lambda \\
&= q^{-\lambda_j} t^{\frac{1}{2}(n-j)-(n-i+1-1)+\frac{1}{2}(n-1)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{i-1} T_i \cdots T_{j-1} E_\lambda \\
&= q^{-\mu_n} t^{-\frac{1}{2}j+i-\frac{1}{2}} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{i-1} t^{\frac{1}{2}(j-i)} E_\lambda \\
&= q^{-\mu_n} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) t^{\frac{1}{2}(i-1)} T_1 \cdots T_{i-1} E_\lambda \\
&= q^{-\mu_n} f_{(\lambda_i, \mu_1, \dots, \mu_{n-1})} = q^{-\mu_n} f_{(\mu_n, \mu_1, \dots, \mu_{n-1})},
\end{aligned}$$

where the next to last equality follows from
$s_1 \cdots s_{i-1}(\lambda_1, \dots, \lambda_n) = (\lambda_i, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$ and
$c_n z c_n^{-1}(\lambda_i, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n) = (\lambda_i, \mu_1, \dots, \mu_{n-1})$. □

1.3.1. *Examples of the elements E_μ and f_μ in $\mathbb{C}[X]^{(2,1,0)}$.*

$$\begin{aligned}
E_{(2,1,0)} &= x_1^2 x_2 + \Big(\frac{1-t}{1-qt^2}\Big) q x_1 x_2 x_3, \\
E_{(2,0,1)} &= x_1^2 x_3 + \Big(\frac{1-t}{1-qt}\Big) x_1^2 x_2 + \Big(\frac{1-t}{1-qt}\Big) q x_1 x_2 x_3, \\
E_{(1,2,0)} &= x_1 x_2^2 + \Big(\frac{1-t}{1-qt}\Big) x_1^2 x_2 + \Big(\frac{1-t}{1-qt}\Big) q x_1 x_2 x_3, \\
E_{(0,2,1)} &= x_2^2 x_3 + \Big(\frac{1-t}{1-qt}\Big) x_1 x_2^2 + \Big(\frac{1-t}{1-q^2t^2}\Big) x_1^2 x_3 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) x_1^2 x_2 \\
&\qquad + \Big(\Big(\frac{1-t}{1-qt}\Big) + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) q\Big) x_1 x_2 x_3, \\
E_{(1,0,2)} &= x_1 x_3^2 + \Big(\frac{1-t}{1-qt}\Big) x_1^2 x_3 + \Big(\frac{1-t}{1-q^2t^2}\Big) x_1 x_2^2 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) x_1^2 x_2 \\
&\qquad + \Big(\Big(\frac{1-t}{1-qt}\Big) + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) q\Big) x_1 x_2 x_3, \\
E_{(0,1,2)} &= x_2 x_3^2 + \Big(\frac{1-t}{1-qt}\Big) x_2^2 x_3 + \Big(\frac{1-t}{1-qt}\Big) x_1 x_3^2 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) x_1^2 x_3 \\
&\qquad + \Big(\frac{1-t}{1-q^2t^2}\Big) t x_1^2 x_2 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) x_1^2 x_2 \\
&\qquad + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) q t x_1 x_2^2 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) x_1 x_2^2 \\
&\qquad + \Big(\frac{1-t}{1-qt}\Big)^2 x_1 x_2 x_3 + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big) q t x_1 x_2 x_3 \\
&\qquad + \Big(\frac{1-t}{1-q^2t^2}\Big)\Big(\frac{1-t}{1-qt}\Big)^2 q x_1 x_2 x_3 + \Big(\frac{1-t}{1-qt}\Big) x_1 x_2 x_3,
\end{aligned}$$

$$
\begin{aligned}
f_{(2,1,0)} &= E_{(2,1,0)} = x_1^2x_2 + q\frac{(1-t)}{(1-qt^2)}x_1x_2x_3, \\
f_{(1,2,0)} &= t^{\frac{1}{2}}T_{s_1}E_{(2,1,0)} = x_1x_2^2 + t^{-1}\frac{(1-t)qt^2}{(1-qt^2)}x_1x_2x_3, \\
f_{(2,0,1)} &= t^{\frac{1}{2}}T_{s_2}E_{(2,1,0)} = x_1^2x_3 + t^{-1}\frac{(1-t)qt^2}{(1-qt^2)}x_1x_2x_3, \\
f_{(1,0,2)} &= t^{\frac{2}{2}}T_{s_2}T_{s_1}E_{(2,1,0)} = x_1x_3^2 + \frac{(1-t)}{(1-qt^2)}x_1x_2x_3, \\
f_{(0,2,1)} &= t^{\frac{2}{2}}T_{s_1}T_{s_2}E_{(2,1,0)} = x_2^2x_3 + \frac{(1-t)}{(1-qt^2)}x_1x_2x_3, \\
f_{(0,1,2)} &= t^{\frac{3}{2}}T_{s_1}T_{s_2}T_{s_1}E_{(2,1,0)} = x_2x_3^2 + t\frac{(1-t)}{(1-qt^2)}x_1x_2x_3.
\end{aligned}
$$

1.3.2. $P_{(2,1,0)}$ *as a symmetrization of* $E_{(2,1,0)}$. When $n=3$ then

$$
W_0(t) = \sum_{w\in S_3} t^{\ell(w)} = (1+t)(1+t+t^2) = \frac{(1-t^2)(1-t^3)}{(1-t)(1-t)},
$$

and

$$
\mathbf{1}_0 = t^{-\frac{3}{2}} + t^{-\frac{2}{2}}T_1 + t^{-\frac{2}{2}}T_2 + t^{-\frac{1}{2}}T_1T_2 + t^{-\frac{1}{2}}T_2T_1 + T_1T_2T_1.
$$

Since $S_{(2,1,0)} = \{1\}$ then $W_{(2,1,0)}(t) = 1$ and

$$
P_{(2,1,0)} = \frac{t^{\frac{3}{2}}}{W_{(2,1,0)}(t)}\mathbf{1}_0E_{(2,1,0)} = t^{\frac{3}{2}}\mathbf{1}_0t^{-\frac{3}{2}}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1},
$$

and, with $f_{(2,1,0)}, f_{(1,2,0)}, \dots, f_{(0,1,2)}$ as in Section 1.3.1,

$$
\begin{aligned}
P_{(2,1,0)} &= (1 + t^{\frac{1}{2}}T_1 + t^{\frac{1}{2}}T_2 + t^{\frac{2}{2}}T_1T_2 + t^{\frac{2}{2}}T_2T_1 + t^{\frac{3}{2}}T_1T_2T_1)E_{(2,1,0)} \\
&= f_{(2,1,0)} + f_{(1,2,0)} + f_{(2,0,1)} + f_{(1,0,2)} + f_{(0,2,1)} + f_{(0,1,2)} \\
&= (x_1^2x_2 + q\frac{(1-t)}{1-qt^2)}x_1x_2x_3) + (x_1x_2^2 + qt\frac{(1-t)}{(1-qt^2)}x_1x_2x_3) \\
&\qquad + (x_1^2x_3 + qt\frac{(1-t)}{(1-qt^2)}x_1x_2x_3) + (x_1x_3^2 + \frac{(1-t)}{(1-qt^2)}x_1x_2x_3) \\
&\qquad + (x_2^2x_3 + \frac{(1-t)}{(1-qt^2)}x_1x_2x_3) + (x_2x_3^2 + t\frac{(1-t)}{(1-qt^2)}x_1x_2x_3) \\
&= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\
&\qquad + \Big(\frac{(1-t^2)}{(1-qt)}\frac{(1-q^2t)}{(1-qt^2)} + \frac{(1-t)}{(1-q)}\frac{(1-q^2)}{(1-qt)}\Big)x_1x_2x_3.
\end{aligned}
$$

1.3.3. *Symmetrizations for μ with distinct parts when $n=3$.* If $n=3$ and $\lambda_1>\lambda_2>\lambda_3$ then $S_\lambda=\{1\}$ and $W_\lambda(t)=1$ and $w_0=s_1s_2s_1$ and $\ell(w_0)=3$. Then

$$\begin{aligned}
F_{(\lambda_1,\lambda_2,\lambda_3)} &= t^{\frac{3}{2}}\mathbf{1}_0E_{(\lambda_1,\lambda_2,\lambda_3)} = P_{(\lambda_1,\lambda_2,\lambda_3)},\\
F_{(\lambda_2,\lambda_1,\lambda_3)} &= t^{\frac{3}{2}}\Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)},\\
F_{(\lambda_1,\lambda_3,\lambda_2)} &= t^{\frac{3}{2}}\Big(\frac{1-tq^{\lambda_2-\lambda_3}t^{3-2}}{1-q^{\lambda_2-\lambda_3}t^{3-2}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)},\\
F_{(\lambda_2,\lambda_3,\lambda_1)} &= t^{\frac{3}{2}}\Big(\frac{1-tq^{\lambda_1-\lambda_3}t^{3-1}}{1-q^{\lambda_1-\lambda_3}t^{3-1}}\Big)\Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)},\\
F_{(\lambda_3,\lambda_1,\lambda_2)} &= t^{\frac{3}{2}}\Big(\frac{1-tq^{\lambda_1-\lambda_3}t^{3-1}}{1-q^{\lambda_1-\lambda_3}t^{3-1}}\Big)\Big(\frac{1-tq^{\lambda_2-\lambda_3}t^{3-2}}{1-q^{\lambda_2-\lambda_3}t^{3-2}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)},\\
F_{(\lambda_3,\lambda_2,\lambda_1)} &= t^{\frac{3}{2}}\Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)\Big(\frac{1-tq^{\lambda_1-\lambda_3}t^{3-1}}{1-q^{\lambda_1-\lambda_3}t^{3-1}}\Big)\Big(\frac{1-tq^{\lambda_2-\lambda_3}t^{3-2}}{1-q^{\lambda_2-\lambda_3}t^{3-2}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)}.
\end{aligned}$$

For example, using $v_\lambda(1)=3,\ v_\lambda(2)=2,\ v_\lambda(3)=1$, and

$$Y_i^{-1}Y_jE_{(\lambda_1,\lambda_2,\lambda_3)} = q^{\lambda_i-\lambda_j}t^{v_\lambda(i)-v_\lambda(j)}E_{(\lambda_1,\lambda_2,\lambda_3)}$$

and $v_\lambda(i)-v_\lambda(j)=(n-i+1)-(n-j+1)=i-j$,

$$\begin{aligned}
F_{(\lambda_2,\lambda_1,\lambda_3)} &= \mathbf{1}_0t^{\frac{1}{2}}\tau_1^\vee E_{(\lambda_1,\lambda_2,\lambda_3)} = \mathbf{1}_0\Big(t^{\frac{1}{2}}T_1+\frac{(1-t)}{1-Y_1^{-1}Y_2}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)}\\
&= \mathbf{1}_0\Big(t+\frac{(1-t)}{1-Y_1^{-1}Y_2}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)} = \mathbf{1}_0\Big(\frac{1-tY_1^{-1}Y_2}{1-Y_1^{-1}Y_2}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)}\\
&= \mathbf{1}_0\Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)} = \Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)}
\end{aligned}$$

and

$$\begin{aligned}
F_{(\lambda_2,\lambda_3,\lambda_1)} &= \mathbf{1}_0t^{\frac{1}{2}}\tau_2^\vee t^{\frac{1}{2}}\tau_1^\vee E_{(\lambda_1,\lambda_2,\lambda_3)} = \mathbf{1}_0\Big(\frac{1-tY_2^{-1}Y_3}{1-Y_2^{-1}Y_3}\Big)t^{\frac{1}{2}}\tau_1^\vee E_{(\lambda_1,\lambda_2,\lambda_3)}\\
&= \mathbf{1}_0t^{\frac{1}{2}}\tau_1^\vee\Big(\frac{1-tY_1^{-1}Y_3}{1-Y_1^{-1}Y_3}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)} = \mathbf{1}_0t^{\frac{1}{2}}\tau_1^\vee\Big(\frac{1-tq^{\lambda_1-\lambda_3}t^{3-1}}{1-q^{\lambda_1-\lambda_3}t^{3-1}}\Big)E_{(\lambda_1,\lambda_2,\lambda_3)}\\
&= \Big(\frac{1-tq^{\lambda_1-\lambda_3}t^{3-1}}{1-q^{\lambda_1-\lambda_3}t^{3-1}}\Big)\Big(\frac{1-tq^{\lambda_1-\lambda_2}t^{2-1}}{1-q^{\lambda_1-\lambda_2}t^{2-1}}\Big)P_{(\lambda_1,\lambda_2,\lambda_3)}.
\end{aligned}$$

1.3.4. *Examples of the gf_μ condition for a KZ-family.* Let $n=3$ and $\lambda=(2,1,0)$. Then $v_\lambda(1)=3$, $v_\lambda(2)=2$ and $v_\lambda(3)=1$ and

$$Y_iE_{(2,1,0)} = q^{-\lambda_i}t^{-(v_\lambda(i)-1)+\frac{1}{2}(n-1)}E_{(2,1,0)}.$$

Then

$$Y_1=gT_2T_1,\qquad Y_2=T_1^{-1}gT_2,\quad Y_3=T_2^{-1}T_1^{-2}g,$$

Since

$$\begin{aligned}
&f_{(2,1,0)}=E_{(2,1,0)}, && f_{(1,2,0)}=t^{\frac{1}{2}}T_1E_{(2,1,0)}, && f_{(2,0,1)}=t^{\frac{1}{2}}T_2E_{(2,1,0)},\\
&f_{(0,2,1)}=t^{\frac{2}{2}}T_1T_2E_{(2,1,0)}, && f_{(1,0,2)}=t^{\frac{2}{2}}T_2T_1E_{(2,1,0)}, && f_{(0,1,2)}=t^{\frac{3}{2}}T_1T_2T_1E_{(2,1,0)},
\end{aligned}$$

then

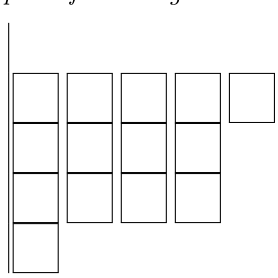
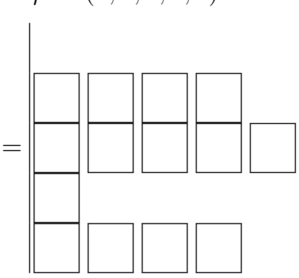
$$\begin{aligned}
gf_{(2,1,0)} &= gE_{(2,1,0)} = T_1T_2(T_2^{-1}T_1^{-1}g)E_{(2,1,0)} = T_1T_2Y_3E_{(2,1,0)} \\
&= q^{-0}t^1T_1T_2E_{(2,1,0)} = f_{(0,2,1)}, \\
gf_{(1,2,0)} &= gt^{\frac12}T_1E_{(2,1,0)} = t^{\frac12}T_2gE_{(2,1,0)} = t^{\frac12}T_2tT_1T_2E_{(2,1,0)} = f_{(0,1,2)}, \\
gf_{(2,0,1)} &= gt^{\frac12}T_2E_{(2,1,0)} = t^{\frac12}T_1T_1^{-1}gT_2E_{(2,1,0)} = t^{\frac12}T_1Y_2E_{(2,1,0)} \\
&= t^{\frac12}T_1q^{-1}t^{-1+1}E_{(2,1,0)} = q^{-1}f_{(1,2,0)}, \\
gf_{(0,2,1)} &= gt^{\frac22}T_1T_2E_{(2,1,0)} = t^{\frac22}T_2gT_2E_{(2,1,0)} = t^{\frac22}T_2T_1q^{-1}t^0E_{(2,1,0)} = q^{-1}f_{(1,0,2)}, \\
gf_{(1,0,2)} &= t^{\frac22}gT_2T_1E_{(2,1,0)} = t^{\frac22}Y_1E_{(2,1,0)} = t^{\frac22}q^{-2}t^{-2+1}E_{(2,1,0)} = q^{-2}f_{(2,1,0)}, \\
gf_{(0,1,2)} &= t^{\frac32}gT_1T_2T_1E_{(2,1,0)} = t^{\frac32}T_1gT_2T_1E_{(2,1,0)} = t^{\frac32}T_1q^{-2}t^{-1}E_{(2,1,0)} \\
&= q^{-2}f_{(1,2,0)}.
\end{aligned}$$

2. Boxes, arms, legs and counting terms

2.0.1. *Common terminology.*

The set of *weak compositions*, $\mathbb{Z}^n_{\geqslant 0} = \{\mu = (\mu_1, \dots, \mu_n) \mid \mu_i \in \mathbb{Z}_{\geqslant 0}\}$,
the set of *strong compositions*, $\mathbb{Z}^n_{>0} = \{\mu = (\mu_1, \dots, \mu_n) \mid \mu_i \in \mathbb{Z}_{>0}\}$,
the *lattice of integral weights*, $\mathbb{Z}^n = \{\mu = (\mu_1, \dots, \mu_n) \mid \mu_i \in \mathbb{Z}\}$,
dominant integral weights, $(\mathbb{Z}^n)^+ = \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n\}$,
partititions of length $\leqslant n$ $(\mathbb{Z}^n_{\geqslant 0})^+ = \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0} \mid \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n\}$.

2.0.2. *Examples of box diagrams.* If $\lambda = (5,4,4,1,0)$ and $\mu = (0,4,5,1,4)$ then

$dg(\lambda) =$  and $dg(\mu) =$ 

To conform to [10, p.2], we draw the box (i,j) as a square in row i and column j using the same coordinates as are usually used for matrices.

The *cylindrical coordinate* of the box (i,j) is the number $i + nj$.

