ALGEBRAIC COMBINATORICS

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Volume 5, issue 5 (2022), p. 849-883.
https://doi.org/10.5802/alco.227

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Comparing formulas for type GL_n Macdonald polynomials

Weiying Guo & Arun Ram

Dedicated to Hélène Barcelo

Abstract The paper compares (and reproves) the alcove walk and the nonattacking fillings formulas for type GL_n Macdonald polynomials which were given in [10], [1] and [18]. The "compression" relating the two formulas in this paper is the same as that of Lenart [13]. We have reformulated it so that it holds without conditions and so that the proofs of the alcove walks formula and the nonattacking fillings formula are parallel. This reformulation highlights the role of the double affine Hecke algebra and Cherednik's intertwiners. An exposition of the type GL_n double affine braid group, double affine Hecke algebra, and all definitions and proofs regarding Macdonald polynomials are provided to make this paper self contained.

0. Introduction

The Macdonald polynomials are an incredible family of orthogonal polynomials which simultaneously generalize Schur functions, Weyl characters, Demazure characters, Askey-Wilson polynomials, Koornwinder polynomials, Hall-Littlewood polynomials, Jack polynomials and spherical functions on p-adic groups. They are eigenfunctions of a family of difference operators which generalize the classical Laplacian and, in this sense, the Macdonald polynomials E_μ are generalizations of spherical harmonics.

This paper is a study of the relationship between combinatorial formulas for GL_n-type Macdonald polynomials:

(a) The nonattacking fillings formulas from [10, Theorem 3.5.1] and [1], and
(b) The alcove walks formula from [18, Theorem 3.1].

Except for Section 4, which contains the recursions and the calculations for the proofs, we have made an effort to try to make the different sections of this paper readable independent of each other. The reader should not hesitate to go directly to Section 5 for an introduction to the double affine Hecke algebra, to Section 2 for an entrée to n-periodic permutations and the affine Weyl group, and to Section 3 for the basics of Macdonald polynomials and some explicit examples of them.

The first half of Section 3 defines the various kinds of Macdonald polynomials, the E_μ, the P_λ and the E^z_μ; the second half of Section 3 computes some examples. In [1], the *relative Macdonald polynomials* $E^z_\mu = (const)T_z E_\mu$ of this paper are called "permuted basement Macdonald polynomials". These "T_z shifted Macdonald polynomials" are useful for all root systems and have an alcove walks formula [18, Theorem 2.2]. In the general root system setting, the notion of a "basement" has a different flavor

Manuscript received 7th April 2021, revised 6th May 2021, accepted 18th January 2022.

Keywords. Macdonald polynomials, affine Hecke algebras, tableaux.

ISSN: 2589-5486 http://algebraic-combinatorics.org/

(the z in the T_z is the "basement") and so we propose the term *relative Macdonald polynomials* for the E_μ^z.

As explained in Macdonald's book [16], the n-periodic permutation u_μ defined in Section 2 is a critical ingredient for the understanding of the combinatorics of Macdonald polynomials and their construction by intertwiners $\tau_i^\vee$. Proposition 2.2 provides a favorite reduced word for u_μ and determines its inversions. The inversions of u_μ provide the "arms" and "legs" that appear in [10] (denoted Narm_μ and Nleg_μ in this paper), and this observation connects those statistics with the roots of the affine root system for type GL_n. Proposition 4.3 derives a box-by-box recursion for computing Macdonald polynomials and Remark 4.4 shows that the statistic that falls out of this derivation (in terms of a comparison of lengths of permutations) counts the coinversion triples that are used in [10]. This observation completes the interpretation of the statistics in the nonattacking fillings formula in terms of the Weyl group and the root system. Let us highlight that using the box-by-box recursion to compute Macdonald polynomials is equivalent to using a special reduced word for the n-periodic permutation u_μ, the *box greedy reduced word for* u_μ.

The proof of the alcove walks formula is obtained by iterating the *step-by-step recursion* for the relative Macdonald polynomials E_μ^z. The proof of the nonattacking fillings formula is obtained by iterating the *box-by-box recursion* for the relative Macdonald polynomials E_μ^z. Except for the effort to normalize the E_μ^z so that $x^{z\mu}$ has coefficient 1, the proof of the step-by-step recursion does not differ from the proof of [18, Theorem 2.2]. The proof of the box-by-box recursion is, at its core, the same as [13, Proposition 4.1] (Lenart's main results are stated for symmetric Macdonald polynomials P_λ where λ has distinct parts, we treat the general relative case E_μ^z). Our reformulation and proof highlights the role of the intertwiners and the connection to the affine root system and pinpoints exactly which intertwiners get "compressed".

Section 5 provides a Type GL_n specific exposition, from scratch, of the double affine Hecke algebra and its use for defining and studying Macdonald polynomials. **In** [9]**, a supplement to this paper, we provide examples and further observations.**

A small warning: Even though they all have a Type A root system, type SL_n Macdonald polynomials, type PGL_n Macdonald polynomals and type GL_n Macdonald polynomials are all *different* (though the relationship is well known and not difficult). We should stress that this paper is specific to the GL_n-case and some results of this paper do not hold for Type SL_n or type PGL_n unless properly modified.

1. Boxes, alcove walks and nonattacking fillings

The goal of this section is to state the main results: the alcove walks formula and the nonattacking fillings formula, and the compression map ψ which relates them. We begin by setting up the combinatorics of boxes, diagrams, alcove walks and nonattacking fillings. Then, after specifying the weights attached to alcove walks and to nonattacking fillings we state the alcove walks formula and the nonattacking fillings formula for Macdonald polynomials as weighted sums of alcove walks and nonattacking fillings, respectively.

1.1. Boxes.

Fix $n \in \mathbb{Z}_{>0}$. A *box* is an element of $\{1, \ldots, n\} \times \mathbb{Z}_{\geqslant 0}$ so that

$$\{\text{boxes}\} = \{(i, j) \mid i \in \{1, \ldots, n\},\ j \in \mathbb{Z}_{\geqslant 0}\}.$$

To conform to [14, p.2], we draw the box (i, j) as a square in row i and column j using the same coordinates as are usually used for matrices.

(1) The *cylindrical coordinate* of the box (i, j) is the number $i + nj$.

The *basement* is the set $\{(i,0) \mid i \in \{1,\dots,n\}\}$, so that the basement is the collection of boxes in the 0th column. Pictorially,

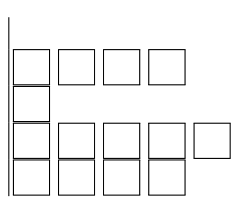
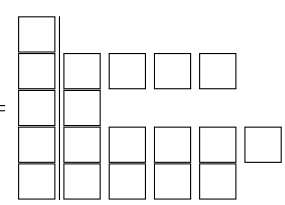
$_1\ (1,0)$	$_6\ (1,1)$	$_{11}(1,2)$	$_{16}\ (1,3)$	$_{23}\ (1,4)$	$\cdots$
$_2\ (2,0)$	$_7\ (2,1)$	$_{12}\ (2,2)$	$_{17}\ (2,3)$	$_{22}\ (2,4)$	$\cdots$
$_3\ (3,0)$	$_8\ (3,1)$	$_{13}\ (3,2)$	$_{18}\ (3,3)$	$_{23}\ (3,4)$	$\cdots$
$_4\ (4,0)$	$_9\ (4,1)$	$_{14}\ (4,2)$	$_{19}\ (4,3)$	$_{24}\ (4,4)$	$\cdots$
$_5\ (5,0)$	$_{10}\ (5,1)$	$_{15}\ (5,2)$	$_{20}\ (5,3)$	$_{25}\ (5,4)$	$\cdots$

with box (i,j) numbered $_{i+nj}$.

Let $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ be an n-tuple of nonnegative integers. The *diagram of* μ is the set $dg(\mu)$ of boxes with μ_i boxes in row i and the *diagram of* μ *with basement* $\widehat{dg}(\mu)$ includes the extra boxes $(i,0)$ for $i \in \{1,\dots,n\}$:

$$dg(\mu) = \{(i,j) \mid i \in \{1,\dots,n\} \text{ and } j \in \{1,\dots,\mu_i\}\} \quad \text{and}$$
$$\widehat{dg}(\mu) = \{(i,j) \mid i \in \{1,\dots,n\} \text{ and } j \in \{0,1,\dots,\mu_i\}\}.$$

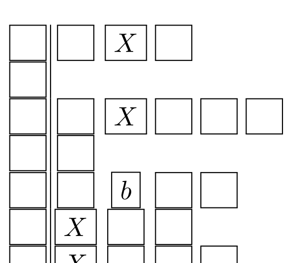
It is often convenient to abuse notation and identify μ, $dg(\mu)$ and $\widehat{dg}(\mu)$ (because these are just different ways of viewing the sequence $(\mu_1,\dots,\mu_n)$). For example, if $\mu = (0,4,1,5,4)$ then

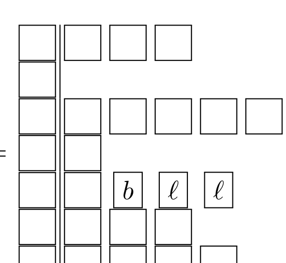
$dg(\mu) =$  and $\widehat{dg}(\mu) =$ 

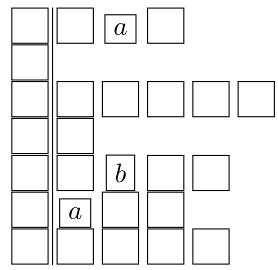
1.2. Alcove walks and nonattacking fillings. Let $\mu \in \mathbb{Z}^n_{\geqslant 0}$. Using cylindrical coordinates for boxes as specified (1), define, for a box $b \in dg(\mu)$,

$$\text{attack}_\mu(b) = \{b-1,\dots,b-n+1\} \cap \widehat{dg}(\mu), \tag{2}$$
$$\text{Nleg}_\mu(b) = (b + n\mathbb{Z}_{>0}) \cap dg(\mu) \quad \text{and} \tag{3}$$
$$\text{Narm}_\mu(b) = \{a \in \text{attack}_\mu(b) \mid \#\text{Nleg}_\mu(a) \leqslant \#\text{Nleg}_\mu(b)\}, \tag{4}$$

where $\#\text{Nleg}_\mu(a)$ denotes the number of elements of $\text{Nleg}_\mu(a)$. For example, with $\mu = (3,0,5,1,4,3,4)$ and $b = (5,2)$, which has cylindrical coordinate $b = 5 + 7\cdot 2 = 19$ the sets $\text{attack}_\mu(b)$, $\text{Narm}_\mu(b)$ and $\text{Nleg}_\mu(b)$ are pictured as

$\text{attack}_\mu(b) =$ 

$\text{Nleg}_\mu(b) =$ 

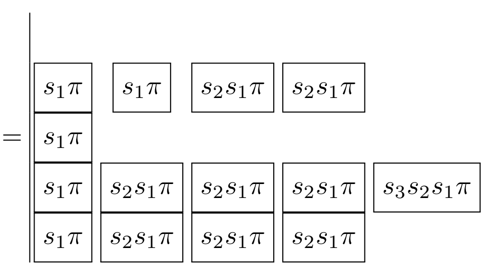
$\mathrm{Narm}_\mu(b) =$ 

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and let u_μ be the n-periodic permutation defined in (33). Letting $u_\mu(i,j) = n - 1 - \#\mathrm{attack}_\mu(i,j)$, the *box-greedy reduced word for* u_μ is

$$u_\mu^\square = \prod_{\text{boxes } (i,j) \text{ in } dg(\mu)} (s_{u_\mu(i,j)} \cdots s_2 s_1 \pi). \tag{5}$$

For the purposes of this section it is only necessary to recognize $u_\mu^\square$ as an abstract word in symbols $s_1, \ldots, s_{n-1}, \pi$. For an example, if $\mu = (0,4,1,5,4)$ then the box-greedy reduced word for u_μ is

$$u_\mu^\square = (s_1\pi)^5 (s_2 s_1 \pi)^8 (s_3 s_2 s_1 \pi) = \tag{6}$$

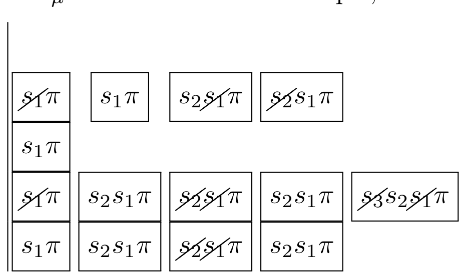


(the reduced word is a product of the boxes read in increasing order by cylindrical coordinate).

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. Let $\vec{u}_\mu = w_1 w_2 \cdots w_\ell$ be a reduced word for u_μ so that $w_1, \ldots, w_\ell$ are the factors of $\vec{u}_\mu$ (a good choice is to let $\vec{u}_\mu = u_\mu^\square$). An *alcove walk* of type $(z, \vec{u}_\mu)$ is a sequence $p = (p_0, p_1, \ldots, p_r)$ of elements of W (see (19), but, as for $u_\mu^\square$, it is sensible just to view the p_k as words in the symbols $s_1, \ldots, s_{n-1}, \pi$) such that

$$p_0 = z, \qquad p_k = p_{k-1}\pi \text{ if } w_k = \pi, \qquad \text{and} \qquad p_k \in \{p_{k-1}, p_{k-1}w_k\} \text{ if } w_k \neq \pi.$$

In other words, an alcove walk of type $(z, \vec{u}_\mu)$ is equivalent to choosing a subset of the s_i factors in $\vec{u}_\mu$ to cross out. For example,

(7) $P =$  is equivalent to the alcove walk

$$p = (p_0, p_1, \ldots, p_{37}) = (z, z, z\pi, z\pi s_1, z\pi s_1 \pi, z\pi s_1 \pi, z\pi s_1 \pi^2, z\pi s_1 \pi^2 s_1, z\pi s_1 \pi^2 s_1 \pi, \ldots)$$

(there is a repeat entry in p each time there is an s_i crossed out in P). In this example, there are $5 + 2 \cdot 8 + 3 = 24$ factors of the form s_i in $u_\mu^\square$ and so there are a total of 2^{24} alcove walks of type $(z, u_\mu^\square)$ (for any fixed permutation $z \in S_n$).

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. A *nonattacking filling* for (z, μ) is $T \colon \widehat{dg}(\mu) \to \{1, \ldots, n\}$ such that

(a) $T(i,0) = z(i)$ for $i \in \{1,\dots,n\}$ and
(b) if $b \in dg(\mu)$ and $a \in \mathrm{attack}_\mu(b)$ then $T(a) \neq T(b)$.

For example,

$$T = \begin{array}{r|l} 1 & \\ 2 & 1\ 1\ 1\ 2 \\ 3 & 3 \\ 4 & 4\ 4\ 5\ 4\ 4 \\ 5 & 5\ 2\ 3\ 3 \end{array} \qquad \begin{array}{l} \text{is a nonattacking filling for } (z,\mu) \\ \\ \quad \text{with } z = \mathrm{id} \in S_5 \text{ and } \mu = (0,4,1,5,4). \end{array} \tag{8}$$

Let b be a box in μ. Starting at b read, in succession, in reverse order by cylindrical coordinate, the entries from T in (earlier) boxes, skipping values that have already been encountered. This process produces, for each box $b \in dg(\mu)$, a permutation $z_T(b)$ in S_n. For example, with T as in (8), box $(4,3)$ in T (row 4, column 3) produces the permutation (in one line notation)

$$(34215) \text{ formed from the circled numbers in } \qquad \begin{array}{r|lllll} 1 & \\ 2 & 1 & 1 & \textcircled{1} & 2 \\ 3 & \textcircled{3} \\ 4 & 4 & \textcircled{4} & \textcircled{5} & 4 & 4 \\ 5 & 5 & \textcircled{2} & 3 & 3 \end{array}$$

and doing this for all boxes in T produces

$$z_T = \begin{array}{|lllll} \\ \boxed{(23451)} & \boxed{(23451)} & \boxed{(35421)} & \boxed{(41532)} \\ \boxed{(24513)} \\ \boxed{(25134)} & \boxed{(23514)} & \boxed{(34215)} & \boxed{(15324)} & \boxed{(15234)} \\ \boxed{(21345)} & \boxed{(35142)} & \boxed{(42153)} & \boxed{(15243)} \end{array} \tag{9}$$

The sequence

$$z_T = (z_T(b) \mid b \in dg(\mu)) \quad \text{is the } \textit{permutation sequence of } T. \tag{10}$$

Let $c_n = s_1 \cdots s_{n-1}$, an n-cycle in S_n. If $b = (i,j)$ is a box in $dg(\mu)$, the permutation $z_T(b')$ in the next box of z_T (by cylindrical coordinate) is

$$z_T(b') = z_T(b) s_r \cdots s_2 s_1 c_n, \qquad \text{where } r \in \{0,1,\dots,u_\mu(b')\}, \tag{11}$$

and $s_r \cdots s_2 s_1 \pi$ is the entry in box b' of the alcove walk $\varphi(T)$ corresponding to the nonattacking filling T. (If this construction of the permutation sequence feels ad hoc, the sentence before Lemma 4.2 may help to provide some insight into its source.)

For example, for z_T as in (9), and with $z = (12345)$ the permutation in the basement of T, then

$$\begin{array}{ll} (23451) = z_T(2,1) = z c_n, & (23451) = z_T(2,2) = z_T(5,1) s_1 c_n, \\ (24513) = z_T(3,1) = z_T(2,1) s_1 c_n, & \\ (25134) = z_T(4,1) = z_T(3,1) s_1 c_n, & (23514) = z_T(4,2) = z_T(2,2) s_2 s_1 c_n, \\ (21345) = z_T(5,1) = z_T(4,1) s_1 c_n, & (35142) = z_T(5,2) = z_T(4,2) c_n, \end{array}$$

and so forth all the way to the last box of μ. Keeping track only of the factor which is the difference between successive boxes produces the alcove walk

$$\varphi(T) = \left| \begin{array}{lllll} \boxed{\cancel{s_1}\pi} & \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{\cancel{s_2}s_1\pi} & \\ \boxed{s_1\pi} & & & & \\ \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{\cancel{s_2}s_1\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{s_3s_2s_1\pi} \\ \boxed{s_1\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{s_2s_1\pi} & \end{array} \right.$$

In summary, letting

$$\begin{aligned} \mathrm{AW}^z_\mu &= \{\text{alcove walks of type } (z, u^\square_\mu)\} \qquad \text{and} \\ \mathrm{NAF}^z_\mu &= \{\text{nonattacking fillings for } (z,\mu)\}, \end{aligned}$$

we have produced an injective map

$$\varphi\colon \mathrm{NAF}^z_\mu \to \mathrm{AW}^z_\mu.$$

By (11), the image of φ consists exactly of the alcove walks such that, each box $b = (i,j) \in dg(\mu)$ contains a suffix $s_r \cdots s_1\pi$ of the entry $s_{u_\mu(i,j)} \cdots s_1\pi$ in box b in $u^\square_\mu$. The *compression map* is the function

$$\psi\colon \mathrm{AW}^z_\mu \to \mathrm{NAF}^z_\mu \tag{12}$$

which, in each box, forces every s_i factor before the last crossed out factor in that box also to be crossed out. For example, if P is the alcove walk in (7) then

$$\psi(P) = \left| \begin{array}{lllll} \boxed{\cancel{s_1}\pi} & \boxed{s_1\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{\cancel{s_2}s_1\pi} & \\ \boxed{s_1\pi} & & & & \\ \boxed{\cancel{s_1}\pi} & \boxed{s_2s_1\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{s_2s_1\pi} & \boxed{\cancel{s_3}\cancel{s_2}\cancel{s_1}\pi} \\ \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{\cancel{s_2}\cancel{s_1}\pi} & \boxed{s_2s_1\pi} & \end{array} \right.$$

Identifying NAF^z_μ with $\operatorname{im}\varphi$, then $\psi\circ\varphi\colon \mathrm{NAF}^z_\mu \to \mathrm{NAF}^z_\mu$ is the identity map on nonattacking fillings.

