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ABSTRACT We review a class of modules for the wreath product $S_m \wr S_n$ of two symmetric groups which are analogous to the Specht modules of the symmetric group, and prove a pair of branching rules for this family of modules. These branching rules describe the behaviour of these wreath product Specht modules under restriction to the wreath products $S_{m-1} \wr S_n$ and $S_m \wr S_{n-1}$. In particular, we see that these restrictions of wreath product Specht modules have Specht module filtrations, and we obtain combinatorial interpretations of the multiplicities in these filtrations.

1. INTRODUCTION

Let k be a field. Recall that the *Specht modules* for the symmetric group S_n (over k) are a family of kS_n-modules (we shall use right modules in this article) which are indexed by the partitions of n. We shall write the Specht module for S_n which is indexed by the partition λ as S^λ. These Specht modules have a close relationship with the simple modules of kS_n, and indeed if kS_n is semisimple then the Specht modules are exactly the simple modules. Because of this and other properties, the Specht modules for kS_n have been the subject of intense study for decades, and a large and varied literature has built up around them.

In this article we consider a class of modules for the *wreath product* $S_m \wr S_n$ of two symmetric groups which are analogous to the Specht modules for the symmetric group. These modules may be obtained from Specht modules for S_m and S_n via a well-known method of constructing modules for wreath products, see for example [2] or [8, chapter 4]. We may alternatively obtain them as the cell modules of a certain cellular structure (in the sense of Graham and Lehrer) on the group algebra $k(S_m \wr S_n)$. This cellularity was originally proved in [3], while an alternative proof via the method of *iterated inflation* was given in [6]. The characterisation of these modules as cell modules allows us to see at once that they bear exactly the same relation to the simple modules for the wreath product as the symmetric group Specht modules bear to the simple modules for that group, and this justifies the name "Specht modules". Although the construction by which these modules may be obtained is well-known, the author is not aware that these modules have previously been studied in the literature as wreath product analogues of the symmetric group Specht modules.

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A key fact in the theory of Specht modules is the result of James which we shall call the "Specht branching rule", which gives a Specht filtration for the restriction of a Specht module from kS_n to kS_{n-1} with an elegant combinatorial description of the set of Specht modules occurring in this filtration. Moreover, these multiplicities are independent of the field k. The main results presented in this paper are two Specht branching rules for the wreath product of two symmetric groups: the first describes the restriction of a wreath product Specht module from $k(S_m \wr S_n)$ to $k(S_{m-1} \wr S_n)$, while the second describes the restriction to $k(S_m \wr S_{n-1})$. In both cases, we obtain a Specht module filtration with multiplicities that do not depend on the field and which moreover have nice combinatorial descriptions.

Note that we are using the name "branching rule" in what is perhaps a slightly non-standard way. Indeed, in general group representation theory, if we have some nested family of finite groups $G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n \leqslant G_{n+1} \leqslant \cdots$, then a *branching rule* is a result describing the simple composition factors of the restriction of a simple module from G_{n+1} to G_n. Such branching rules are known in general for groups of the form $G \wr S_n$ for G a finite group, see in particular [11] for a modular branching rule for $G \wr S_n$.

However, the results we present here are concerned specifically with *Specht* modules, which do not always coincide with the simple modules, and thus the results of this paper are not consequences of the known general results. However, our results are of a very similar nature, and indeed our Specht modules are in fact the simple modules when the group algebra is semisimple (in both the symmetric group and the wreath product case) and so in the semisimple case our Specht branching rules are in fact branching rules in the more usual sense (and thus coincide with the known general results). The proofs of our results are however rather different to those of the known general branching rules, since by specialising to the symmetric groups and focussing on the Specht modules, we can use combinatorial methods rather than needing to appeal to, for example, Clifford theory as in the general case.

2. Background

We shall work over a field k in this article. We shall often need to deal with tensor products of k-vector spaces, and we shall abbreviate $\otimes_k$ to $\otimes$.

If G is a group and k is a field, then we shall write kG for the group algebra of G over k. By a *kG-module*, we shall mean a right kG-module of finite k-dimension. If G is a group with a subgroup H, then for a field k we shall write the operations of induction and restriction of modules between the group algebras kG and kH as $\uparrow_H^G$ and $\downarrow_H^G$, with the field being implicit.

Mackey's theorem is a fundamental result in finite group theory which describes the interaction of the operations of induction and restriction. If G is a finite group and H is a subgroup of G, then for $g \in G$ we define H^g to be the subgroup $\{g^{-1}hg \mid h \in H\}$ of G, and we call this the *conjugate subgroup of H by g* (note that H^g is isomorphic to H). Further, if X is a kH-module, then we define X^g to be the kH^g-module with underlying vector space X and action given by $x(g^{-1}hg) = xh$ for $x \in X$ and $h \in H$. We call this the *conjugate module of X by g*.

Theorem 2.1. (*Mackey's Theorem* [1, Theorem 3.3.4]) *Let G be a finite group with subgroups H and K, let $\mathcal{U}$ be a complete non-redundant system of (H,K)-double coset representatives in G, and let X be a right kH-module. Then we have a decomposition of right kK-modules*

$$X\uparrow_H^G\downarrow_K^G \;\cong\; \bigoplus_{u\in\mathcal{U}} X^u \downarrow_{H^u\cap K}^{H^u}\uparrow_{H^u\cap K}^{K} .$$