2.0.3. *Formulas for* $\#\mathrm{Nleg}_\mu(i,j)$ *and* $\#\mathrm{Narm}_\mu(i,j)$. Using cylindrical coordinates for boxes define, for a box $b \in dg(\mu)$,

$$\mathrm{attack}_\mu(b) = \{b-1, \dots, b-n+1\} \cap \widehat{dg}(\mu), \tag{12}$$

$$\mathrm{Nleg}_\mu(b) = (b + n\mathbb{Z}_{>0}) \cap dg(\mu) \quad \text{and} \tag{13}$$

$$\mathrm{Narm}_\mu(b) = \{a \in \mathrm{attack}_\mu(b) \mid \#\mathrm{Nleg}_\mu(a) \leqslant \#\mathrm{Nleg}_\mu(b)\}. \tag{14}$$

As in [6, (15)], the number of elements of $\mathrm{Nleg}_\mu(i,j)$ and $\mathrm{Narm}_\mu(i,j)$ are

$$\begin{aligned}
\#\mathrm{Nleg}_\mu(i,j) &= \#\{(i,j') \in dg(\mu) \mid j' > j\} = \mu_i - j, \\
\#\mathrm{Narm}_\mu(i,j) &= \#\{(i',j) \in dg(\mu) \mid i' < i \text{ and } \mu_{i'} \leqslant \mu_i\} \\
&\qquad + \#\{(i', j-1) \in \widehat{dg}(\mu) \mid i' > i \text{ and } \mu_{i'} < \mu_i\},
\end{aligned}$$

where $\widehat{dg}(\mu) = dg(\mu) \cup \{(1,0), \dots, (n,0)\}$.

2.0.4. *Relating HHL arms and legs to Macdonald arms and legs.* If μ is decreasing so that $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ then μ is a partition and

$$\#\mathrm{Narm}_\mu(i,j) = \mu'_{j-1} - i = \mathrm{leg}_\mu(i,j-1) \quad \text{and} \quad \#\mathrm{Nleg}_\mu(i,j) = \mu_i - j = \mathrm{arm}_\mu(i,j).$$

If μ is increasing so that $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ then $w_0\mu = (\mu_n, \dots, \mu_1)$ is a partition and

$$\begin{aligned}\#\mathrm{Narm}_\mu(i,j) &= (w_0\mu)'_j - (n-i)\\ &= \mathrm{leg}_{w_0\mu}(n-i,j)\end{aligned} \quad \text{and} \quad \begin{aligned}\#\mathrm{Nleg}_\mu(i,j) &= \mu_i - j = (w_0\mu)_{n-i} - j\\ &= \mathrm{arm}_{w_0\mu}(n-i,j)\end{aligned}$$

(see [6, remarks before (17)] and [7, p. 136, remarks before Figure 6]).

2.0.5. *Formulas for the number of alcove walks and nonattacking fillings.* The motivation for computing $\#\mathrm{AW}^z_\mu$ and $\#\mathrm{NAF}^z_\mu$ is that the alcove walks formula and the nonattacking fillings formulas for the relative Macdonald polynomial E^z_μ are, respectively,

$$E^z_\mu = \sum_{p \in AW^z_\mu} \mathrm{wt}(p) \qquad \text{and} \qquad E^z_\mu = \sum_{T \in \mathrm{NAF}^z_\mu} \mathrm{wt}(T).$$

(see [5, Theorem 1.1]). The number of terms in the first formula is $\#\mathrm{AW}^z_\mu$ and the number of terms in the second formula is $\#\mathrm{NAF}^z_\mu$.

For a box $(i,j) \in dg(\mu)$ define $u_\mu(i,j)$ by the equation

$$u_\mu(i,j) + 1 = n - \#\mathrm{attack}_\mu(i,j).$$

Since $\#\mathrm{attack}_\mu(i,j) = \#\{i' \in \{1,\dots,i-1\} \mid \mu_{i'} \geqslant j\} + \#\{i' \in \{i+1,\dots,n\} \mid \mu_{i'} \geqslant j-1\}$ then

$$u_\mu(i,j) = \#\{i' \in \{1,\dots,i-1\} \mid \mu_{i'} < j \leqslant \mu_i\} + \#\{i' \in \{i+1,\dots,n\} \mid \mu_{i'} < j-1 < \mu_i\}.$$

Let $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. By [5, Proposition (2.2)] and the definition of alcove walks and nonattacking fillings in [5, (1.11) and (1.7)],

$$\#\mathrm{AW}^z_\mu = 2^{\ell(u_\mu)} = \prod_{(i,j)\in\mu} 2^{u_\mu(i,j)} \qquad \text{and} \qquad \#\mathrm{NAF}^z_\mu = \prod_{(i,j)\in\mu} (u_\mu(i,j)+1). \tag{15}$$

(The right hand side does not depend on the choice of z.) For example (as in [4, Table 1]),

$$\#\mathrm{NAF}^z_{(4,3,3,3,2,2,1,1,0,0)} = \begin{pmatrix} 1\cdot 3\cdot 5\cdot 7 \\ \cdot 1\cdot 3\cdot 5 \\ \cdot 1\cdot 3\cdot 5 \\ \cdot 1\cdot 3\cdot 5 \\ \cdot 1\cdot 3 \\ \cdot 1\cdot 3 \\ \cdot 1 \\ \cdot 1 \end{pmatrix} = 3189375, \qquad \text{for } z \in S_{10}.$$

2.1. The column strict tableaux formula for P_λ. Let λ and μ be partitions such that $\lambda \supseteq \mu$ and λ/μ is a horizontal strip. Following [10, Ch. VI §7 Ex. 2(b)], define

$$\psi_{\lambda/\mu} = \prod_{1\leqslant i<j\leqslant \ell(\mu)} \frac{\left(\dfrac{(q^{\mu_i-\mu_j}t^{j-i+1};q)_\infty (q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q)_\infty}{(q^{\mu_i-\mu_j+1}t^{j-i};q)_\infty (q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q)_\infty}\right)}{\left(\dfrac{(q^{\lambda_i-\mu_j}t^{j-i+1};q)_\infty (q^{\mu_i-\lambda_{j+1}}t^{j-i+1};q)_\infty}{(q^{\lambda_i-\mu_j+1}t^{j-i};q)_\infty (q^{\mu_i-\lambda_{j+1}+1}t^{j-i};q)_\infty}\right)},$$

where the infinite product $(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\cdots$.

A *column strict tableau* of shape λ is a filling $T\colon dg(\lambda) \to \{1,\dots,n\}$ such that

$$T(i,j) \leqslant T(i,j+1) \qquad \text{and} \qquad T(i,j) < T(i+1,j).$$

For a column strict tableau T define

$$\psi_T = \prod_{i=1}^{r} \psi_{\lambda^{(i)}/\lambda^{(i-1)}} \qquad \text{where} \quad \lambda^{(i)} = \{u \in dg(\lambda) \mid T(u) \leqslant i\}.$$

Then [10, Ch. VI (7.13′)] gives

$$P_\lambda = \sum_T \psi_T x^T, \qquad \text{where} \quad x^T = x_1^{\#(1\text{s in } T)} \cdots x_n^{\#(n\text{s in } T)}. \tag{16}$$

By [10, Ch. 1 §3 Ex. 4], this formula for P_λ has

$$\prod_{b\in\lambda} \frac{n+c(b)}{h(b)} \quad \text{terms,} \quad \text{where} \quad \begin{array}{l} c(b) \text{ is the content of the box } b, \\ h(b) \text{ is the hook length at the box } b. \end{array}$$

2.1.1. *Comparing numbers of terms in formulas for* P_λ. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and write $\lambda = (0^{m_0} 1^{m_1} 2^{m_2} \cdots)$ so that m_i is the number of rows of λ of length i. Then number of elements of the orbit $S_n\lambda$ (the number of rearrangements of λ) is

$$\mathrm{Card}(S_n\lambda) = \frac{n!}{m_\lambda!}, \qquad \text{where} \qquad m_\lambda! = m_0! m_1! m_2! \cdots .$$

By (5), the symmetric Macdonald polynomial is given by $P_\lambda = \sum_{\nu\in S_n\lambda} E^z_\lambda$, and using the alcove walks formula for E^z_λ and the nonattacking fillings formulas for E^z_λ provide formulas for P_λ with

$$\frac{n!}{m_\lambda!} \cdot \#\mathrm{AW}^z_\lambda \text{ terms,} \qquad \text{and} \qquad \frac{n!}{m_\lambda!} \cdot \#\mathrm{NAF}^z_\lambda \text{ terms,} \quad \text{respectively.}$$

Alternatively, by Proposition 1.1, there is a constant $(const)$ such that

$$P_\lambda = (const) \sum_{\nu\in S_n\lambda} E^z_{rev(\lambda)}, \qquad \text{where} \quad \begin{array}{l} \text{if } \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0) \text{ with } \lambda_k \neq 0 \\ \text{then } rev(\lambda) = (\lambda_k, \dots, \lambda_2, \lambda_1, 0, \dots, 0). \end{array}$$

Then using the alcove walks formula for $E^z_{rev(\lambda)}$ and the nonattacking fillings formulas for $E^z_{rev(\lambda)}$ provide formulas for P_λ with

$$\frac{n!}{m_\lambda!} \cdot \#\mathrm{AW}^z_{rev(\lambda)} \text{ terms,} \qquad \text{and} \qquad \frac{n!}{m_\lambda!} \cdot \#\mathrm{NAF}^z_{rev(\lambda)} \text{ terms,} \quad \text{respectively.}$$

Let λ be a partition. Let $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ be the conjugate partition to λ so that λ'_j is the length of the jth column of λ. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0)$ with $\lambda_k \neq 0$ let $rev(\lambda) = (\lambda_k, \dots, \lambda_2, \lambda_1, 0, \dots, 0)$. Then $u_\lambda(i,1) = u_{rev(\lambda)}(i,1) = 0$ and if $j > 1$ then $u_\lambda(i,j) = n - \lambda'_{j-1}$ and $u_{rev(\lambda)}(i,j) = n - \lambda'_j$. Thus

$$\#\mathrm{AW}_\lambda = \prod_{\substack{(i,j)\in\lambda \\ j>1}} 2^{n-\lambda'_{j-1}},$$

$$\#\mathrm{NAF}_\lambda = \prod_{\substack{(i,j)\in\lambda \\ j>1}} (n - \lambda'_{j-1} + 1), \qquad \#\mathrm{NAF}_{rev(\lambda)} = \prod_{\substack{(i,j)\in\lambda \\ j>1}} (n - \lambda'_j + 1),$$

and

$$t(\lambda) = n! \cdot \prod_{\substack{(i,j)\in\lambda \\ j>1}} (n - \lambda'_{j-1} + 1),$$

$$c(\lambda) = \prod_{\substack{(i,j)\in\lambda \\ j>1}} \frac{2^{n-\lambda'_{j-1}}}{n - \lambda'_{j-1} + 1}, \qquad r(\lambda) = \prod_{\substack{(i,j)\in\lambda \\ j>1}} \frac{n - \lambda'_j + 1}{n - \lambda'_{j-1} + 1}$$

are formulas for the values provided in the table in [9, end of §3] (Lenart assumes that the parts of λ are distinct so that $m_\lambda! = 1$). For example, if $\lambda = (5,4,2,1,0)$ as in the last row of Lenart's table then

$$t(\lambda) = 5! \cdot \begin{pmatrix} 1\cdot 2\cdot 3\cdot 4\cdot 4 \\ 1\cdot 2\cdot 3\cdot 4 \\ 1\cdot 2 \\ 1 \end{pmatrix},$$

$$c(\lambda) = \frac{\begin{pmatrix} 2^0\cdot 2^1\cdot 2^2\cdot 2^3\cdot 2^3 \\ 2^0\cdot 2^1\cdot 2^2\cdot 2^3 \\ 2^0\cdot 2^1 \\ 2^0 \end{pmatrix}}{\begin{pmatrix} 1\cdot 2\cdot 3\cdot 4\cdot 4 \\ 1\cdot 2\cdot 3\cdot 4 \\ 1\cdot 2 \\ 1 \end{pmatrix}}, \qquad r(\lambda) = \frac{\begin{pmatrix} 1 \\ 1\cdot 3 \\ 1\cdot 3\cdot 4\cdot 4 \\ 1\cdot 3\cdot 4\cdot 4\cdot 5 \end{pmatrix}}{\begin{pmatrix} 1\cdot 2\cdot 3\cdot 4\cdot 4 \\ 1\cdot 2\cdot 3\cdot 4 \\ 1\cdot 2 \\ 1 \end{pmatrix}},$$

so that $t(\lambda) = 552960$, $c(\lambda) = \frac{128}{9} \approx 14.222$ and $r(\lambda) = \frac{15}{2} = 7.5$. To compare this with the number of column strict tableaux of shape $\lambda = (5,4,2,1,0)$ (the number of terms in the formula for P_λ in (16)),

$$\prod_{b\in\lambda} \frac{n+c(b)}{h(b)} = \frac{\begin{pmatrix} 5\cdot 6\cdot 7\cdot 8\cdot 9 \\ 4\cdot 5\cdot 6\cdot 7 \\ 3\cdot 4 \\ 2 \end{pmatrix}}{\begin{pmatrix} 8\cdot 6\cdot 4\cdot 3\cdot 1 \\ 6\cdot 4\cdot 2\cdot 1 \\ 3\cdot 1 \\ 1 \end{pmatrix}} = 5\cdot 7\cdot 3\cdot 5\cdot 7 = 3675,$$

and $\dfrac{552960}{3675} = 150.465$.

3. Converting fillings and alcove walks to paths and pipe dreams

3.0.1. *Hyperplanes and alcoves.* Let $\mathbb{R}^n = \mathfrak{a}^*_{\mathbb{R}} = \mathbb{R}\varepsilon_1 + \cdots + \mathbb{R}\varepsilon_n$. For $i,j,k \in \{1,\ldots,n\}$ with $i<j$ and $\ell \in \mathbb{Z}$ define

$$\begin{aligned} \mathfrak{a}^{\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K} &= \{(\mu_1,\ldots,\mu_n) \in \mathbb{R}^n \mid \mu_i - \mu_j = -\ell\}, \quad \text{and} \\ \mathfrak{a}^{\varepsilon_k^\vee + \ell K} &= \{(\mu_1,\ldots,\mu_n) \in \mathbb{R}^n \mid \mu_k = -\ell\}. \end{aligned} \tag{17}$$

The union of these hyperplanes is

$$\mathcal{H} = \{(\mu_1,\ldots,\mu_n) \in \mathbb{R}^n \mid \text{if } i,j \in \{1,\ldots,n\} \text{ and } i \neq j \text{ then } \mu_i \notin \mathbb{Z} \text{ and } \mu_i - \mu_j \notin \mathbb{Z}\}.$$

An *alcove* is a connected component of

$$\mathbb{R}^n - \mathcal{H}, \quad \text{the complement of the hyperplanes listed in (17).}$$

The *fundamental alcove* is

$$A_1 = \left\{ \mu = (\mu_1,\ldots,\mu_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \mu_1 - \mu_n \in \mathbb{R}_{>0} \text{ and} \\ \text{if } i \in \{1,\ldots,n\} \text{ then } \mu_i \in \mathbb{R}_{(-1,0)} \end{array} \right\}.$$

For $n=2$, some pictures of these hyperplanes and paths in $\mathfrak{a}^*_{\mathbb{R}} \cong \mathbb{R}^2$ are in section 3.0.8.

3.0.2. *Bijection $W \leftrightarrow W\cdot \frac{1}{n}\rho \leftrightarrow \{alcoves\}$.* Let W be the group of n-periodic permutations and define an action of W_{GL_n} on $\mathbb{R}^n$ by

$$\begin{aligned}(18)\qquad & \pi(\mu_1,\dots,\mu_n) = (\mu_n+1,\mu_1,\dots,\mu_n)\\ \text{and}\qquad & s_i(\mu_1,\dots,\mu_n) = (\mu_1,\dots,\mu_{i-1},\mu_{i+1},\mu_i,\mu_{i+1},\dots,\mu_n),\end{aligned}$$

for $i\in\{1,\dots,n-1\}$. Let

$$(19)\qquad \rho = (\tfrac{n-1}{2},\tfrac{n-3}{2},\dots,\tfrac{-(n-1)}{2}) = (n-1,n-2,\dots,1,0) - \tfrac{n-1}{2}(1,1,\dots,1).$$

Then the maps

$$(20)\qquad \begin{matrix} W & \longleftrightarrow & W\cdot\frac{1}{n}\rho & \longleftrightarrow & \{\text{alcoves}\} \\ w & \longmapsto & \frac{1}{n}w\rho & \longmapsto & wA_1\end{matrix} \qquad \text{are bijections,}$$

and so we can identify W with the set of alcoves and with the orbit $W\cdot\frac{1}{n}\rho$. The statement in (20) holds because the stabilizer of $\frac{1}{n}\rho$ under the action of W on $\mathbb{R}^n$ is $\{1\}$.

3.0.3. *Reflections in W.* For any pair $(j,k)\in\mathbb{Z}\times\mathbb{Z}$ with $j\neq k$ define

$$s_{jk}(j)=k,\quad s_{jk}(k)=(j),\quad s_{jk}(i)=i \text{ if } i\neq j \bmod n \text{ and } i\neq k \bmod n..$$

If $i\in\{1,\dots,n-1\}$ and $t_\mu v = ((\mu_1)_{v(1)},(\mu_2)_{v(2)},\dots,(\mu_n)_{v(n)})$ then

$$s_it_\mu v = ((\mu_1)_{v(1)},\dots,(\mu_{i-1})_{v(i-1)},(\mu_{i+1})_{v(i+1)},(\mu_i)_{v(i)},(\mu_{i+2})_{v(i+2)},\dots,(\mu_n)_{v(n)}),$$

so that, in extended one-line notation, s_i acts by switching the ith and $(i+1)$st components. The hyperplane

$$\mathfrak{a}^{\beta^\vee} \text{ between } t_\mu vA_1 \text{ and } s_it_\mu vA_1 \text{ has root} \qquad \beta^\vee = \varepsilon^\vee_{v(i+1)} - \varepsilon^\vee_{v(i)} + (\mu_i-\mu_{i+1})K.$$

3.0.4. *Paths.* A *path* is a piecewise linear function $\gamma\colon\mathbb{R}_{[0,a]}\to\mathbb{R}^n$, where $a\in\mathbb{R}_{>0}$ and $\mathbb{R}_{[0,a]}=\{t\in\mathbb{R}\mid 0\leqslant t\leqslant a\}$. The *concatenation of paths* $\gamma_1\colon\mathbb{R}_{[0,a]}\to\mathfrak{h}^*_{\mathbb{R}}$ *and* $\gamma_2\colon\mathbb{R}_{[0,b]}\to\mathfrak{h}^*_{\mathbb{R}}$ is the path

$$\gamma_1\gamma_2\colon\mathbb{R}_{[0,a+b]}\to\mathfrak{h}^*_{\mathbb{R}} \quad\text{given by}\quad (\gamma_1\gamma_2)(t) = \begin{cases}\gamma_1(t), & \text{if } i\in\mathbb{R}_{[0,a]},\\ \gamma_1(a)+\gamma_2(t-a), & \text{if } t\in\mathbb{R}_{[a,a+b]}.\end{cases}$$

3.0.5. *Paths corresponding to nonattacking fillings.* The *straight line path* $0\to\varepsilon_i$ is

$$\begin{aligned} x_i\colon\ \mathbb{R}_{[0,1]} &\to \mathbb{R}^n\\ t\ &\mapsto t\varepsilon_i.\end{aligned}$$

If T is a nonattacking filling of type (z,μ) then the *word, or path, of* T is

$$\vec{x}_T = \prod_{u\in\mu} x_{T(u)} \qquad \text{taken in increasing order of cylindrical coordinate.}$$

The path, or word,

$$\vec{x}_T = x_{i_1}x_{i_2}\cdots x_{i_\ell} \qquad\text{is}\qquad 0\to\varepsilon_{i_1}\to(\varepsilon_{i_1}+\varepsilon_{i_2})\to\cdots\to\varepsilon_{i_1}+\cdots+\varepsilon_{i_\ell}$$

as a sequence of straight line segments.