1.3. Formulas for the relative Macdonald polynomials E^z_μ.

In this subsection we state the alcove walks formula and the nonattacking fillings formula for E^z_μ. The proofs are by the step-by-step recursion (Proposition 4.1) and the box-by-box recursion (Proposition 4.3), respectively. The statistics $\mathrm{sh}(-\beta^\vee_k)$, $\mathrm{ht}(-\beta^\vee_k)$, $\mathrm{norm}(p_k)$ on alcove walks which are introduced below are read off of the step-by-step recursion, Proposition 4.1. Similarly, the statistics $\#\mathrm{Nleg}_\mu(b)+1$, $\#\mathrm{Narm}_\mu(b)+1$, and $\#\mathrm{bwn}_T(b)$ on nonattacking fillings which are introduced below are read off the box-by-box recursion, Proposition 4.3, and Remark 4.4.

Equations (13)–(15) use the notations of Section 2.3 so that W is the group of n-periodic permutations defined in (19) the root sequence for $\vec{u}_\mu$ corresponds to the inversions of u_μ as in (31) and the shift and height of an affine coroot are as given in (27).

Let $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. Let $s_\pi = \pi$ and let $\vec{u}_\mu = s_{i_1}\cdots s_{i_r}$ be a reduced word for u_μ (a good choice is to let $\vec{u}_\mu = u^\square_\mu$). An *alcove walk* of type $(z,\vec{u}_\mu)$ is

$$\text{a sequence} \quad p = (p_0,p_1,\ldots,p_r) \quad \text{of elements of } W \quad \text{such that} \tag{13}$$

$p_0 = z$; if $s_{i_k} = \pi$ then $p_k = p_{k-1}\pi$; and if $s_{i_k} \neq \pi$ then $p_k \in \{p_{k-1}, p_{k-1}s_{i_k}\}$. The *permutation sequence of* p is the sequence of elements of S_n,

$$\vec{z}_p = (z_0, \ldots, z_r), \qquad \text{given by} \qquad z_j = \overline{p_j}, \tag{14}$$

where $\overline{}\colon W \to S_n$ is the homomorphism given by $\overline{t_\nu v} = v$ (see (24)). The *root sequence* for $\vec{u}_\mu$ is

$$\text{the sequence} \quad (\beta_k^\vee \mid i_k \neq \pi) \quad \text{given by} \quad \beta_j^\vee = s_{i_r}^{-1} s_{i_{r-1}}^{-1} \cdots s_{i_{k+1}}^{-1} \alpha_{i_k}^\vee.$$

Define

$$\mathrm{ht}(\varepsilon_i^\vee - \varepsilon_j^\vee - \ell K) = j - i, \qquad \text{and} \qquad \mathrm{sh}(\varepsilon_i^\vee - \varepsilon_j^\vee - \ell K) = \ell.$$

For $k \in \{1, \ldots, r\}$ with $p_{k-1} = p_k$ define

$$\mathrm{norm}(p_k) = \tfrac{1}{2}\big(\ell(z_{k-1}s_{i_k}v_{z_{k-1}\mu}^{-1}) - \ell(z_{k-1}v_{z_{k-1}\mu}^{-1}) - \ell(s_{i_k})\big),$$

where $\ell(s_{i_k}) = 1$. Let q and t and $x_1, \ldots, x_n$ be variables. For a step $k \in \{1, \ldots, \ell\}$ of the alcove walk $p = (p_0, \ldots, p_\ell)$ define the *weight of* p_k by

$$\mathrm{wt}_p(k) = \begin{cases} \Big(\dfrac{1-t}{1-q^{\mathrm{sh}(-\beta_k^\vee)}t^{\mathrm{ht}(-\beta_k^\vee)}}\Big)t^{\mathrm{norm}(p_k)}, & \text{if } p_k = p_{k-1} \text{ and } p_{k-1}s_{i_k} < p_{k-1}, \\ \Big(\dfrac{(1-t)q^{\mathrm{sh}(-\beta_k^\vee)}t^{\mathrm{ht}(-\beta_k^\vee)}}{1-q^{\mathrm{sh}(-\beta_k^\vee)}t^{\mathrm{ht}(-\beta_k^\vee)}}\Big)t^{\mathrm{norm}(p_k)}, & \text{if } p_k = p_{k-1} \text{ and } p_{k-1}s_{i_k} > p_{k-1}, \\ 1, & \text{if } p_k = p_{k-1}s_{i_k}, \\ x_{z_{k-1}(1)} & \text{if } p_k = p_{k-1}\pi, \end{cases}$$

and define the *weight of* p by

$$\mathrm{wt}(p) = \prod_{k=1}^{\ell} \mathrm{wt}_p(k), \qquad \text{a product over the steps of } p. \tag{15}$$

Let $\mu \in \mathbb{Z}_{\geqslant 0}^n$ and $z \in S_n$ and let T be a nonattacking filling of shape (z, μ). For $b \in dg(\mu)$ let

$$\mathrm{bwn}_T(b) = \Big\{ a \in \mathrm{Narm}_\mu(b) \;\Big|\; \begin{array}{l} T(b-n) > T(a) > T(b) \\ \text{or } T(b-n) < T(a) < T(b) \end{array} \Big\}. \tag{16}$$

The *weight of* b *in* T is

(17)

$$\mathrm{wt}_T(b) = \begin{cases} \Big(\dfrac{1-t}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)+1}}\Big)t^{\#\mathrm{bwn}_T(b)}x_{T(b)}, & \text{if } T(b-n) > T(b), \\ \Big(\dfrac{(1-t)q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)}}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)+1}}\Big)t^{-\#\mathrm{bwn}_T(b)}x_{T(b)}, & \text{if } T(b-n) < T(b), \\ x_{T(b)}, & \text{if } T(b-n) = T(b), \end{cases}$$

and the *weight of* T is

$$\mathrm{wt}(T) = \prod_{b \in dg(\mu)} \mathrm{wt}_T(b), \qquad \text{a product over the boxes of } T. \tag{18}$$

The following theorem summarizes (and slightly generalizes) [18, Theorem 3.1], [1, Def. 5 and Prop. 6] and [10, Theorem 3.5.1].

Theorem 1.1. *Let* $\mu \in \mathbb{Z}_{\geqslant 0}^n$ *and* $z \in S_n$. *Let* E_μ^z *be the relative* (*or permuted basement*) *nonsymmetric Macdonald polynomial defined in* (43). *Let* $\vec{u}_\mu$ *be a reduced word for* u_μ *and let*

$$\mathrm{AW}_\mu^z = \{\text{alcove walks of type } (z, \vec{u}_\mu)\} \qquad \text{and}$$
$$\mathrm{NAF}_\mu^z = \{\text{nonattacking fillings for } (z, \mu)\}$$

(a) *Alcove walks formula:* $E^z_\mu = \sum_{p\in AW^z_\mu} \mathrm{wt}(p)$.

(b) *Nonattacking fillings formula:* $E^z_\mu = \sum_{T\in \mathrm{NAF}^z_\mu} \mathrm{wt}(T)$.

Proof. (a) is obtained by successive applications of the step-by-step recursion (Proposition 4.1), and (b) is obtained by successive applications of the box-by-box recursion (Proposition 4.3). The weight of each box $\mathrm{wt}_T(b)$ comes from the coefficient of the correpsonding term in Proposition 4.3 and, by Remark 4.4 and Remark 2.3, these weights can be stated in the form (16) and (17). □

The following is a corollary of Lemma 4.2 (specifically, the step in line (52)). Lemma 4.2 is a version of [13, Proposition 4.1], which forms the core of the proof of the box-by-box recursion Proposition 4.3.

Corollary 1.2. *Let $\mu \in \mathbb{Z}^n_{\geqslant 0}$ and $z \in S_n$. Let $\psi\colon \mathrm{AW}^z_\mu \to \mathrm{NAF}^z_\mu$ be the compression function defined in* (12) *and let $T \in \mathrm{NAF}^z_\mu$. Then*

$$\mathrm{wt}(T) = \sum_{p\in\psi^{-1}(T)} \mathrm{wt}(p).$$

The following example illustrates the proof of the nonattacking fillings formula by iterating the box-by-box recursion to produce the nonattacking fillings expansion of the Macdonald polynomial $E_{(2,2,1,1,0,0)} = E^{(123456)}_{(2,2,1,1,0,0)}$. The first four applications of Proposition 4.3 give

$$\begin{aligned}E^{(123456)}_{(2,2,1,1,0,0)} &= x_1E^{(234561)}_{(2,1,1,0,0,1)} = x_1x_2E^{(345612)}_{(1,1,0,0,1,1)} = x_1x_2x_3E^{(456123)}_{(1,0,0,1,1,0)}\\ &= x_1x_2x_3x_4E^{(561234)}_{(0,0,1,1,0,0)}.\end{aligned}$$

The fifth box is produced by applying Proposition 4.3 to $E^{(561234)}_{(0,0,1,1,0,0)}$ to obtain

$$E^{(561234)}_{(0,0,1,1,0,0)} = x_1E^{(562341)}_{(0,0,1,0,0,0)} + \Big(\frac{1-t}{1-qt^{5-2}}\Big)\big(qx_6E^{(512346)}_{(0,0,1,0,0,0)} + qx_5E^{(612345)}_{(0,0,1,0,0,0)}\big).$$

The last box is obtained by applying Proposition 4.3 to each of the terms $E^{(562341)}_{(0,0,1,0,0,0)}$, $E^{(512346)}_{(0,0,1,0,0,0)}$ and $E^{(612345)}_{(0,0,1,0,0,0)}$ which have been generated in the previous step:

$$\begin{aligned}E^{(562341)}_{(0,0,1,0,0,0)} &= x_2 + \Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(qtx_6+qtx_5\big),\\ E^{(512346)}_{(0,0,1,0,0,0)} &= x_2 + \Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(x_1+qtx_5\big),\\ E^{(612345)}_{(0,0,1,0,0,0)} &= x_2 + \Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(x_1+qx_6\big), \quad \text{since } E^z_{(0,0,0,0,0,0)} = 1 \text{ for } z\in S_n.\end{aligned}$$

Compiling these produces an expansion of $E_{(2,2,1,1,0,0)}$ with 9 terms,

$$\begin{aligned}E^{(123456)}_{(2,2,1,1,0,0)} &= x_1x_2x_3x_4E^{(561234)}_{(0,0,1,1,0,0)}\\ &= x_1x_2x_3x_4\left(x_1E^{(562341)}_{(0,0,1,0,0,0)} + \Big(\frac{1-t}{1-qt^{5-2}}\Big)\big(qx_6E^{(512346)}_{(0,0,1,0,0,0)} + qx_5E^{(612345)}_{(0,0,1,0,0,0)}\big)\right)\\ &= x_1x_2x_3x_4\begin{pmatrix} x_1\Big(x_2+\Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(qtx_6+qtx_5\big)\Big)\\ +\Big(\frac{1-t}{1-qt^{5-2}}\Big)qx_6\Big(x_2+\Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(x_1+qtx_5\big)\Big)\\ +\Big(\frac{1-t}{1-qt^{5-2}}\Big)qx_5\Big(x_2+\Big(\frac{1-t}{1-qt^{6-2}}\Big)\big(x_1+qx_6\big)\Big)\end{pmatrix}.\end{aligned}$$

These 9 terms are exactly the 9 nonattacking fillings of $\mu = (2,2,1,1,0,0)$ as follows

$$\begin{array}{c|ll} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4x_1x_2$$

$$\begin{array}{c|ll} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4x_1\left(\frac{1-t}{1-qt^{6-2}}\right)qtx_6$$

$$\begin{array}{c|ll} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4x_1\left(\frac{1-t}{1-qt^{6-2}}\right)qtx_5$$

$$\begin{array}{c|ll} 1 & 1 & 6 \\ 2 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_6x_2$$

$$\begin{array}{c|ll} 1 & 1 & 6 \\ 2 & 2 & 1 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_6\left(\frac{1-t}{1-qt^{6-2}}\right)x_1$$

$$\begin{array}{c|ll} 1 & 1 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_6\left(\frac{1-t}{1-qt^{6-2}}\right)qtx_5$$

$$\begin{array}{c|ll} 1 & 1 & 5 \\ 2 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_5x_2$$

$$\begin{array}{c|ll} 1 & 1 & 5 \\ 2 & 2 & 1 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_5\left(\frac{1-t}{1-qt^{6-2}}\right)x_1$$

$$\begin{array}{c|ll} 1 & 1 & 5 \\ 2 & 2 & 6 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \\ 6 & & \end{array}$$
$$x_1x_2x_3x_4\left(\frac{1-t}{1-qt^{5-2}}\right)qx_5\left(\frac{1-t}{1-qt^{6-2}}\right)qx_6$$

In this table, the weight $\mathrm{wt}(T)$ of the nonattacking filling is shown directly below the filling. These are exactly the weights produced by iterating the box-by-box recursion.

2. The affine Weyl group and the element u_μ

The underlying permutation combinatorics that controls Macdonald polynomials is that of n-periodic permutations. In this section we define the group of n-periodic permutations (the affine Weyl group), and establish notations and facts about inversions and lengths of n-periodic permutations. At the end of this section we introduce the special n-periodic permutation u_μ, which is used for the construction of the Macdonald polynomial E_μ. Proposition 2.2 provides a favorite reduced word for u_μ (the box-greedy reduced word) and determines the inversions of u_μ.

2.1. The finite Weyl group W_{fin} and the affine Weyl group W. Let $n \in \mathbb{Z}_{>1}$. The *finite Weyl group* is

$$W_{\mathrm{fin}} = S_n, \quad \text{the symmetric group of bijections } v\colon \{1,\dots,n\} \to \{1,\dots,n\}$$

with operation of composition of functions. The *type GL_n affine Weyl group* W is the group of *n-periodic permutations* $w\colon \mathbb{Z} \to \mathbb{Z}$ i.e.,

$$\text{bijective functions } w\colon \mathbb{Z} \to \mathbb{Z} \text{ such that } w(i+n) = w(i) + n. \tag{19}$$

Any n-periodic permutation w is determined by its values $w(1),\dots,w(n)$. Using $w(i+n) = w(i) + n$, any permutation $v\colon \{1,\dots,n\} \to \{1,\dots,n\}$ in S_n extends to an n-periodic permutation in W, and so $S_n \subseteq W$.

Define $\pi \in W$ by

$$\pi(i) = i + 1, \quad \text{for } i \in \mathbb{Z}. \tag{20}$$

Define $s_0, s_1, \dots, s_{n-1} \in W$ by

$$\begin{matrix} s_i(i) = i+1, \\ s_i(i+1) = i, \end{matrix} \qquad \text{and} \quad s_i(j) = j \ \text{ for } j \in \{0, 1, \dots, i-1, i+2, \dots, n-1\}. \tag{21}$$

The finite Weyl group S_n is the subgroup of W generated by $s_1, \dots, s_{n-1}$.

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define $t_\mu \in W$ by

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \dots, \quad t_\mu(n) = n + n\mu_n. \tag{22}$$

Then

$$W = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\} \qquad \text{with} \qquad v t_\mu = t_{v\mu} v \ \text{ for } v \in S_n \text{ and } \mu \in \mathbb{Z}^n. \tag{23}$$

The map

$$\overline{}\colon W \to S_n \quad \text{given by} \qquad \overline{t_\mu v} = v, \quad \text{for } \mu \in \mathbb{Z}^n \text{ and } v \in S_n, \tag{24}$$

is a surjective group homomorphism. If $v \in S_n$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ then $(t_\mu v)(i) = v(i) + n\mu_{v(i)}$ for $i \in \{1, \dots, n\}$. The *two-line notation* for $w = t_\mu v$ is

$$t_\mu v = \begin{pmatrix} 1 & 2 & \cdots & n \\ v(1) + n\mu_{v(1)} & v(2) + n\mu_{v(2)} & \cdots & v(n) + n\mu_{v(n)} \end{pmatrix}. \tag{25}$$

Another useful notation for n-periodic permutations is an extended *one-line-notation*: If $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ and $v \in S_n$ write

$$t_\mu v = ((\mu_1)_{v^{-1}(1)}, (\mu_2)_{v^{-1}(2)}, \dots, (\mu_n)_{v^{-1}(n)}). \tag{26}$$

For example, if $\mu = (0, 4, 5, 1, 4)$ with $n = 5$ and $v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$ then

$$t_\mu v = (0_1, 4_3, 5_5, 1_2, 4_4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4+n & 2+4n & 5+4n & 3+5n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 9 & 22 & 25 & 28 \end{pmatrix}.$$

2.2. Inversions of n-periodic permutations. Let $w \in W$ be an n-periodic permutation. An *inversion of* w is

$$(j,k) \quad \text{with} \quad j<k \text{ and } w(j)>w(k).$$

If (j,k) is an inversion of w then $(j+\ell n, k+\ell n)$ is an inversion of w for $\ell \in \mathbb{Z}$ and so it is sensible to assume $j \in \{1,\dots,n\}$ and define

$$\mathrm{Inv}(w) = \{(j,k) \mid j \in \{1,\dots,n\},\, k \in \mathbb{Z},\, j<k \text{ and } w(j)>w(k)\}.$$

The number of elements of $\mathrm{Inv}(w)$,

$$\ell(w) = \#\mathrm{Inv}(w), \quad \text{is the } \textit{length of } w.$$

For notational convenience when working with reduced words, let $s_\pi = \pi$. Then

$$\ell(s_\pi) = \ell(\pi) = 0 \quad \text{and} \qquad \ell(s_i) = 1 \;\; \text{for } i \in \{1,\dots,n-1\}.$$

Let $w \in W$. A *reduced word for* w is an expression of w as a product of $s_1,\dots,s_{n-1}$ and s_π,

$$w = s_{i_1}\dots s_{i_\ell} \quad \text{such that} \qquad \ell(w) = \ell(s_{i_1}) + \cdots + \ell(s_{i_\ell}),$$

with $i_1,\dots,i_\ell \in \{1,\dots,n-1,\pi\}$.