2.1. Filtrations and series. Let kG be a group algebra over a field, let M be a kG-module and $X_1, \ldots, X_t$ also be kG-modules. A *series for M over $X_1, \ldots, X_t$* is a series of submodules

$$M = M_n \supseteq M_{n-1} \supseteq M_{n-2} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0 \tag{1}$$

such that each quotient $\frac{M_l}{M_{l-1}}$ is isomorphic to some X_i. If such a series exists, we say that M has a *filtration by the modules $X_1, \ldots, X_t$*. If $\frac{M_1}{M_0} = M_1$ is isomorphic to X_l, then we say that X_l occurs at the *bottom* of the filtration or series. Further, let $f : \{1, \ldots, n\} \longrightarrow \{1, \ldots, t\}$ be any function such that for each l, $X_{f(l)}$ is isomorphic to the quotient of M_l by M_{l-1}, and let $\alpha_i = |f^{-1}(i)|$. We then say that (1) is a *series for M over $X_1, \ldots, X_t$ admitting the multiplicities $(\alpha_i)_i$*, and that M has a *filtration by the modules $X_1, \ldots, X_t$ admitting the multiplicities $(\alpha_i)_i$*. Note that we have not assumed that the modules X_i are pairwise non-isomorphic. If there are isomorphisms between the modules X_i, then the multiplicities in a filtration are *not uniquely determined* by the series of submodules, and so the same series of submodules will admit different multiplicities. Even if the modules X_i are pairwise non-isomorphic, so that the multiplicities *are* uniquely determined by the series, the multiplicities are *not* in general uniquely determined by the module, as the same module can have two series of submodules where the X_i occur with different multiplicities.

2.2. Combinatorics. We now review a few combinatorial concepts. We assume that the reader is already familiar with these notions, and so our treatment will be brief.

Recall that a *composition* of n is a tuple of non-negative integers adding up to n. We call n the *size* of α and write $n = |\alpha|$. We call the elements of a composition its *parts*, and the number of parts in a composition is its *length*. We shall adopt the common shorthand of using exponent notation for repeated parts in a composition, so that for example we might write $(3, 2^2, 1^3)$ for $(3, 2, 2, 1, 1, 1)$. A *partition* of n is a composition of n whose parts are all positive and appear in non-increasing order. We shall write $\lambda \vdash n$ to mean that λ is a partition of n. We note that the *empty partition* $()$ is the unique partition of size 0, and also the unique partition of length 0.

A simple total order on the partitions of an integer n is the *lexicographic order*, in which partitions are sorted by the size of their first part, then by the size of their second part, and so on. Thus in this order, (n) is the largest and (1^n) the least partition. A more sophisticated (partial) order in the partitions of n is the *dominance order*, where $\lambda \trianglerighteq \mu$ if and only if the sum of the first t parts of λ equals or exceeds the sum of the first t parts of μ for all t.

The *Young diagram* of a composition α is an arrangement of rows of boxes with a number of boxes on the i^{th} row (counting downward) equal to the i^{th} part of α. If α, γ are compositions of n, then a *tableau of shape α and type γ* is a Young diagram of shape α where each box contains a positive integer i such that for each $i \in \{1, \ldots, t\}$ where t is the length of γ, i occurs exactly γ_i times. Note that since we allow zero parts in compositions, a Young diagram or tableau can have empty rows.

Now if λ, α are compositions such that $|\alpha| \leqslant |\lambda|$ and the Young diagram of α lies wholly inside the Young diagram of λ (i.e. the length of α is at most the length of λ and $\alpha_i \leqslant \lambda_i$ for all i from 1 to the length of α), then for γ a composition of $|\lambda| - |\alpha|$, we define a *skew tableau of shape $\lambda \smallsetminus \alpha$ and type γ* to be a diagram obtained by removing the boxes of the Young diagram of α from λ and then filling the remaining boxes with positive integers such that each i occurs γ_i times, as for non-skew tableaux. Note that the boxes of a skew tableau may be non-contiguous, as in the example below.

If a tableau (skew or non-skew) has its entries strictly increasing down each column and weakly increasing from left to right across each row, we say that it is *semistandard* (note that there may be gaps in the columns of a skew tableau). Thus for example

$$\begin{array}{cccccc} & & & \boxed{1} & \boxed{3} & \boxed{3} \\ & & & & & \\ & & \boxed{1} & & & \\ & \boxed{2} & \boxed{2} & & & \\ \boxed{3} & & & & & \end{array}$$

is a semistandard skew tableau of shape $(6,4,3,3,1) \setminus (3,4,2,1)$ and type $(2,2,3)$.

A *multicomposition* of n is a tuple of compositions whose sizes add up to n. The number of elements in the tuple is called its *length*. A multicomposition whose components are all partitions is called a *multipartition*. We typically use underlined symbols to denote multicompositions and index their components with superscripts, so that for example a multicomposition of length t might be written $\underline{\alpha}$, with $\underline{\alpha} = (\alpha^1, \dots, \alpha^t)$, and α^i_j being the j^{th} part of the composition α^i. The *size* of a multicomposition is the sum of the sizes of its parts, and if $\underline{\alpha} = (\alpha^1, \dots, \alpha^t)$ is a multicomposition of n, then we let $|\underline{\alpha}|$ be the composition $(|\alpha^1|, \dots, |\alpha^t|)$ of n. Note that we do allow the empty partition $() \vdash 0$ to occur as an entry in a multicomposition or multipartition, and that the unique multipartition of length 0 is $()$, the *empty multipartition*.

2.3. Symmetric groups. If α is a composition of n, then we shall write S_α for the *Young subgroup* of S_n associated to α. We shall be making frequent use of the operations of induction and restriction between group algebras of symmetric groups and of their Young subgroups, for example $X{\uparrow}^{kS_n}_{kS_\alpha}$ and $Y{\downarrow}^{kS_n}_{kS_\alpha}$. To de-clutter such expressions, we shall abbreviate the notation by replacing the full symbols for the group algebras with the subscripts used to identify the various subgroups of S_n involved, so for example the above would be abbreviated to $X{\uparrow}^n_\alpha$ and $Y{\downarrow}^n_\alpha$.

Recall that we have for each $n > 0$ a natural embedding of the symmetric group S_{n-1} into S_n by letting $\sigma \in S_{n-1}$ act on $1, \dots, n$ by fixing n and permuting the other elements as it does in S_{n-1}. Thus we can regard S_{n-1} as a subgroup of S_n and hence we may induce a module X from kS_{n-1} to kS_n, or restrict a module Y from kS_n to kS_{n-1}. We shall write these operations as $X{\uparrow}^n_{n-1}$ and $Y{\downarrow}^n_{n-1}$.