3.0.6. *Paths corresponding to alcove walks.* Define paths $\omega\colon \mathbb{R}_{[0,1]} \to \mathbb{R}^n$ and $c_\alpha\colon \mathbb{R}_{[0,1]} \to \mathbb{R}^n$ and $f_\alpha\colon \mathbb{R}_{[0,1]} \to \mathbb{R}^n$ by

$$\omega(t) = \frac{t}{n}(1,1,\dots,1), \qquad c_\alpha(t) = t\alpha \quad \text{and} \quad f_\alpha(t) = \begin{cases} t\alpha, & \text{if } 0 \leqslant t \leqslant \frac{1}{2},\\ (1-t)\alpha, & \text{if } \frac{1}{2} \leqslant t \leqslant 1.\end{cases}$$

Let $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. Let $s_\pi = \pi$ and let $\vec{u}_\mu = s_{i_1}\cdots s_{i_r}$ be a reduced word for u_μ. An *alcove walk* of type $(z, \vec{u}_\mu)$ is

$$\text{(21)} \qquad \text{a sequence} \quad p = (p_0, p_1, \dots, p_r) \quad \text{of elements of } W \quad \text{such that}$$

$p_0 = z$; if $s_{i_k} = \pi$ then $p_k = p_{k-1}\pi$; and if $s_{i_k} \neq \pi$ then $p_k \in \{p_{k-1}, p_{k-1}s_{i_k}\}$. The path corresponding to p is

$$\text{(22)} \qquad \gamma_{\beta_1}\cdots\gamma_{\beta_\ell}, \qquad \text{where} \qquad \gamma_{\beta_j} = \begin{cases} f_{p_{k-1}\alpha_{i_k}}, & \text{if } p_k = p_{k-1},\\ c_{p_{k-1}\alpha_{i_k}}, & \text{if } p_k = p_{k-1}s_{i_k},\\ \omega, & \text{if } p_k = p_{k-1}\pi,\end{cases}$$

See §6.0.3 for pictures in $\mathbb{R}^2$, for $n=2$. The pictures of paths for $n=3$ in sections 3.0.9 and 3.0.9 are projections from $\mathbb{R}^3$ to the plane $\{(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3 \mid \gamma_1+\gamma_2+\gamma_3 = 0\}$.

3.0.7. *Pipe dreams corresponding to nonattacking fillings.* Let $\mu \in \mathbb{Z}^n_{\geqslant 0}$. A *filling of* $dg(\mu)$ is a function $T\colon dg(\mu) \to \{1,\dots,n\}$. If the filling is nonattacking then it satisfies the *column distinct condition*,

$$\text{(CD)} \qquad \text{if } j \in \mathbb{Z}_{\geqslant 0} \text{ and } (i,j),(i',j) \in D \text{ then } T(i,j) \neq T(i',j),$$

and so the filling T can be converted into a *pipe dream* $P\colon \{1,\dots,n\} \times \mathbb{Z}_{\geqslant 0} \to \{1,\dots,n\}$ by setting

$$\text{(23)} \qquad P(k,j) = i \qquad \text{if and only if} \qquad T(i,j) = k,$$

and putting $P(k,j) = 0$ if there does not exist $i \in \{1,\dots,n\}$ such that $T(i,j) = k$. (This bijection is given in [1, (5.10)] and [4, Definition A.6]. In [4, Definition A.6] the pipe dreams are the *multiline queues* and the fillings are the Queue Tableaux and in [1, (5.10)] the pipe dreams are the μ-legal configurations.) The column distinct condition on T is exactly the condition that P obtained in this way is a function.

For example,

$$\begin{array}{c|cc} 1 & 1 & 1\\ 2 & 2 & 2\\ 3 & & \end{array} \qquad \begin{array}{c|cc} 1 & 1 & 1\\ 2 & 2 & 3\\ 3 & & \end{array} \qquad \begin{array}{c|cc} 1 & 1 & 3\\ 2 & 2 & 1\\ 3 & & \end{array} \qquad \begin{array}{c|cc} 1 & 1 & 3\\ 2 & 2 & 2\\ 3 & & \end{array}$$

are the 4 nonattacking fillings of $\mu = (2,2,0)$. Converting these to pipe dreams gives

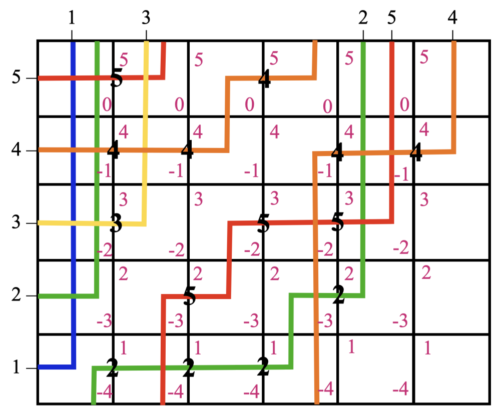
$$\begin{pmatrix} 1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1\\ 2 & 2 & 0\\ 3 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2\\ 2 & 2 & 1\\ 3 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0\\ 2 & 2 & 2\\ 3 & 0 & 1 \end{pmatrix}$$

The example in [1, Figure 5] has

$$\text{filling} \quad \begin{array}{c|ccccc} 1 & & & & & \\ 2 & 1 & 1 & 1 & 2 & \\ 3 & 3 & & & & \\ 4 & 4 & 4 & 5 & 4 & 4\\ 5 & 5 & 2 & 3 & 3 & \end{array} \qquad \text{with corresponding pipe dream} \qquad \left(\begin{array}{c|ccccc} 1 & 2 & 2 & 2 & 0 & 0\\ 2 & 0 & 5 & 0 & 2 & 0\\ 3 & 3 & 0 & 5 & 5 & 0\\ 4 & 4 & 4 & 0 & 4 & 4\\ 5 & 5 & 0 & 4 & 0 & 0 \end{array}\right)$$

and the picture of this pipe dream from [1, Figure 5] is

(24)

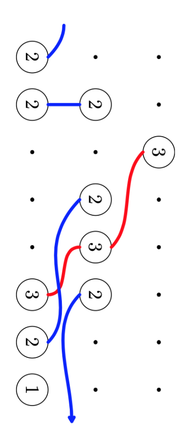


([1] index rows bottom to top instead of top to bottom). The example in [4, Figures 3 and 12] has

$$\text{nonattacking filling}\quad T = \begin{array}{c|ccc} 6 & 6 & 5 & 3 \\ 1 & 1 & 6 & \\ 2 & 2 & 2 & \\ 7 & 7 & 4 & \\ 8 & 8 & & \\ 3 & & & \\ 4 & & & \\ 5 & & & \end{array} \qquad \text{and pipe dream}\quad P = \left(\begin{array}{c|ccc} 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 6 & 0 & 0 & 1 \\ 7 & 0 & 4 & 0 \\ 8 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 4 & 4 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{array}\right)$$

and the picture of this pipe dream (multiline queue in the terminology of [4]) from [4, Fig. 3] is

the multiline queue



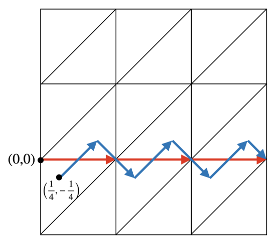
3.0.8. *Alcove walks, nonattacking fillings and paths for $E_{(3,0)}$.* The explicit expansion of $E_{(3,0)}$ is

$$E_{(3,0)} = x_1^3 + \Big(\frac{1-t}{1-q^2t}\Big)q^2x_1x_2^2 + \Big(\Big(\frac{1-t}{1-qt}\Big)q + \Big(\frac{1-t}{1-q^2t}\Big)\Big(\frac{1-t}{1-qt}\Big)q^2\Big)x_1^2x_2.$$

The nonattacking fillings, words, paths, alcove walks and corresponding weights for $E_{(3,0)}$ are

$$\begin{array}{c|l} 1 & 1\ 1\ 1 \\ 2 & \end{array}$$

$x_1x_1x_1$

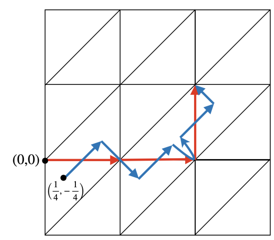


$\pi s_1 \pi s_1$

1

$$\begin{array}{c|l} 1 & 1\ 1\ 2 \\ 2 & \end{array}$$

$x_1x_1x_2$

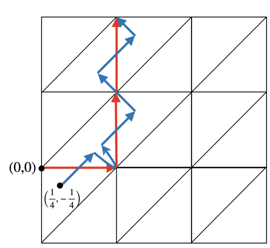


$\pi s_1 \pi 1 \pi$

$q\left(\frac{1-t}{1-qt}\right)$

$$\begin{array}{c|l} 1 & 1\ 2\ 2 \\ 2 & \end{array}$$

$x_1x_2x_2$

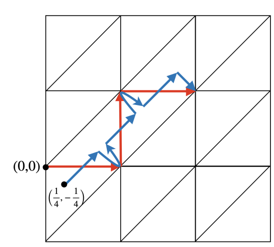


$\pi 1 \pi s_1 \pi$

$q^2\left(\frac{1-t}{1-q^2t}\right)$

$$\begin{array}{c|l} 1 & 1\ 2\ 1 \\ 2 & \end{array}$$

$x_1x_2x_1$



$\pi 1 \pi 1 \pi$

$q^2\left(\frac{1-t}{1-qt}\right)\left(\frac{1-t}{1-qt}\right)$

The first row contains the nonattacking fillings. The second row contains the words of the nonattacking fillings. The red paths are the paths corresponding to the words of the nonattacking fillings, and the blue paths are the paths corresponding to the alcove walks. We used a shortened notation for the alcove walks so that

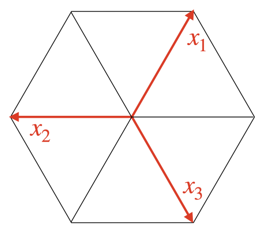
$$\begin{array}{ll}
\pi s_1 \pi s_1 \pi & \text{represents the alcove walk } (1, \pi, \pi s_1, \pi s_1 \pi, \pi s_1 \pi s_1, \pi s_1 \pi s_1 \pi), \\
\pi s_1 \pi 1 \pi & \text{represents the alcove walk } (1, \pi, \pi s_1, \pi s_1 \pi, \pi s_1 \pi, \pi s_1 \pi^2), \\
\pi 1 \pi s_1 \pi & \text{represents the alcove walk } (1, \pi, \pi, \pi^2, \pi^2 s_1, \pi^2 s_1 \pi), \\
\pi 1 \pi 1 \pi & \text{represents the alcove walk } (1, \pi, \pi, \pi^2, \pi^2, \pi^3).
\end{array}$$

The last row contains the weights of the alcove walks (which are the same as the weights of the nonattacking fillings to illustrate that the factors of the form $\left(\frac{1-t}{1-q^a t^b}\right)$ are in bijection with the folds of the blue path.

3.0.9. *Alcove walks, nonattacking fillings and pipe dreams for $E_{(2,0.1)}$.* In the orthogonal projection from $\mathbb{R}^3$ to the plane

$$\{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \mid \gamma_1 + \gamma_2 + \gamma_3 = 0\}$$

(so that we can draw 2-dimensional pictures), the straight line paths x_1, x_2, x_3 to ε_1, ε_2, ε_3, respectively, are pictured as

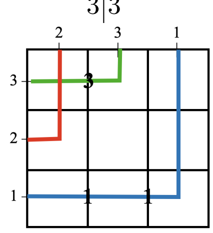
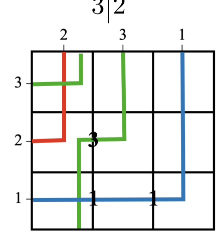


The explicit expansion of $E_{(2,0,1)}$ is

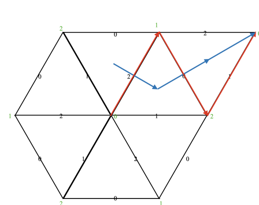
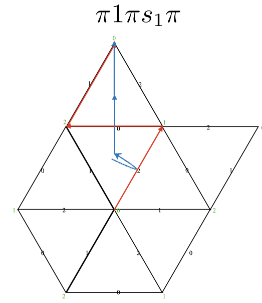
$$E_{(2,0,1)} = x_1x_3x_1 + \frac{1-t}{1-qt}x_1x_2x_1 + qt\frac{1-t}{1-qt^2}x_1x_3x_2 + q\frac{1-t}{1-qt}\frac{1-t}{1-qt^2}x_1x_2x_3$$

The nonattacking fillings, words, paths, alcove walks and corresponding weights for $E_{(2,0,1)}$ are

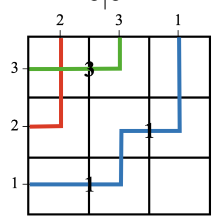
$$\begin{array}{c|cc} 1 & 1 & 1 \\ 2 & & \\ 3 & 3 & \end{array} \qquad\qquad \begin{array}{c|cc} 1 & 1 & 1 \\ 2 & & \\ 3 & 2 & \end{array}$$

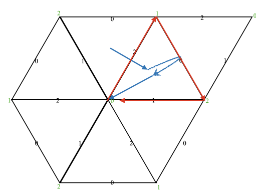
$\pi s_1 \pi s_1 \pi$ $\qquad$ $\pi 1 \pi s_1 \pi$

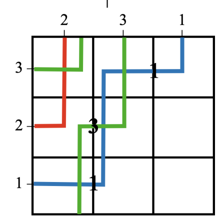
$$\begin{array}{c|l} 1 & 1\ 2 \\ 2 & \\ 3 & 3 \end{array}$$



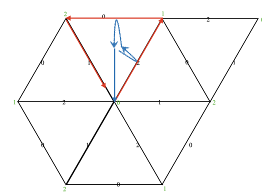
$\pi s_1 \pi 1 \pi$



$$\begin{array}{c|l} 1 & 1\ 3 \\ 2 & \\ 3 & 2 \end{array}$$



$\pi 1 \pi 1 \pi$



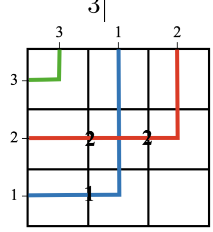
where we have used the same shortened notation for alcove walks as in the table in Section 3.0.8. The sections of type ω in the paths corresponding to the alcove walks (see (22)) are not visible in these pictures since the pictures are in a projection orthogonal to the direction of ω.

3.0.10. *Alcove walks, nonattacking fillings and pipe dreams for $E_{(1,2,0)}$.* The explicit expansion of $E_{(1,2,0)}$ is

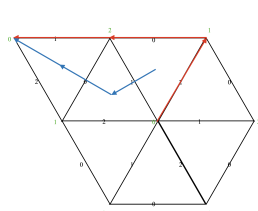
$$E_{(1,2,0)} = x_1x_2x_2 + \frac{1-t}{1-qt}x_1x_2x_1 + q\frac{(1-qt^2)}{(1-qt)}\frac{(1-t)}{(1-qt^2)}x_1x_2x_3$$

The nonattacking fillings, words, paths, alcove walks and corresponding weights for $E_{(1,2,0)}$ are

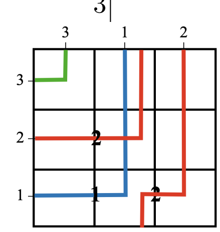
$$\begin{array}{c|l} 1 & 1 \\ 2 & 2\ 2 \\ 3 & \end{array}$$



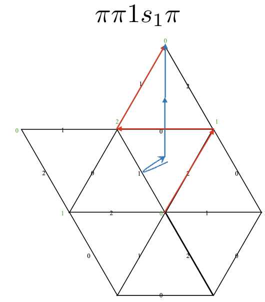
$\pi\pi s_2 s_1 \pi$



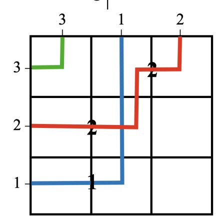
$$\begin{array}{c|l} 1 & 1 \\ 2 & 2\ 1 \\ 3 & \end{array}$$



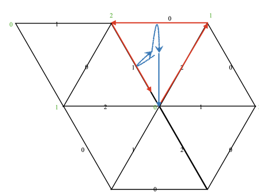
$\pi\pi 1 s_1 \pi$



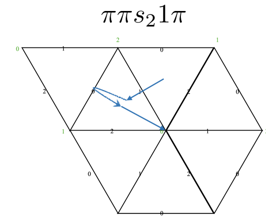
$$\begin{array}{c|l} 1 & 1 \\ 2 & 2\ 3 \\ 3 & \end{array}$$



$\pi\pi 11\pi$



$\pi\pi s_2 1\pi$



where we have used the same shortened notation for alcove walks as in the table in Section 3.0.8. The sections of type ω in the paths corresponding to the alcove walks (see (22)) are not visible in these pictures since the pictures are in a projection orthogonal to the direction of ω. For this example, there are 4 alcove walks and 3 nonattacking fillings.