2.3. Affine coroots and the root sequence of a reduced word. Let $\mathfrak{a}_\mathbb{Z}$ be the set of $\mathbb{Z}$-linear combinations of symbols $\varepsilon_1^\vee,\dots,\varepsilon_n^\vee, K$. The *affine coroots* are

$$\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K \qquad \text{with } i,j \in \{1,\dots,n\} \text{ and } i \neq j \text{ and } \ell \in \mathbb{Z}$$

(in the context of the corresponding affine Lie algebra the symbol K is the central element). The *shift* and *height* of an affine coroot are given by

$$\mathrm{sh}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = -\ell \qquad \text{and} \qquad \mathrm{ht}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = j-i. \tag{27}$$

The affine coroot corresponding to an inversion

$$(i,k) = (i, j+\ell n) \quad \text{with } i,j \in \{1,\dots,n\} \text{ and } \ell \in \mathbb{Z}, \qquad \text{is} \quad \beta^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K. \tag{28}$$

Define a $\mathbb{Z}$-linear action of the affine Weyl group W on $\mathfrak{a}_\mathbb{Z}$ by

$$\pi^{-1}\varepsilon_1^\vee = \varepsilon_n^\vee + K, \quad \pi^{-1}\varepsilon_i^\vee = \varepsilon_{i-1}^\vee \;\; \text{for } i \in \{2,\dots,n\}, \tag{29}$$

$$s_i\varepsilon_i^\vee = \varepsilon_{i+1}^\vee, \quad s_i\varepsilon_{i+1}^\vee = \varepsilon_i^\vee, \quad s_i\varepsilon_j = \varepsilon_j^\vee \;\; \text{if } j \in \{1,\dots,n\} \text{ and } j \notin \{i,i+1\}.$$

If $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{Z}^n$ then $t_\mu \varepsilon_i^\vee = \varepsilon_i^\vee - \mu_i K$. This action matches the action from the double affine Hecke algebra results in Proposition 5.5 and equation (75).

Let

$$\alpha_0^\vee = \varepsilon_n^\vee - \varepsilon_1^\vee + K, \qquad \text{and} \qquad \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee \;\; \text{for } i \in \{1,\dots,n-1\}.$$

Let $w \in W$ and let $w = s_{i_1}\cdots s_{i_\ell}$ be a reduced word for w. The *root sequence* of the reduced word $w = s_{i_1}\cdots s_{i_\ell}$ (recall that $s_\pi = \pi$) is

$$\text{the sequence } (\beta_k^\vee \mid k \in \{1,\dots,\ell\} \text{ and } i_k \neq \pi) \text{ given by} \qquad \beta_k^\vee = s_{i_\ell}^{-1}\cdots s_{i_{k+1}}^{-1}\alpha_{i_k}^\vee. \tag{30}$$

Then, identifying inversions with affine coroots as in (28),

$$\mathrm{Inv}(w) = \{\beta_k^\vee \mid k \in \{1,\dots,\ell\} \text{ and } k \neq \pi\} \tag{31}$$

(see [16, (2.2.9)] or [3, Ch. VI §1 no. 6 Cor. 2]).

2.4. The element u_μ in the affine Weyl group. Define an action of W on $\mathbb{Z}^n$ by

(32) $$\pi(\mu_1,\dots,\mu_n)=(\mu_n+1,\mu_1,\dots,\mu_{n-1})\qquad\text{and}$$
$$s_i(\mu_1,\dots,\mu_n)=(\mu_1,\dots,\mu_{i-1},\mu_{i+1},\mu_i,\mu_{i+2},\dots,\mu_n),$$

for $i\in\{1,\dots,n-1\}$. Let u_μ be the minimal length element of W such that

(33) $$u_\mu(0,0,\dots,0)=(\mu_1,\dots,\mu_n)\qquad\text{and define}\quad v_\mu\in S_n\ \text{by}\quad u_\mu=t_\mu v_\mu^{-1},$$

where $t_\mu\in W$ is as defined in (22). As noted in [16, (2.4.3)], u_μ is the minimal length element of the coset $t_\mu S_n$ in W and the choice of the notation u_μ and v_μ for these elements follows that lead. Let $\lambda=(\lambda_1\geqslant\cdots\geqslant\lambda_n)$ be the decreasing rearrangement of μ and let

(34) $$z_\mu\in S_n\quad\text{be minimal length such that } \mu=z_\mu\lambda.$$

The following result is the translation of [16, (2.4.1)-(2.4.5) and (2.4.14)(i) and (2.4.12)] to our current setting.

Proposition 2.1. *Let $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n_{\geqslant 0}$. Let u_μ, v_μ, λ and z_μ be as defined in* (33) *and* (34).

(a) *v_μ is the minimal length element of S_n such that $v_\mu\mu$ is (weakly) increasing.*

(b) *The permutation $v_\mu\colon\{1,\dots,n\}\to\{1,\dots,n\}$ is given by*
$$v_\mu(i)=1+\#\{i'\in\{1,\dots,i-1\}\mid\mu_{i'}\leqslant\mu_i\}+\#\{i'\in\{i+1,\dots,n\}\mid\mu_{i'}<\mu_i\}.$$

(c) *The n-periodic permutations $u_\mu\colon\mathbb{Z}\to\mathbb{Z}$ and $u_\mu^{-1}\colon\mathbb{Z}\to\mathbb{Z}$ are given by*
$$u_\mu(i)=v_\mu^{-1}(i)+n\mu_i\qquad\text{and}\qquad u_\mu^{-1}(i)=v_\mu(i)-n\mu_{v_\mu(i)}\qquad\text{for } i\in\{1,\dots,n\}.$$

(d) *Let λ be the decreasing rearrangement of μ. The lengths of t_μ, u_μ and v_μ are given by*
$$\ell(t_\mu)=\ell(t_\lambda)=\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\lambda_i-\lambda_j,$$
$$\ell(v_\mu)=\#\{i<j\mid\mu_i>\mu_j\}\qquad\text{and}\qquad\ell(u_\mu)=\ell(t_\mu)-\ell(v_\mu).$$

(e) *Let $i\in\{1,\dots,n-1\}$. If $\mu_i\neq\mu_{i+1}$ so that $s_i\mu\neq\mu$ then $u_{s_i\mu}=s_iu_\mu$ and $v_{s_i\mu}=v_\mu s_i$ and*
$$\ell(u_{s_i\mu})=\begin{cases}\ell(u_\mu)+1, & \text{if } \mu_i>\mu_{i+1},\\ \ell(u_\mu)-1, & \text{if } \mu_i<\mu_{i+1},\end{cases}\quad\text{and}\quad \ell(v_{s_i\mu})=\begin{cases}\ell(v_\mu)-1, & \text{if } \mu_i>\mu_{i+1},\\ \ell(v_\mu)+1, & \text{if } \mu_i<\mu_{i+1},\end{cases}$$

(f) *With π as in* (20), *then $u_{\pi\mu}=\pi u_\mu$ and $\ell(u_{\pi\mu})=\ell(u_\mu)$ and*
$$\ell(v_\mu)-\ell(v_{\pi\mu})=(n-1)-2(v_\mu(n)-1).$$

Proof. (c) The first formula follows from $u_\mu=t_\mu v_\mu^{-1}$ and (22). To verify the second formula:
$$u_\mu^{-1}u_\mu(i)=u_\mu^{-1}(v_\mu^{-1}(i)+n\mu_i)=u_\mu^{-1}(v_\mu^{-1}(i))+n\mu_i=v_\mu(v_\mu^{-1}(i))-n\mu_{v_\mu v_\mu^{-1}(i)}+n\mu_i=i.$$

(d) From the definition of t_μ and $\mathrm{Inv}(w)$,
$$\mathrm{Inv}(t_\mu)=\Big(\bigcup_{\substack{i<j\\ \mu_i\geqslant\mu_j}}\{(i,j),(i,j+n),\dots,(i,j+n(\mu_j-\mu_i-1))\}\Big)$$
$$\cup\Big(\bigcup_{\substack{i<j\\ \mu_j<\mu_i}}\{(j,i+n),\dots,(j,i+n(\mu_i-\mu_j))\}\Big)$$

and so $\ell(t_\mu) = \#\mathrm{Inv}(t_\mu) = \sum_{i<j} |\mu_i - \mu_j|$, which gives the first statement. More generally,

$$\begin{aligned}\mathrm{Inv}(t_\mu v) = &\Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} \geqslant \mu_{v(j)}}} \{(i,j),(i,j+n),\ldots,(i,j+n(\mu_j-\mu_i-1))\}\Big) \\ &\cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} \geqslant \mu_{v(j)}}} \{(i,j),(i,j+n),\ldots,(i,j+n(\mu_j-\mu_i))\}\Big) \\ &\cup \Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \{((j,i+n),\ldots,(j,i+n(\mu_i-\mu_j))\}\Big) \\ &\cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \{((j,i+n),\ldots,(j,i+n(\mu_i-\mu_j-1))\}\Big).\end{aligned}$$

The length of $t_\mu v$ is $\ell(t_\mu v) = \#\mathrm{Inv}(t_\mu v)$. Thus the minimal length element of the coset $t_\mu S_n$ is the element $t_\mu v_\mu^{-1}$ where, if $i<j$ then $v_\mu^{-1}(i) > v_\mu^{-1}(j)$ if $\mu_{v_\mu^{-1}(i)} < \mu_{v_\mu^{-1}(j)}$ and $v_\mu^{-1}(i) < v_\mu^{-1}(j)$ if $\mu_{v_\mu^{-1}(i)} \geqslant \mu_{v_\mu^{-1}(j)}$. Thus $v_\mu\mu = v_\mu(\mu_1,\ldots,\mu_n) = (\mu_{v_\mu^{-1}(1)},\ldots,\mu_{v_\mu^{-1}(n)})$ is in weakly increasing order and $\ell(t_\mu) = \ell(u_\mu)+\ell(v_\mu)$.

(a) and (e): These now follow from the last line of the proof of (d).

(b) In order for v_μ to rearrange μ into increasing order v_μ must move the ith part of μ to the position just to the right of the number of parts of μ which are less than μ_i, or equal to μ_i and to the left of μ_i.

(f) Write $\gamma = (\mu_1,\ldots,\mu_{n-1})$. Then

$$\begin{aligned}\ell(v_\mu) &= \ell(v_\gamma) + \#\{i\in\{1,\ldots,n-1\} \mid \mu_i > \mu_n\} \qquad \text{and} \\ \ell(v_{\pi\mu}) &= \ell(v_\gamma) + \#\{i\in\{1,\ldots,n-1\} \mid \mu_i < \mu_n+1\}, \qquad \text{giving} \\ \ell(v_\mu) - \ell(v_{\pi\mu}) &= \#\{i\in\{1,\ldots,n-1\} \mid \mu_i > \mu_n\} \\ &\quad - \#\{i\in\{1,\ldots,n-1\} \mid \mu_i < \mu_n+1\} \\ &= (n-1) - \#\{i\in\{1,\ldots,n-1\} \mid \mu_i \leqslant \mu_n\} \\ &\quad - \#\{i\in\{1,\ldots,n-1\} \mid \mu_i \leqslant \mu_n\} \\ &= (n-1) - 2(v_\mu(n)-1),\end{aligned}$$

where the third equality follows from the description of $v_\mu(n)$ in (b). □

2.5. The box-greedy reduced word for u_μ. Let $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0}$ and let u_μ be as defined in (33). The *box-greedy reduced word* for the element u_μ is the sequence $u_\mu^\square$ defined inductively by the conditions $u^\square_{(0,\ldots,0)} = 1$ and, when $\mu_k \neq 0$,

$$u^\square_{(0,\ldots,0,0,\mu_k,\mu_{k+1},\ldots,\mu_n)} = s_{k-1}\cdots s_2 s_1 \pi u^\square_{(0,\ldots,0,0,\mu_{k+1},\ldots,\mu_n,\mu_k-1)}. \tag{35}$$

This is the reduced word for u_μ that is used implicitly in [11, 12, 19]. Under the action in (32), the factor $s_{k-1}\cdots s_2 s_1\pi$ which appears in (35) is an element of W of minimal length which moves $(0,\ldots,0,0,\mu_k,\mu_{k+1},\ldots,\mu_n)$ to a composition with one less box.

Proposition 2.2. *For a box $(i,j) \in dg(\mu)$ (i.e. $i\in\{1,\ldots,n\}$ and $j\in\{1,\ldots,\mu_i\}$) define*

$$\begin{aligned}u_\mu(i,j) &= \#\{i'\in\{1,\ldots,i-1\} \mid \mu_{i'} < j \leqslant \mu_i\} \\ &\quad + \#\{i'\in\{i+1,\ldots,n\} \mid \mu_{i'} < j-1 < \mu_i\}), \quad \textit{and}\end{aligned} \tag{36}$$

$$R_\mu(i,j) = \left\{ \begin{matrix} \varepsilon^\vee_{v_\mu(i)} - \varepsilon^\vee_1 + (\mu_i-j+1)K, \ldots, \\ \varepsilon^\vee_{v_\mu(i)} - \varepsilon^\vee_{u_\mu(i,j)} + (\mu_i-j+1)K \end{matrix} \right\}. \tag{37}$$

(a) *The box greedy reduced word for* u_μ *is*

$$u_\mu^\square = \prod_{(i,j)\in dg(\mu)} (s_{u_\mu(i,j)} \cdots s_1\pi),$$

where the product is over the boxes of μ in increasing cylindrical wrapping order*.*

(b) *The inversion set of* u_μ *is*

$$\mathrm{Inv}(u_\mu) = \bigcup_{(i,j)\in dg(\mu)} R_\mu(i,j). \qquad \textit{and} \qquad \ell(u_\mu) = \sum_{(i,j)\in\mu} u_\mu(i,j).$$

Proof. Let $\mu = (0,\ldots,0,\mu_k,\ldots,\mu_n)$ and let

$$\nu = \pi^{-1}s_1s_2\cdots s_{k-1}\mu = (0,\ldots,0,\mu_{k+1},\ldots,\mu_n,\mu_k-1).$$

From the definition of $u_\mu(i,j)$ in (36),

$$\begin{aligned} u_\mu(k,1) &= k-1, \\ u_\mu(i,j) &= u_\nu(i-1,j) \text{ for } i\in\{k+1,\ldots,n\}, \quad \text{and} \\ u_\mu(k,j) &= u_\nu(n,j-1), \text{ if } j\in\{2,\ldots,\mu_k\}, \end{aligned}$$

which already establishes (a). Then, using Proposition 2.1 gives $v_\mu(i) = i$ for $i \in \{1,\ldots,k-1\}$,

$$v_\mu(i) = v_\nu(i-1) \text{ for } i\in\{k+1,\ldots,n\} \qquad \text{and} \qquad v_\mu(k) = v_\nu(n).$$

These expressions for $u_\mu(i,j)$ and $v_\mu(i)$ in terms of $u_\nu(i,j)$ and $v_\nu(i)$ establish that

$$\begin{aligned} R_\mu(i,j) &= R_\nu(i-1,j), &&\text{if } i\neq k, \text{ and} \\ R_\mu(k,j) &= R_\nu(n,j-1), &&\text{if } j\in\{2,\ldots,\mu_k\}. \end{aligned}$$

It remains to compute $R_\mu(k,1)$. Since $u_\nu^{-1}\varepsilon_i^\vee = v_\nu t_\nu^{-1}\varepsilon_i^\vee = \varepsilon^\vee_{v_\nu(i)-n\nu_i} = \varepsilon^\vee_{v_\nu(i)} + \nu_i K$ then

$$\begin{aligned} R_\mu(k,1) &= \{u_\nu^{-1}\pi^{-1}\alpha_1^\vee,\ldots,u_\nu^{-1}\pi^{-1}s_1s_2\cdots s_{k-2}\alpha_{k-1}^\vee\} \\ &= \{u_\nu^{-1}\pi^{-1}(\varepsilon_1^\vee-\varepsilon_2^\vee),\ldots,u_\nu^{-1}\pi^{-1}s_1s_2\cdots s_{k-2}(\varepsilon_{k-1}^\vee-\varepsilon_k^\vee)\} \\ &= \{u_\nu^{-1}((\varepsilon_n^\vee+K)-\varepsilon_1^\vee),\ldots,u_\nu^{-1}((\varepsilon_n^\vee+K)-\varepsilon_{k-1}^\vee)\} \\ &= \left\{ \begin{array}{c} (\varepsilon^\vee_{v_\nu(n)}+\nu_nK+K)-(\varepsilon_1^\vee+\nu_1K),\ldots, \\ (\varepsilon^\vee_{v_\nu(n)}+\nu_nK+K)-(\varepsilon_{k-1}^\vee+\nu_{k-1}K) \end{array} \right\} \\ &= \{\varepsilon^\vee_{v_\mu(k)}-\varepsilon_1^\vee+(\mu_k-1)K+K,\ldots,\varepsilon^\vee_{v_\mu(k)}-\varepsilon_{k-1}^\vee+(\mu_k-1)K+K\}, \end{aligned}$$

where the next to last equality uses $\nu_1=\cdots=\nu_{k-1}=0$ and $\nu_n=\mu_k-1$. □

Remark 2.3. **Relating affine roots to** $\#\mathrm{Nleg}_\mu(b)$ **and** $\#\mathrm{Narm}_\mu(b)$**.** In the derivation of box-by-box recursion for relative Macdonald polynomials (Proposition 4.3), the last root in each box in the expression of $\mathrm{Inv}(u_\mu)$ in Proposition 2.2(b) gets picked out (this is the $d_{-\beta^\vee_{j-1}}$ and $f_{-\beta^\vee_{j-1}}$ in the proof of Lemma 4.2). More precisely, for $(i,j)\in dg(\mu)$, let $\beta_\mu^\vee(i,j)$ be the last element of $R_\mu(i,j)$ in (37):

$$\beta_\mu^\vee(i,j) = \varepsilon^\vee_{v_\mu(i)} - \varepsilon^\vee_{u_\mu(i,j)} + (\mu_i-j+1)K.$$

With the *shift and height of an affine coroot* as defined in (27), then

$$\mathrm{sh}(-\beta_\mu^\vee(i,j)) = \#\mathrm{Nleg}_\mu(i,j)+1, \quad \text{and} \quad \mathrm{ht}(-\beta_\mu^\vee(i,j)) = \#\mathrm{Narm}_\mu(i,j)+1,$$

since, by (36) and Proposition 2.1(b),

$$\begin{aligned} v_\mu(i) - u_\mu(i,j) &= 1 + \#\{i' \in \{1,\dots,i-1\} \mid j \leqslant \mu_{i'} \leqslant \mu_i\} \\ &\qquad + \#\{i' \in \{i+1,\dots,n\} \mid j-1 \leqslant \mu_{i'} < \mu_i\} \\ &= \#\mathrm{Narm}_\mu(i,j) + 1. \end{aligned}$$

3. Type GL_n Macdonald polynomials

In this section we define the Macdonald polynomials E_μ and provide explicit fomulas for all E_μ for μ with less than 3 boxes. These examples are helpful for getting a feel for what Macdonald polynomials actually look like. Although we have hidden the double affine Hecke algebra (DAHA) from our exposition in this section, Section 5 derives, from scratch, all the formulas for the operators T_i and Y_i and the Macdonald polynomials E_μ which are efficiently pulled out of a hat in this section.

3.1. The polynomial representation and Cherednik-Dunkl operators. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ let

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}. \tag{38}$$

The Laurent polynomial ring $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ has basis $\{x^\mu \mid \mu \in \mathbb{Z}^n\}$ and the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ has basis $\{x^\mu \mid \mu \in \mathbb{Z}^n_{\geqslant 0}\}$, indexed by the set $\mathbb{Z}^n_{\geqslant 0}$ of compositions. The symmetric group

S_n acts on $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ and $\mathbb{C}[x_1, \dots, x_n]$ by permuting the variables $x_1, \dots, x_n$.

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the positions of the entries so that $wx^\mu = x^{w\mu}$ for $w \in S_n$ and $\mu \in \mathbb{Z}^n$.