For a partition λ of n, we write S^λ for the associated Specht module. We refer the reader to [7, Ch. 4] for the construction of S^λ, as we shall not need any details of this construction here. A few key properties of S^λ will suffice. If a kS_n-module M has a filtration by the Specht modules S^λ for $\lambda \vdash n$, then we say that M has a *Specht module filtration*, or just a *Specht filtration*.

Theorem 2.2. (*Specht branching rule*) *([7], Theorem 9.3) Let $\lambda \vdash n$ where $n > 0$, and let k be a field. Then the $kS_{(n-1)}$-module $S^\lambda{\downarrow}^n_{n-1}$ has a Specht filtration, as follows. Let $\nu_t \rhd \nu_{t-1} \rhd \cdots \rhd \nu_1$ be all the (distinct) partitions of $n-1$ which can be obtained by removing a single box from λ, arranged in dominance order. Then we have a series of submodules*

$$S^\lambda{\downarrow}^n_{n-1} = M_t \supseteq M_{t-1} \supseteq M_{t-2} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$$

such that $\frac{M_l}{M_{l-1}} \cong S^{\nu_l}$.

We shall make use below of *Littlewood–Richardson coefficients*. These numbers appear in many different places in combinatorics and representation theory and have an extensive literature, but we shall only need a few basic facts. The reader is referred to the literature, for example [10, Chapter 7], for more details. Indeed, if λ is a partition of n and α, β are partitions whose sizes add up to n, then we have a non-negative integer $c^{\lambda}_{\alpha,\beta}$ called a *Littlewood–Richardson coefficient*. Moreover, if $(\alpha_1,\dots,\alpha_t)$ for $t \geqslant 1$ is a tuple of partitions whose sizes add up to n, then we may define a more general Littlewood–Richardson coefficient $c(\lambda;\underline{\alpha})$. Indeed, for the case $t=1$, we let $c(\lambda;\underline{\alpha}) = c\big(\lambda;(\alpha^1)\big)$ be 1 if $\underline{\alpha} = (\lambda)$ and zero otherwise. For the case $t=2$, we let $c(\lambda;\underline{\alpha}) = c\big(\lambda;(\alpha^1,\alpha^2)\big) = c^{\lambda}_{\alpha^1,\alpha^2}$. For $t>2$, we define $c(\lambda;\underline{\alpha})$ by induction on t by setting

$$c\big(\lambda;\underline{\alpha}\big) = \sum_{\beta\vdash n-|\alpha^1|} c^{\lambda}_{\alpha^1,\beta}\, c\big(\beta;(\alpha^2,\dots,\alpha^t)\big). \tag{2}$$

The *Littlewood–Richardson rule* [10, Theorem A1.3.3] states that $c^{\lambda}_{\alpha,\beta}$ is equal to the number of skew semistandard tableaux of shape $\lambda\smallsetminus\alpha$ and type β where the sequence obtained by concatenating its reversed rows is a *lattice word* (if $|\beta| \neq |\lambda| - |\alpha|$, then this number is zero). Here a lattice word is a finite sequence of integers, allowing repetitions, such that if for any $r \geqslant 0$ and any $i \geqslant 0$ we let $\#^i_r$ be the number of times i appears in the first r places of the sequence, then for each r we have $\#^1_r \geqslant \#^2_r \geqslant \#^3_r \geqslant \cdots$. In particular, every Littlewood–Richardson coefficient is in fact a non-negative integer.

2.4. Wreath products. We shall now review some key definitions and constructions connected to wreath products. For more details, see [2] and [8, Chapter 4]. Let n and m be non-negative integers. The *wreath product* of S_n on S_m is the group whose underlying set is the Cartesian product of S_n with n copies of S_m. We shall write elements of $S_m \wr S_n$ as $(\sigma;\alpha_1,\alpha_2,\dots,\alpha_n)$ for $\alpha_1,\alpha_2,\dots,\alpha_n \in S_m$ and $\sigma\in S_n$. Multiplication is given by the formula

$$(\sigma;\alpha_1,\alpha_2,\dots,\alpha_n)(\pi;\beta_1,\beta_2,\dots,\beta_n) = \Big(\sigma\pi;(\alpha_{(1)\pi^{-1}}\beta_1),(\alpha_{(2)\pi^{-1}}\beta_2),\dots,(\alpha_{(n)\pi^{-1}}\beta_n)\Big).$$

If G is a subgroup of S_m and H a subgroup of S_n, we shall write $G\wr H$ for the subgroup of $S_m\wr S_n$ consisting of all elements $(\sigma;\alpha_1,\alpha_2,\dots,\alpha_n)$ for $\alpha_1,\alpha_2,\dots,\alpha_n \in G$ and $\sigma \in H$. We shall make frequent use of such groups where G and H are each either the full symmetric group or a Young subgroup thereof, and we shall often restrict or induce modules between such groups, for example $X{\uparrow}^{k(S_m\wr S_n)}_{k(S_m\wr S_\gamma)}$ and $Y{\downarrow}^{k(S_m\wr S_n)}_{k(S_m\wr S_\gamma)}$ where γ is some composition of n. As with the symmetric group, we shall de-clutter such expressions where possible by suppressing the field and replacing the full symbols for subgroups of S_m and S_n with the subscript used to identify them, so for example the above would be abbreviated to $X{\uparrow}^{m\wr n}_{m\wr\gamma}$ and $Y{\downarrow}^{m\wr n}_{m\wr\gamma}$.

We now extend the notion of a Young subgroup to encompass multicompositions. Let $\underline{\gamma} = (\gamma^1,\dots,\gamma^t)$ be a t-multicomposition of n (t some non-negative integer) and let $\hat{\gamma}$ be the composition of n obtained by concatenating the compositions $\gamma^1,\dots,\gamma^t$ in that order (so $\hat{\gamma}$ consists of the parts of γ^1, followed by the parts of γ^2, and so on). We define the *Young subgroup* of S_n associated to $\underline{\gamma}$ to be the Young subgroup $S_{\hat{\gamma}}$ associated to $\hat{\gamma}$, and we write $S_{\underline{\gamma}}$ for this subgroup. Thus we have a canonical isomorphism $S_{\underline{\gamma}} \cong S_{\gamma^1}\times S_{\gamma^2}\times\cdots\times S_{\gamma^t}$. Further, we note that $S_{\underline{\gamma}}$ is a subgroup of $S_{|\underline{\gamma}|}$.