4. Reduced words and inversions

4.0.1. *Examples of the inversion set* $\mathrm{Inv}(w)$. Define n-periodic permutations π and $s_0, s_1, \ldots, s_{n-1} \in W$ by

$$\pi(i) = i+1, \quad \text{for } i \in \mathbb{Z}, \tag{25}$$

$$\begin{matrix} s_i(i) = i+1, \\ s_i(i+1) = i, \end{matrix} \qquad \text{and} \quad s_i(j) = j \ \text{ for } j \in \{0, 1, \ldots, i-1, i+2, \ldots, n-1\}. \tag{26}$$

An *inversion* of a bijection $w\colon \mathbb{Z} \to \mathbb{Z}$ is

$$(j,k) \in \mathbb{Z} \times \mathbb{Z} \quad \text{with} \quad j<k \text{ and } w(j) > w(k).$$

and the affine root corresponding to an inversion

$$(i,k) = (i, j+\ell n) \ \text{ with } i,j \in \{1,\ldots,n\} \text{ and } \ell \in \mathbb{Z}, \qquad \text{is} \quad \beta^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K. \tag{27}$$

Let $n=3$. The element

$$w = s_1 s_2 \quad \text{has} \quad w(1) = 2,\ w(2) = 3,\ w(3) = 1,$$

and $w(1) > w(3)$ and $w(2) > w(3)$ and

$$\mathrm{Inv}(w) = \{\alpha_2^\vee, s_2\alpha_1^\vee\} = \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee\}.$$

The element

$$w = s_2 s_1 \quad \text{has} \quad w(1) = 3,\ w(2) = 1,\ w(3) = 2,$$

and $w(1) > w(2)$ and $w(1) > w(3)$ and

$$\mathrm{Inv}(w) = \{\alpha_1^\vee, s_1\alpha_2^\vee\} = \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee\}.$$

These are examples of [5, (2.11)].

4.0.2. *Relations in the affine Weyl group W.* The following relations are useful when working with n-periodic permutations.

Proposition 4.1. *Then*

$$s_0 = t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}, \qquad t_{\varepsilon_1^\vee} = \pi s_{n-1} \cdots s_2 s_1, \tag{28}$$

$$\textit{and} \qquad t_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i, \qquad \pi s_i \pi^{-1} = s_{i+1}, \tag{29}$$

for $i \in \{1, \ldots, n-1\}$.

Proof. Proof of (28): If $i \notin \{1, n\}$

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(i) t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(i) = i = s_0(i).$$

If $i = 1$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(1) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(n) = n - n = 0 = s_0(1),$$

and, if $i = n$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(n) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(1) = 1 + n = s_0(n),$$

For $i \in \{2, \ldots, n\}$

$$\begin{aligned} \pi s_{n-1} \cdots s_1(i) &= \pi(i-1) = i = t_{\varepsilon_1}(i), \qquad \text{and} \\ \pi s_{n-1} \cdots s_1(1) &= \pi(n) = n + 1 = t_{\varepsilon_1}(1). \end{aligned}$$

Proof of (29):

$$\begin{aligned} s_i t_{\varepsilon_i^\vee} s_i(i) &= s_i t_{\varepsilon_i^\vee}(i+1) = s_i(i+1) = i = t_{\varepsilon_{i+1}^\vee}(i), \\ s_i t_{\varepsilon_i^\vee} s_i(i+1) &= s_i t_{\varepsilon_i^\vee}(i) = s_i(i+n) = i+1+n, = t_{\varepsilon_{i+1}^\vee}(i+1), \text{ and} \\ s_i t_{\varepsilon_i^\vee} s_i(j) &= s_i t_{\varepsilon_i^\vee}(j) = s_i(j) = j = t_{\varepsilon_{i+1}^\vee}(j), \end{aligned}$$

if $j \in \{1, \ldots, n\}$ and $j \notin \{i, i+1\}$. Finally,

$$\begin{aligned} \pi s_i \pi^{-1}(i) &= \pi s_i(i-1) = \pi(i) = i+1 = s_{i+1}(i), \quad \text{and} \\ \pi s_i \pi^{-1}(i+1) &= \pi s_i(i) = \pi(i+1) = i+2 = s_{i+1}(i+1). \end{aligned}$$

□

4.0.3. *The "affine Weyl group" and the "extended affine Weyl group".* The type GL_n affine Weyl group W is generated by $s_1, \ldots, s_n$ and π. The group W contains also s_0 and all the elements t_μ for $\mu \in \mathbb{Z}^n$. The *projection homomorphism* is the group homomorphism $\overline{}\colon W \to S_n$ given by

$$\overline{t_\mu v} = v, \qquad \text{for } \mu \in \mathbb{Z}^n \text{ and } v \in S_n. \tag{30}$$

The subgroup W_{PGL_n} generated by $s_0, s_1, \ldots, s_{n-1}$ is the *type PGL_n-affine Weyl group.*

$$\begin{aligned} W_{PGL_n} &= \{t_\mu v \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \text{ with } \mu_1 + \cdots + \mu_n = 0 \text{ and } v \in S_n\}, \quad \text{and} \\ W_{GL_n} &= W = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\} = \{\pi^h w \mid h \in \mathbb{Z}, w \in W_{PGL_n}\}. \end{aligned}$$

Then

$$W_{GL_n} = \mathbb{Z}^n \rtimes S_n = \Omega \ltimes W_{PGL_n}, \qquad \text{where} \quad \Omega = \{\pi^h \mid h \in \mathbb{Z}\} \quad \text{with} \quad \Omega \cong \mathbb{Z}.$$

The symbols $\ltimes$ and $\rtimes$ are brief notations whose purpose is to indicate that the relations in (29) hold.

The group W_{PGL_n} is also a quotient of W_{GL_n}, by the relation $\pi = 1$. The *type SL_n affine Weyl group* is the quotient of W_{GL_n} by the relation $\pi^n = 1$. This is equivalent to putting a relation requiring

$$t_\mu = t_\nu \qquad \text{if } \mu_i = \nu_i \bmod n \text{ for } i \in \{1, \dots, n\}.$$

As explained in [13, Ch. 3, Exercise after Corollary 5], there is a Chevalley group G_d for each positive integer d dividing n. The group G_d is a central extension of PGL_n by $\mathbb{Z}/d\mathbb{Z}$ (so that $G_1 = PGL_n$ and $G_n = SL_n$). Each of these groups G_d has an affine Weyl group W_{G_d}. The group W_{G_d} is the quotient of W_{GL_n} by the relation $\pi^d = 1$, and is an extension of W_{PGL_n} by $\mathbb{Z}/d\mathbb{Z}$. The group W_{PGL_n} is sometimes called the "affine Weyl group of type A" and the groups W_{GL_n} and W_{G_d} for $d \neq 1$ are sometimes called the "extended affine Weyl groups of type A". We prefer the more specific terminologies "affine Weyl group of type PGL_n" for W_{PGL_n}, "affine Weyl group of type SL_n" for W_{SL_n}, "affine Weyl group of type GL_n" for W_{GL_n}, and "affine Weyl group of type $PGL_n \veebar (\mathbb{Z}/d\mathbb{Z})$" for W_{G_d} (the symbol $\veebar$ indicates a central extension).

4.0.4. *The elements u_μ, v_μ, z_μ and t_μ.* Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and let u_μ be the minimal length n-periodic permutation such that

$$u_\mu(0, 0, \dots, 0) = (\mu_1, \dots, \mu_n).$$

Let $\lambda = (\lambda, \dots, \lambda_n)$ be the weakly decreasing rearrangement of μ and let

$$\begin{aligned} &z_\mu \in S_n \quad \text{be minimal length such that} \quad z_\mu \lambda = \mu, \quad \text{and let} \\ &v_\mu \in S_n \quad \text{be minimal length such that } v_\mu \mu \text{ is weakly increasing.} \end{aligned}$$

Let $t_\mu \colon \mathbb{Z} \to \mathbb{Z}$ be the n-periodic permutation determined by

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \dots, \quad t_\mu(n) = n + n\mu_n. \tag{31}$$

4.0.5. *Relating u_μ, v_μ, z_μ to $u_\lambda, v_\lambda, z_\lambda$.* Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Let $S_\lambda = \{w \in S_n \mid w\lambda = \lambda\}$ be the stabilizer of λ in S_n. Let

w_0 be the longest element in S_n,
w_λ the longest length element in S_λ, and
w^λ the minimal length element in the coset $w_0 S_\lambda$,

so that

$$w_0 = w^\lambda w_\lambda \qquad \text{and} \qquad \binom{n}{2} = \ell(w_0) = \ell(w^\lambda) + \ell(w_\lambda).$$

Let $\mu \in \mathbb{Z}^n$ and let λ be the decreasing rearrangement of λ. Let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu \lambda$. Then $z_\lambda = 1$,

$$t_\mu = u_\mu v_\mu = (z_\mu u_\lambda) v_\mu \qquad \text{and} \quad t_\lambda = u_\lambda v_\lambda = u_\lambda (w^\lambda)^{-1}, \quad \text{with}$$

$$\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu) = \ell(z_\mu) + \ell(u_\lambda) + \ell(v_\mu) \quad \text{and} \quad \ell(t_\lambda) = \ell(u_\lambda) + \ell((w^\lambda)^{-1}).$$

Using that $z_\mu t_\lambda z_\mu^{-1} = t_{z_\mu \lambda} = t_\mu$ gives that the elements u_μ and v_μ are given in terms of z_μ, u_λ and w^λ by

$$u_\mu = z_\mu u_\lambda \quad \text{and} \quad v_\mu = v_\lambda z_\mu^{-1} = (w^\lambda)^{-1} z_\mu^{-1} = (z_\mu w^\lambda)^{-1} = (z_\mu w_0 w_\lambda)^{-1} = w_\lambda w_0 z_\mu^{-1},$$

since $v_\lambda = (w^\lambda)^{-1}$ and $v_\lambda = v_\mu z_\mu$ with $\ell((w_\lambda)^{-1}) = \ell(v_\lambda) = \ell(v_\mu) + \ell(z_\mu)$.

4.0.6. *Inversions of t_{ε_1}, $t_{-\varepsilon_1}$ and t_{ε_2}.* Let t_μ be as in (31) and let $\varepsilon_i = (0,\dots,0,1,0,\dots,0)$ where the 1 appears in the ith position. Then

$$\begin{aligned}
t_{\varepsilon_1} &= (1_1, 0_2, \dots, 0_n) = \begin{pmatrix} 1 & 2 \cdots & n \\ n+1 & 2 \cdots & n \end{pmatrix} = \pi s_{n-1}\cdots s_1, \\
t_{-\varepsilon_1} &= (-1_1, 0_2, \dots, 0_n) = \begin{pmatrix} 1 & 2 \cdots & n \\ 1-n & 2 \cdots & n \end{pmatrix} = s_1\cdots s_{n-1}\pi^{-1}, \\
t_{\varepsilon_1}s_1 &= (0_2, 1_1, 0_3, \dots, 0_n) = \begin{pmatrix} 1 & 2 & 3 \cdots & n \\ 2 & 1+n & 3 \cdots & n \end{pmatrix} = \pi s_{n-1}\cdots s_2, \\
s_1 t_{\varepsilon_1} &= (1_2, 0_1, 0_3, \dots, 0_n) = \begin{pmatrix} 1 & 2 & 3 \cdots & n \\ 2+n & 1 & 3 \cdots & n \end{pmatrix} = s_1\pi s_{n-1}\cdots s_1, \\
t_{\varepsilon_2} &= s_1 t_{\varepsilon_1} s_1 = (0_1, 1_2, 0_3, \dots, 0_n) = \begin{pmatrix} 1 & 2 & 3 \cdots & n \\ 1 & 2+n & 3 \cdots & n \end{pmatrix} = s_1\pi s_{n-1}\cdots s_2,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Inv}(t_{\varepsilon_1}) &= \{(1,2),(1,3),\dots,(1,n)\} \\
&= \{\alpha_1^\vee, s_1\alpha_2^\vee, \dots, s_1\cdots s_{n-2}\alpha_{n-1}^\vee\} = \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee, \dots, \varepsilon_1^\vee - \varepsilon_n^\vee\} \\
\mathrm{Inv}(t_{-\varepsilon_1}) &= \{(2-n,1),(3-n,1),\dots,(n-n,1)\} \\
&= \{(n,1+n),(n-1,1+n),\dots,(2,1+n)\} \\
&= \{\pi\alpha_{n-1}^\vee, \pi s_{n-1}\alpha_{n-2}^\vee, \dots, \pi s_{n-1}\cdots s_2\alpha_1^\vee\} \\
&= \{\varepsilon_n^\vee - (\varepsilon_1^\vee - K), \varepsilon_{n-1}^\vee - (\varepsilon_1^\vee - K), \dots \varepsilon_2^\vee - (\varepsilon_1^\vee - K)\} \\
\mathrm{Inv}(t_{\varepsilon_1}s_1) &= \{(2,3),\dots,(2,n)\} \\
&= \{\alpha_2^\vee, s_2\alpha_3^\vee, \dots, s_2\cdots s_{n-2}\alpha_{n-1}^\vee\} = \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \dots, \varepsilon_2^\vee - \varepsilon_n^\vee\} \\
\mathrm{Inv}(s_1t_{\varepsilon_1}) &= \{(1,2),(1,3),\dots,(1,n),(1-n,2)\} = \{(1,2),(1,3),\dots,(1,n),(1,2+n)\} \\
&= \{\alpha_1^\vee, s_1\alpha_2^\vee, \dots, s_1\cdots s_{n-2}\alpha_{n-1}^\vee, s_1\cdots s_{n-2}s_{n-1}\pi^{-1}\alpha_1^\vee\} \\
&= \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee, \dots, \varepsilon_1^\vee - \varepsilon_n^\vee, (\varepsilon_1^\vee + K) - \varepsilon_2^\vee\} \\
\mathrm{Inv}(t_{\varepsilon_2}) &= \{((2,3),\dots,(2,n),(2-n,1)\} = \{((2,3),\dots,(2,n),(2,1+n)\} \\
&= \{\alpha_2^\vee, s_2\alpha_3^\vee, \dots, s_2\cdots s_{n-2}\alpha_{n-1}^\vee, s_2\cdots s_{n-2}s_{n-1}\pi^{-1}\alpha_1^\vee\} \\
&= \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \dots, \varepsilon_2^\vee - \varepsilon_n^\vee, (\varepsilon_2^\vee + K) - \varepsilon_1^\vee\},
\end{aligned}$$

where we have used

$$\begin{aligned}
s_1\cdots s_{n-1}\pi^{-1}\alpha_1^\vee &= s_1\cdots s_{n-1}\pi^{-1}(\varepsilon_1^\vee - \varepsilon_2^\vee) \\
&= s_1\cdots s_{n-1}((\varepsilon_n^\vee + K) - \varepsilon_1^\vee) = (\varepsilon_1^\vee + K) - \varepsilon_2^\vee
\end{aligned}$$

and

$$s_2\cdots s_{n-1}\pi^{-1}\alpha_1^\vee = s_2\cdots s_{n-1}((\varepsilon_n^\vee + K) - \varepsilon_1^\vee) = (\varepsilon_2^\vee + K) - \varepsilon_1^\vee.$$

4.0.7. *The elements u_μ and v_μ for $\mu = (0,4,5,1,4)$.* Let u_μ, v_μ, z_μ and t_μ be as in Section 4.0.4. If $\mu = (0,4,5,1,4)$ then

$$\lambda = (5,4,4,1,0) \qquad \text{and} \qquad z_\mu = s_2s_4s_1s_2s_3s_4,$$

since $(5,4,4,1,0) \overset{s_1s_2s_3s_4}{\longrightarrow} (0,5,4,4,1) \overset{s_4}{\to} (0,5,4,1,4) \overset{s_2}{\to} (0,4,5,1,4)$. Also

$$v_\mu = s_4s_2s_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}, \qquad \text{with} \qquad \begin{array}{l} v_\mu(1) = 1 = 1, \\ v_\mu(2) = 3 = 1 + \#\{1\}, \\ v_\mu(3) = 5 = 1 + \#\{1,2\} + \#\{4\}, \\ v_\mu(4) = 2 = 1 + \#\{1\}, \\ v_\mu(5) = 4 = 1 + \#\{2,4\}. \end{array}$$

Then $v_\mu = (0_1, 0_3, 0_5, 0_3, 0_4)$ and

$$\begin{aligned}\mathrm{Inv}(v_\mu) &= \{(2,4),(3,4),(3,5)\} = \{\alpha_3^\vee, s_3\alpha_2^\vee, s_3s_2\alpha_4^\vee\} \\ &= \{\varepsilon_3^\vee - \varepsilon_4^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \varepsilon_3^\vee - \varepsilon_5^\vee\}.\end{aligned}$$

Then, with $n = 5$,

$$\begin{aligned} v_\mu^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} = (0_1, 0_4, 0_2, 0_5, 0_3) \quad \text{and} \\ u_\mu &= t_\mu v_\mu^{-1} = (0_1, 4_3, 5_5, 1_2, 4_4) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4+n & 2+4n & 5+4n & 3+5n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 9 & 22 & 25 & 28 \end{pmatrix}. \end{aligned}$$

Then

$$\ell(t_\lambda) = \begin{pmatrix} \ \ (5-4)+(5-4)+(5-1)+(5-0) \\ +(4-4)+(4-1)+(4-0) \\ +(4-1)+(4-0) \\ +(1-0) \end{pmatrix} = 26 = \ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu),$$

with

$$\ell(u_\mu) = 6 + 7\cdot 2 + 3 = 23, \quad \ell(v_\mu) = 3, \quad \ell(z_\mu) = 6.$$

The decreasing rearrangement of $\mu = (0,4,5,1,4)$ is $\lambda = (5,4,4,1,0)$ and

$$z_\lambda = 1, \quad w_\lambda = s_2, \quad v_\lambda = w_0 s_2$$

4.0.8. *The box greedy reduced word for u_μ.* If $\mu = (0,4,5,1,4)$ then the box greedy reduced word for u_μ is

$$u_\mu^\square = (s_1\pi)^6 (s_2s_1\pi)^7 (s_3s_2s_1\pi) = \left| \begin{array}{lllll} \boxed{s_1\pi} & \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \\ \boxed{s_1\pi} & \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_3s_2s_1\pi} \\ \boxed{s_1\pi} & & & & \\ \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \end{array} \right. \tag{32}$$

and the length of u_μ is

$$\ell(u_\mu) = 6 + 14 + 3 = 23, \qquad \text{since} \quad \ell(\pi) = 0 \quad \text{and} \quad \ell(s_i) = 1.$$