Let $q, t^{\frac12} \in \mathbb{C}^\times$. Following the notation of [14, Ch. VI (3.1)], let T_{q^{-1},x_n} be the operator on $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ given by

$$(T_{q^{-1},x_n} h)(x_1, \dots, x_n) = h(x_1, \dots, x_{n-1}, q^{-1}x_n).$$

For $i \in \{1, \dots, n-1\}$ let s_i be the transposition which switches i and $i+1$. Define operators $T_1, \dots, T_{n-1}$, g and $g^\vee$ on $\mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ by

$$T_i = t^{-\frac12}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big), \tag{39}$$

$$g = s_1 s_2 \cdots s_{n-1} T_{q^{-1},x_n} \qquad \text{and} \qquad g^\vee = x_1 T_1 \cdots T_{n-1}.$$

In §5.6 we give the derivation of these operators from the type GL_n double affine Hecke algebra (DAHA). Except for the factor of $t^{-\frac12}$, T_i is the operator in [2, (2.3)], which appears in the form (79) in [10, (7)]). The *Cherednik-Dunkl operators* are

$$Y_1 = gT_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad \dots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}. \tag{40}$$

3.2. Macdonald polynomials. Let $g^\vee$, T_i and Y_i be as in (39) and (40) and define

$$\tau_\pi^\vee = g^\vee, \quad \text{and} \quad \tau_i^\vee = T_i + \frac{t^{-\frac12}(1-t)}{1 - Y_i^{-1} Y_{i+1}} \quad \text{for } i \in \{1, \dots, n-1\}. \tag{41}$$

Using the action of $s_1, \dots, s_n, \pi$ on $\mathbb{Z}^n$ given in (32), the *(nonsymmetric) Macdonald polynomials* E_μ, for $\mu \in \mathbb{Z}^n$, are determined by $E_0 = 1$,

$$\begin{aligned} E_{\pi\mu} &= t^{\#\{i \in \{1,\dots,n\} \mid \mu_i > \mu_n\} - \frac12(n-1)}\, \tau_\pi^\vee E_\mu, \quad \text{and} \\ E_{s_i\mu} &= t^{\frac12}\, \tau_i^\vee E_\mu \quad \text{if } i \in \{1, \dots, n-1\} \text{ and } \mu_i > \mu_{i+1}. \end{aligned} \tag{42}$$

Remark 3.1. The source of the strange coefficients in (42) is Proposition 2.1(e) and (f) which gives that $-\frac{1}{2}(\ell(v_{s_i\mu}) - \ell(v_\mu)) = \frac{1}{2}$ and $-\frac{1}{2}(\ell(v_{\pi\mu}) - \ell(v_\mu)) = \frac{1}{2}(n-1) - (v_\mu(n)-1) = \frac{1}{2}(n-1) - \#\{i \in \{1,\dots,n-1\} \mid \mu_i \leqslant \mu_n\}$. The role of these coefficients is to force the coefficient of x^μ in E_μ to be 1.

The following theorem, the type GL_n case of [4, Theorem 4.1 and Proposition 4.2], shows that the E_μ are simultaneous eigenvectors of the Cherednik-Dunkl operators. We provide a proof in Theorem 5.7 of this paper.

Theorem 3.2. *Let $\mu \in \mathbb{Z}^n$ and let $v_\mu \in S_n$ be the minimal length permutation that rearranges μ into weakly increasing order. Then E_μ is the unique element of $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that*

$$\text{if } i \in \{1,\dots,n\} \quad \text{then} \quad Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu,$$

and the coefficient of x^μ in E_μ is 1.

Let $\mu = (\mu_1, \dots, \mu_n)$ and let $z \in S_n$. Define $T_z = T_{i_1} \cdots T_{i_r}$ if $z = s_{i_1} \cdots s_{i_r}$ is a reduced word for z.

(43) The *relative Macdonald polynomial* E_μ^z is $\quad E_\mu^z = t^{-\frac{1}{2}(\ell(zv_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu.$

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n) \in \mathbb{Z}^n$.

(44) The *symmetric Macdonald polynomial* P_λ is $\quad P_\lambda = \sum_{\nu \in S_n\lambda} t^{\frac{1}{2}\ell(z_\nu)} T_{z_\nu} E_\lambda,$

where the sum is over rearrangements ν of λ and $z_\nu \in S_n$ is minimal length such that $\nu = z_\nu \lambda$. These definitions follow [15, Remarks after (6.8)], [16, (5.7.6), (5.7.7)], [8, Definition 4.4.2], [1, Definition 5] and [7, (2.8)] (Ferreira references private communication with Haglund). In [1], the E_μ^z are called *permuted basement Macdonald polynomials.*

Remark 3.3. The following properties of the E_μ are proved in Proposition 5.8:

$$\begin{aligned} E_{(\mu_n+1,\mu_1,\dots,\mu_{n-1})} &= q^{\mu_n} x_1 E_\mu(x_2,\dots,x_n,q^{-1}x_1), \\ E_{(\mu_1+1,\dots,\mu_n+1)} &= x_1 \cdots x_n E_{(\mu_1,\dots,\mu_n)}, \\ E_{(-\mu_n,\dots,-\mu_1)}(x_1,\dots,x_n;q,t) &= E_\mu(x_n^{-1},\dots,x_1^{-1};q,t). \end{aligned}$$

Remark 3.4. In generalization of (43), one could, for any $\mu \in \mathbb{Z}^n$ and any n-periodic permutation $z \in W$, define $E_\mu^z = (const) T_z E_\mu$, where $(const)$ is a constant determined by requiring the coefficient of $x^{z\mu}$ in E_μ^z to be 1. A more useful alternative might be to define $E_\mu^z = X^z E_\mu = X^z \tau_\mu \mathbf{1}$ in the notation of [18, (2.26) and Theorem 2.2].

3.3. Explicit E_μ with less than three boxes. The following explicit formulas for E_μ with 1 and 2 boxes already provide enough data that one might have a chance at guessing the nonattacking fillings formula.

Proposition 3.5. *Let $\varepsilon_i = (0,\dots,0,1,0,\dots,0)$ be the sequence with 1 in the ith component.*

(a) *If $i \in \{1,\dots,n\}$ then*

$$E_{\varepsilon_i} = x_i + \frac{(1-t)}{(1-qt^{n-(i-1)})}(x_{i-1} + \cdots + x_1).$$

(b) *If $i \in \{1,\ldots,n\}$ then*

$$
\begin{aligned}
E_{2\varepsilon_i} = x_i^2 &+ \Big(\frac{1-t}{1-q^2t^{n-(i-1)}}\Big) \sum_{k\in\{1,\ldots,i-1\}} x_k^2 + \Big(\frac{1-t}{1-qt}\Big) q \sum_{\ell\in\{i+1,\ldots,n\}} x_i x_\ell \\
&+ \Big(\frac{1-t}{1-qt}\Big)\Big(1+\Big(\frac{1-t}{1-q^2t^{n-(i-1)}}\Big)q\Big) \sum_{k\in\{1,\ldots,i-1\}} x_k x_i \\
&+ \Big(\frac{1-t}{1-qt}\Big)\Big(\frac{1-t}{1-q^2t^{n-(i-1)}}\Big)(1+q) \sum_{\{k,\ell\}\subseteq\{1,\ldots,i-1\}} x_k x_\ell \\
&+ \Big(\frac{1-t}{1-q^2t^{n-(i-1)}}\Big)\Big(\frac{1-t}{1-qt}\Big) q \sum_{k\in\{1,\ldots,i-1\}} \sum_{\ell\in\{i+1,\ldots,n\}} x_k x_\ell .
\end{aligned}
$$

(c) *If $j_1, j_2 \in \{1,\ldots,n\}$ with $j_1 < j_2$ then*

$$
\begin{aligned}
E_{\varepsilon_{j_1}+\varepsilon_{j_2}} = x_{j_1}x_{j_2} &+ \Big(\frac{1-t}{1-qt^{n-j_1}}\Big) \sum_{k=1}^{j_1-1} x_k x_{j_2} + \Big(\frac{1-t}{1-qt^{n-(j_2-2)}}\Big) \sum_{\ell=j_1+1}^{j_2-1} x_{j_1} x_\ell \\
&+ \Big(\frac{1-t}{1-qt^{n-(j_2-2)}}\Big)\Big(\frac{1-t}{1-qt^{n-j_1}}+t\Big) \sum_{k=1}^{j_1-1} x_k x_{j_1} \\
&+ \Big(\frac{1-t}{1-qt^{n-(j_2-2)}}\Big)\Big(\frac{1-t}{1-qt^{n-j_1}}\Big) \sum_{k=1}^{j_1-1}\sum_{\ell=j_1+1}^{j_2-1} x_k x_\ell \\
&+ \Big(\frac{1-t}{1-qt^{n-(j_2-2)}}\Big)\Big(\frac{1-t}{1-qt^{n-j_1}}\Big)(1+t) \sum_{\{k,\ell\}\subseteq\{1,\ldots,j_1-1\}} x_k x_\ell .
\end{aligned}
$$

Proof. Using the first identity in (39), if $r \in \{1,\ldots,n\}$ then

$$
t^{\frac12} T_i(x_r) = \begin{cases} x_{i+1}, & \text{if } r = i, \\ t x_i + (t-1)x_{i+1}, & \text{if } r = i+1, \\ t x_r, & \text{otherwise.} \end{cases} \tag{45}
$$

Assuming $r, s \in \{1,\ldots,n\}$ with $r < s$ then

$$
t^{\frac12} T_i(x_r x_s) = \begin{cases} x_r x_{i+1}, & \text{if } s = i, \\ t x_r x_i + (t-1)x_r x_{i+1}, & \text{if } s = i+1 \text{ and } r < i, \\ t x_i x_{i+1}, & \text{if } r = i \text{ and } s = i+1, \\ x_{i+1} x_s, & \text{if } r = i \text{ and } s > i+1, \\ (t-1)x_{i+1}x_s + t x_i x_s, & \text{if } r = i+1 \text{ and } s > i+1, \\ t x_r x_s, & \text{otherwise.} \end{cases}
$$

If $r \in \{1,\ldots,n\}$ then

$$
t^{\frac12} T_i(x_r^2) = \begin{cases} x_{i+1}^2 + (1-t)x_i x_{i+1}, & \text{if } r = i, \\ t x_i^2 + (t-1)x_{i+1}^2 + (1-t)x_i x_{i+1}, & \text{if } r = i+1, \\ t x_r^2, & \text{otherwise.} \end{cases} \tag{46}
$$

(a) The proof is by induction on i. The base case $i = 1$ is

$$
E_{\varepsilon_1} = t^{-\frac12(n-1)} \tau_\pi^\vee \mathbf{1} = t^{-\frac12(n-1)} X_1 T_1 \cdots T_{n-1} \mathbf{1} = x_1 .
$$

For the induction step (note that $Y_i^{-1} Y_{i+1} E_{\varepsilon_i} = q^{1-0} t^{n-i} E_{\varepsilon_i}$) and use

$$
E_{\varepsilon_{i+1}} = t^{\frac12} \tau_i^\vee E_{\varepsilon_i} = \Big(t^{\frac12} T_i + \frac{1-t}{1-qt^{n-i}}\Big) E_{\varepsilon_i} .
$$

(b) The proof is by induction on i. Using part (a) and the first identity in Remark 3.3 applied to E_{ε_n},

$$E_{2\varepsilon_1} = x_1^2 + \frac{1-t}{1-qt}q(x_1x_n + \cdots + x_1x_2),$$

and this provides the base of the induction. Then use $Y_i^{-1}Y_{i+1}E_{2\varepsilon_i} = q^{2-0}t^{n-i}E_{2\varepsilon_i}$ and

$$E_{2\varepsilon_{i+1}} = \Big(t^{\frac{1}{2}}T_i + \frac{1-t}{1-Y_i^{-1}Y_{i+1}}\Big)E_{2\varepsilon_i} = \Big(t^{\frac{1}{2}}T_i + \frac{1-t}{1-q^2t^{n-i}}\Big)E_{2\varepsilon_i}.$$

(c) The proof is by induction on j_1. From part (a) and the first identity in Remark 3.3 applied to $E_{\varepsilon_{j_2-1}}$,

$$E_{\varepsilon_1+\varepsilon_{j_2}} = x_1x_{j_2} + \Big(\frac{1-t}{1-qt^{n-(j_2-2)}}\Big)(x_1x_{j_2-1} + \cdots + x_1x_3 + x_1x_2),$$

and this provides the base of the induction. Then use

$$E_{\varepsilon_{j_1}+\varepsilon_{j_2}} = \Big(t^{\frac{1}{2}}T_{j_1-1} + \frac{1-t}{1-qt^{n-j_1}}\Big)E_{\varepsilon_{j_1-1}+\varepsilon_{j_2}}. \qquad \square$$

4. Recursions for computing E_μ^z

In this section we derive the recursions which are used to produce expansions of Macdonald polynomials in terms of monomials. These computations are extensions of the defining recursions given in (42). It will be helpful to record carefully the action of $t^{\frac{1}{2}}\tau_i^\vee$ and $t^{\frac{1}{2}}T_i$ on the Macdonald polynomials E_μ as follows.

Let $\mu \in \mathbb{Z}^n$ and, with notations as in Theorem 3.2, let

$$\begin{matrix} a_\mu = q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)}, \\ a_{s_i\mu} = q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}, \end{matrix} \qquad \text{and} \qquad D_\mu = \frac{(1-ta_\mu)(1-ta_{s_i\mu})}{(1-a_\mu)(1-a_{s_i\mu})}.$$

Assume that $\mu_i > \mu_{i+1}$. Using the identity $E_{s_i\mu} = t^{\frac{1}{2}}\tau_i^\vee E_\mu$ if $\mu_i > \mu_{i+1}$ from (42), the eigenvalue from Theorem 3.2, and (74) gives

$$\begin{matrix} Y_i^{-1}Y_{i+1}E_\mu = a_\mu E_\mu, & t^{\frac{1}{2}}\tau_i^\vee E_\mu = E_{s_i\mu}, \\ Y_i^{-1}Y_{i+1}E_{s_i\mu} = a_{s_i\mu}E_{s_i\mu}, & t^{\frac{1}{2}}\tau_i^\vee E_{s_i\mu} = D_\mu E_\mu, \end{matrix}$$

$$t^{\frac{1}{2}}T_iE_\mu = -\frac{1-t}{1-a_\mu}E_\mu + E_{s_i\mu} \quad \text{and} \quad t^{\frac{1}{2}}T_iE_{s_i\mu} = D_\mu E_\mu + \frac{1-t}{1-a_{s_i\mu}}E_{s_i\mu}. \tag{47}$$

Now assume that $\mu_i = \mu_{i+1}$. Then $v_\mu(i+1) = v_\mu(i)+1$ and $a_\mu = t^{-1}$ so that

$$Y_i^{-1}Y_{i+1}E_\mu = t^{-1}E_\mu, \qquad (t^{\frac{1}{2}}\tau_i^\vee)E_\mu = 0, \qquad \text{and} \qquad (t^{\frac{1}{2}}T_i)E_\mu = tE_\mu. \tag{48}$$

4.1. Step-by-step recursion for computing E_μ^z. Proposition 4.1(a) is used to reduce the number of boxes in μ and part (b) is used to reduce the computation to decreasing μ. Iterating these steps delivers a monomial expansion of E_μ^z as a weighted sum of alcove walks p. The permutation sequence $\vec{z}_p$ of the alcove walk which appears in (14) is the sequence $z_0, z_1, \ldots$ of permutations which arise as superscripts of the $E_\nu^{z_i}$ which occur in the intermediate applications of the step-by-step recursion to obtain the monomial expansion.

Proposition 4.1. *Let $\mu \in \mathbb{Z}^n$ and let $z \in S_n$. Let v_μ be the minimal length element of S_n such that v_μ rearranges μ to be weakly increasing.*

(a) *If $\mu_1 \neq 0$ then*

$$E_\mu^z = x_{z(1)}E^{zc_n}_{(\mu_2,\ldots,\mu_n,\mu_1-1)}, \qquad \text{where } c_n = s_1\cdots s_{n-1} \text{ (an } n\text{-cycle in } S_n).$$

(b) *Let $i \in \{1, \ldots, n-1\}$ such that $\mu_i < \mu_{i+1}$ and let*

$$\beta^\vee = \varepsilon^\vee_{v_\mu(i+1)} - \varepsilon^\vee_{v_\mu(i)} + (\mu_{i+1} - \mu_i)K$$

so that $q^{\mathrm{sh}(-\beta^\vee)} t^{\mathrm{ht}(-\beta^\vee)} = q^{\mu_{i+1}-\mu_i} t^{v_\mu(i+1)-v_\mu(i)}$.
Let $\mathrm{norm}^z_\mu(i) = \frac{1}{2}\big(\ell(zs_iv_\mu^{-1}) - \ell(zv_\mu^{-1}) - \ell(s_i)\big)$. Then

$$E^z_\mu = \begin{cases} E^{zs_i}_{s_i\mu} + \Big(\dfrac{1-t}{1-q^{\mathrm{sh}(-\beta^\vee)}t^{\mathrm{ht}(-\beta^\vee)}}\Big) t^{\mathrm{norm}^z_\mu(i)} E^z_{s_i\mu}, & \text{if } z(i) < z(i+1), \\ E^{zs_i}_{s_i\mu} + \Big(\dfrac{(1-t)q^{\mathrm{sh}(-\beta^\vee)}t^{\mathrm{ht}(-\beta^\vee)}}{1-q^{\mathrm{sh}(-\beta^\vee)}t^{\mathrm{ht}(-\beta^\vee)}}\Big) t^{\mathrm{norm}^z_\mu(i)} E^z_{s_i\mu}, & \text{if } z(i) > z(i+1). \end{cases}$$

Proof. (a) By the second identity in Proposition 5.3, $T_z g^\vee = x_{z(1)} T_{zc_n}$ giving

$$T_z \tau_{u_\mu} \mathbf{1} = T_z \tau^\vee_\pi \tau^\vee_{u_{\pi^{-1}\mu}} \mathbf{1} = T_z g^\vee \tau^\vee_{u_{\pi^{-1}\mu}} \mathbf{1} = x_{z(1)} T_{zc_n} \tau^\vee_{u_{\pi^{-1}\mu}} \mathbf{1}.$$

Then (a) follows by using $E^z_\mu = t^{-\frac{1}{2}\ell(zv_\mu^{-1})} T_z \tau^\vee_{u_\mu} \mathbf{1}$ to rewrite each side, and computing

$$\begin{aligned} \ell(zv_\mu^{-1}) - \ell(zc_n v^{-1}_{\pi^{-1}\mu}) &= \ell(\overline{zu_\mu}) - \ell(\overline{zc_n u_{\pi^{-1}\mu}}) = \ell(\overline{zu_\mu}) - \ell(\overline{zc_n\pi^{-1}u_\mu}) \\ &= \ell(\overline{zu_\mu}) - \ell(\overline{zc_nc_n^{-1}u_\mu}) = 0, \end{aligned}$$

where $\overline{}\colon W \to S_n$ is the homomorphism defined in (24).