We now recall several standard methods for constructing modules for wreath products, as described in [8, Section 4.3] and [2, Section 3]. Recall that we are using *right* modules.

Firstly, let G be a subgroup of S_m, and let X be a kG-module. We define $X^{\widetilde{\boxtimes} n}$ to be the $k(G\wr S_n)$-module obtained by equipping the k-vector space $X^{\otimes n}$ (that is, the tensor product over k of n copies of X) with the action given by the formula

$$(x_1 \otimes \cdots \otimes x_n)(\sigma; \alpha_1, \ldots, \alpha_n) = (x_{(1)\sigma^{-1}}\alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}}\alpha_n)$$

for $x_1, \ldots, x_n \in X$, $\alpha_1, \ldots, \alpha_n \in G$, $\sigma \in S_n$. More generally, let $X_1, \ldots, X_t$ be kG-modules, and $\gamma = (\gamma_1, \ldots, \gamma_t)$ a composition of n of length t. We form a $k(G \wr S_\gamma)$-module by equipping the k-vector space $\left(X_1^{\otimes \gamma_1}\right) \otimes \left(X_2^{\otimes \gamma_2}\right) \otimes \cdots \otimes \left(X_t^{\otimes \gamma_t}\right)$ with the action given by the formula

$$(x_1 \otimes \cdots \otimes x_n)(\sigma; \alpha_1, \ldots, \alpha_n) = (x_{(1)\sigma^{-1}}\alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}}\alpha_n)$$

where each x_i lies in the appropriate X_j, $\alpha_1, \ldots, \alpha_n \in G$, and $\sigma \in S_\gamma$. We denote this module by $\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma}$, and we note that $X^{\widetilde{\boxtimes} n}$ is the special case of this construction where γ has an n in one place and all the other parts are 0.

Now let G be a subgroup of S_m, H a subgroup of S_n, and Y a kH-module. It is easy to check that we may make Y into a $k(G \wr H)$-module via the formula

$$y(\sigma; \alpha_1, \ldots, \alpha_n) = y\sigma \tag{3}$$

for $y \in Y$, $\alpha_1, \ldots, \alpha_n \in G$, and $\sigma \in H$. This module may be understood by noting that $G \wr H$ is the semidirect product of the normal subgroup consisting of all elements $(e; \alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in G$ with the subgroup consisting of all elements $(\sigma; e, \ldots, e)$ for $\sigma \in H$. This latter subgroup is canonically isomorphic to H, and hence we see that the module obtained from Y via (3) is the *inflation* of Y from H to $G \wr H$ with respect to the semidirect product structure. Hence we shall denote this module by $\mathrm{Inf}_H^{G \wr H} Y$. Now let H be a subgroup of S_n, G be a subgroup of S_m, Y be a kH-module, and further let Z be a $k(G \wr H)$-module. Then we define a $k(G \wr H)$-module $Z \oslash Y$ as follows: the underlying k-vector space is $Z \otimes Y$, and the action is given by the formula

$$(z \otimes y)(\sigma; \alpha_1, \ldots, \alpha_n) = (z(\sigma; \alpha_1, \ldots, \alpha_n)) \otimes (y\sigma)$$

for $z \in Z$, $y \in Y$, $\alpha_1, \ldots, \alpha_n \in G$, $\sigma \in H$. Thus we see that we have an equality of $k(G \wr H)$-modules $Z \oslash Y = Z \otimes \mathrm{Inf}_H^{G \wr H} Y$ where the module on the right-hand side is the internal tensor product of the $k(G \wr H)$-modules Z and $\mathrm{Inf}_H^{G \wr H} Y$. Since taking the (internal) tensor product of group modules and inflating group modules are both exact functors, it follows that the operation $- \oslash -$ "preserves submodule series" in both places, in the sense that if Z has a submodule series with quotients $X_1, \ldots, X_t$ in order from top to bottom, then $Z \oslash Y$ will have a submodule series with quotients $X_1 \oslash Y, \ldots, X_t \oslash Y$ in order from top to bottom, with a symmetrical result for a submodule series for Y.

We can combine the above constructions as follows: if G is a subgroup of S_m, $X_1, \ldots, X_t$ are kG-modules and Y is a kS_γ-module for γ a composition of n, then we obtain a $k(G \wr S_\gamma)$-module $\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma} \oslash Y$ with underlying vector space $\left(X_1^{\otimes \gamma_1}\right) \otimes \left(X_2^{\otimes \gamma_2}\right) \otimes \cdots \otimes \left(X_t^{\otimes \gamma_t}\right) \otimes Y$ and action given by the formula

$$(x_1 \otimes \cdots \otimes x_n \otimes y)(\sigma; \alpha_1, \ldots, \alpha_n) = (x_{(1)\sigma^{-1}}\alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}}\alpha_n) \otimes (y\sigma) \tag{4}$$

for $x_i \in X$, $\alpha_i \in G$, $y \in Y$, $\sigma \in S_\gamma$.

We now recall an elementary construction for producing kS_γ-modules Y for use in the above constructions. Indeed, for each $i \in \{1, \ldots, t\}$, let Y_i be a right kS_{γ_i}-module. Now recall that we have a canonical identification of the group S_γ with the direct product $S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t}$ of groups. Thus any module for $k(S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t})$ may be regarded as a kS_γ-module in a canonical way, and vice versa. In particular, if Y_i is a kS_{γ_i}-module for each i, then the external tensor product $Y_1 \boxtimes Y_2 \boxtimes \cdots \boxtimes Y_t$, which is a $k\left(S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t}\right)$-module, may be regarded as a kS_γ-module.