Using one-line notation for n-periodic permutations, the computation verifying the expression for $u_\mu^\square$ is

$$
\begin{aligned}
&(0_1,4_3,5_5,1_2,4_4) \xrightarrow{s_1} (4_3,0_1,5_5,1_2,4_4) \xrightarrow{\pi^{-1}} \\
&(0_1,5_5,1_2,4_4,3_3)) \xrightarrow{s_1} (5_5,0_1,1_2,4_4,3_3)) \xrightarrow{\pi^{-1}} \\
&(0_1,1_2,4_4,3_3,4_5)) \xrightarrow{s_1} (1_2,0_1,4_4,3_3,4_5)) \xrightarrow{\pi^{-1}} \\
&(0_1,4_4,3_3,4_5,0_2)) \xrightarrow{s_1} (4_4,0_1,3_3,4_5,0_2)) \xrightarrow{\pi^{-1}} \\
&(0_1,3_3,4_5,0_2,3_4)) \xrightarrow{s_1} (3_3,0_1,4_5,0_2,3_4)) \xrightarrow{\pi^{-1}} \\
&(0_1,4_5,0_2,3_4,2_3)) \xrightarrow{s_1} (4_5,0_1,0_2,3_4,2_3)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,3_4,2_3,3_5)) \xrightarrow{s_2} (0_1,3_4,0_2,2_3,3_5)) \xrightarrow{s_1} (3_4,0_1,0_2,2_3,3_5)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,2_3,3_5,2_4)) \xrightarrow{s_2} (0_1,2_3,0_2,3_5,2_4)) \xrightarrow{s_1} (2_3,0_1,0_2,3_5,2_4)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,3_5,2_4,1_3)) \xrightarrow{s_2} (0_1,3_5,0_2,2_4,1_3)) \xrightarrow{s_1} (3_5,0_1,0_2,2_4,1_3)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,2_4,1_3,2_5)) \xrightarrow{s_2} (0_1,2_4,0_2,1_3,2_5)) \xrightarrow{s_1} (2_4,0_1,0_2,1_3,2_5)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,1_3,2_5,1_4)) \xrightarrow{s_2} (0_1,1_3,0_2,2_5,1_4)) \xrightarrow{s_1} (1_3,0_1,0_2,2_5,1_4)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,2_5,1_4,0_3)) \xrightarrow{s_2} (0_1,2_5,0_2,1_4,0_3)) \xrightarrow{s_1} (2_5,0_1,0_2,1_4,0_3)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,1_4,0_3,1_5)) \xrightarrow{s_2} (0_1,1_4,0_2,0_3,1_5)) \xrightarrow{s_1} (1_4,0_1,0_2,0_3,1_5)) \xrightarrow{\pi^{-1}} \\
&(0_1,0_2,0_3,1_5,0_4)) \xrightarrow{s_3} (0_1,0_2,1_5,0_3,0_4)) \xrightarrow{s_2} (0_1,1_5,0_2,0_3,0_4)) \\
&\qquad \xrightarrow{s_1} (1_5,0_1,0_2,0_3,0_4)) \xrightarrow{\pi^{-1}} (0_1,0_2,0_3,0_4,0_5))
\end{aligned}
$$

4.0.9. *Inversions of u_μ.* If $\mu = (0,4,5,1,4)$ then the inversion set of u_μ is

$$
\mathrm{Inv}(u_\mu) = \left|\;
\begin{array}{|c|c|c|c|c|}
\hline
\alpha_{31}^\vee + 4K & \alpha_{31}^\vee + 3K & \begin{matrix}\alpha_{31}^\vee + 2K \\ \alpha_{32}^\vee + 2K\end{matrix} & \begin{matrix}\alpha_{31}^\vee + K \\ \alpha_{32}^\vee + K\end{matrix} & \\
\hline
\alpha_{51}^\vee + 5K & \alpha_{51}^\vee + 4K & \begin{matrix}\alpha_{51}^\vee + 3K \\ \alpha_{52}^\vee + 3K\end{matrix} & \begin{matrix}\alpha_{51}^\vee + 2K \\ \alpha_{52}^\vee + 2K\end{matrix} & \begin{matrix}\alpha_{51}^\vee + K \\ \alpha_{52}^\vee + K \\ \alpha_{53}^\vee + K\end{matrix} \\
\hline
\alpha_{21}^\vee + K & & & & \\
\hline
\alpha_{41}^\vee + 4K & \begin{matrix}\alpha_{41}^\vee + 3K \\ \alpha_{42}^\vee + 3K\end{matrix} & \begin{matrix}\alpha_{41}^\vee + 2K \\ \alpha_{42}^\vee + 2K\end{matrix} & \begin{matrix}\alpha_{41}^\vee + K \\ \alpha_{42}^\vee + K\end{matrix} & \\
\hline
\end{array}
\right.
$$

where $\alpha_{ij}^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee$. The following is an example that executes the last line of the proof of [5, Proposition 2.2]. The factor of s_1 in the factorization $u_\mu = s_1 \pi u_{(0,5,1,4,3)}$

gives the root

$$\begin{aligned}
u^{-1}_{(0,5,1,4,3)}\pi^{-1}(\varepsilon^\vee_1-\varepsilon^\vee_2) &= u^{-1}_{(0,5,1,4,3)}\pi^{-1}(\varepsilon^\vee_1-\varepsilon^\vee_2) = u^{-1}_{(0,5,1,4,3)}((\varepsilon^\vee_5+K)-\varepsilon^\vee_1) \\
&= v_{(0,5,1,4,3)}t^{-1}_{(0,5,1,4,3)}(\varepsilon^\vee_5-\varepsilon^\vee_1+K) = v_{(0,5,1,4,3)}(\varepsilon^\vee_5+3K-(\varepsilon^\vee_1+0K)+K) \\
&= \varepsilon^\vee_3-\varepsilon^\vee_1+4K, \qquad \text{since } v_{(0,5,1,4,3)}(5)=3.
\end{aligned}$$

4.0.10. *The column-greedy reduced word for u_μ.* Let $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n_{\geqslant 0}$. Let $J=(j_1<\dots<j_r)$ be the sequence of positions of the nonzero entries of μ and let ν be the composition defined by

$$\nu_j=\mu_j-1 \quad \text{if } j\in J \qquad \text{and} \qquad \nu_k=0 \quad \text{if } k\notin J,$$

so that ν is the composition which has one fewer box than μ in each (nonempty) row. Define the *column-greedy reduced word* for the element u_μ inductively by setting

$$u^\downarrow_\mu=\Big(\prod_{m=1}^{r} s_{j_m-1}\cdots s_{m+1}s_m\Big)\pi^r u^\downarrow_\nu, \tag{33}$$

where the product is taken in increasing order.

For example, if $\lambda=(5,4,4,1,0)$ then $z_\lambda=1$, $w_\lambda=s_2$, $v_\lambda=w_0s_2$ and the column greedy reduced word for u_λ is

$$u^\downarrow_\lambda=\pi^4s_1s_2s_3\pi^3(s_2s_1s_3s_2s_4s_3\pi^3)^2s_2s_1\pi = \left|\begin{array}{ccccc}
\boxed{} & \boxed{s_1} & \boxed{s_2s_1} & \boxed{s_2s_1} & \boxed{s_2s_1} \\
\boxed{} & \boxed{s_2} & \boxed{s_3s_2} & \boxed{s_3s_2} & \\
\boxed{} & \boxed{s_3} & \boxed{s_4s_3} & \boxed{s_4s_3} & \\
\boxed{} & & & & \\
\pi^4 & \pi^3 & \pi^3 & \pi^3 & \pi
\end{array}\right.$$

The computation verifying the expression for $u^\downarrow_\lambda$ is

$$\begin{array}{rcl}
 & & (5,4,4,1,0) \xrightarrow{\pi^{-4}} \\
(0,4,3,3,0) & \xrightarrow{s_1s_2s_3} & (4,3,3,0,0) \xrightarrow{\pi^{-3}} \\
(0,0,3,2,2) & \xrightarrow{s_2s_1s_3s_2s_4s_3} & (3,2,2,0,0) \xrightarrow{\pi^{-3}} \\
(0,0,2,1,1) & \xrightarrow{s_2s_1s_3s_2s_4s_3} & (2,1,1,0,0) \xrightarrow{\pi^{-3}} \\
(0,0,2,0,0) & \xrightarrow{s_2s_1} & (1,0,0,0,0) \xrightarrow{\pi^{-1}} (0,0,0,0,0)
\end{array}$$

If $\mu=(0,4,5,1,4)$ then the column greedy reduced word for u_μ is

$$u^\downarrow_\mu=s_1s_2s_3s_4\pi^4\cdot s_1s_2s_4s_3\pi^3\cdot s_2s_1s_3s_2s_4s_3\pi^3\cdot s_2s_1s_3s_2s_4s_3\pi^3\cdot s_3s_2s_1\pi.$$

This follows from (32) by using that $\pi s_i\pi^{-1}=s_{i+1}$.

5. The step-by-step and box-by-box recursions

5.0.1. *Examples of the step-by-step recursion.* Examples illustrating [5, Proposition 4.1(a)] are

$$E^{(156234)}_{(1,0,0,1,0,0)}=x_1E^{(562341)}_{(0,0,1,0,0,0)}, \qquad E^{(516234)}_{(1,0,0,1,0,0)}=x_5E^{(162345)}_{(0,0,1,0,0,0)},$$

$$E^{(651234)}_{(1,0,0,1,0,0)}=x_6E^{(512346)}_{(0,0,1,0,0,0)}.$$

An example illustrating [5, Proposition 4.1(b)] with $zs_i < z$ is

$$\begin{aligned}E^{(561234)}_{(0,0,1,1,0,0)} &= E^{(516234)}_{(0,1,0,1,0,0)} + \Big(\frac{1-t}{1-qt^{5-2}}\Big)qt^{5-2}t^{-3}E^{(561234)}_{(0,1,0,1,0,0)} \\ &= E^{(516234)}_{(0,1,0,1,0,0)} + \Big(\frac{1-t}{1-qt^{5-2}}\Big)qE^{(561234)}_{(0,1,0,1,0,0)},\end{aligned}$$

with $\mu = (0,0,1,1,0,0)$ and $z = (561234)$,

$$zv_\mu^{-1} = (563412), \quad v_\mu^{-1} = (125634), \quad zv_{s_2\mu}^{-1} = (513462), \quad v_{s_2\mu}^{-1} = (135624),$$

and

$$-\tfrac{1}{2}\big(\ell(zv_\mu^{-1}) - \ell(v_\mu^{-1}) - \ell(zv_{s_2\mu}^{-1}) + \ell(v_{s_2\mu}^{-1})\big) = -\tfrac{1}{2}\big(12-4-7+5\big) = -\tfrac{1}{2}\cdot 6 = -3.$$

An example illustrating [5, Proposition 4.1(b)] with $zs_i > z$ is

$$E^{(561234)}_{(0,1,0,1,0,0)} = E^{(651234)}_{(1,0,0,1,0,0)} + \Big(\frac{1-t}{1-qt^{5-1}}\Big)E^{(561234)}_{(1,0,0,1,0,0)}$$

with $\mu = (0,1,0,1,0,0)$ and $z = (561234)$,

$$zv_\mu^{-1} = (513462), \quad v_\mu^{-1} = (135624), \quad zv_{s_1\mu}^{-1} = (613452), \quad v_{s_1\mu}^{-1} = (235614),$$

and

$$-\tfrac{1}{2}\big(\ell(zv_\mu^{-1}) - \ell(v_\mu^{-1}) - \ell(zv_{s_1\mu}^{-1}) + \ell(v_{s_1\mu}^{-1})\big) = -\tfrac{1}{2}\big(7-5-8+6\big) = 0.$$

5.0.2. *Examples of the box by box recursion.* An example executing the box-by-box recursion is provided just after Theorem 1.1. in [5].

5.0.3. *An example of a 2^{j-1} to j term compression when $j=3$.* In order to check the powers of t in [5, Lemma 4.2] compute $\tau_2^\vee\tau_1^\vee E_\gamma$,

$$\begin{aligned}\tau_2^\vee\tau_1^\vee E_\gamma &= C_{-\beta_2^\vee}(T_1 + f_{-\beta_1^\vee})E_\gamma = C_{-\beta_2^\vee}T_1E_\gamma + f_{-\beta_1^\vee}C_{-\beta_2^\vee}E_\gamma \\ &= C_{-\beta_2^\vee}T_1E_\gamma + c_{-\beta_2^\vee}f_{-\beta_1^\vee}E_\gamma = (T_2 + f_{-\beta_2^\vee})T_1E_\gamma + c_{-\beta_2^\vee}f_{-\beta_1^\vee}E_\gamma \\ &= T_2T_1E_\gamma + f_{-\beta_2^\vee}T_1E_\gamma + t^{-\frac{1}{2}}f_{-\beta_2^\vee}E_\gamma \\ &= T_2T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}(t^{-\frac{1}{2}}T_1E_\gamma + t^{-\frac{2}{2}}E_\gamma).\end{aligned}$$

Now replace $T_2 = T_2^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ to get

$$\begin{aligned}\tau_2^\vee\tau_1^\vee E_\gamma &= (T_2^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}(t^{-\frac{1}{2}}T_1E_\gamma + t^{-\frac{2}{2}}E_\gamma) \\ &= T_2^{-1}T_1E_\gamma + (t-1+t^{\frac{1}{2}}f_{-\beta_2^\vee})t^{-\frac{1}{2}}T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}t^{-\frac{2}{2}}E_\gamma \\ &= T_2^{-1}T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2^\vee}t^{-\frac{1}{2}}T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}t^{-\frac{2}{2}}E_\gamma,\end{aligned}$$

and then replacing T_1 in the first term by $T_1 = T_1^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$

$$\begin{aligned}\tau_2^\vee\tau_1^\vee E_\gamma &= T_2^{-1}(T_1^{-1} + t^{-\frac{1}{2}}(t-1))E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2}t^{-\frac{1}{2}}T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}t^{-\frac{2}{2}}E_\gamma \\ &= T_2^{-1}T_1^{-1}E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2}t^{-\frac{1}{2}}T_1E_\gamma + t^{-\frac{1}{2}}(1-t)t^{-\frac{1}{2}}E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}t^{-\frac{2}{2}}E_\gamma \\ &= T_2^{-1}T_1^{-1}E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2}t^{-\frac{1}{2}}T_1E_\gamma + (t-1+t^{\frac{1}{2}}f_{-\beta_2^\vee})t^{-\frac{2}{2}}E_\gamma \\ &= T_2^{-1}T_1^{-1}E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2}t^{-\frac{1}{2}}T_1E_\gamma + t^{\frac{1}{2}}f_{-\beta_2^\vee}d_{-\beta_2^\vee}t^{-\frac{2}{2}}E_\gamma.\end{aligned}$$

5.0.4. *Check of the norm statistic in the step by step recursion.* This is an example which is helpful for checking the coefficients in [5, Proposition 4.3] and its proof. Let

$$\mu = (0,0,1,1,0,0), \qquad \gamma = (1,0,0,1,0,0), \qquad \nu = (0,0,1,0,0,0)$$

and $z = y = (561234)$. Then

$$\begin{array}{ll} v_\mu^{-1} = (125634), & \ell(v_\mu^{-1}) = 2+2 = 4, \\ yv_\mu^{-1} = (563412), & \ell(yv_\mu^{-1}) = 4+4+2+2 = 12, \\ v_\gamma^{-1} = (235614), & \ell(v_\mu^{-1}) = 1+1+2+2 = 6, \\ ys_2s_1v_\gamma^{-1} = (563412) & \ell(ys_2s_1v_\gamma^{-1}) = 4+4+2+2 = 12, \\ ys_1v_\gamma^{-1} = (513462) & \ell(ys_1v_\gamma^{-1}) = 4+1+1+1 = 7, \\ yv_\gamma^{-1} = (613452) & \ell(yv_\gamma^{-1}) = 5+1+1+1 = 8. \end{array}$$

Then $j = 3$ and

$$\begin{aligned} E_\mu^y &= t^{-\frac{1}{2}(\ell(yv_\mu)-\ell(v_\mu^{-1})-(3-1)}T_y\tau_2^\vee\tau_1^\vee E_\gamma = t^{-\frac{1}{2}(12-4-2)}T_y\tau_2^\vee\tau_1^\vee E_\gamma, \\ E_\gamma^{ys_2s_1} &= t^{-\frac{1}{2}(\ell(ys_2s_1v_\gamma^{-1})-\ell(v_\gamma^{-1})}T_{ys_2s_1}E_\gamma = t^{-\frac{1}{2}(12-6)}T_{ys_2s_1}E_\gamma = t^{-\frac{6}{2}}T_yT_2^{-1}T_1^{-1}E_\gamma \\ E_\gamma^{ys_1} &= t^{-\frac{1}{2}(\ell(ys_1v_\gamma^{-1})-\ell(v_\gamma^{-1})}T_{ys_1}E_\gamma = t^{-\frac{1}{2}(7-6)}T_{ys_1}E_\gamma = t^{-\frac{1}{2}}T_yT_1E_\gamma \\ E_\gamma^{y} &= t^{-\frac{1}{2}(\ell(yv_\gamma^{-1})-\ell(v_\gamma^{-1})}T_{y}E_\gamma = t^{-\frac{1}{2}(8-6)}T_{y}E_\gamma = t^{-\frac{2}{2}}T_yE_\gamma \end{aligned}$$

so that

$$\begin{aligned} t^{\frac{6}{2}}E_\mu^y &= t^{\frac{6}{2}}E_\gamma^{ys_2s_1} + d_{-\beta_1^\vee}f_{-\beta_1^\vee}t^{\frac{1}{2}}E_\gamma^{ys_1} + t^{-\frac{1}{2}}d_{-\beta_1^\vee}f_{-\beta_1^\vee}t^{\frac{2}{2}}E_\gamma^y \\ &= t^{\frac{6}{2}}E_\gamma^{ys_2s_1} + \frac{1-t}{1-qt^{5-2}}qt^{5-2}E_\gamma^{ys_1} + \frac{1-t}{1-qt^{5-2}}qt^{5-2}E_\gamma^y \end{aligned}$$

giving

$$E_\mu^y = E_\gamma^{ys_2s_1} + \frac{1-t}{1-qt^{5-2}}qE_\gamma^{ys_1} + \frac{1-t}{1-qt^{5-2}}qE_\gamma^y$$

as in the second line of the example in 5.0.2.