(b) Let $\nu = s_i\mu$ and let $a_\nu = q^{\nu_i - \nu_{i+1}} t^{v_\nu(i) - v_\nu(i+1)} = q^{\mathrm{sh}(-\beta^\vee)} t^{\mathrm{ht}(-\beta^\vee)}$. Using (42), (76) and the eigenvalue formula from (3.2), then

$$\begin{aligned} T_z \tau^\vee_{u_\mu} \mathbf{1} = T_z \tau^\vee_i \tau^\vee_{u_{s_i\mu}} \mathbf{1} &= \begin{cases} T_z\Big(T_i + \Big(\frac{t^{-\frac{1}{2}}(1-t)}{1-a_\nu}\Big)\Big) \tau^\vee_{u_{s_i\mu}} \mathbf{1}, & \text{if } zs_i > z, \\ T_z\Big(T_i^{-1} + \Big(\frac{t^{-\frac{1}{2}}(1-t)}{1-a_\nu}\Big) a_\nu\Big) \tau^\vee_{u_{s_i\mu}} \mathbf{1}, & \text{if } zs_i < z, \end{cases} \\ &= \begin{cases} T_{zs_i} \tau^\vee_{u_{s_i\mu}} \mathbf{1} + \Big(\frac{t^{-\frac{1}{2}}(1-t)}{1-a_\nu}\Big) T_z \tau^\vee_{u_{s_i\mu}} \mathbf{1}, & \text{if } zs_i > z, \\ T_{zs_i} \tau^\vee_{u_{s_i\mu}} \mathbf{1} + \Big(\frac{t^{-\frac{1}{2}}(1-t)}{1-a_\nu}\Big) a_\nu T_z \tau^\vee_{u_{s_i\mu}} \mathbf{1}, & \text{if } zs_i < z, \end{cases} \end{aligned}$$

Then, using $E^z_\mu = t^{-\frac{1}{2}\ell(zv_\mu^{-1})} T_z \tau^\vee_\mu \mathbf{1}$ to obtain

$$E^z_\mu = \begin{cases} t^{\frac{1}{2}(\ell(zs_iv^{-1}_{s_i\mu}) - \ell(zv_\mu^{-1}))} E^{zs_i}_{s_i\mu} \\ \qquad + \Big(\dfrac{1-t}{1-a_\nu}\Big) t^{\frac{1}{2}(-1+\ell(zs_iv_\mu^{-1}) - \ell(zv_\mu^{-1}))} E^z_{s_i\mu}, & \text{if } zs_i > z, \\ t^{\frac{1}{2}(\ell(zs_iv^{-1}_{s_i\mu}) - \ell(zv_\mu^{-1}))} E^{zs_i}_{s_i\mu} \\ \qquad + \Big(\dfrac{1-t}{1-a_\nu}\Big) a_\nu t^{\frac{1}{2}(-1+\ell(zs_iv_\mu^{-1}) - \ell(zv_\mu^{-1}))} E^z_{s_i\mu}, & \text{if } zs_i < z. \end{cases}$$

By Proposition 2.1(e), $v^{-1}_{s_i\mu} = s_i v_\mu^{-1}$ and so

$$t^{\frac{1}{2}(\ell(zs_iv^{-1}_{s_i\mu}) - \ell(zv_\mu^{-1}))} = t^{\frac{1}{2}(\ell(zs_is_iv_\mu^{-1}) - \ell(zv_\mu^{-1}))} = t^0 = 1. \qquad \square$$

4.2. Box-by-box recursion for computing E^z_μ. Proposition 4.3 executes several steps of Proposition 4.1 at once to provide a recursion for computing E^z_μ which removes a box at each application of the recursion. Iterating this recursion delivers a monomial expansion of E^z_μ as a weighted sum of nonattacking fillings T. The permutation sequence $\vec{z}_T$ of the nonattacking filling T (see (10)) is the sequence of permutations $z_0, z_1, \ldots$ which arise as superscripts of the $E^{z_i}_\nu$ which occur in the intermediate applications of the box-by-box recursion.

Lemma 4.2. (*Compressing 2^{j-1} terms to j terms*) *Let $j\in\{1,\dots,n\}$ and let*

$$\mu=(0,\dots,0,\mu_j,\dots,\mu_n)\in\mathbb{Z}^n_{\geqslant 0}\qquad \text{with } \mu_1=0,\dots,\mu_{j-1}=0 \text{ and } \mu_j\neq 0.$$

and let $\gamma=\pi\nu=(\mu_j,0,\dots,0,\mu_{j+1},\dots,\mu_n)$. Let $\tau_1^\vee,\dots,\tau_{n-1}^\vee$ be the intertwiners of (41) *acting on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ by* (39) *and* (40).

(a) $$\tau_{j-1}^\vee\cdots\tau_1^\vee E_\gamma=T_{j-1}\cdots T_1E_\gamma+\frac{1-t}{1-q^{\mu_j}t^{v_\mu(j)-(j-1)}}\sum_{a=1}^{j-1}T_{a-1}\cdots T_1t^{-\frac12(j-a)}E_\gamma.$$

(b) *Let $i\in\{1,\dots,j-1\}$. Then*

$$\begin{aligned}\tau_{j-1}^\vee\cdots\tau_1^\vee E_\gamma&=T_{j-1}^{-1}T_{j-2}^{-1}\cdots T_i^{-1}T_{i-1}\cdots T_1E_\gamma\\&\quad+\frac{1-t}{1-q^{\mu_j}t^{v_\mu(j)-(j-1)}}q^{\mu_j}t^{v_\mu(j)-(j-1)}\sum_{a=i}^{j-1}T_{a-1}\cdots T_1t^{-\frac12(j-a)}E_\gamma\\&\quad+\frac{1-t}{1-q^{\mu_j}t^{v_\mu(j)-(j-1)}}\sum_{a=1}^{i-1}T_{a-1}\cdots T_1t^{-\frac12(j-a)}E_\gamma.\end{aligned}$$

Proof. Let $\mathrm{ev}^\rho_\gamma\colon\mathbb{C}(Y)\to\mathbb{C}(q,t)$ be the homomorphism given by

$$\mathrm{ev}^\rho_\gamma(Y_i)=q^{-\gamma_i}t^{-(v_\gamma(i)-1)+\frac12(n-1)}\qquad\text{so that}\qquad fE_\gamma=\mathrm{ev}^\rho_\gamma(f)E_\gamma,\quad\text{for } f\in\mathbb{C}(Y).$$

(we shall only apply this to rational expressions in $Y_1,\dots,Y_n$ where the denominator does not evaluate to 0.) For $i\in\{1,\dots,n-1\}$ set $\alpha_i^\vee=\varepsilon_i^\vee-\varepsilon_{i+1}^\vee$ and let

$$\beta_1^\vee=\alpha_1^\vee=\varepsilon_1^\vee-\varepsilon_2^\vee,\quad \beta_2^\vee=s_1\alpha_2^\vee=\varepsilon_1^\vee-\varepsilon_3^\vee,\quad\dots\quad \beta_{j-1}^\vee=s_1\cdots s_{j-2}\alpha_{j-1}^\vee=\varepsilon_1^\vee-\varepsilon_j^\vee.$$

For $i\in\{1,\dots,j-1\}$ let

$$Y^{-\beta_i^\vee}=Y_1^{-1}Y_{i+1},\qquad F_{-\beta_i^\vee}=\frac{t^{-\frac12}(1-t)}{1-Y^{-\beta_i^\vee}},\qquad C_{-\beta_i^\vee}=T_i+f_{-\beta_i^\vee}\qquad\text{and}$$

$$d_{-\beta_i^\vee}=\mathrm{ev}^\rho_\gamma(Y^{-\beta_i^\vee}),\qquad f_{-\beta_i^\vee}=\mathrm{ev}^\rho_\gamma(F_{-\beta_i^\vee}),\qquad c_{-\beta_i^\vee}=\mathrm{ev}^\rho_\gamma(t^{\frac12}+F_{-\beta_i^\vee})$$

If $i\in\{1,\dots,j-1\}$ then

$$\begin{aligned}c_{-\beta_i^\vee}&=\mathrm{ev}^\rho_\gamma\Big(\frac{t^{-\frac12}(1-tY^{-\beta_i^\vee})}{1-Y^{-\beta_i^\vee}}\Big)=\mathrm{ev}^\rho_\gamma(t^{\frac12}+F_{-\beta_i^\vee})=t^{\frac12}+f_{-\beta_i^\vee},\\ d_{-\beta_i^\vee}&=q^{\gamma_1-\gamma_{i+1}}t^{v_\gamma(1)-v_\gamma(i+1)}=q^{\mu_j}t^{v_\gamma(1)-i}=td_{-\beta_{i+1}^\vee},\\ c_{\beta_{j-1}}\cdots c_{\beta_{a+1}}f_{\beta_a}&=\Big(\frac{t^{-\frac12}(1-td_{-\beta_{j-1}^\vee})}{1-d_{-\beta_{j-1}^\vee}}\Big)\cdots\Big(\frac{t^{-\frac12}(1-t)}{1-d_{-\beta_a^\vee}}\Big)\end{aligned}\tag{49}$$

$$=t^{-\frac12(j-1-a)}\Big(\frac{t^{-\frac12}(1-t)}{1-d_{-\beta_{j-1}^\vee}}\Big)=t^{-\frac12(j-a)}t^{\frac12}f_{\beta_{j-1}^\vee},\tag{50}$$

$$t-1+t^{\frac12}f_{-\beta_i^\vee}=d_{-\beta_i^\vee}t^{\frac12}f_{-\beta_i^\vee},\tag{51}$$

where the last equality follows from

$$(t^{\frac12}-t^{-\frac12})+F_{-\beta_i^\vee}=(t^{\frac12}-t^{-\frac12})+\frac{t^{-\frac12}(1-t)}{1-Y^{-\beta_i^\vee}}=\frac{t^{-\frac12}(1-t)Y^{-\beta_i^\vee}}{1-Y^{-\beta_i^\vee}}=Y^{-\beta_i^\vee}F_{-\beta_i^\vee}.$$

Since

$$\begin{aligned}\tau_i^\vee\tau_{i-1}^\vee\cdots\tau_1^\vee E_\gamma&=(T_i+F_{-\alpha_i})\tau_{i-1}^\vee\cdots\tau_1^\vee E_\gamma=(T_i+\mathrm{ev}^\rho_\gamma(F_{-s_1\cdots s_{i-1}\alpha_i^\vee})\tau_{i-1}^\vee\cdots\tau_1^\vee E_\gamma\\&=(T_i+f_{-\beta_i^\vee})\tau_{i-1}^\vee\cdots\tau_1^\vee E_\gamma=C_{-\beta_i^\vee}\tau_{i-1}^\vee\cdots\tau_1^\vee E_\gamma\end{aligned}$$

then $\tau_{j-1}^\vee\cdots\tau_1^\vee E_\gamma=C_{-\beta_{j-1}^\vee}\cdots C_{-\beta_1^\vee}E_\gamma$.

For $i\in\{2,\dots,j-1\}$, $\quad C_{-\beta_i^\vee}E_\gamma=(T_i+f_{-\beta_i^\vee})E_\gamma=(t^{\frac12}+f_{-\beta_i^\vee})E_\gamma=c_{-\beta_i^\vee}E_\gamma.$

Thus,

$$
\begin{aligned}
\tau^\vee_{j-1}\cdots\tau^\vee_1 E_\gamma &= C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_1}E_\gamma = C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_2}(T_1+f_{-\beta^\vee_1})E_\gamma \\
&= C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_2}T_1E_\gamma + f_{-\beta^\vee_1}C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_2}E_\gamma \\
&= C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_2}T_1E_\gamma + c_{-\beta^\vee_{j-1}}\cdots c_{-\beta^\vee_2}f_{-\beta^\vee_1}E_\gamma \\
&= C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_3}(T_2+f_{-\beta^\vee_2})T_1E_\gamma + c_{-\beta^\vee_{j-1}}\cdots c_{-\beta^\vee_2}f_{-\beta^\vee_1}E_\gamma \\
&= C_{-\beta^\vee_{j-1}}\cdots C_{-\beta^\vee_3}T_2T_1E_\gamma + c_{-\beta^\vee_{j-1}}\cdots c_{-\beta^\vee_3}f_{-\beta^\vee_2}T_1E_\gamma \\
&\qquad + c_{-\beta^\vee_{j-1}}\cdots c_{-\beta^\vee_2}f_{-\beta^\vee_1}E_\gamma,
\end{aligned}
\tag{52}
$$

and continuing this process and using (50) gives (a).

Let R_i be the right hand side of the expression in statement of (b), so that the identity in (a) can be considered as $\tau^\vee_{j-1}\cdots\tau^\vee_1E_\gamma = R_j$. Then, canceling the common terms in R_{i+1} and R_i gives

$$
\begin{aligned}
R_{i+1}-R_i &= T^{-1}_{j-1}\cdots T^{-1}_{i+1}T_i\cdots T_1E_\gamma + T_{i-1}\cdots T_1 t^{-\frac12(j-i)}t^{\frac12}f_{-\beta^\vee_{j-1}}E_\gamma \\
&\qquad - T^{-1}_{j-1}T^{-1}_{j-2}\cdots T^{-1}_iT_{i-1}\cdots T_1E_\gamma \\
&\qquad - t^{\frac12}f_{-\beta^\vee_{j-1}}d_{-\beta^\vee_j}t^{-\frac12(j-i)}T_{i-1}\cdots T_1E_\gamma \\
&= T^{-1}_{j-1}\cdots T^{-1}_{i+1}(T^{-1}_i + t^{-\frac12}(t-1))T_{i-1}\cdots T_1E_\gamma + T_{i-1}\cdots T_1t^{-\frac12(j-i)}t^{\frac12}f_{-\beta^\vee_{j-1}}E_\gamma \\
&\qquad - T^{-1}_{j-1}T^{-1}_{j-2}\cdots T^{-1}_iT_{i-1}\cdots T_1E_\gamma - t^{\frac12}f_{-\beta^\vee_{j-1}}d_{-\beta^\vee_j}t^{-\frac12(j-i)}T_{i-1}\cdots T_1E_\gamma \\
&= t^{-\frac12}(t-1)T_{i-1}\cdots T_1T^{-1}_{j-1}\cdots T^{-1}_{i+1}E_\gamma + t^{-\frac12(j-i)}t^{\frac12}f_{-\beta^\vee_{j-1}}(1-d_{-\beta^\vee_j})T_{i-1}\cdots T_1E_\gamma \\
&= t^{-\frac12(j-i)}((t-1)+t^{\frac12}f_{-\beta^\vee_{j-1}} - t^{\frac12}f_{-\beta^\vee_{j-1}}d_{-\beta^\vee_j})T_{i-1}\cdots T_1E_\gamma = 0,
\end{aligned}
$$

by (51). Thus $\tau^\vee_{j-1}\cdots\tau^\vee_1E_\gamma = R_j = R_{j-1} = \ldots = R_1$ and this establishes (b). □

Using Propostion 4.1(a) and adjusting for the normalization in the definition of E^z_ν in (43) produces the following box-by-box recursion for the relative Macdonald polynomials E^z_μ.

Proposition 4.3. *Let $z \in S_n$. Let*

$$\mu = (0,\ldots,0,\mu_j,\ldots,\mu_n) \in \mathbb{Z}^n_{\geqslant 0} \qquad \textit{with } \mu_1 = 0,\ldots,\mu_{j-1}=0 \textit{ and } \mu_j \neq 0.$$

and let $\nu = (0,\ldots,0,\mu_{j+1},\ldots,\mu_n,\mu_j-1)$. For $m\in\{0,\ldots,n\}$ let $c^{-1}_m = s_{m-1}\cdots s_2s_1$ (which is an m-cycle in S_n). Let $y = (y(1),\ldots,y(n))$ be the permutation which has $y(k) = z(k)$ for $k\in\{j,\ldots,n\}$

$$\textit{and}\quad \{y(1),\ldots,y(j-1)\} = \{z(1),\ldots,z(j-1)\} \quad\textit{and}\quad y(1)<\cdots<y(j-1).$$

Then

$$E^z_\mu = x_{y(j)}E^{yc^{-1}_jc_n}_\nu + \frac{(1-t)}{1-q^{\mu_j}t^{v_\mu(j)-(j-1)}}\sum_{a=1}^{j-1} q^{\mathrm{maj}^y_\mu(a)}t^{\mathrm{covid}^y_\mu(a)}x_{y(a)}E^{yc^{-1}_ac_n}_\nu,$$

where, for $a\in\{1,\ldots,j-1\}$,

$$\mathrm{maj}^y_\mu(a) = \begin{cases} 0, & \textit{if } y(j)>y(a),\\ \mu_j, & \textit{if } y(j)<y(a),\end{cases} \qquad \textit{and}$$

$$
\begin{aligned}
&\mathrm{covid}^y_\mu(a) \\
&\quad = \begin{cases} \frac12(\ell(yc^{-1}_ac_jv^{-1}_\mu) - \ell(yv^{-1}_\mu) - \ell(c^{-1}_ac_j)), & \textit{if } y(j)>y(a),\\ v_\mu(j)-(j-1)+\frac12(\ell(yc^{-1}_ac_jv^{-1}_\mu) - \ell(yv^{-1}_\mu) - \ell(c^{-1}_ac_j)), & \textit{if } y(j)<y(a).\end{cases}
\end{aligned}
$$

Proof. Write $z = y\sigma$ with $\sigma \in S_{j-1}$ and y minimal length in the coset zS_{j-1}. Then $y(j) = z(j)$ and, by the last identity in (48),

$$T_z E_\mu = T_y T_\sigma E_\mu = T_y t^{\frac{1}{2}\ell(\sigma)} E_\mu, \quad \text{so that} \quad E^z_\mu = E^y_\mu.$$

To control the spacing let $c_a = s_1 \cdots s_{a-1}$ (which is an a-cycle in S_n) and let

$$d_{-\beta^\vee_{j-1}} = q^{\mu_j} t^{v_\gamma(1)-(j-1)} = q^{\mu_j} t^{v_\mu(j)-(j-1)} \quad \text{and} \qquad t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} = \frac{1-t}{1-q^{\mu_j} t^{v_\mu(j)-(j-1)}}.$$

Let $\gamma = \pi\nu = (\mu_j, 0, \dots, 0, \mu_{j+1}, \dots, \mu_n)$ as in Lemma 4.2 and note that

$$v_\gamma = v_\mu s_{j-1} \cdots s_1 = v_\mu c_j^{-1}.$$

If $y(j) > y(j-1)$ then $T_{yc_j^{-1}} = T_y T_{j-1} \cdots T_1$, and using Lemma 4.2(a) gives

$$T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma = T_{yc_j^{-1}} E_\gamma + t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{j-1} t^{-\frac{1}{2}(j-a)} T_{yc_a^{-1}} E_\gamma,$$

If $y(j) < y(j-1)$ then $T_{yc_j^{-1}} = T_y T^{-1}_{j-1} \cdots T_i^{-1} T_{i-1} \cdots T_1$ with

$$i = \min\{r \in \{1, \dots, j-1\} \mid y(r) > y(j)\},$$

and using Lemma 4.2(b) gives

$$\begin{aligned} T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma = T_{yc_j^{-1}} E_\gamma &+ t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=i}^{j-1} t^{-\frac{1}{2}(j-a)} d_{-\beta^\vee_{j-1}} T_{yc_a^{-1}} E_\gamma \\ &+ t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{i-1} t^{-\frac{1}{2}(j-a)} T_{yc_a^{-1}} E_\gamma. \end{aligned}$$

For $a \in \{1, \dots, j\}$, let

$$\begin{aligned} \mathrm{norm}^y_\mu(a) &= (\ell(yc_a^{-1} c_j v_\mu^{-1}) - \ell(c_j v_\mu^{-1})) - (\ell(yv_\mu^{-1}) - \ell(v_\mu^{-1}) - (j-1)) \\ &= (\ell(yc_a^{-1} c_j v_\mu^{-1}) - (\ell(v_\mu^{-1}) + j - 1) - \ell(yv_\mu^{-1}) + (\ell(v_\mu^{-1}) + (j-1)) \\ &= \ell(yc_a^{-1} c_j v_\mu^{-1}) - \ell(yv_\mu^{-1}). \end{aligned}$$

With this notation, the identities

$$\begin{aligned} E^y_\mu &= t^{-\frac{1}{2}(\ell(yv_\mu^{-1}) - \ell(v_\mu^{-1}))} T_y E_\mu = t^{-\frac{1}{2}(\ell(yv_\mu^{-1}) - \ell(v_\mu^{-1}))} T_y (t^{\frac{1}{2}} \tau^\vee_{j-1}) \cdots (t^{\frac{1}{2}} \tau^\vee_1) E_\gamma \\ &= t^{-\frac{1}{2}(\ell(yv_\mu^{-1}) - \ell(v_\mu^{-1}) - (j-1))} T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma, \qquad \text{and} \\ E^{yc_a^{-1}}_\gamma &= t^{-\frac{1}{2}(\ell(yc_a^{-1} v_\gamma^{-1}) - \ell(v_\gamma^{-1}))} T_{yc_a^{-1}} E_\gamma = t^{-\frac{1}{2}(\ell(yc_a^{-1} v_\gamma^{-1}) - \ell(v_\gamma^{-1}))} T_y T_{a-1} \cdots T_1 E_\gamma \\ &= t^{-\frac{1}{2}(\ell(yc_a^{-1} c_j v_\mu^{-1}) - \ell(c_j v_\mu^{-1}))} T_y T_{a-1} \cdots T_1 E_\gamma, \qquad \text{for } a \in \{1, \dots, j\}, \end{aligned}$$

then give

$$E^y_\mu = t^{\frac{1}{2}\mathrm{norm}^y_\mu(j)} E^{yc_j^{-1}}_\gamma + t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{j-1} t^{-\frac{1}{2}(j-a)} t^{\frac{1}{2}\mathrm{norm}^y_\mu(a)} E^{yc_a^{-1}}_\gamma$$

for $y(j) > y(j-1)$; and

$$\begin{aligned} E^y_\mu = t^{\frac{1}{2}\mathrm{norm}^y_\mu(a)} E^{yc_j^{-1}}_\gamma &+ t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=i}^{j-1} d_{-\beta^\vee_{j-1}} t^{-\frac{1}{2}(j-a)} t^{\frac{1}{2}\mathrm{norm}^y_\mu(a)} E^{yc_a^{-1}}_\gamma \\ &+ t^{\frac{1}{2}} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{i-1} t^{-\frac{1}{2}(j-a)} t^{\frac{1}{2}\mathrm{norm}^y_\mu(a)} E^{yc_a^{-1}}_\gamma, \end{aligned}$$

for $y(j) < y(j-1)$ and $i = \min\{r \in \{1, \dots, j-1\} \mid y(r) > y(j)\}$.