Now if G is a subgroup of S_m and γ is a composition of n, then we have an obvious isomorphism between $G \wr S_\gamma$ and $(G \wr S_{\gamma_1}) \times (G \wr S_{\gamma_2}) \times \cdots \times (G \wr S_{\gamma_t})$, and hence we have a canonical identification of algebras between $k(G \wr S_\gamma)$ and $k\,(G \wr S_{\gamma_1}) \otimes k\,(G \wr S_{\gamma_2}) \otimes \cdots \otimes k\,(G \wr S_{\gamma_t})$. With this identification, it is now easy to see that we have an isomorphism of modules

$$(5)\quad \big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\gamma} \oslash \big(Y_1 \boxtimes Y_2 \boxtimes \cdots \boxtimes Y_t\big) \cong \big(X_1^{\widetilde{\boxtimes}\gamma_1} \oslash Y_1\big) \boxtimes \big(X_2^{\widetilde{\boxtimes}\gamma_2} \oslash Y_2\big) \boxtimes \cdots \boxtimes \big(X_t^{\widetilde{\boxtimes}\gamma_t} \oslash Y_t\big)$$

(this isomorphism was given in [2, Lemma 3.2 (1)]).

Proposition 2.3. *Let $G_1 \subseteq G_2$ be subgroups of S_m and X a kG_2-module. Then we have an isomorphism of $k(G_1 \wr S_n)$-modules*

$$\Big[X^{\widetilde{\boxtimes}n}\Big]\Big\downarrow^{G_2 \wr S_n}_{G_1 \wr S_n} \cong \Big[X\big\downarrow^{G_2}_{G_1}\Big]^{\widetilde{\boxtimes}n}.$$

Proof. This is immediate from the definition of $(-)^{\widetilde{\boxtimes}n}$. $\square$

Proposition 2.4. [2, Lemma 3.2] *Let G be a subgroup of S_m. Let $\alpha = (\alpha_1, \ldots, \alpha_t)$ be a composition of n and let V be a $k(G \wr S_n)$-module, W a $k(G \wr S_\alpha)$-module, X a kS_n-module, and Y a kS_α-module. Then we have module isomorphisms*

(1) $\big[V \oslash X\big]\big\downarrow^{G \wr n}_{G \wr \alpha} \cong \big(V\big\downarrow^{G \wr n}_{G \wr \alpha}\big) \oslash \big(X\big\downarrow^{n}_{\alpha}\big)$

(2) $V \oslash \big(Y\big\uparrow^{n}_{\alpha}\big) \cong \Big[\big(V\big\downarrow^{G \wr n}_{G \wr \alpha}\big) \oslash Y\Big]\Big\uparrow^{G \wr n}_{G \wr \alpha}$

(3) $\big(W\big\uparrow^{G \wr n}_{G \wr \alpha}\big) \oslash X \cong \big[W \oslash \big(X\big\downarrow^{n}_{\alpha}\big)\big]\big\uparrow^{G \wr n}_{G \wr \alpha}$

where the symbols n and α represent the subgroups S_n and S_α of S_n, respectively.

3. Wreath product Specht modules

We now define analogues for the wreath product $S_m \wr S_n$ of the Specht modules of the symmetric group using the above constructions. As mentioned in the introduction, although these constructions are well-known, the author is not aware that these modules have previously been considered as analogues of the symmetric group Specht modules.

Firstly we define some useful notation. If $Y_1, \ldots, Y_s$ are kS_m-modules and $\underline{\eta} = (\eta^1, \ldots, \eta^s)$ is an s-component multipartition of n, then we define the $k(S_m \wr S_n)$-module $S^{\underline{\eta}}(Y_1, \ldots, Y_s)$ by setting

$$S^{\underline{\eta}}(Y_1, \ldots, Y_s) \;=\; \Big[\big(Y_1, \ldots, Y_s\big)^{\widetilde{\boxtimes}|\underline{\eta}|} \oslash \big(S^{\eta^1} \boxtimes \cdots \boxtimes S^{\eta^s}\big)\Big]\Big\uparrow^{m \wr n}_{m \wr |\underline{\eta}|}.$$

We take r to be the number of distinct partitions of m, and we enumerate them in the lexicographic order as follows

$$(m) = \mu^1 > \mu^2 > \ldots > \mu^r = (1^m).$$

Then for $\underline{\nu} = (\nu^1, \ldots, \nu^r)$ an r-multipartition of n, we define a $k(S_m \wr S_n)$-module

$$S^{\underline{\nu}} = S^{\underline{\nu}}(S^{\mu^1}, \ldots, S^{\mu^r})$$

and we call $S^{\underline{\nu}}$ the the *Specht module* for $S_m \wr S_n$ associated to $\underline{\nu}$. For later convenience, we also define a $k(S_m \wr S_{|\underline{\nu}|})$-module

$$T^{\underline{\nu}} = \big(S^{\mu^1}, \ldots, S^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\nu}|} \oslash \big(S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r}\big)$$

so that $S^{\underline{\nu}} = T^{\underline{\nu}}\Big\uparrow_{m\wr|\underline{\nu}|}^{m\wr n}$. As mentioned above, the use of the name "Specht module" here is justified by the fact that these modules have the same relationship with the simple modules of $k(S_m \wr S_n)$ as the Specht modules of kS_n have with the simple of kS_n. This may be demonstrated by noting that the wreath product Specht modules occur as the cell modules of a cellular structure on $k(S_m \wr S_n)$, in the sense of Graham and Lehrer [4]. For details, see [6] and [5, Section 5.5].