5.0.5. *Check of the statistic for $E_{\varepsilon_j}^z$ where $z(j) = j+k$.* This is an example of [5, Proposition 4.3] with

$$\mu = \varepsilon_j, \quad \gamma = \varepsilon_1, \quad y = s_{j+(k-1)}\cdots s_j.$$

Then

$$v_\mu = s_{n-1}\cdots s_j, \quad v_\gamma = s_{n-1}\cdots s_1, \quad v_\mu^{-1} = s_j\cdots s_{n-1}, \quad v_\gamma^{-1} = s_1\cdots s_{n-1}.$$

Then $yv_\mu^{-1} = s_{j+k}\cdots s_{n-1}$ and $\ell(yv_\mu^{-1}) = (n-1)-(j-1)-k$ and

$$\begin{aligned} \ell(yv_\mu^{-1}) - \ell(v_\mu^{-1}) - (j-1) &= ((n-1)-(j-1)-k) - ((n-1)-(j-1)) \\ &= -k-(j-1). \end{aligned}$$

Next, $yc_a^{-1}c_jv_\mu^{-1} = ((s_{j+(k-1)}\cdots s_j)(s_a\cdots s_{j-1})(s_j\cdots s_{n-1})$ and

$$\begin{aligned} \ell(yc_a^{-1}c_jv_\mu^{-1}) &= (j-1+k-(j-1)) + ((j-1)-(a-1)) + (n-1-(j-1)) \\ &= (n-1)-(a-1)+k. \end{aligned}$$

So

$$\begin{aligned} &\ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j) \\ &\qquad = (n-1)-(a-1)+k - ((n-1)-(j-1)-k) - ((j-1)-(a-1)) \\ &\qquad = 2k. \end{aligned}$$

Thus

$$\begin{aligned}
E_\mu^z = E_\mu^y &= x_{y(j)} E_\nu^{yc_n} + \frac{(1-t)}{1-q^{\mu_j} t^{v_\mu(j)-(j-1)}} \sum_{a=0}^{j-1} t^{\frac{1}{2}\cdot 2k} x_{y(a)} E_\nu^{yc_a^{-1}c_n} \\
&= x_{y(j)} + \frac{(1-t)}{1-q^{\mu_j} t^{v_\mu(j)-(j-1)}} \sum_{a=0}^{j-1} t^k x_{y(a)}.
\end{aligned}$$

6. Type GL_n DAArt, DAHA and the polynomial representation

6.0.1. *Example to check the eigenvalues of Y_i on E_μ.* The box greedy reduced words for $u_{(2,1,0)}$, $u_{(2,0,1)}$ and $u_{(1,2,0)}$ are

$$u^{\square}_{(2,1,0)} = \left|\begin{array}{ll} \boxed{\pi} & \boxed{s_1\pi} \\ \boxed{\pi} & \\ & \end{array}\right. \qquad u^{\square}_{(2,0,1)} = \left|\begin{array}{ll} \boxed{\pi} & \boxed{s_1\pi} \\ & \\ \boxed{s_1\pi} & \end{array}\right. \qquad u^{\square}_{(1,2,0)} = \left|\begin{array}{ll} \boxed{\pi} & \\ \boxed{\pi} & \boxed{s_2 s_1\pi} \\ & \end{array}\right.$$

Using $u_\mu = t_\mu v_\mu^{-1}$ to carefully compute v_μ^{-1}:

$$\begin{aligned}
u_{(2,1,0)} &= \pi^2 s_1 \pi = t_{\varepsilon_1} s_1 s_2 t_{\varepsilon_1} s_1 s_2 s_1 t_{\varepsilon_1} s_1 s_2 \\
&= t_{\varepsilon_1} t_{\varepsilon_2} s_1 s_2 s_1 s_2 s_1 t_{\varepsilon_1} s_1 s_2 \\
&= t_{\varepsilon_1} t_{\varepsilon_2} s_2 t_{\varepsilon_1} s_1 s_2 \\
&= t_{2\varepsilon_1+\varepsilon_2} s_2 s_1 s_2, \quad \text{so} \quad v^{-1}_{(2,1,0)} = s_2 s_1 s_2.
\end{aligned}$$

$$\begin{aligned}
u_{(2,0,1)} &= \pi s_1 \pi s_1 \pi = t_{\varepsilon_1} s_1 s_2 s_1 t_{\varepsilon_1} s_1 s_2 s_1 t_{\varepsilon_1} s_1 s_2 \\
&= t_{\varepsilon_1} t_{\varepsilon_3} s_1 s_2 s_1 s_1 s_2 s_1 t_{\varepsilon_1} s_1 s_2 \\
&= t_{2\varepsilon_1+\varepsilon_3} s_1 s_2, \quad \text{so} \quad v^{-1}_{(2,0,1)} = s_1 s_2.
\end{aligned}$$

$$\begin{aligned}
u_{(1,2,0)} &= \pi^2 s_2 s_1 \pi = t_{\varepsilon_1} s_1 s_2 t_{\varepsilon_1} s_1 s_2 s_2 s_1 t_{\varepsilon_1} s_1 s_2 \\
&= t_{\varepsilon_1+2\varepsilon_2} s_1 s_2 s_1 s_2 \\
&= t_{\varepsilon_1+2\varepsilon_2} s_2 s_1, \quad \text{so} \quad v^{-1}_{(1,2,0)} = s_2 s_1.
\end{aligned}$$

Using

$$\begin{array}{ll}
u_{(2,1,0)} = t_{(2,1,0)} s_1 s_2 s_1 = t_{(2,1,0)} v^{-1}_{(2,1,0)}, & u_{(2,0,1)} = t_{(2,0,1)} s_1 s_2 = t_{(2,0,1)} v^{-1}_{(2,0,1)}, \\
u_{(1,2,0)} = t_{(1,2,0)} s_2 s_1 = t_{(1,2,0)} v^{-1}_{(1,2,0)}, & u_{(0,2,1)} = t_{(0,2,1)} s_2 = t_{(0,2,1)} v^{-1}_{(0,2,1)}, \\
u_{(1,0,2)} = t_{(1,0,2)} s_2, = t_{(1,0,2)} v^{-1}_{(1,0,2)}, & u_{(0,1,2)} = t_{(0,1,2)} = t_{(0,1,2)} v^{-1}_{(0,1,2)},
\end{array}$$

and the relations

$$Y_1 \tau_\pi^\vee = q^{-1} \tau_\pi^\vee Y_3, \qquad Y_2 \tau_\pi^\vee = \tau_\pi^\vee Y_1, \qquad Y_3 \tau_\pi^\vee = \tau_\pi^\vee Y_2,$$

then

$$\begin{aligned}
Y_1E_{(2,1,0)} &= t^{-\frac{3}{2}}Y_1\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee Y_3\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee Y_2\tau_1^\vee\tau_\pi^\vee\mathbf{1}\\
&= t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee Y_1\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-2}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee Y_3\mathbf{1}\\
&= q^{-2}t^{-(3-1)+\frac{1}{2}(3-1)}E_{(2,1,0)},\\
Y_2E_{(2,1,0)} &= t^{-\frac{3}{2}}Y_2\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}\tau_\pi^\vee Y_1\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee Y_3\tau_1^\vee\tau_\pi^\vee\mathbf{1}\\
&= t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee Y_3\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee Y_2\mathbf{1}\\
&= q^{-1}t^{-(2-1)+\frac{1}{2}(3-1)}E_{(2,1,0)},\\
Y_3E_{(2,1,0)} &= t^{-\frac{3}{2}}Y_3\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}\tau_\pi^\vee Y_2\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee Y_1\tau_1^\vee\tau_\pi^\vee\mathbf{1}\\
&= t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee Y_2\tau_\pi^\vee\mathbf{1} = t^{-\frac{3}{2}}q^{-1}\tau_\pi^\vee\tau_\pi^\vee\tau_1^\vee\tau_\pi^\vee Y_1\mathbf{1}\\
&= t^{-(1-1)+\frac{1}{2}(3-1)}E_{(2,1,0)}.
\end{aligned}$$

Then

$$\begin{aligned}
Y_1E_{(1,2,0)} &= t^{\frac{1}{2}}Y_1\tau_1^\vee E_{(2,1,0)} = t^{\frac{1}{2}}\tau_1^\vee Y_2E_{(2,1,0)} = q^{-1}t^{-(2-1)+\frac{1}{2}(3-1)}E_{(1,2,0)},\\
Y_2E_{(1,2,0)} &= t^{\frac{1}{2}}Y_2\tau_1^\vee E_{(2,1,0)} = t^{\frac{1}{2}}\tau_1^\vee Y_1E_{(2,1,0)} = q^{-2}t^{-(3-1)+\frac{1}{2}(3-1)}E_{(1,2,0)},\\
Y_3E_{(1,2,0)} &= t^{\frac{1}{2}}Y_3\tau_1^\vee E_{(2,1,0)} = t^{\frac{1}{2}}\tau_1^\vee Y_3E_{(2,1,0)} = q^{-0}t^{-(1-1)+\frac{1}{2}(3-1)}E_{(1,2,0)},
\end{aligned}$$

and $v_{(1,2,0)}(1) = s_1s_2(1) = s_1(1) = 2$, $v_{(1,2,0)}(2) = s_1s_2(2) = s_1(3) = 3$ and $v_{(1,2,0)}(3) = s_1s_2(3) = s_1(2) = 1$.

6.0.2. *The elements X^{ω_r}.* For $i \in \{1, \dots, n\}$ let $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$. Then

$$X^{\omega_i} = X^{\varepsilon_1+\dots+\varepsilon_i} = (g^\vee)^i T_{w_i}^{-1}, \qquad \text{where} \quad w_i = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & n \\ i+1 & \cdots & n & 1 & \cdots & i \end{pmatrix}$$

In W, the element $t_{\omega_i} = \pi^i w_i$. There are two favorite choices of reduced word for w_i, which are

$$\begin{aligned}
w_i &= (s_i \cdots s_{n-1})(s_{i-1} \cdots s_{n-2}) \cdots (s_1 \cdots s_{n-i})\\
&= (s_i \cdots s_1)(s_{i+1} \cdots s_2) \cdots (s_{n-1} \cdots s_{n-i}).
\end{aligned}$$

For example, if $n = 6$ then

$$\begin{aligned}
w_1 &= s_5s_4s_3s_2s_1,\\
w_2 &= (s_4s_3s_2s_1)(s_5s_4s_3s_2) = (s_4s_5)(s_3s_4)(s_2s_3)(s_1s_2)\\
w_3 &= (s_3s_2s_1)(s_4s_3s_2)(s_5s_4s_3) = (s_3s_4s_5)(s_2s_3s_4)(s_1s_2s_3)\\
w_4 &= (s_2s_1)(s_3s_2)(s_4s_3)(s_5s_4) = (s_2s_3s_4s_5)(s_1s_2s_3s_4)\\
w_5 &= s_1s_2s_3s_4s_5\\
w_6 &= 1,
\end{aligned}$$

and

$$\begin{aligned}
X^{\omega_1} &= g^\vee T_5^{-1}T_4^{-1}T_3^{-1}T_2^{-1}T_1^{-1},\\
X^{\omega_2} &= (g^\vee)^2(T_4^{-1}T_3^{-1}T_2^{-1}T_1^{-1})(T_5^{-1}T_4^{-1}T_3^{-1}T_2^{-1})\\
&= (g^\vee)^2(T_4^{-1}T_5^{-1})(T_3^{-1}T_4^{-1})(T_2^{-1}T_3^{-1})(T_1^{-1}T_2^{-1})\\
X^{\omega_3} &= (g^\vee)^3(T_3^{-1}T_2^{-1}T_1^{-1})(T_4^{-1}T_3^{-1}T_2^{-1})(T_5^{-1}T_4^{-1}T_3^{-1})\\
&= (g^\vee)^3(T_3^{-1}T_4^{-1}T_5^{-1})(T_2^{-1}T_3^{-1}T_4^{-1})(T_1^{-1}T_2^{-1}T_3^{-1})\\
X^{\omega_4} &= (g^\vee)^4(T_2^{-1}T_1^{-1})(T_3^{-1}T_2^{-1})(T_4^{-1}T_3^{-1})(T_5^{-1}T_4^{-1})\\
&= (g^\vee)^4(T_2^{-1}T_3^{-1}T_4^{-1}T_5^{-1})(T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1})\\
X^{\omega_5} &= (g^\vee)^5T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1}T_5^{-1}\\
X^{\omega_6} &= (g^\vee)^6.
\end{aligned}$$

6.0.3. *Type GL_2.* For type GL_2, $X_1 = g^\vee T_1^{-1}$ and $X_2 = T_1X_1T_1 = T_1g^\vee$ and

$$X_1X_2 = (g^\vee)^2, \quad X_1^{k+1}T_1 = (g^\vee T_1^{-1})^k g^\vee, \quad (T_1g^\vee)^k = X_2^k.$$

The box greedy reduced words for the first few cases are

$$u^{\square}_{(1,0)} = \left|\begin{array}{l}\boxed{\pi}\\ \\ \end{array}\right. \qquad u^{\square}_{(0,1)} = \left|\begin{array}{l} \\ \boxed{s_1\pi}\end{array}\right.$$

$$u^{\square}_{(2,0)} = \left|\begin{array}{ll}\boxed{\pi} & \boxed{s_1\pi}\\ & \end{array}\right. \qquad u^{\square}_{(1,1)} = \left|\begin{array}{l}\boxed{\pi}\\ \boxed{\pi}\end{array}\right. \qquad u^{\square}_{(0,2)} = \left|\begin{array}{ll} & \\ \boxed{s_1\pi} & \boxed{s_1\pi}\end{array}\right.$$

$$u^{\square}_{(3,0)} = \left|\begin{array}{lll}\boxed{\pi} & \boxed{s_1\pi} & \boxed{s_1\pi}\\ & & \end{array}\right.$$

In this case the construction of E_μ as $E_\mu = t^{\frac{1}{2}\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}\mathbf{1}$ in [5, Proposition 5.7] is

$$E_{(k+h,k)} = t^{-\frac{1}{2}}(\tau_\pi^\vee)^{2k}(\tau_\pi^\vee\tau_1^\vee)^{h-1}\tau_\pi^\vee\mathbf{1} \quad \text{and} \quad E_{(k,k+h)} = (\tau_\pi^\vee)^{2k}(\tau_1^\vee\tau_\pi^\vee)^h\mathbf{1},$$

with $\tau_\pi^\vee = g^\vee$.

Let $h \in \mathbb{Z}_{>0}$. The nonattacking fillings and words for $E_{(h,0)}$ and $E_{(0,h)}$ are

$$\begin{array}{c}\begin{array}{r|l}1 & 1\ i_1\ \cdots\ i_{h-1}\ i_h\\ 2 & \end{array}\\[2ex] x_1x_{i_2}\cdots x_{i_h}\end{array} \qquad \text{and} \qquad \begin{array}{c}\begin{array}{r|l}1 & \\ 2 & i_1\ \cdots\ i_{h-1}\ i_h\end{array}\\[2ex] x_{i_1}\cdots x_{i_h}\end{array} \qquad \text{with } i_1,\ldots,i_h \in \{1,2\}.$$

7. Additional examples

7.0.1. *Formulas for E_μ when $n=2$.*

$$\begin{aligned}
E_{(0,0)} &= 1,\\
E_{(1,0)} &= x_1,\\
E_{(0,1)} &= x_2 + \Big(\frac{1-t}{1-qt}\Big)x_1,\\
E_{(1,1)} &= x_1x_2,\\
E_{(2,0)} &= x_1^2 + \Big(\frac{1-t}{1-qt}\Big)qx_1x_2,\\
E_{(0,2)} &= x_2^2 + \Big(\frac{1-t}{1-q^2t}\Big)x_1^2 + \Big(\Big(\frac{1-t}{1-qt}\Big) + \Big(\frac{1-t}{1-q^2t}\Big)\Big(\frac{1-t}{1-qt}\Big)q\Big)x_1x_2,\\
E_{(3,0)} &= x_1^3 + \Big(\frac{1-t}{1-q^2t}\Big)q^2x_1x_2^2 + \Big(\Big(\frac{1-t}{1-qt}\Big)q + \Big(\frac{1-t}{1-q^2t}\Big)\Big(\frac{1-t}{1-qt}\Big)q^2\Big)x_1^2x_2.
\end{aligned}$$

Then [12, (6.2.7) and (6.28)] provides the general formula as follows. Let

$$(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\cdots, \qquad (x;q)_r = \frac{(x;q)_\infty}{(q^rx;q)_\infty},$$

and

$$\begin{bmatrix} s\\ r\end{bmatrix} = \frac{(q;q)_s}{(q;q)_r(q;q)_{s-r}}.$$

Let $k\in\mathbb{Z}_{>0}$ and let $t=q^k$. Then

$$\begin{aligned}
E_{(0,m)} &= \begin{bmatrix} k+m\\ m\end{bmatrix}^{-1} \sum_{i+j=m} \begin{bmatrix} k+i-1\\ i\end{bmatrix}\begin{bmatrix} k+j\\ j\end{bmatrix} x_1^jx_2^i \qquad \text{and}\\
E_{(m+1,0)} &= \begin{bmatrix} k+m\\ m\end{bmatrix}^{-1} \sum_{i+j=m} \begin{bmatrix} k+i-1\\ i\end{bmatrix}\begin{bmatrix} k+j\\ j\end{bmatrix} q^ix_1^{j+1}x_2^i.
\end{aligned}$$