If $a = j$ then $\mathrm{norm}^y_\mu(j) = 0$. Since

$$\ell(c_a^{-1}c_j) = \ell(s_{a-1}\cdots s_1 s_1 \cdots s_{j-1}) = \ell(s_a \cdots s_{j-1}) = (j-a)$$

then

$$\mathrm{norm}^y_\mu(a) - (j-a) = \ell(yc_a^{-1}c_j v_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j).$$

Applying Proposition 4.1(a) to the right hand side of the expressions that have been obtained for E^y_μ and substituting

$$d_{-\beta^\vee_{j-1}} = q^{\mu_j} t^{v_\gamma(1)-(j-1)} = q^{\mu_j} t^{v_\mu(j)-(j-1)} = q^{\mu_j} t^{\frac{1}{2}\cdot 2(v_\mu(j)-(j-1))}$$

gives

$$E^z_\mu = E^y_\mu = x_{y(j)} E_\nu^{yc_j^{-1}c_n} + \frac{(1-t)}{1-q^{\mu_j}t^{v_\mu(j)-(j-1)}} \sum_{a=1}^{j-1} q^{\mathrm{maj}^y_\mu(a)} t^{\mathrm{covid}^y_\mu(a)} x_{y(a)} E_\nu^{yc_a^{-1}c_n},$$

where, if $i = \min\{r \in \{1,\dots,j\} \mid y(r) \geqslant y(j)\}$ then

$$\begin{aligned}\mathrm{covid}^y_\mu(a) &= \begin{cases} \frac{1}{2}\mathrm{norm}^y_\mu(a) - \frac{1}{2}(j-a), & \text{if } a < i,\\ v_\mu(j) - (j-1) + \frac{1}{2}\mathrm{norm}^y_\mu(a) - \frac{1}{2}(j-a), & \text{if } a \geqslant i,\end{cases}\\ &= \begin{cases} \frac{1}{2}(\ell(yc_a^{-1}c_j v_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j)), & \text{if } y(j) > y(a),\\ v_\mu(j) - (j-1) + \frac{1}{2}(\ell(yc_a^{-1}c_j v_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j)), & \text{if } y(j) < y(a).\end{cases}\end{aligned}$$

□

REMARK 4.4. **Relating $\mathrm{covid}^y_\mu(a)$ to coinversion triples.** To give an alternate point of view on the statistic $\mathrm{covid}^y_\mu(a)$ which fell out of the computation in the proof of Proposition 4.3, let us analyze how the inversions of yv_μ^{-1} change when the factor $c_a^{-1}c_j$ is inserted to form $yc_a^{-1}c_j v_\mu^{-1}$. To do this note that

$$yc_a^{-1}c_j v_\mu^{-1} = yv_\mu^{-1}(s_{v_\mu(j)-1}s_{v_\mu(j)-2}\cdots s_j)(s_a\cdots s_{j-2})s_{j-1}(s_j\cdots s_{v_\mu(j)-1}).$$

and analyze the effect of each of the factors on the right hand side.

(a) Since $y(a) < y(a+1) < \cdots < y(j-1)$ then $(s_a \cdots s_{j-2})$ creates $(j-1-a)$ inversions in $yc_a^{-1}c_j v_\mu^{-1}$ which do not occur in xv_μ^{-1}.
(b) The factor s_{j-1} creates an inversion if $y(j) > y(a)$ and removes an inversion if $y(j) < y(a)$.
(c) The factor $(s_j \cdots s_{v_\mu(j)-1})$
undoes inversions $yv_\mu^{-1}(k) < yv_\mu^{-1}(a)$ for $k \in \{j,\dots,v_\mu(j)-1\}$,
adds inversions $yv_\mu^{-1}(k) > yv_\mu^{-1}(a)$ for $k \in \{j,\dots,v_\mu(j)-1\}$,
(d) The factor $(s_{v_\mu(j)-1}s_{v_\mu(j)-2}\cdots s_j)$
undoes inversions $yv_\mu^{-1}(k) > yv_\mu^{-1}(v_\mu(j))$ for $k \in \{j,\dots,v_\mu(j)-1\}$,
adds inversions $yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))$ for $k \in \{j,\dots,v_\mu(j)-1\}$,

Thus, if $yv_\mu^{-1}(v_\mu(j)) > yv_\mu^{-1}(a)$ (so that $y(j) > y(a)$) then

$$\begin{aligned}
&\ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j) = \ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - (j-a)\\
&\quad = (j-1-a)+1\\
&\qquad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(a)\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) > yv_\mu^{-1}(a)\})\\
&\qquad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k)\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) > yv_\mu^{-1}(k)\})\\
&\qquad - (j-a)\\
&\quad = -(\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j))\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k)\})\\
&\qquad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k)\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j))\})\\
&\quad = 2\cdot(\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\}).
\end{aligned}$$

Then, if $yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(a)$ (so that $y(j) < y(a)$) then

$$\begin{aligned}
&\ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j) = \ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - (j-a)\\
&\quad = (j-1-a)-1\\
&\qquad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(a)\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k)\})\\
&\qquad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k)\})\\
&\qquad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\})\\
&\qquad - (j-a)
\end{aligned}$$

$$\begin{aligned}
&= -1 - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(a)\})\\
&\quad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k) < yv_\mu^{-1}(a)\})\\
&\quad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(a) < yv_\mu^{-1}(k)\})\\
&\quad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k) < yv_\mu^{-1}(a)\})\\
&\quad - (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(a) < yv_\mu^{-1}(k)\})\\
&\quad + (\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(a)\})\\
&= -2 - 2\cdot(\#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\}).
\end{aligned}$$

If $y(j) > y(a)$ then

$$\begin{aligned}
&\tfrac{1}{2}(\ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j))\\
&\qquad = \#\{k\in\{j,\dots,v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\}\\
&\qquad = \#\{k\in\{j,\dots,v_\mu(j)-1\} \mid y(a) < yv_\mu^{-1}(k) < y(j)\}\\
&\qquad = \#\{v_\mu(b)\in\{j,\dots,v_\mu(j)-1\} \mid y(a) < yv_\mu^{-1}(v_\mu(b)) < y(j)\}\\
&\qquad = \#\{b\in\{v_\mu^{-1}(j),\dots,v_\mu^{-1}(v_\mu(j)-1)\} \mid y(a) < y(b) < y(j)\}
\end{aligned}$$

and, if $y(j) < y(a)$ then

$$\begin{aligned}
&(v_\mu(j) - (j-1)) + \tfrac{1}{2}\big(\ell(yc_a^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1}) - \ell(c_a^{-1}c_j)\big) \\
&\quad = (v_\mu(j) - j + 1) - 1 \\
&\qquad - \#\{k \in \{j, \dots, v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\} \\
&\quad = ((v_\mu(j) - 1) - (j-1)) \\
&\qquad - \#\{k \in \{j, \dots, v_\mu(j)-1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\} \\
&\quad = \#\left\{k \in \{j, \dots, v_\mu(j)-1\} \;\middle|\; \begin{matrix} yv_\mu^{-1}(k) < yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j)) \\ \text{or } yv_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j)) < yv_\mu^{-1}(k) \end{matrix}\right\} \\
&\quad = \#\left\{b \in \{v_\mu^{-1}(j), \dots, v_\mu^{-1}(v_\mu(j)-1)\} \;\middle|\; \begin{matrix} y(b) < y(a) < y(j) \\ \text{or } y(a) < y(j) < y(b) \end{matrix}\right\}.
\end{aligned}$$

These last two expressions are exactly the numbers of coninversion triples that appear in [10, Lemma 3.6.3] for the box $(j,1)$ filled with $y(a)$ in a filling of shape μ with basement $(y(1), \dots, y(n))$.

5. Type GL_n DAArt, DAHA and the polynomial representation

The power tools that enable us to construct and manipulate Macdonald polynomials with ease are the polynomial generators $X_1, \dots, X_n$, the Cherednik-Dunkl operators $Y_1, \dots, Y_n$ and the intertwiners $\tau_1^\vee, \dots, \tau_{n-1}^\vee, \tau_\pi^\vee$ which all live inside the double affine Hecke algebra $\tilde{H}_{GL_n}$. In this section we will build the Macdonald polynomials E_μ by first constructing the double affine Artin group $\tilde{\mathcal{B}}_{GL_n}$, then the elements $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, then the DAHA $\tilde{H}_{GL_n}$ and the intertwiners $\tau_1^\vee, \dots, \tau_{n-1}^\vee, \tau_\pi^\vee$. Let us begin by defining the DAArt $\tilde{\mathcal{B}}_{GL_n}$ and establishing its primary dualities. The definition is by generators and relations and the dualities are automorphisms of $\tilde{\mathcal{B}}_{GL_n}$. The double affine Hecke algebra $\tilde{H}_{GL_n}$ is constructed as a quotient of the group algebra of $\tilde{\mathcal{B}}_{GL_n}$ by the Hecke relations $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$. Alternative expositions of the material in this section are found in [5] and [16].

Use Coxeter diagram shorthand for relations so that

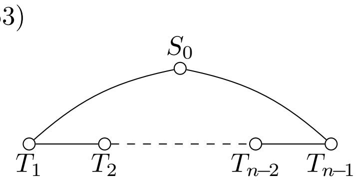
indicates $aba = bab$, and


indicates $ab = ba$,

5.1. The type GL_n double affine Artin group (DAArt). The element q will be a parameter in the Macdonald polynomials. In the definition of the DAArt by generators and relations the elememt q appears as a central element of the group, but in Section 5.6 the element q will get specialized to be a complex parameter.

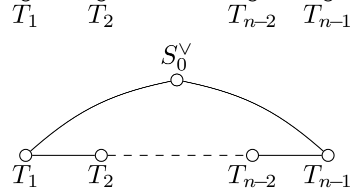
The *type GL_n double affine Artin group (DAArt)* $\tilde{\mathcal{B}}_{GL_n}$ is generated by $q, g^\vee, g, S_0^\vee, S_0, T_1, \dots, T_{n-1}$ with the relations

(53)



$$gS_0g^{-1} = T_1,$$

$$gT_ig^{-1} = T_{i+1}, \qquad gT_{n-1}g^{-1} = S_0,$$



$$g^\vee S_0^\vee (g^\vee)^{-1} = T_1,$$

$$g^\vee T_i (g^\vee)^{-1} = T_{i+1}, \qquad g^\vee T_{n-1}(g^\vee)^{-1} = S_0^\vee,$$

$$q \in Z(\widetilde{\mathcal{B}}_{GL_n}) \qquad \text{and}$$

(54) $\quad T_1 g^\vee g = g g^\vee T_{n-1}^{-1} \qquad \text{and} \qquad T_{n-1}^{-1}\cdots T_1^{-1} g (g^\vee)^{-1} = q (g^\vee)^{-1} g T_{n-1}\cdots T_1.$

for $i \in \{1,\dots,n-2\}$.

The two visible symmetries in this defnition, switching the Coxeter diagram containing S_0 and the Coxeter diagram containing $S_0^\vee$, and flipping the Coxeter diagrams about the middle, form two important dualities. These dualities are expressed as involutive automorphisms of the DAArt $\tilde{\mathcal{B}}_{GL_n}$.

Theorem 5.1. (a) ($\vee$*Duality*) *There is an involution* $\iota\colon \tilde{\mathcal{B}}_{GL_n} \to \tilde{\mathcal{B}}_{GL_n}$ *with*

$$\iota(q) = q^{-1}, \qquad \iota(T_i) = T_i^{-1}, \qquad \iota(S_0^\vee) = S_0^{-1}, \qquad \iota(g) = g^\vee.$$

(b) ($\vdash\dashv$*Duality*) *There is an involution* $\eta\colon \tilde{\mathcal{B}}_{GL_n} \to \tilde{\mathcal{B}}_{GL_n}$ *with*

$$\eta(q) = q, \qquad \eta(T_i) = T_{n-i}, \qquad \eta(g) = g^{-1}, \qquad \eta(g^\vee) = (g^\vee)^{-1}.$$

Proof. (a) Applying ι to the relations in (53) switches the upper (nonchecked) relations with the lower (checked) relations. Applying ι to the relations in (54) produces the relations $q^{-1} \in Z(\widetilde{\mathcal{B}}_{GL_n})$,

$$T_1^{-1} g g^\vee = g^\vee g T_{n-1} \quad \text{and} \quad T_{n-1}\cdots T_1 g^\vee g^{-1} = q^{-1} g^{-1} g^\vee T_{n-1}^{-1}\cdots T_1^{-1},$$

respectively. Thus the relations in (54) are preserved under ι.
(b) The involution η preserves the relations in (53). Applying η to the relations in (54) produces the relations $q \in Z(\widetilde{\mathcal{B}}_{GL_n})$,

$$T_{n-1}(g^\vee)^{-1} g^{-1} = g^{-1}(g^\vee)^{-1} T_1^{-1} \quad \text{and} \quad T_1^{-1}\cdots T_{n-1}^{-1} g^{-1} g^\vee = q g^\vee g^{-1} T_1 \cdots T_{n-1}$$

which are equivalent to the original relations in (54) by taking inverses. □

5.2. The elements $X^{\varepsilon_1},\dots,X^{\varepsilon_n}$ and $Y^{\varepsilon_1^\vee},\dots,Y^{\varepsilon_n^\vee}$. The elements $X^{\varepsilon_1},\dots,X^{\varepsilon_n}$ will be used as the generators for a polynomial ring (inside the group algebra of $\tilde{\mathcal{B}}_{GL_n}$), and the Macdonald polynomials are polynomials in these variables. Inside the DAArt, these elements form a large commutative subgroups and, because of duality, there is *another* large commutative subgroup generated by elements $Y^{\varepsilon_1^\vee},\dots,Y^{\varepsilon_n^\vee}$. In this section we define these elements and give alternate presentations of $\tilde{\mathcal{B}}_n$ in terms of these elements.

Define $Y^{\varepsilon_1^\vee},\dots,Y^{\varepsilon_n^\vee}$ and $X^{\varepsilon_1},\dots,X^{\varepsilon_n}$ in $\tilde{\mathcal{B}}_{GL_n}$ by

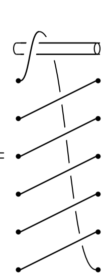
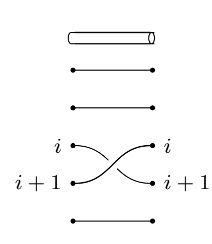
$$(55) \qquad \begin{array}{lcl} Y^{\varepsilon_1^\vee} = g T_{n-1}\cdots T_1 & \text{and} & Y^{\varepsilon_{j+1}^\vee} = T_j^{-1} Y^{\varepsilon_j^\vee} T_j^{-1}, \\ X^{\varepsilon_1} = g^\vee T_{n-1}^{-1}\cdots T_1^{-1}, & \text{and} & X^{\varepsilon_{j+1}} = T_j X^{\varepsilon_j} T_j, \end{array}$$

for $j \in \{1,\dots,n-1\}$. If $\iota\colon \tilde{\mathcal{B}}_{GL_n} \to \tilde{\mathcal{B}}_{GL_n}$ and $\eta\colon \tilde{\mathcal{B}}_{GL_n} \to \tilde{\mathcal{B}}_{GL_n}$ are the involutions in Theorem 5.1 then

(56) $\quad \iota(X^{\varepsilon_i}) = Y^{\varepsilon_i} \qquad \text{and} \qquad \eta(X^{\varepsilon_i}) = X^{-\varepsilon_{n-i+1}} \quad \text{and} \quad \eta(Y^{\varepsilon_i}) = Y^{-\varepsilon_{n-i+1}},$

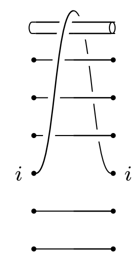
for $i \in \{1,\dots,n\}$.

The subgroup generated by $g^\vee, T_1,\dots,T_{n-1}$ has a pictorial representation given by

$g^\vee =$  and $T_i =$ 

for $i = 1,\dots,n-1$,

so that

(57) $$X^{\varepsilon_i} = T_{i-1}\cdots T_1 g^\vee T_{n-1}^{-1}\cdots T_i^{-1} =$$ 

for $i \in \{1,\dots,n\}$.

Use the notation $X_i = X^{\varepsilon_i}$,

and let $\quad X^\mu = X_1^{\mu_1}\cdots X_n^{\mu_n} = X^{\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n}, \qquad$ for $\mu = (\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$.

Similarly use the notation $Y_i = Y^{\varepsilon_i^\vee}$,

and let $\quad Y^{\lambda^\vee} = Y_1^{\lambda_1}\cdots Y_n^{\lambda_n} = Y^{\lambda_1\varepsilon_1^\vee+\cdots+\lambda_n\varepsilon_n^\vee}, \qquad$ for $\lambda^\vee = (\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n$.