We now consider how filtrations of the kS_m-modules $Y_1, \dots, Y_s$ induce filtrations of the $k(S_m \wr S_n)$-module $S^{\underline{\eta}}(Y_1, \dots, Y_s)$. This question was answered by Chuang and Tan in [2], and the results we now present are taken from there. However, we shall present these results in a very slightly modified form, using the notion of a *multipartition matrix*, which is simply a matrix whose entries are multipartitions. We shall typically denote the multipartition matrix whose $(i, j)^{\text{th}}$ entry is the multipartition $\underline{\epsilon}^{ij}$ as $[\underline{\epsilon}]$. Thus a multipartition matrix is simply a matrix whose entries are tuples of tuples of integers. Now let s and t be positive integers, let α, β be compositions of the same integer n and with lengths s and t respectively, and let L be an $s\times t$ matrix with non-negative integer entries. We define $\mathrm{Mat}_{\underline{\Lambda}}(L; \alpha\times\beta)$ to be the set of all $s\times t$ multipartition matrices $[\underline{\epsilon}]$ such that:

(1) for each $i = 1, \dots, s$, the sum of all of the integers occurring in the i^{th} row of $[\underline{\epsilon}]$ is equal to the i^{th} part of α;
(2) for each $j = 1, \dots, t$, the sum of all of the integers occurring in the j^{th} column of $[\underline{\epsilon}]$ is equal to the j^{th} part of β;
(3) the length of the $(i, j)^{\text{th}}$ entry of $[\underline{\epsilon}]$ is equal to the $(i, j)^{\text{th}}$ entry of L.

Note that we do allow entries of L to be zero, meaning that the corresponding entry of $[\underline{\epsilon}]$ is the empty multipartition (). Note also that we allow multipartitions to contain the empty partition () as an entry, which means, for example, that we can have a multipartition of length 1 but size 0 in $[\underline{\epsilon}]$, namely the multipartition (()) whose sole entry is an empty partition.

For $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(L; \alpha\times\beta)$, we define $R_i[\underline{\epsilon}]$ to be the multipartition obtained by concatenating the multipartitions on the i^{th} row of $[\underline{\epsilon}]$, and we similarly define $C_j[\underline{\epsilon}]$ to be the multipartition obtained by concatenating the multipartitions on the j^{th} column of $[\underline{\epsilon}]$ (ordered from top to bottom).

From [2] we have the following result. Now [2] formally works only with the semisimple case, but the arguments given there work in the general case, if one speaks of filtrations rather than direct sum decompositions and makes use of the modular Littlewood–Richardson result provided by [9]. Further, [2] formally makes the assumption that the modules $X_1, \dots, X_t$ are pairwise non-isomorphic, but this is not in fact needed for the proof if we speak in terms of filtrations *admitting* multiplicities (as explained in Section 2.1) to allow for non-uniqueness. For a very detailed proof of the result in essentially the form below (taking account of the aforementioned minor discrepancies between our situation and that of [2]), see Section 6.4 of the author's PhD thesis, [5].

Proposition 3.1. ([2, Lemma 4.4, (1)]; *see also* [5, Proposition 6.4.1]) *Let $Y_1, \dots, Y_s$ and $X_1, \dots, X_t$ be kS_m-modules such that for each $i = 1, \dots, s$, Y_i has a filtration by the modules $X_1, \dots, X_t$ admitting the multiplicities $(a^i_j)_j$ (where we allow $a^i_j = 0$). Let $\underline{\eta}$ be an s-component multipartition of n. Then $S^{\underline{\eta}}(Y_1, \dots, Y_s)$ has a filtration by the modules $S^{\underline{\nu}}(X_1, \dots, X_t)$ for $\underline{\nu}$ a t-multipartition of n, admitting the multiplicities*

$\left(B_{\underline{\nu}}\right)_{\underline{\nu}}$, *with*

$$B_{\underline{\nu}} = \sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A;|\underline{\eta}| \times |\underline{\nu}|)} \left(\prod_{i=1}^{s} c(\eta^i; R_i[\underline{\epsilon}])\right)\left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])\right)$$

where we define A to be the $s \times t$ integer matrix whose $(i,j)^{th}$ entry is a^i_j (since $a^i_j = 0$ is allowed, we note that the empty multipartition $()$ may occur as an entry of $[\underline{\epsilon}]$). Further, suppose that we have $s = t$ and moreover for $i = 1, \ldots, t$, X_i occurs at the bottom of the filtration of Y_i. Then the module occurring at the bottom of this filtration is $S^{\underline{\eta}}(X_1, \ldots, X_t)$.

We remark that Proposition 3.1 is rather more general that is strictly required for proving the branching rules below, but we feel that it is worth stating the result in this more general form, as it is of interest in its own right.

4. First Specht branching rule for wreath products

For $m > 0$, we can embed $S_{m-1} \wr S_n$ into $S_m \wr S_n$ using the canonical embedding of S_{m-1} into S_m, thus identifying $S_{m-1} \wr S_n$ with the subgroup of $S_m \wr S_n$ consisting of all elements $(\sigma; \alpha_1, \ldots, \alpha_n)$ where $\sigma \in S_n$ and each α_i is an element of the subgroup S_{m-1} of S_m. Hence for $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ an r-multipartition of n, we can consider the $k(S_{m-1} \wr S_n)$-module

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n} \cong T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{(m-1)\wr n}$$

obtained by restricting $S^{\underline{\lambda}}$ from $k(S_m \wr S_n)$ to $k(S_{m-1} \wr S_n)$. By Mackey's Theorem, we have

$$T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{(m-1)\wr n} \cong \bigoplus_{u \in \mathcal{U}} \left(T^{\underline{\lambda}}\right)^u \big\downarrow^{(m\wr|\underline{\lambda}|)^u}_{(m\wr|\underline{\lambda}|)^u \cap (m-1)\wr n}\big\uparrow^{(m-1)\wr n}_{(m\wr|\underline{\lambda}|)^u \cap (m-1)\wr n}$$

where $\mathcal{U}$ represents a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_{(m-1)} \wr S_n)$-double coset representatives in $S_m \wr S_n$, and where we allow ourselves a slight abuse of notation by writing $(m \wr |\underline{\lambda}|)^u$ to represent the subgroup $(S_m \wr S_{|\underline{\lambda}|})^u$ conjugate to $S_m \wr S_{|\underline{\lambda}|}$ by u, and $(m \wr |\underline{\lambda}|)^u \cap (m-1) \wr n$ for the intersection of this subgroup with $S_{(m-1)} \wr S_n$. But it turns out that in fact the group $S_m \wr S_n$ is a single $(S_m \wr S_{|\underline{\lambda}|}, S_{(m-1)} \wr S_n)$-double coset. Indeed, choosing $(\sigma; \alpha_1, \ldots, \alpha_n) \in S_m \wr S_n$, we have equalities of double cosets