Since $t=q^k$, it appears that t must be a power of q. But this is not really the case since we may rewrite these formulas using

$$\begin{bmatrix} k+m\\ m\end{bmatrix} = \frac{(q;q)_{k+m}}{(q;q)_m(q;q)_k} = \frac{(q;q)_\infty(q^m;q)_\infty(q^k;q)_\infty}{(q^{k+m};q)_\infty(q;q)_\infty(q;q)_\infty} = \frac{(q^m;q)_\infty(t;q)_\infty}{(tq^m;q)_\infty(q;q)_\infty} = \frac{(t;q)_m}{(q;q)_m}$$

and

$$\begin{aligned}
\begin{bmatrix} k+i-1\\ i\end{bmatrix}\begin{bmatrix} k+j\\ j\end{bmatrix} &= \frac{(q^i;q)_\infty(tq^{-1};q)_\infty}{(tq^{i-1};q)_\infty(q;q)_\infty}\,\frac{(q^j;q)_\infty(t;q)_\infty}{(tq^j;q)_\infty(q;q)_\infty}\\
&= \frac{(q^i;q)_\infty(q^j;q)_\infty(tq^{-1};q)_\infty(t;q)_\infty}{(q;q)_\infty(q;q)_\infty(tq^{i-1};q)_\infty(tq^j;q)_\infty}.
\end{aligned}$$

7.0.2. *Some small E_μ for $n=3$.*

$$
\begin{aligned}
E_{(0,0,0)} &= 1,\\
E_{(1,0,0)} &= x_1,\\
E_{(0,1,0)} &= x_2 + \Big(\frac{1-t}{1-qt^2}\Big)x_1,\\
E_{(0,0,1)} &= x_3 + \Big(\frac{1-t}{1-qt}\Big)(x_2+x_1)\\
E_{(1,1,0)} &= x_1x_2,\\
E_{(1,0,1)} &= x_1x_3 + \Big(\frac{1-t}{1-qt^2}\Big)x_1x_2,\\
E_{(0,1,1)} &= x_2x_3 + \Big(\frac{1-t}{1-qt}\Big)(x_1x_3+x_1x_2),\\
E_{(2,0,0)} &= x_1^2 + \Big(\frac{1-t}{1-qt}\Big)q(x_1x_3+x_1x_2),\\
E_{(2,2,0)} &= x_1^2x_2^2 + \Big(\frac{1-t}{1-qt^2}\Big)qx_1^2x_2x_3 + \Big(\frac{1-t}{1-qt^2}\Big)qx_1x_2^2x_3,
\end{aligned}
$$

and $E_{(2,1,0)}$, $E_{(2,0,1)}$, $E_{(1,2,0)}$, $E_{(0,2,1)}$, $E_{(1,0,2)}$, $E_{(0,1,2)}$ are given in section 1.3.1. Additionally,

$$
\begin{aligned}
P_{(1,0,0)} &= m_1 = x_1+x_2+x_3,\\
P_{(2,0,0)} &= m_2 + \frac{(1-q^2)(1-t)}{(1-q)(1-tq)}m_{1^2},\\
P_{(1,1,0)} &= m_{1^2} = x_1x_2+x_1x_3+x_2x_3,
\end{aligned}
$$

where $m_\lambda = \sum_{\mu\in S_n\lambda} x^\mu$ is the monomial symmetric function so that $m_2 = x_1^2+x_2^2+x_3^2$.

7.0.3. *E_λ and P_λ when λ is a partition with 3 boxes.* Letting $x^\gamma = x_1^{\gamma_1}\cdots x_n\gamma_n$ if $\gamma=(\gamma_1,\dots,\gamma_n)$, let

$$
m_\lambda = \sum_{\gamma\in S_n\lambda} x^\gamma, \quad \text{be the monomial symmetric function (orbit sum).}
$$

Proposition 7.1. *Let $\varepsilon_i = (0,\dots,0,1,0,\dots,0)$ where the 1 appears in the ith spot. Then*

$$
\begin{aligned}
E_{3\varepsilon_1} &= x_1^3 + \Big(\frac{1-t}{1-q^2t}\Big)q^2 \sum_{k\in\{2,\dots,n\}} x_1x_k^2\\
&\quad + \Big(\frac{1-t}{1-qt}\Big)\Big(1+\Big(\frac{1-t}{1-q^2t}\Big)q\Big)q \sum_{k\in\{2,\dots,n\}} x_1^2x_k\\
&\quad + \Big(\frac{1-t}{1-qt}\Big)\Big(\frac{1-t}{1-q^2t}\Big)(1+q)q^2 \sum_{\{k,\ell\}\subseteq\{2,\dots,n\}} x_1x_kx_\ell,\\
E_{2\varepsilon_1+\varepsilon_2} &= x_1^2x_n + \Big(\frac{1-t}{1-qt^2}\Big)q(x_1x_2x_n+\cdots+x_1x_2x_4+x_1x_2x_3),\\
E_{\varepsilon_1+\varepsilon_2+\varepsilon_3} &= x_1x_2x_3,
\end{aligned}
$$

$$
\begin{aligned}
P_{3\varepsilon_1} &= m_3 + \frac{(1-q^3)}{(1-tq^2)}\Big(\frac{1-t}{1-q}\Big)m_{21} + \frac{(1-q^3)}{(1-tq^2)}\frac{(1-q^2)}{(1-tq)}\Big(\frac{1-t}{1-q}\Big)^2 m_{1^3},\\
P_{2\varepsilon_1+\varepsilon_2} &= m_{21} + \Big(\frac{(1-t^2)}{(1-qt)}\frac{(1-q^2t)}{(1-qt^2)} + \frac{(1-t)}{(1-q)}\frac{(1-q^2)}{(1-qt)}\Big)m_{1^3}, \ \textit{and}\\
P_{\varepsilon_1+\varepsilon_2+\varepsilon_3} &= m_{1^3} = e_3, \ \textit{where } e_r \textit{ denotes the elementary symmetric function.}
\end{aligned}
$$

Proof. From [5, Proposition 3.5(b)],

$$
\begin{aligned}
E_{2\varepsilon_n} = x_n^2 &+ \Big(\frac{1-t}{1-q^2t}\Big)\sum_{k\in\{1,\dots,n-1\}} x_k^2 + \Big(\frac{1-t}{1-qt}\Big)\Big(1+\Big(\frac{1-t}{1-q^2t}\Big)q\Big)\sum_{k\in\{1,\dots,n-1\}} x_kx_n\\
&+ \Big(\frac{1-t}{1-qt}\Big)\Big(\frac{1-t}{1-q^2t}\Big)(1+q)\sum_{\{k,\ell\}\subseteq\{1,\dots,n-1\}} x_kx_\ell,
\end{aligned}
$$

and applying [5, Proposition 5.8(c)] gives the formula for $E_{3\varepsilon_1} = E_{\pi 2\varepsilon_n}$. Similarly, from [5, Proposition 3.5(c)],

$$
E_{\varepsilon_1+\varepsilon_n} = x_1x_n + \Big(\frac{1-t}{1-qt^2}\Big)(x_1x_{n-1} + \cdots + x_1x_3 + x_1x_2),
$$

and applying [5, Proposition 5.8(c)] gives the formula for $E_{2\varepsilon_1+\varepsilon_2} = E_{\pi(\varepsilon_1+\varepsilon_n)}$ in the statement. The formula for $E_{\varepsilon_1+\varepsilon_2+\varepsilon_3}$ follows from the first statement of Proposition 7.2.

For $r\in\mathbb{Z}_{\geqslant 0}$ and $\mu\in\mathbb{Z}^n_{\geqslant 0}$ define $(x;q)_r = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{r-1})$

$$
\text{and}\qquad (x;q)_\mu = (x;q)_{\mu_1}\cdots(x;q)_{\mu_n}
$$

(when $r=0$ then $(x;q)_0 = 1$). As proved in [10, Ch. VI equation (4.9) and Ch. VI §2 Ex. 1], if $r\in\mathbb{Z}_{>0}$ then

$$
P_{\varepsilon_1+\cdots+\varepsilon_r} = e_r = m_{1^r} \qquad\text{and}\qquad P_{r\varepsilon_1} = \sum_{|\mu|=r}\frac{(q;q)_r}{(t;q)_r}\frac{(t;q)_\mu}{(q;q)_\mu}m_\mu.
$$

By [10, Ch. VI (4.3) and (4.10)], the formula for $P_{2\varepsilon_1+\varepsilon_2}$ follows from the formula for $P_{(2,1,0)}$ in 3 variables given at the end of section 1.3.1. □

7.0.4. *Macdonald polynomials E^z_μ and P_μ when μ is a single column.*

Proposition 7.2. *Let $r\in\{1,\dots,n\}$ and let $\omega_r = \varepsilon_1+\cdots+\varepsilon_r$.*

$$
E_{\varepsilon_1+\cdots+\varepsilon_r} = x_1x_2\cdots x_i.
$$

Let W^{ω_r} be the set of $z\in S_n$ such that z is the minimal length element of its coset $z(S_r\times S_{n-r})$ in S_n. If $z\in W^{\omega_r}$ then

$$
z = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & n\\ i_1 & i_2 & \cdots & i_r & j_1 & \cdots & j_{n-r}\end{pmatrix} \qquad \textit{with}\quad \begin{matrix} i_1<i_2<\cdots<i_r \ \textit{and}\\ j_1<j_2<\cdots<j_{n-r}\end{matrix}
$$

and

$$
t^{\frac12\ell(z)}T_zE_{\omega_r} = x_{i_1}\dots x_{i_r} \qquad \textit{and}\qquad P_{\omega_r} = \sum_{z\in W^{\omega_r}} t^{\frac12\ell(z)}T_zE_{\omega_r} = e_r,
$$

where e_r is the rth elementary symmetric function.

Proof. Since

$$
v^{-1}_{\varepsilon_1+\cdots+\varepsilon_r} = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n\\ r+1 & \cdots & n & 1 & \cdots & r\end{pmatrix} \qquad\text{with}\quad \ell(v^{-1}_{\varepsilon_1+\cdots+\varepsilon_r}) = (n-r)r,
$$

and $u_{\varepsilon_1+\cdots+\varepsilon_r} = \pi^r$ then

$$\begin{aligned} E_{\varepsilon_1+\cdots+\varepsilon_r} &= t^{-\frac{1}{2}(n-r)r}(\tau_\pi^\vee)^r\mathbf{1} = t^{-\frac{1}{2}(n-r)r}(g^\vee)^r\mathbf{1} \\ &= t^{-\frac{1}{2}(n-r)r}X_1\cdots X_r T_{v^{-1}_{\varepsilon_1+\cdots+\varepsilon_r}}\mathbf{1}, = x_1\cdots x_r. \end{aligned}$$

A reduced word for z is $z = (s_{i_1-1}\cdots s_1)(s_{i_2-1}\cdots s_2)\cdots(s_{i_r-1}\cdots s_r)$. Then

$$\begin{aligned} t^{\frac{1}{2}\ell(z)}T_z E_{\omega_r} &= ((t^{\frac{1}{2}}T_{i_1-1})\cdots(t^{\frac{1}{2}}T_1))\cdot((t^{\frac{1}{2}}T_{i_2-1})\cdots(t^{\frac{1}{2}}T_2)) \\ &\qquad \cdots((t^{\frac{1}{2}}T_{i_r-1})\cdots(t^{\frac{1}{2}}T_r))(x_1x_2\cdots x_r) \\ &= ((t^{\frac{1}{2}}T_{i_1-1})\cdots(t^{\frac{1}{2}}T_1))\cdot((t^{\frac{1}{2}}T_{i_2-1})\cdots(t^{\frac{1}{2}}T_2)) \\ &\qquad \cdots((t^{\frac{1}{2}}T_{i_{r-1}-1})\cdots(t^{\frac{1}{2}}T_{r-1}))(x_1x_2\cdots x_{r-1}x_{i_r}) \\ &= ((t^{\frac{1}{2}}T_{i_1-1})\cdots(t^{\frac{1}{2}}T_1))\cdot((t^{\frac{1}{2}}T_{i_2-1})\cdots(t^{\frac{1}{2}}T_2)) \\ &\qquad \cdots((t^{\frac{1}{2}}T_{i_{r-2}-1})\cdots(t^{\frac{1}{2}}T_{r-2}))(x_1x_2\cdots x_{r-2}x_{i_{r-1}}x_{i_r}) \\ &= \cdots = x_{i_1}x_{i_2}\cdots x_{i_r}. \end{aligned}$$

The last equality then follows from (5). □

7.0.5. *E^z_μ for a single box.*

Proposition 7.3. *Let $j \in \{1,\dots,n\}$ and let $z \in S_n$. Then*

$$E^z_{\varepsilon_j} = c_j x_{z(j)} + \cdots + c_2 x_{z(2)} + c_1 x_{z(1)}$$

where

$$c_a = \begin{cases} \Big(\dfrac{1-t}{1-qt^{n-j+1}}\Big)qt^{C(a)}, & \text{if } z(j) < z(a), \\ \Big(\dfrac{1-t}{1-qt^{n-j+1}}\Big)t^{C(a)}, & \text{if } z(j) > z(a), \\ 1, & \text{if } z(j) = z(a). \end{cases}$$

with

$$C(a) = \begin{cases} \left\{k \in \{j+1,\dots,n\} \;\middle|\; \begin{array}{l} z(k) < z(j) < z(a) \\ \text{or } z(j) < z(a) < z(k) \end{array}\right\}, & \text{if } z(j) < z(a), \\ \\ \{k \in \{j+1,\dots,n\} \mid z(j) > z(k) > z(a)\}, & \text{if } z(j) > z(a). \end{cases}$$

Proof. The proof is by induction on $\ell(z)$. If $z = 1$ then $T_z = 1$ and the formula is the same as given in [5, Proposition 3.5(a)] for E_{ε_j}. Let $r \in \{1,\dots,n-1\}$ such that $s_r z > z$. Recall

$$t^{\frac{1}{2}}T_r(x_\ell) = \begin{cases} x_{r+1}, & \text{if } \ell = r, \\ tx_r + (t-1)x_{r+1}, & \text{if } \ell = r+1, \\ tx_\ell, & \text{otherwise.} \end{cases} \tag{34}$$

$$t^{-\frac{1}{2}}T_r(x_\ell) = \begin{cases} t^{-1}x_{r+1}, & \text{if } r = \ell, \\ x_r + (1-t^{-1})x_{r+1}, & \text{if } \ell = r+1, \\ x_\ell, & \text{otherwise.} \end{cases} \tag{35}$$

Write

$$t^{-\frac{1}{2}(\ell(zv^{-1}_{\varepsilon_j})-\ell(v^{-1}_{\varepsilon_j}))}E^z_{\varepsilon_j} = \sum_{i=1}^n c^z_i x_{z(i)}.$$

If we multiply by $t^{\frac12}T_r$ then $t^{\frac12}T_r(c^z_a x_r + c^z_b x_{r+1}) = c^z_a x_{r+1} + c^z_b(tx_r + (t-1)x_{r+1}) = tc^z_b x_r + (c^z_b(t-1) + c^z_a)x_{r+1}$ giving

$$c^{s_r z}_a = tc^z_b \qquad \text{and} \qquad c^{s_r z}_b = c^z_b(t-1) + c^z_a.$$

If we multiply by $t^{-\frac12}T_r$ then $t^{-\frac12}T_r(c^z_a x_r + c^z_b x_{r+1}) = t^{-1}c^z_a x_{r+1} + c^z_b(x_r + (1-t^{-1})x_{r+1}) = c^z_b x_r + (c^z_b(1-t^{-1}) + t^{-1}c^z_a)x_{r+1}$ giving

$$c^{s_r z}_a = c^z_b \qquad \text{and} \qquad c^{s_r z}_b = c^z_b(1-t^{-1}) + t^{-1}c^z_a.$$

Let

$$a = z^{-1}(r) \text{ and } b = z^{-1}(r+1) \quad \text{ so that } \quad b = (s_r z)^{-1}(r) \text{ and } a = (s_r z)^{-1}(r+1).$$

Assume $s_r z > z$ so that $a < b$. In each of the cases

(lll)	$z(j) < r$	$a < j$	$b < j$	$c^z_a = c^z_b$
(llg)	$z(j) < r$	$a < j$	$b > j$	$c^z_b = 0$
(lgg)	$z(j) < r$	$a > j$	$b > j$	$c^z_a = c^z_b = 0$
(ele)	$z(j) = r$	$a = j$	$b > j$	$c^z_a = 1,\ c^z_b = 0$
(flf)	$z(j) = r+1$	$a < j$	$b = j$	$c^z_b = 1$
(gll)	$z(j) > r+1$	$a < j$	$b < j$	$c^z_a = c^z_b$
(glg)	$z(j) > r+1$	$a < j$	$b > j$	$c^z_b = 0$
(ggg)	$z(j) > r+1$	$a > j$	$b > j$	$c^z_a = c^z_b = 0$

(lll)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = c^{s_z}_b = c^z_a$
(llg)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = 0,\ c^{s_r z}_b = t^{-1}c^z_a$
(lgg)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = c^{s_r z}_b = 0$
(ele)	multiply by $t^{\frac12}T_r$	to get	$c^{s_r z}_a = 0,\ c^{s_r z}_b = 1,$
(flf)	multiply by $t^{-\frac12}T_r$		
(gll)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = c^{s_z}_b = c^z_a$
(glg)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = 0,\ c^{s_r z}_b = t^{-1}c^z_a$
(ggg)	multiply by $t^{-\frac12}T_r$	to get	$c^{s_r z}_a = c^{s_r z}_b = 0$

Now we need to show that the statistics $C(a)$ provide the same recursions. For example, in the case (flf), $r+1 = z(j) > z(a) = r$ with $C(a) = 0$ and $r = (s_r z)(j) < (s_r z)(a) = r+1$ and $C(a) = n-j$. So

$$c^z_j = 1, \quad c^z_a = \Big(\frac{1-t}{1-qt^{n-j+1}}\Big)t^0 \quad \text{and} \qquad c^{s_r z}_j = 1, \quad c^{s_r z}_a = \Big(\frac{1-t}{1-qt^{n-j+1}}\Big)qt^{n-j}$$

since

$$\begin{aligned} c^{s_r z}_a &= (1-t^{-1}) + t^{-1}\Big(\frac{1-t}{1-qt^{n-j+1}}\Big)t^0 \\ &= \Big(\frac{1-t}{1-qt^{n-j+1}}\Big)(-t^{-1}(1-qt^{n-j+1}) + t^{-1}) = \Big(\frac{1-t}{1-qt^{n-j+1}}\Big)qt^{n-j}. \end{aligned}$$ □