The pictorial representation provides an easy check of the relations

(58) $$(g^\vee)^n = X^{\varepsilon_1+\cdots+\varepsilon_n} \quad \text{and} \quad X^{\varepsilon_i}X^{\varepsilon_j} = X^{\varepsilon_j}X^{\varepsilon_i},$$

(59) $$\text{and similarly,}\qquad g^n = Y^{\varepsilon_1^\vee+\cdots+\varepsilon_n^\vee} \quad \text{and} \quad Y^{\varepsilon_i^\vee}Y^{\varepsilon_j^\vee} = Y^{\varepsilon_j^\vee}Y^{\varepsilon_i^\vee},$$

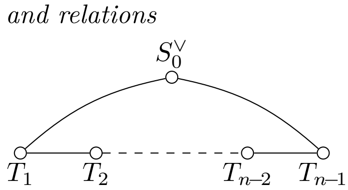
for $i,j \in \{1,\dots,n\}$. The pictorial perspective also verifies the relations

(60) $$S_0 T_{s_\varphi} = Y^{\varepsilon_1^\vee-\varepsilon_n^\vee} \quad \text{and} \quad (S_0^\vee)^{-1}T_{s_\varphi}^{-1} = X^{\varepsilon_1-\varepsilon_n},$$

where $T_{s_\varphi} = T_{n-1}\cdots T_1\cdots T_{n-1}$.

Theorem 5.2.

(a) *The group $\tilde{\mathcal{B}}_{GL_n}$ is presented by generators $q, g^\vee, S_0^\vee, T_1,\dots,T_{n-1}, Y^{\varepsilon_1^\vee},\dots,Y^{\varepsilon_n^\vee}$ and relations*



$$g^\vee T_i (g^\vee)^{-1} = T_{i+1}, \quad g^\vee T_{n-1}(g^\vee)^{-1} = S_0^\vee,$$

(61) $$q \in Z(\tilde{\mathcal{B}}_{GL_n}), \qquad Y^{\varepsilon_k^\vee}Y^{\varepsilon_j^\vee} = Y^{\varepsilon_j^\vee}Y^{\varepsilon_k^\vee} \quad \textit{for } k,j\in\{1,\dots,n\},$$

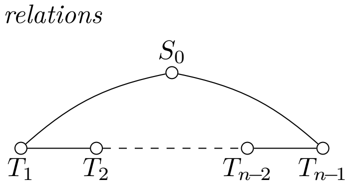
(62) $$Y^{\varepsilon_{i+1}^\vee} = T_i^{-1}Y^{\varepsilon_i^\vee}T_i^{-1} \quad \textit{and} \quad T_iY^{\varepsilon_j^\vee} = Y^{\varepsilon_j^\vee}T_i$$

for $i\in\{1,\dots,n-1\}$ and $j\neq i,i+1$, and

(63) $$g^\vee Y^{\varepsilon_i^\vee}(g^\vee)^{-1} = Y^{\varepsilon_{i+1}^\vee} \ \textit{for } i\in\{1,\dots,n-1\}, \quad \textit{and} \quad g^\vee Y^{\varepsilon_n^\vee}(g^\vee)^{-1} = qY^{\varepsilon_1^\vee}.$$

(b) *The group $\tilde{\mathcal{B}}_{GL_n}$ is presented by generators $q, g, T_0, T_1,\dots,T_{n-1}$ and $X^{\varepsilon_1},\dots,X^{\varepsilon_n}$*

and relations



$$gT_ig^{-1} = T_{i+1}, \quad gT_{n-1}g^{-1} = S_0,$$

(64) $$q\in Z(\tilde{\mathcal{B}}_{GL_n}), \qquad X^{\varepsilon_k}X^{\varepsilon_j} = X^{\varepsilon_j}X^{\varepsilon_k} \ \textit{for } k,j\in\{1,\dots,n\},$$

(65) $$X^{\varepsilon_{i+1}} = T_iX^{\varepsilon_i}T_i \qquad \textit{and} \qquad T_iX^{\varepsilon_j} = X^{\varepsilon_j}T_i$$

for $i \in \{1, \dots, n-1\}$ *and* $j \neq i, i+1$, *and*

(66) $$gX^{\varepsilon_i}g^{-1} = X^{\varepsilon_{i+1}}, \quad \textit{for } i \in \{1, 2, \dots, n-1\} \quad \textit{and} \qquad gX^{\varepsilon_n}g^{-1} = q^{-1}X^{\varepsilon_1}.$$

Proof. The proof is by showing that the relations in (61), (62) and (63) follow from the defining relations of $\tilde{\mathcal{B}}_{GL_n}$, and vice versa.

(53)&(54) $\Longrightarrow$ (a): Using (55) to define $Y^{\varepsilon_i^\vee}$, the pictorial perspective establishes the relations in (61) and (62). The proof of the relations in (63) is completed by

$$\begin{aligned}
g^\vee Y^{\varepsilon_1^\vee} &= g^\vee g T_{n-1}\cdots T_1 = T_1^{-1} g g^\vee T_{n-2}\cdots T_1 = T_1^{-1} g T_{n-1}\cdots T_2 g^\vee \\
&= T_1^{-1} g T_{n-1}\cdots T_2 T_1 T_1^{-1} g^\vee = T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1} g^\vee = Y^{\varepsilon_2^\vee} g^\vee, \\
g^\vee Y^{\varepsilon_i^\vee}(g^\vee)^{-1} &= g^\vee T_{i-1}^{-1}\cdots T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{i-1}^{-1}(g^\vee)^{-1} \\
&= T_i^{-1}\cdots T_2^{-1} Y^{\varepsilon_2^\vee} T_2^{-1}\cdots T_i^{-1} = Y^{\varepsilon_{i+1}^\vee}, \qquad \text{and} \\
g^\vee Y^{\varepsilon_n^\vee}(g^\vee)^{-1} &= g^\vee T_{n-1}^{-1}\cdots T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1}(g^\vee)^{-1} \\
&= g^\vee T_{n-1}^{-1}\cdots T_1^{-1} g T_{n-1}\cdots T_1 T_1^{-1}\cdots T_{n-1}^{-1}(g^\vee)^{-1} = g^\vee T_{n-1}^{-1}\cdots T_1^{-1} g (g^\vee)^{-1} \\
&= q g^\vee (g^\vee)^{-1} g T_{n-1}\cdots T_1 = q Y^{\varepsilon_1^\vee}.
\end{aligned}$$

(a) $\Rightarrow$ (53)&(54): Use (60) to define g and S_0 in terms of T_i and $Y^{\varepsilon_i^\vee}$s. If $i \in \{1, \dots, n-2\}$ then

$$\begin{aligned}
gT_ig^{-1} &= Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1} T_i T_{n-1}\cdots T_1 Y^{-\varepsilon_1^\vee} = Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{i+1}^{-1} T_i T_{i+1}\cdots T_1 Y^{-\varepsilon_1^\vee} \\
&= Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_i^{-1} T_i T_{i+1} T_i^{-1} T_i\cdots T_1 Y^{-\varepsilon_1^\vee} = Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{i-1}^{-1} T_{i+1} T_{i-1}^{-1}\cdots T_1 Y^{-\varepsilon_1^\vee} \\
&= Y^{\varepsilon_1^\vee} T_{i+1} Y^{-\varepsilon_1^\vee} = T_{i+1},
\end{aligned}$$

and

$$\begin{aligned}
gT_{n-1}g^{-1} &= gT_{n-1}T_{n-2}\cdots T_1 T_1^{-1}\cdots T_{n-2}^{-1} g^{-1} = Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-2}^{-1} g^{-1} \\
&= Y^{\varepsilon_1^\vee} g^{-1} T_2^{-1}\cdots T_{n-1}^{-1} = Y^{\varepsilon_1^\vee} g^{-1} T_1\cdots T_{n-1} T_{n-1}^{-1}\cdots T_1^{-1} T_2^{-1}\cdots T_{n-1}^{-1} \\
&= Y^{\varepsilon_1^\vee - \varepsilon_n^\vee} T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1} = S_0,
\end{aligned}$$

and

$$\begin{aligned}
gS_0g^{-1} &= Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1} Y^{\varepsilon_1^\vee - \varepsilon_n^\vee} T_{n-1}^{-1}\cdots T_1^{-1}\cdots T_{n-1}^{-1} T_{n-1}\cdots T_1 Y^{-\varepsilon_1^\vee} \\
&= Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_n^{-1} Y^{-\varepsilon_n^\vee} Y^{\varepsilon_1^\vee} T_{n-1}^{-1}\cdots T_2^{-1} Y^{-\varepsilon_1^\vee} \\
&= Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1} T_{n-1}\cdots T_1 Y^{-\varepsilon_1^\vee} T_1\cdots T_{n-1} Y^{\varepsilon_1^\vee} Y^{-\varepsilon_1^\vee} T_{n-1}^{-1}\cdots T_2^{-1} = T_1,
\end{aligned}$$

which establishes the relations in (53).

To prove the first relation in (54):

$$\begin{aligned}
T_1 g^\vee g &= T_1 g^\vee Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1} = T_1 Y^{\varepsilon_2^\vee} g^\vee T_1^{-1}\cdots T_{n-1}^{-1} \\
&= T_1 T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1} g^\vee T_1^{-1}\cdots T_{n-1}^{-1} = g T_{n-1}\cdots T_2 T_1 T_1^{-1} g^\vee T_1^{-1}\cdots T_{n-1}^{-1} \\
&= g T_{n-1}\cdots T_2 g^\vee T_1^{-1}\cdots T_{n-1}^{-1} = g g^\vee T_{n-2}\cdots T_1 T_1^{-1}\cdots T_{n-1}^{-1} = g g^\vee T_{n-1}^{-1},
\end{aligned}$$

and to prove the second relation in (54):

$$\begin{aligned}
T_{n-1}^{-1}\cdots T_1^{-1} g (g^\vee)^{-1} &= T_{n-1}^{-1}\cdots T_1^{-1} g T_{n-1}\cdots T_1 T_1^{-1}\cdots T_{n-1}^{-1}(g^\vee)^{-1} \\
&= T_{n-1}^{-1}\cdots T_1^{-1} Y^{\varepsilon_1^\vee} T_1^{-1}\cdots T_{n-1}^{-1}(g^\vee)^{-1} = Y^{\varepsilon_n^\vee}(g^\vee)^{-1} \\
&= (g^\vee)^{-1} g^\vee Y^{\varepsilon_n^\vee}(g^\vee)^{-1} = (g^\vee)^{-1} q Y^{\varepsilon_1^\vee} = q (g^\vee)^{-1} g T_{n-1}\cdots T_1.
\end{aligned}$$

Part (b) follows from part (a) by applying the duality involution ι. $\square$

5.3. The elements $X^{t_\mu w}$. Use the notation $X_i = X^{\varepsilon_i}$,

and let $\qquad X^\mu = X_1^{\mu_1}\cdots X_n^{\mu_n} = X^{\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n}, \qquad$ for $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$.

Using the notation of the affine Weyl group $W = \{t_\mu w \mid \mu\in\mathbb{Z}^n, w\in S_n\}$ from Section 2.1, for $\mu\in\mathbb{Z}^n$ and $w\in S_n$ define

$$X^{t_\mu w} = X^\mu T_w, \qquad \text{where} \quad T_w = T_{i_1}\cdots T_{i_\ell} \tag{67}$$

if $w = s_{i_1}\cdots s_{i_\ell}$ is a reduced word. The following proposition establishes how these elements are affected by right multiplication by the generators $T_1,\dots,T_n$ and $g^\vee$ of $\tilde{\mathcal{B}}_{GL_n}$.

Proposition 5.3. *Let $\mu\in\mathbb{Z}^n$ and $w\in S_n$. Then*

$$X^\mu T_{ws_i} = \begin{cases} X^\mu T_w T_i, & \text{if } \ell(ws_i)>\ell(w),\\ X^\mu T_w T_i^{-1}, & \text{if } \ell(ws_i)<\ell(w),\end{cases} \qquad \text{and}$$
$$X^\mu T_w g^\vee = X^\mu X_{w(1)} T_{ws_1\cdots s_{n-1}}.$$

Proof. The first equality follows from the fact that if $z\in S_n$ and $\ell(zs_i)>\ell(z)$ then $T_zT_i = T_{zs_i}$. For the second equality: Let $k=w(1)$ and write $w = s_{k-1}\cdots s_1 z$ with z in the subgroup of S_n that is generated by $s_2,\dots,s_{n-1}$. Letting $c_n = s_1\cdots s_{n-1}$ and using $g^\vee T_i (g^\vee)^{-1} = T_{i+1}$ then $(g^\vee)^{-1}T_z g^\vee = T_{c_n^{-1}zc_n}$ and

$$\begin{aligned} T_w g^\vee &= T_{k-1}\cdots T_1 T_z g^\vee = T_{k-1}\cdots T_1 g^\vee((g^\vee)^{-1}T_z g^\vee)\\ &= T_{k-1}\cdots T_1 g^\vee T_{c_n^{-1}zc_n} = (T_{k-1}\cdots T_1 g^\vee T_{n-1}^{-1}\cdots T_k^{-1})T_k\cdots T_{n-1}T_{c_n^{-1}zc_n}\\ &= X_k T_{s_k\cdots s_{n-1}c_n^{-1}zc_n} = X_k T_{s_{k-1}\cdots s_1 z c_n} = X_k T_{wc_n}. \end{aligned}$$

□

5.4. The type GL_n double affine Hecke algebra (DAHA). The *type GL_n double affine Hecke algebra $\tilde{H}_{GL_n}$* is the quotient of the group algebra of $\tilde{\mathcal{B}}_{GL_n}$ by the relations

$$(T_i - t^{\frac12})(T_i + t^{-\frac12}) = 0, \qquad \text{for } i\in\{1,\dots,n-1\}. \tag{68}$$

The involutions $\iota\colon \tilde{\mathcal{B}}_{GL_n}\to\tilde{\mathcal{B}}_{GL_n}$ and $\eta\colon \tilde{\mathcal{B}}_{GL_n}\to\tilde{\mathcal{B}}_{GL_n}$ from Theorem 5.1 preserve the relations in (68) to provide involutions

$$\iota\colon \tilde{H}_{GL_n}\to\tilde{H}_{GL_n} \qquad\text{and}\qquad \eta\colon \tilde{H}_{GL_n}\to\tilde{H}_{GL_n}. \tag{69}$$

The following proposition explains how $g, T_1,\dots,T_n$ move past the $X_1,\dots,X_n$ inside the affine Hecke algebra.

Proposition 5.4. *Let $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$ and define $X^\mu = X^{\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n}$. The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting coordinates. Let $s_1,\dots,s_{n-1}$ be the simple reflections in S_n. Then, as elements of $\tilde{H}_{GL_n}$,*

$$gX^\mu = q^{-\mu_n}X^{s_1s_2\cdots s_{n-1}\mu}g = q^{-\mu_n}X^{(\mu_n,\mu_1,\dots,\mu_{n-1})}g,$$

and

$$T_iX^\mu = (s_iX^\mu)T_i + \frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\mu, \quad \text{for } i\in\{1,\dots,n-1\}.$$

Proof. Start with $X_{i+1} = T_iX_iT_i$ and use $T_i^{-1} = T_i - (t^{\frac12}-t^{-\frac12})$ to get

$$\begin{aligned} T_iX_i = X_{i+1}T_i^{-1} &= X_{i+1}(T_i-(t^{\frac12}-t^{-\frac12})) = X^{s_i\varepsilon_i}T_i + (t^{\frac12}-t^{-\frac12})\frac{X_i-X_{i+1}}{1-X_iX_{i+1}^{-1}}\\ &= X^{s_i\varepsilon_i}T_i + \frac{(t^{\frac12}-t^{-\frac12})}{1-X_iX_{i+1}^{-1}}(1-s_i)X_i. \end{aligned}$$

and

$$\begin{aligned}T_iX_{i+1} &= T_i^2X_iT_i = ((t^{\frac12}-t^{-\frac12})T_i+1)X_iT_i = X_iT_i+(t^{\frac12}-t^{-\frac12})X_{i+1}\\ &= X^{s_i\varepsilon_{i+1}}T_i+(t^{\frac12}-t^{-\frac12})\frac{X_{i+1}-X_i}{1-X_iX_{i+1}^{-1}}\\ &= (s_iX_{i+1})T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X_{i+1}\end{aligned}$$

and, for $j\notin\{i,i+1\}$,

$$T_iX_j = X_jT_i = (s_iX_j)T_i+0 = (s_iX_j)T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X_j.$$

If

$$T_iX^\mu = (s_iX^\mu)T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\mu \quad\text{and}$$
$$T_iX^\nu = (s_iX^\nu)T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\nu$$

then

$$\begin{aligned}T_iX^{\mu+\nu} &= T_iX^\mu X^\nu = \left((s_iX^\mu)T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\mu\right)X^\nu\\ &= (s_iX^\mu)\left((s_iX^\nu)T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\nu\right)+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(X^\mu-(s_iX^\mu))X^\nu\\ &= (s_iX^{\mu+\nu})T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(X^{s_i\mu+\nu}-X^{s_i(\mu+\nu)}+X^{\mu+\nu}-X^{s_i\mu+\nu})\\ &= (s_iX^{\mu+\nu})T_i+\frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^{\mu+\nu}.\end{aligned}$$

Since $gX_ng^{-1}=q^{-1}X_1$ and $gX_ig^{-1}=X_{i+1}$ then

$$gX^{(\mu_1,\dots,\mu_n)} = gX_1^{\mu_1}\cdots X_n^{\mu_n} = X_2^{\mu_1}\cdots X_n^{\mu_{n-1}}gX_n^{\mu_n} = q^{-\mu_n}X_2^{\mu_1}\cdots X_n^{\mu_{n-1}}X_1^{\mu_n}g. \qquad \square$$

5.5. Intertwiners. Structurally, the elements $X^{\varepsilon_1},\dots,X^{\varepsilon_n}$ are playing the role of generators of a polynomial ring inside of the double affine Hecke algebra $\tilde H_{GL_n}$. The next key point is that we can produce elements $\tau_1^\vee,\dots,\tau_{n-1}^\vee$ and $\tau_\pi^\vee$ which are "replacements" for the generators $T_1,\dots,T_n$ and $g^\vee$, and which move past the elements $Y^{\varepsilon_1^\vee},\dots,Y^{\varepsilon_n^\vee}$ in the best possible way, by permuting the Y_i, as seen in (75).

Define Y_i for $i\in\mathbb{Z}$ by setting

(70) $$Y_i = Y^{\varepsilon_i^\vee}\ \text{ for } i\in\{1,\dots,n\}\qquad\text{and}\qquad Y_{j+n}=qY_j\ \text{ for } j\in\mathbb{Z}.$$

Letting $Y^K=q^{-1}$ and $\varepsilon_0^\vee=\varepsilon_n^\vee+K$ then

(71) $$Y_0 = Y^{\varepsilon_0^\vee} = Y^{\varepsilon_n^\vee+K} = Y^KY^{\varepsilon_n^\vee} = q^{-1}Y_n.$$

Let

(72) $$\tau_\pi^\vee = g^\vee \qquad\text{and}\qquad \tau_i^\vee = T_i+\frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}}\quad\text{for } i\in\{1,\dots,n-1\}$$

(the $\tau_i^\vee$ lie in a localization of the double affine Hecke algebra $\tilde H_{GL_n}$ which allows the denominators $1-Y_iY_j^{-1}$ for $i,j\in\{1,\dots,n\}$ with $i\neq j$). For $w\in W$ define

$$\tau_w^\vee = \tau_{i_1}^\vee\cdots\tau_{i_\ell}^\vee\qquad\text{for a reduced word } w=s_{i_1}\cdots s_{i_\ell}.$$

The following proposition establishes that the $\tau_i^\vee$ satisfy the braid relations so that the element $\tau_w^\vee$ does not depend on the choice of reduced word for w. The $\tau_i^\vee$ do not quite generate a symmetric group, because $(\tau_i^\vee)^2$ is not the identity.