$$\begin{aligned} &S_m \wr S_{|\underline{\lambda}|}(\sigma; \alpha_1, \ldots, \alpha_n) S_{(m-1)} \wr S_n \\ &\qquad = S_m \wr S_{|\underline{\lambda}|}(e; \alpha_{(1)\sigma}, \ldots, \alpha_{(n)\sigma})(e; e, \ldots, e)(\sigma; e, \ldots, e) S_{(m-1)} \wr S_n \\ &\qquad = S_m \wr S_{|\underline{\lambda}|}(e; e, \ldots, e) S_{(m-1)} \wr S_n \end{aligned}$$

and so we may take $\mathcal{U} = \{(e; e, \ldots, e)\}$. We thus have

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n} \cong T^{\underline{\lambda}}\big\downarrow^{m\wr|\underline{\lambda}|}_{m\wr|\underline{\lambda}| \cap (m-1)\wr n}\big\uparrow^{(m-1)\wr n}_{m\wr|\underline{\lambda}| \cap (m-1)\wr n}$$

and clearly $\left(S_m \wr S_{|\underline{\lambda}|}\right) \cap \left(S_{(m-1)} \wr S_n\right) = S_{(m-1)} \wr S_{|\underline{\lambda}|}$ (note that formally these are subgroups of $S_m \wr S_n$, so that $S_{(m-1)} \wr S_{|\underline{\lambda}|}$ is the subgroup of $S_m \wr S_n$ consisting of all elements $(\sigma; \alpha_1, \ldots, \alpha_n)$ for $\sigma \in S_{|\underline{\lambda}|}$ and $\alpha_i \in S_{(m-1)} \leqslant S_m$). Thus we have

$$\begin{aligned}
S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1)\wr n} &\cong T^{\underline{\lambda}}\big\downarrow^{m\wr|\underline{\lambda}|}_{(m-1)\wr|\underline{\lambda}|}\big\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\cong \left[\mathop{\boxtimes}_{i=1}^{r} \left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right]\Bigg\downarrow^{m\wr|\underline{\lambda}|}_{(m-1)\wr|\underline{\lambda}|}\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\cong \left[\mathop{\boxtimes}_{i=1}^{r} \left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|}\Bigg\downarrow^{m\wr|\lambda^i|}_{(m-1)\wr|\lambda^i|}\right] \oslash S^{\lambda^i}\right]\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\qquad \text{(it is easy to prove this directly)} \\
&\cong \left[\mathop{\boxtimes}_{i=1}^{r} \left(S^{\mu^i}\Big\downarrow^{m}_{m-1}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right]\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\qquad \text{(by Proposition 2.3)} \\
&\cong S^{\underline{\lambda}}\left(S^{\mu^1}\big\downarrow^{m}_{m-1}, \ldots, S^{\mu^r}\big\downarrow^{m}_{m-1}\right) \\
&\qquad \text{(using the isomorphism (5))}.
\end{aligned}$$

Now let us fix the partitions of $m-1$ just as we have done for m. Indeed, let t be the number of distinct partitions of $m-1$, and let

$$(m-1) = \theta^1 > \theta^2 > \ldots > \theta^t = (1^{m-1})$$

be the partitions of $m-1$ in lexicographic order. Then by Theorem 2.2, we have for any $i \in \{1, \ldots, r\}$ a filtration of $S^{\mu^i}\big\downarrow^{m}_{m-1}$ by the modules S^{θ^j} admitting the multiplicities $(a^i_j)_j$, where we define a^i_j to be 1 if θ^j can be obtained by removing a box from μ^i, and zero otherwise. It now follows by Proposition 3.1 that we have a filtration of $S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n}$ by the modules $S^{\underline{\nu}}$ for $\underline{\nu}$ a t-multipartition of n, admitting the multiplicities $(D_{\underline{\nu}})_{\underline{\nu}}$ with

$$D_{\underline{\nu}} = \sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)} \left(\prod_{i=1}^{r} c(\lambda^i; R_i[\underline{\epsilon}]) \cdot \prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])\right)$$

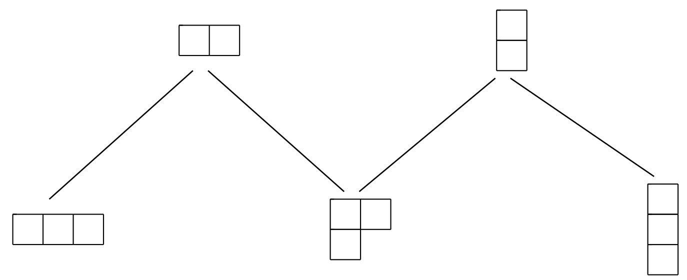
where A is the $r \times t$ integer matrix whose $(i,j)^{\text{th}}$ entry is a^i_j. This filtration is the basis of our desired Specht branching rule, but we would like some kind of combinatorial interpretation of the multiplicities which occur. Our task is now to find such an interpretation.

So with $\underline{\lambda}$ as above and $\underline{\nu}$ a t-multipartition of n, consider, for a given multipartition matrix $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ the coefficient

$$\prod_{i=1}^{r} c(\lambda^i; R_i[\underline{\epsilon}]) \cdot \prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]). \tag{6}$$

Now the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$ is a multipartition of length 1, say (ϵ^{ij}), if θ^j can be obtained by removing a box from μ^i, and $()$ otherwise. This gives us an alternative way to think of such multipartition matrices and calculate the associated coefficient (6), as we shall now explain.