Some examples are

$$(t^{\frac12}T_{i+(k-1)})\cdots(t^{\frac12}T_i)E_{\varepsilon_i} = x_{i+k} + \frac{(1-t)}{(1-qt^{n-(i-1)})}t^k(x_{i-1} + \cdots + x_1),$$

$$\begin{aligned} &(t^{-\frac12}T_{i-k})\cdots(t^{-\frac12}T_{i-1})E_{\varepsilon_i} \\ &\qquad = x_{i-k} + \frac{(1-t)}{(1-qt^{n-(i-1)})}\begin{pmatrix} qt^{n-i}(x_i + x_{i-1} + \cdots + x_{i-(k-1)}) \\ +(x_{i-(k+1)} + \cdots + x_1) \end{pmatrix}. \end{aligned}$$

7.0.6. *The nonattacking fillings for E_{ε_i}.* The box greedy reduced word for u_{ε_i} is

$$u^{\square}_{\varepsilon_i} = \begin{array}{c|}\square\\ \vdots\\ \square\\ \vdots\\ \square\end{array}\boxed{s_{i-1}\cdots s_1\pi} \qquad \text{with } i \text{ non-attacking fillings,} \qquad \begin{array}{r|l}1&\\ \vdots&\\ i&i\\ \vdots&\\ n&\end{array} \qquad \begin{array}{c}\begin{array}{r|l}1&\\ \vdots&\\ i&k\\ \vdots&\\ n&\end{array}\\ i>k\\ \left(\frac{1-t}{1-qt^{n-(i-1)}}\right)\end{array}$$

7.0.7. *The nonattacking fillings for $E^z_{\varepsilon_i}$.* If $z(i)=i+k$ then the i non-attacking fillings are

$$\begin{array}{c}\begin{array}{r|l}z(1)&\\ \vdots&\\ i+k&i+k\\ \vdots&\\ z(n)&\end{array}\\ \\ t^{-k}\end{array} \qquad \begin{array}{c}\begin{array}{r|l}z(1)&\\ \vdots&\\ i+k&j\\ \vdots&\\ z(n)&\end{array}\\ i>j\geqslant 1\\ \left(\frac{1-t}{1-qt^{n-(i-1)}}\right)\end{array}$$

If $z(i)=i-k$ then the i non-attacking fillings are

$$\begin{array}{c}\begin{array}{r|l}z(1)&\\ \vdots&\\ i-k&j\\ \vdots&\\ z(n)&\end{array}\\ i\geqslant j>i-k\\ \left(\frac{(1-t)qt^{n-i}}{1-qt^{n-(i-1)}}\right)\end{array} \qquad \begin{array}{c}\begin{array}{r|l}z(1)&\\ \vdots&\\ i-k&i-k\\ \vdots&\\ z(n)&\end{array}\\ \\ 1\end{array} \qquad \begin{array}{c}\begin{array}{r|l}z(1)&\\ \vdots&\\ i-k&j\\ \vdots&\\ z(n)&\end{array}\\ i-k>j\geqslant 1\\ \left(\frac{1-t}{1-qt^{n-(i-1)}}\right)\end{array}$$

7.0.8. *The nonattacking fillings for $E_{2\varepsilon_i}$.* The box greedy reduced word for $u_{2\varepsilon_i}$ is

$$u^{\square}_{2\varepsilon_i} = (s_{i-1}\cdots s_1\pi)(s_{n-1}\cdots s_1\pi) = \begin{array}{c|}\square\\ \vdots\\ \square\\ \vdots\\ \square\end{array}\boxed{s_{i-1}\cdots s_1\pi}\,\boxed{s_{n-1}\cdots s_1\pi}$$

The case $E_{2\varepsilon_i}$ has $i\cdot n$ nonattacking fillngs and 2^{n+i-2} alcove walks. There are no covid triples for any of the nonattacking fillings so that $t^{covid(T)}=t^0=1$, and $q^{maj(T)}=q^1=q$ exactly when $T(i,1)<T(i,2)$.

$$\begin{array}{c}\begin{array}{r|ll}1&&\\ \vdots&&\\ i&i&i\\ \vdots&&\\ n&&\end{array}\\ \\ \\ \end{array} \quad \begin{array}{c}\begin{array}{r|ll}1&&\\ \vdots&&\\ i&k&k\\ \vdots&&\\ n&&\end{array}\\ k<i\\ \left(\frac{1-t}{1-q^2t^{n-(i-1)}}\right)\end{array} \quad \begin{array}{c}\begin{array}{r|ll}1&&\\ \vdots&&\\ i&i&\ell\\ \vdots&&\\ n&&\end{array}\\ \ell>i\\ \left(\frac{1-t}{1-qt}\right)q\end{array} \quad \begin{array}{c}\begin{array}{r|ll}1&&\\ \vdots&&\\ i&i&k\\ \vdots&&\\ n&&\end{array}\\ k<i\\ \left(\frac{1-t}{1-qt}\right)\end{array} \quad \begin{array}{c}\begin{array}{r|ll}1&&\\ \vdots&&\\ i&k&i\\ \vdots&&\\ n&&\end{array}\\ k<i\\ \left(\frac{1-t}{1-q^2t^{n-(i-1)}}\right)\left(\frac{1-t}{1-qt}\right)q\end{array}$$

$$
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
i & k\ \ell \\
\vdots & \\
n &
\end{array} \\
k<i,\ \ell>i \\
\left(\frac{1-t}{1-q^2t^{n-(i-1)}}\right)\left(\frac{1-t}{1-qt}\right)q
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
i & \ell\ k \\
\vdots & \\
n &
\end{array} \\
\{k,\ell\}\subseteq\{1,\dots,i-1\} \\
\left(\frac{1-t}{1-q^2t^{n-(i-1)}}\right)\left(\frac{1-t}{1-qt}\right)
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
i & k\ \ell \\
\vdots & \\
n &
\end{array} \\
\{k,\ell\}\subseteq\{1,\dots,i-1\} \\
\left(\frac{1-t}{1-q^2t^{n-(i-1)}}\right)\left(\frac{1-t}{1-qt}\right)q
\end{array}
$$

7.0.9. *The nonattacking fillings for $E_{\varepsilon_{j_1}+\varepsilon_{j_2}}$.* Let $j_1, j_2 \in \{1,\dots,n\}$ with $j_1 < j_2$. The box greedy reduced word for $u_{\varepsilon_{j_1}+\varepsilon_{j_2}}$ is

$$
u^{\square}_{\varepsilon_{j_1}+\varepsilon_{j_2}} =
\begin{array}{c|l}
\square & \\
\vdots & \\
\square & \boxed{s_{j_1-1}\cdots s_1\pi} \\
\vdots & \\
\square & \boxed{s_{j_2-2}\cdots s_1\pi} \\
\vdots & \\
\square &
\end{array}
$$

$E_{\varepsilon_{j_1}+\varepsilon_{j_2}}$ has $j_1(j_2-1)$ nonattacking fillings and $2^{j_1-1}2^{j_2-2}$ alcove walks.

$$
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
j_1 & j_1 \\
\vdots & \\
j_2 & j_2 \\
\vdots & \\
n &
\end{array} \\
\\
1
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
j_1 & k \\
\vdots & \\
j_2 & j_2 \\
\vdots & \\
n &
\end{array} \\
1\leqslant k\leqslant j_1-1 \\
\left(\frac{1-t}{1-qt^{n-j_1}}\right)
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
j_1 & j_1 \\
\vdots & \\
j_2 & \ell \\
\vdots & \\
n &
\end{array} \\
j_1+1\leqslant \ell\leqslant j_2-1 \\
\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
j_1 & k \\
\vdots & \\
j_2 & j_1 \\
\vdots & \\
n &
\end{array} \\
1\leqslant k\leqslant j_1-1 \\
\left(\frac{1-t}{1-qt^{n-j_1}}\right)\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c|l}
1 & \\
\vdots & \\
j_1 & j_1 \\
\vdots & \\
j_2 & k \\
\vdots & \\
n &
\end{array} \\
1\leqslant k\leqslant j_1-1 \\
t\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{c|c}
1 & \\
\vdots & \\
j_1 & k \\
\vdots & \\
j_2 & \ell \\
\vdots & \\
n &
\end{array} \\
k \in \{1, \dots, j_1 - 1\} \\
\ell \in \{j_1 + 1, \dots, j_2 - 1\} \\
\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-j_1}}\right)
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c}
1 & \\
\vdots & \\
j_1 & k \\
\vdots & \\
j_2 & \ell \\
\vdots & \\
n &
\end{array} \\
\{k, \ell\} \subseteq \{1, \dots, j_1 - 1\} \\
\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-j_1}}\right)
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c}
1 & \\
\vdots & \\
j_1 & \ell \\
\vdots & \\
j_2 & k \\
\vdots & \\
n &
\end{array} \\
\{k, \ell\} \subseteq \{1, \dots, j_1 - 1\} \\
\left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-j_1}}\right)t?
\end{array}
$$

8. Queue tableaux

8.0.1. *An instance of compression of NAFs – Motivation for Queue Tableaux.* In [5, Proposition 3.5(c)], if $j_1 = j_2 - 1$ then the third and fifth summands disappear to give

$$
\begin{aligned}
E_{\varepsilon_{j_2-1}+\varepsilon_{j_2}} &= x_{j_2-1}x_{j_2} + \left(\frac{1-t}{1-qt^{n-(j_2-1)}}\right)\sum_{k=1}^{j_2-2} x_k x_{j_2} \\
&\qquad + \left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-(j_2-1)}} + t\right)\sum_{k=1}^{j_2-2} x_k x_{j_2-1} \\
&\qquad + \left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-j_1}}\right)(1+t)\sum_{\{k,\ell\}\subseteq\{1,\dots,j_2-2\}} x_k x_\ell \\
&= x_{j_2-1}x_{j_2} + \left(\frac{1-t}{1-qt^{n-(j_2-1)}}\right)\sum_{k=1}^{j_2-2} x_k x_{j_2} \\
&\qquad + \left(\frac{1-t}{\cancel{1-qt^{n-(j_2-2)}}}\right)\left(\frac{\cancel{1-qt^{n-(j_2-2)}}}{1-qt^{n-(j_2-1)}}\right)\sum_{k=1}^{j_2-2} x_k x_{j_2-1} \\
&\qquad + \left(\frac{1-t}{1-qt^{n-(j_2-2)}}\right)\left(\frac{1-t}{1-qt^{n-(j_2-1)}}\right)(1+t)\sum_{\{k,\ell\}\subseteq\{1,\dots,j_2-2\}} x_k x_\ell,
\end{aligned}
$$

which is an example of the additional cancellation that occurs when there are adjacent rows of equal length and illustrates the the difference between nonattacking fillings and queue tableaux.

8.0.2. *Queue tableaux.* Following (and slightly generalizing) [4, Definition A.1], a *queue tableau* of shape (z, μ) is a nonattacking filling T of (z, μ) such that

(QT) If $\mu_i = \mu_{i-1} = \cdots = \mu_{i-r}$ then $T(i,j) \notin \{T(i-1, j-1), \dots, T(i-r, j-1)\}$.

If the parts of μ are distinct then a queue tableau is no different than a nonattacking filling. More generally, if $\mu_i \neq \mu_{i+1}$ for $i \in \{1, \dots, n-1\}$ then a queue tableau is no different than a nonattacking filling.

8.0.3. *Multiline queues.* The *multiline queue* corresponding to a queue tableau T is the pipe dream P corresponding to T under the map given in (23), namely

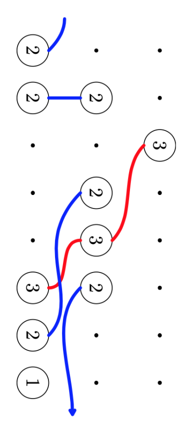
$$P(k,j) = i \qquad \text{if and only if} \qquad T(i,j) = k,$$

The example in [4, Figures 3 and 12] has

$$\text{queue tableau}\quad T=\begin{array}{c|ccc} 6&6&5&3\\ 1&1&6&\\ 2&2&2&\\ 7&7&4&\\ 8&8&&\\ 3&&&\\ 4&&&\\ 5&&& \end{array}\qquad \text{and pipe dream}\quad P=\begin{pmatrix} 2&\vline&2&0&0\\ 3&\vline&3&3&0\\ 6&\vline&0&0&1\\ 7&\vline&0&4&0\\ 8&\vline&0&1&0\\ 1&\vline&1&2&0\\ 4&\vline&4&0&0\\ 5&\vline&5&0&0 \end{pmatrix}$$

The picture of this pipe dream from [4, Figures 3] is

the multiline queue



8.0.4. *Compression not captured by NAFs or QT.* Let $\mathrm{AW}_\mu=\mathrm{AW}_\mu^{\mathrm{id}}$, $\mathrm{NAF}_\mu=\mathrm{NAF}_\mu^{\mathrm{id}}$, and $\mathrm{QT}_\mu=\mathrm{QT}_\mu^{\mathrm{id}}$. The example

$$\#\mathrm{AW}_{(2,2,1,1,0,0)}=16,\quad \#\mathrm{NAF}_{(2,2,1,1,0,0)}=9\quad\text{and}\quad \#\mathrm{QT}_{(2,2,1,1,0,0)}=7.$$

is provided in [4, Figure 4]). The equalities (see (see [5, Proposition 5.8])

$$E_{(2,0,1)}(x_1,x_2,x_3;q,t)=(x_1x_2x_3)^2E_{(1,2,0)}(x_3^{-1},x_2^{-1},x_1^{-1};q,t),\quad\text{and}$$
$$E_{(2,2,0)}(x_1,x_2,x_3;q,t)=q^{-1}E_{(2,0,1)}(x_3,x_1,x_2;q,t)$$

indicate that if one provides a formula for $E_{(1,2,0)}$ then there are formulas for $E_{(2,0,1)}$ and $E_{(2,2,0)}$ with exactly the same number of terms. For these cases,

$$\begin{aligned} &\#\mathrm{AW}_{(1,2,0)}=4,\quad \#\mathrm{NAF}_{(1,2,0)}=3,\quad \#\mathrm{QT}_{(1,2,0)}=3.\\ &\#\mathrm{AW}_{(2,0,1)}=4,\quad \#\mathrm{NAF}_{(2,0,1)}=4,\quad \#\mathrm{QT}_{(2,0,1)}=4.\\ &\#\mathrm{AW}_{(2,2,0)}=4,\quad \#NAF_{(2,2,0)}=4,\quad \#\mathrm{QT}_{(2,2,0)}=3. \end{aligned}$$

Thus $\mu=(2,0,1)$ is a case where possible compression is not realized by either the NAFs or the QT.

8.0.5. *Comparing* $\#\mathrm{NAF}$ *and* $\#\mathrm{QT}$ *for* $(r,0,\dots,0)$ *and* $(r,\dots,r,0)$. Since $u_{(r,0,\dots,0)}=\pi(s_{n-1}\cdots s_1\pi)^{r-1}$ and $u_{(r,r,\dots,r,0)}=\pi^{n-1}(s_1\pi)^{(n-1)(r-1)}$ then

$$\begin{aligned} &\#\mathrm{AW}_{(r,0,0,\dots,0)}=(2^{n-1})^{r-1},\qquad \#\mathrm{NAF}_{(r,0,0,\dots,0)}=n^{r-1},\\ &\#\mathrm{AW}_{(r,r,\dots,r,0)}=(2^{n-1})^{r-1},\qquad \#\mathrm{NAF}_{(r,r,\dots,r,0)}=(2^{n-1})^{r-1}, \end{aligned}$$

$$\#\mathrm{QT}_{(r,0,0,\dots,0)}=n^{r-1}\qquad\text{and}\qquad \#\mathrm{QT}_{(r,r,\dots,r,0)}=n^{r-1}.$$

To see the last equality: In a queue tableau of shape $(r,r,\dots,r,0)$, for each column after the first, we get to choose the position of the $j\in\{1,\dots,n\}$ that did not appear in the column before (n choices total for each column).

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References

[1] Alexei Borodin and Michael Wheeler, *Nonsymmetric Macdonald polynomials via integrable vertex models*, 2019, `https://arxiv.org/abs/arXiv:1904.06804`.

[2] Luigi Cantini, Jan de Gier, and Michael Wheeler, *Matrix product formula for Macdonald polynomials*, J. Phys. A **48** (2015), no. 38, 384001, 25 pages.

[3] ———, *Matrix product and sum rule for Macdonald polynomials*, in 28th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2016), Discrete Math. Theor. Comput. Sci. Proc., BC, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, [2016] ©2016, pp. 311–321.

[4] Sylvie Corteel, Olya Mandelshtam, and Lauren Williams, *From multiline queues to Macdonald polynomials via the exclusion process*, Sém. Lothar. Combin. **82B** (2020), Art. 97, 12 pages.

[5] Weiying Guo and Arun Ram, *Comparing formulas for type GL_n Macdonald polynomials*, Algebr. Comb. **5** (2022), no. 5, 849–883.

[6] J. Haglund, M. Haiman, and N. Loehr, *A combinatorial formula for nonsymmetric Macdonald polynomials*, Amer. J. Math. **130** (2008), no. 2, 359–383.

[7] James Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008, With an appendix on the combinatorics of Macdonald polynomials.

[8] Masahiro Kasatani and Yoshihiro Takeyama, *The quantum Knizhnik-Zamolodchikov equation and non-symmetric Macdonald polynomials*, Funkcial. Ekvac. **50** (2007), no. 3, 491–509.

[9] Cristian Lenart, *On combinatorial formulas for Macdonald polynomials*, Adv. Math. **220** (2009), no. 1, 324–340.

[10] I. G. Macdonald, *Symmetric functions and Hall polynomials. With contributions by A. Zelevinsky*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.

[11] ———, *Affine Hecke algebras and orthogonal polynomials*, Astérisque (1996), no. 237, Exp. No. 797, 4, 189–207, Séminaire Bourbaki, Vol. 1994/95.

[12] ———, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, Cambridge, 2003.

[13] Robert Steinberg, John Faulkner, and Robert Wilson, *Lectures on Chevalley groups*, Yale University New Haven, 1967, Notes prepared by John Faulkner and Robert Wilson. Revised and corrected edition of the 1968 original.

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