PROPOSITION 5.5. *For $i \in \{1,\dots,n-2\}$ and $j,k \in \{1,\dots,n-1\}$ with $k \notin \{j+1, j-1\}$,*

$$\tau_\pi^\vee \tau_i^\vee = \tau_{i+1}^\vee \tau_\pi^\vee, \qquad \tau_i^\vee \tau_{i+1}^\vee \tau_i^\vee = \tau_{i+1}^\vee \tau_i^\vee \tau_{i+1}^\vee \qquad \textit{and} \qquad \tau_k^\vee \tau_j^\vee = \tau_j^\vee \tau_k^\vee; \tag{73}$$

$$(t^{\frac12}\tau_i^\vee)^2 = \frac{(1-tY_i^{-1}Y_{i+1})(1-tY_iY_{i+1}^{-1})}{(1-Y_i^{-1}Y_{i+1})(1-Y_iY_{i+1}^{-1})}, \qquad \textit{for } i \in \{1,\dots,n-1\}; \tag{74}$$

$$Y_i\tau_w^\vee = \tau_w^\vee Y_{w^{-1}(i)}, \qquad \textit{for } w \in W \textit{ and } i \in \mathbb{Z}. \tag{75}$$

Proof. Using $T_i = T_i^{-1} + (t^{\frac12} - t^{-\frac12})$,

$$\begin{aligned}\tau_i^\vee &= T_i + \frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}} = (T_i^{-1} + (t^{\frac12} - t^{-\frac12}) + \frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}} \\ &= T_i^{-1} + \frac{(Y_i^{-1}Y_{i+1} - 1 + 1)t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}} = T_i^{-1} + \frac{t^{-\frac12}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}.\end{aligned} \tag{76}$$

To prove (75), prove that

$$Y_1\tau_\pi^\vee = q^{-1}\tau_\pi^\vee Y_n \quad \text{and} \quad Y_i\tau_\pi^\vee = \tau_\pi^\vee Y_{i-1} \quad \text{for } i \in \{2,\dots n\}, \quad \textit{and} \tag{77}$$

$$Y_i\tau_i^\vee = \tau_i^\vee Y_{i+1}, \qquad Y_{i+1}\tau_i^\vee = \tau_i^\vee Y_i \qquad \text{and} \qquad Y_k\tau_i^\vee = \tau_i^\vee Y_k,$$

for $i \in \{1,\dots,n-1\}$ and $k \in \{1,\dots,n\}$ with $k \notin \{i, i+1\}$. By (63) and (20),

$$\tau_\pi^\vee Y^{\varepsilon_n^\vee} = qY^{\varepsilon_1}\tau_\pi^\vee \quad \text{gives} \quad Y_1\tau_\pi^\vee = \tau_\pi^\vee q^{-1}Y_n = \tau_\pi^\vee Y_0 = \tau_\pi^\vee Y_{\pi^{-1}(1)}.$$

and $Y^{\varepsilon_{i+1}^\vee} g^\vee = g^\vee Y^{\varepsilon_i^\vee}$ for $i \in \{1,\dots,n-1\}$. Using $Y_{i+1} = T_i^{-1}Y_iT_i^{-1}$,

$$\tau_i^\vee Y_i = \left(T_i^{-1} + \frac{t^{-\frac12}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right)Y_i = \left(Y_{i+1}T_i + \frac{t^{-\frac12}(1-t)Y_{i+1}}{1-Y_{i+1}}\right) = Y_{i+1}\tau_i^\vee,$$

and

$$\tau_i^\vee Y_{i+1} = \left(T_iY_{i+1} + \frac{t^{-\frac12}(1-t)Y_{i+1}}{1-Y_{i+1}}\right) = Y_i\left(T_i^{-1} + \frac{t^{-\frac12}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right) = Y_i\tau_i^\vee.$$

If $k \notin \{i, i+1\}$ then $T_iY_k = Y_kT_i$ and $Y_iY_k = Y_kY_i$ and $Y_{i+1}Y_k = Y_kY_{i+1}$ and so

$$\tau_i^\vee Y_k = \left(T_i + \frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)Y_k = Y_k\left(T_i + \frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) = Y_k\tau_i^\vee.$$

Using (76),

$$\begin{aligned}(\tau_i^\vee)^2 &= \left(T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)\tau_i^\vee = T_i\tau_i^\vee + \tau_i^\vee\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_{i+1}^{-1}Y_i} \\ &= T_i\left(T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right) + \left(T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_{i+1}^{-1}Y_i} \\ &= 1 + T_i\frac{t^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} + T_i\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_{i+1}^{-1}Y_i} + \left(\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right)\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_{i+1}^{-1}Y_i} \\ &= \frac{(1-Y_i^{-1}Y_{i+1})(1-Y_{i+1}^{-1}Y_i) + t^{-1} - 2 + t}{(1-Y_i^{-1}Y_{i+1})(1-Y_{i+1}^{-1}Y_i)} \\ &= \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}}Y_i^{-1}Y_{i+1})(t^{-\frac{1}{2}} - t^{\frac{1}{2}}Y_iY_{i+1}^{-1})}{(1-Y_i^{-1}Y_{i+1})(1-Y_iY_{i+1}^{-1})}.\end{aligned}$$

The proof of the relations in (73) can be done by comparing the brute force expansion of each side using the relations in (62) and (63). An alternative, often used, argument is to note that the action of each side on the polynomial representation (which is a faithful representation of $\widetilde{H}$) produces the same output (see Proposition 2.14(e) of [17]). □

5.6. The polynomial representation. In this section we build the action of the double affine Hecke algebra on Laurent polynomials in $X_1,\dots,X_n$. The elements $Y_1,\dots,Y_n$ are then a large family of commuting elements acting on the polynomial ring $\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$. The Macdonald polynomials are the simultaneous eigenvectors for the family of commuting elements $Y_1,\dots,Y_n$.

Let $q,t\in\mathbb{C}^\times$ such that $1\notin\{q^at^b \mid a,b\in\mathbb{Z}$ and a and b not both $0\}$ (alternatively, one may let q and t be formal parameters). The polynomial representation is

$$\mathbb{C}[X] = \mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}] = \mathbb{C}\text{-span}\{X^\mu\mathbf{1} \mid \mu\in\mathbb{Z}^n\}$$

with the action of DAHA determined by $T_i\mathbf{1} = t^{\frac{1}{2}}\mathbf{1}$ and $g\mathbf{1}=\mathbf{1}$ so that, by (55),

$$Y^{\varepsilon_1^\vee}\mathbf{1} = t^{\frac{1}{2}(n-1)}\mathbf{1} \qquad\text{and}\qquad Y^{\varepsilon_i}\mathbf{1} = t^{\frac{1}{2}(n-1)-2(i-1)\frac{1}{2}}\mathbf{1} = t^{-(i-1)+\frac{1}{2}(n-1)}\mathbf{1}. \tag{78}$$

Following the notation of [14, Ch. VI (3.1)], let T_{q^{-1},X_n} be the operator on $\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$ given by

$$(T_{q^{-1},X_n}h)(X_1,\dots,X_n) = h(X_1,\dots,X_{n-1},q^{-1}X_n).$$

Proposition 5.6. *As operators on the polynomial representation*

$$g = s_1s_2\cdots s_{n-1}T_{q^{-1},X_n} \qquad \textit{and} \qquad T_i = t^{-\frac{1}{2}}\big(t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1-s_i)\big),$$

for $i\in\{1,\dots,n-1\}$.

Proof. The first statement in Proposition 5.4 gives

$$gX^\mu\mathbf{1} = (s_1s_2\cdots s_{n-1}T_{q^{-1}X_n}X^\mu)\mathbf{1},$$

since $T_{q^{-1},X_n}X^\mu = q^{-\mu_n}X^\mu$. Using the second statement in Proposition 5.4,

$$T_iX^\mu\mathbf{1} = \Big((s_iX^\mu)t^{\frac12} + \frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)X^\mu\Big)\mathbf{1} \tag{79}$$
$$= \left(t^{\frac12} - t^{\frac12}(1-s_i) + \frac{t^{\frac12}-t^{-\frac12}}{1-X_iX_{i+1}^{-1}}(1-s_i)\right)X^\mu\mathbf{1}$$
$$= \left(t^{\frac12} + \frac{1}{1-X_iX_{i+1}^{-1}}(X_iX_{i+1}^{-1}t^{\frac12} - t^{-\frac12})(1-s_i)\right)X^\mu\mathbf{1}$$
$$= \left(t^{\frac12} + \frac{1}{X_i-X_{i+1}}(-X_it^{\frac12} + X_{i+1}t^{-\frac12})(1-s_i)\right)X^\mu\mathbf{1}$$
$$= t^{-\frac12}\left(t - \frac{tX_i - X_{i+1}}{X_i-X_{i+1}}(1-s_i)\right)X^\mu\mathbf{1}.$$

□

5.7. Constructing the nonsymmetric Macdonald polynomials E_μ. The Macdonald polynomials are the simultaneous eigenvectors for the action of $Y_1,\dots,Y_n$ on the polynomial ring $\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$. Because the intertwiners $\tau_1^\vee,\dots,\tau_{n-1}^\vee,\tau_\pi^\vee$ move past $Y_1,\dots,Y_n$ in the best possible way, they are the perfect tools for explicitly computing the Macdonald polynomials E_μ.

Proposition 5.7. *Let $\mu\in\mathbb{Z}^n_{\geqslant 0}$ and let u_μ and v_μ be as in (33) and let $\ell(v_\mu)$ be the number of inversions of v_μ. Choose a reduced word $\vec{u}_\mu = s_{i_1}\cdots s_{i_\ell}$ (where $i_1,\dots,i_\ell\in\{\pi,1,\dots,n-1\}$) and let $\tau^\vee_{u_\mu} = \tau^\vee_{i_1}\cdots\tau^\vee_{i_\ell}$. Define*

$$E_\mu = t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}\mathbf{1}. \qquad \textit{Then}\quad Y_iE_\mu = q^{-\mu_i}t^{-(v_\mu(i)-1)+\frac12(n-1)}E_\mu,$$

for $i\in\{1,\dots,n\}$, and the coefficient of x^μ in E_μ is 1.

Proof. Compute the eigenvalue as follows:

$$\begin{aligned}
Y_iE_\mu &= Y_it^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}\mathbf{1} && \text{(by definition of } E_\mu)\\
&= t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}Y_{u_\mu^{-1}(i)}\mathbf{1} && \text{(by (75))}\\
&= t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}Y_{v_\mu t_\mu^{-1}(i)}\mathbf{1} && \text{(by (33))}\\
&= t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}Y_{v_\mu(i-n\mu_i)}\mathbf{1} && \text{(by (22))}\\
&= t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}Y_{v_\mu(i)-n\mu_i}\mathbf{1} && \text{(by (19))}\\
&= t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}q^{-\mu_i}Y_{v_\mu(i)}\mathbf{1} && \text{(by (70))}\\
&= q^{-\mu_i}t^{-(v_\mu(i)-1)+\frac12(n-1)}(t^{-\frac12\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}\mathbf{1}) && \text{(by (78))}\\
&= q^{-\mu_i}t^{-(v_\mu(i)-1)+\frac12(n-1)}E_\mu && \text{(by definition of } E_\mu).
\end{aligned}$$

Using (67) and (5.3), the top term of the expansion of in $t^{-\frac12\ell(v_\mu)}\tau^\vee_{u_\mu}\mathbf{1}$ is

$$\begin{aligned}
t^{-\frac12\ell(v_\mu^{-1})}X^{u_\mu}\mathbf{1} &= t^{-\frac12\ell(v_\mu^{-1})}X^{t_\mu v_\mu^{-1}}\mathbf{1} = t^{-\frac12\ell(v_\mu^{-1})}X^\mu T_{v_\mu^{-1}}\mathbf{1}\\
&= t^{-\frac12\ell(v_\mu^{-1})}X^\mu t^{\frac12\ell(v_\mu^{-1})}\mathbf{1} = X^\mu\mathbf{1} = x^\mu.
\end{aligned}$$

□

5.8. Steps and symmetries of E_μ. The following Proposition establishes the inductive construction of the E_μ and the symmetries in (3.3). For examples of the E_μ see Proposition 3.5, which provides explicit formulas for the cases when μ has 1 or 2 boxes.

Proposition 5.8. *Let* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$.

(a) *If* $i \in \{1, \dots, n-1\}$ *and* $\mu_i > \mu_{i+1}$ *then* $E_{s_i\mu} = t^{\frac{1}{2}}\tau_i^\vee E_\mu$.

(b) $E_{(\mu_n+1,\mu_1,\dots,\mu_{n-1})} = t^{\#\{i\in\{1,\dots,n\}\ \mid\ \mu_i>\mu_n\}-\frac{1}{2}(n-1)}\tau_\pi^\vee E_{(\mu_1,\dots,\mu_n)}$.

(c) $E_{(\mu_n+1,\mu_1,\dots,\mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \dots, x_n, q^{-1}x_1)$.

(d) $E_{(\mu_1+1,\dots,\mu_n+1)} = x_1 \cdots x_n E_{(\mu_1,\dots,\mu_n)}$.

(e) $E_{(-\mu_n,\dots,-\mu_1)}(x_1, \dots, x_n; q, t) = E_\mu(x_n^{-1}, \dots, x_1^{-1}; q, t)$.

Proof. (a) Let $\mu = (\mu_1, \dots, \mu_n)$ and let i be such that $\mu_i > \mu_{i+1}$.
By Proposition 2.1(e), $\ell(v_{s_i\mu}) - \ell(v_\mu) = -1$, giving

$$E_{s_i\mu} = t^{-\frac{1}{2}\ell(v_{s_i\mu}^{-1})}\tau^\vee_{u_{s_i\mu}}\mathbf{1} = t^{-\frac{1}{2}\ell(v_{s_i\mu}^{-1})}\tau_i^\vee\tau^\vee_{u_\mu}\mathbf{1} = t^{-\frac{1}{2}(\ell(v_{s_i\mu}^{-1})-\ell(v_\mu^{-1}))}\tau_i^\vee E_\mu = (t^{\frac{1}{2}}\tau_i^\vee)E_\mu.$$

(b) The left hand side is $E_{\pi\mu}$ and

$$\begin{aligned}E_{\pi\mu} &= t^{-\frac{1}{2}\ell(v_{\pi\mu}^{-1})}\tau^\vee_{u_{\pi\mu}}\mathbf{1} = t^{-\frac{1}{2}\ell(v_{\pi\mu}^{-1})}\tau_\pi^\vee\tau^\vee_{u_\mu}\mathbf{1} = t^{-\frac{1}{2}(\ell(v_{\pi\mu}^{-1})-\ell(v_\mu^{-1}))}\tau_\pi^\vee t^{-\frac{1}{2}\ell(v_\mu^{-1})}\tau^\vee_{u_\mu}\mathbf{1}\\ &= t^{-(v_\mu(n)-1)+\frac{1}{2}(n-1)}\tau_\pi^\vee E_\mu.\end{aligned}$$

The result then follows from Proposition 2.1(b).

(c) The second relation in (54) and the second relation in (63) give $X_1 g = g^\vee Y_n$. Beginning with the right hand side of (b) and using $g^\vee Y_n = X_1 g$ gives

$$\begin{aligned}t^{-(v_\mu(n)-1+\frac{1}{2}(n-1)}\tau_\pi^\vee E_\mu &= t^{-(v_\mu(n)-1)+\frac{1}{2}(n-1)}g^\vee E_\mu = g^\vee q^{\mu_n}Y_n E_\mu\\ &= q^{\mu_n}x_1 g E_\mu = q^{\mu_n}x_1 s_1\cdots s_{n-1}T_{q^{-1},x_n}E_\mu = q^{\mu_n}x_1E_\mu(x_2,\dots,x_n,q^{-1}x_1).\end{aligned}$$

(d) By (58), $(\tau_\pi^\vee)^n = (g^\vee)^n = X_1 \cdots X_n$ and so

$$E_{(\mu_1+1,\dots,\mu_n+1)} = (\tau_\pi^\vee)^n E_{(\mu_1,\dots,\mu_n)} = x_1\cdots x_n E_{(\mu_1,\dots,\mu_n)}.$$

(e) Let $\eta\colon \tilde{H}_{GL_n} \to \tilde{H}_{GL_n}$ be the involution in (69). Let w_0 be the longest element of S_n so that $w_0(i) = n-i+1$ for $i \in \{1, \dots, n\}$. Using the last relation in (56),

$$\begin{aligned}Y_i\eta(E_\mu(X_1,\dots,X_n))\mathbf{1} &= \eta(Y_{n-i+1}^{-1}E_\mu(X_1,\dots,X_n))\mathbf{1}\\ &= q^{\mu_{n-i+1}}t^{(v_\mu(n-i+1)-1)-\frac{1}{2}(n-1)}\eta(E_\mu(X_1,\dots,X_n))\mathbf{1}\\ &= q^{\mu_{n-i+1}}t^{(w_0v_{-w_0\mu}w_0(n-i+1)-1)-\frac{1}{2}(n-1)}\eta(E_\mu(X_1,\dots,X_n))\mathbf{1},\\ &= q^{-(-w_0\mu)_i}t^{(w_0v_{-w_0\mu}(i)-1)-\frac{1}{2}(n-1)}\eta(E_\mu(X_1,\dots,X_n))\mathbf{1},\\ &= q^{-(-w_0\mu)_i}t^{(n-v_{-w_0\mu}(i)+1)-1-\frac{1}{2}(n-1)}\eta(E_\mu(X_1,\dots,X_n))\mathbf{1},\\ &= q^{-(-w_0\mu)_i}t^{-(v_{-w_0\mu}(i)-1)+\frac{1}{2}(n-1)}\eta(E_\mu(X_1,\dots,X_n))\mathbf{1},\end{aligned}$$

so that $\eta(E_\mu(X_1, \dots, X_n))\mathbf{1} = E_\mu(x_n^{-1}, \dots, x_1^{-1})$ satisfies the conditions (from Theorem 5.7) determining $E_{-w_0\mu}(x_1, \dots, x_n)$. ☐

Acknowledgements. We thank L. Williams and M. Wheeler for bringing our attention to [6] and [2], both of which were important stimuli during our work. We are also very grateful for the encouragement, questions, and discussions from A. Hicks, S. Mason, O. Mandelshtam, Z. Daugherty, Y. Naqvi, S. Assaf, and especially A. Garsia and S. Corteel, which helped so much in getting going and keeping up the energy. We thank S. Billey, Z. Daugherty, C. Lenart, J. Saied, and S. Viswanath for very useful specific comments for improving the exposition. A. Ram extends a very special and heartfelt thank you to P. Diaconis who has provided unfailing support and advice and honesty and encouragement.

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