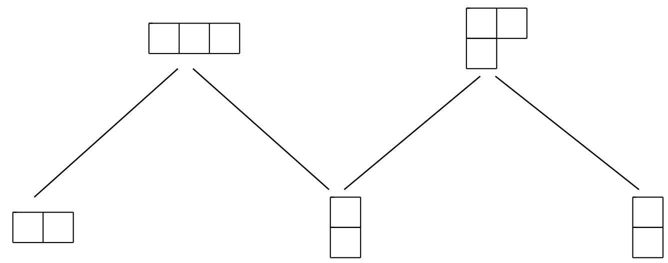
Recall that we can arrange the set of all partitions of all non-negative integers in a graphical structure called the *Young graph*, by arranging the partitions in layers, with the partitions of size s forming the s^{th} layer, and then for each partition $\lambda \vdash s$ in the s^{th} layer, drawing an edge from λ to each partition of $s-1$ in the $(s-1)^{\text{th}}$ layer which can be obtained from λ by removing a single box. For example, the second and third rows of the Young graph, together with the edges connecting them, look like this



(7)

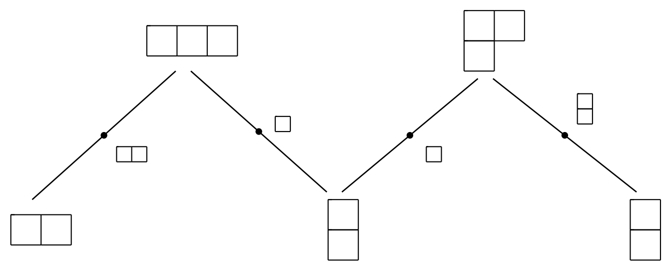
For our purposes, we are interested in the subgraph of the Young graph consisting of the m^{th} and $(m-1)^{\text{th}}$ layers together with the edges connecting them. Let us call this subgraph $\mathcal{Y}_m$. So for example if $m=3$, $\mathcal{Y}_3$ is the graph (7). We see that there is a natural one-to-one correspondence between the 1's in the matrix A and the edges in $\mathcal{Y}_m$. Indeed, a 1 in the $(i,j)^{\text{th}}$ place of A corresponds to an edge linking $\theta^j \vdash m-1$ and $\mu^i \vdash m$ in $\mathcal{Y}_m$. We now see that a multipartition matrix $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ may be identified with a labelling of the edges in $\mathcal{Y}_m$ by partitions. Indeed, to obtain such a labelling from such a matrix $[\underline{\epsilon}]$, we label the edge linking θ^j and μ^i in $\mathcal{Y}_m$, if it exists, with the partition ϵ^{ij} which is the unique entry of the length 1 multipartition which is the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$. We may easily see that we have now established a one-to-one correspondence between on the one hand the set $\mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ and on the other hand labellings of the edges of $\mathcal{Y}_m$ by integer partitions, such that for each $i=1,\dots,r$ the sizes of the partitions labelling the edges touching the node $\mu^i \vdash m$ of $\mathcal{Y}_m$ add up to $|\lambda^i|$, and similarly for each $j=1,\dots,t$ the sizes of the partitions labelling the edges touching the node $\theta^j \vdash m-1$ of $\mathcal{Y}_m$ add up to $|\nu^i|$. We shall henceforth call such a labelling of $\mathcal{Y}_m$ a *labelling of shape* $|\underline{\lambda}| \times |\underline{\nu}|$. The diagram (9) below is an example of such a labelling.

We now explain how to calculate the coefficient (6) associated to a labelling of $\mathcal{Y}_m$ of shape $|\underline{\lambda}| \times |\underline{\nu}|$. In order to do this, we need to introduce a graph which is a modified version of $\mathcal{Y}_m$. Indeed, recall that we have multipartitions $\underline{\lambda} = (\lambda^1,\dots,\lambda^r)$ and $\underline{\nu} = (\nu^1,\dots,\nu^t)$ of n. We define $\mathcal{Y}_m(\underline{\lambda},\underline{\nu})$ to be the graph obtained by replacing each partition $\mu^i \vdash m$ with λ^i, and each partition $\theta^j \vdash m-1$ with ν^j. Thus for example if $m=3$ (so that $r=3$ and $t=2$) and $n=6$, and we take $\underline{\lambda} = \big((2),(1,1),(1,1)\big)$ and $\underline{\nu} = \big((3),(2,1)\big)$, then $\mathcal{Y}_3(\underline{\lambda},\underline{\nu})$ is the graph



(8)

We now see that a labelling of $\mathcal{Y}_m$ of shape $|\underline{\lambda}| \times |\underline{\nu}|$ corresponds to a labelling of the edges $\mathcal{Y}_m(\underline{\lambda},\underline{\nu})$ by partitions in such a way that, for each partition γ lying at a node of $\mathcal{Y}_m(\underline{\lambda},\underline{\nu})$, the sizes of the partitions labelling all the edges touching γ add up to $|\gamma|$. We call such a labelling of $\mathcal{Y}_m(\underline{\lambda},\underline{\nu})$ a *good labelling* of $\mathcal{Y}_m(\underline{\lambda},\underline{\nu})$. To continue our example, one good labelling of the graph $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$ depicted in (8) is

(9) 

Looking back through our arguments, we see that this labelling corresponds to the multipartition matrix

$$\begin{array}{c} \\ (2) \\ (1,1) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cc} \quad(3)\quad & \quad(2,1)\quad \end{array} \\ \begin{pmatrix} \big((2)\big) & () \\ \big((1)\big) & \big((1)\big) \\ () & \big((1,1)\big) \end{pmatrix} \end{array}$$

(where we have labelled the rows and columns with the entries of $\underline{\lambda}$ and $\underline{\nu}$ respectively) and further we see that the coefficient (6) associated to this multipartition matrix is

$$c\big((2); \big((2)\big)\big) \cdot c\big((1,1); \big((1),(1)\big)\big) \cdot c\big((1,1); \big((1,1)\big)\big) \cdot \\ c\big((3); \big((2),(1)\big)\big) \cdot c\big((2,1); \big((1),(1,1)\big)\big).$$

By using our definition of the Littlewood–Richardson coefficient $c(\lambda; \underline{\alpha})$ and the Littlewood–Richardson rule, we may see that each of these Littlewood–Richardson coefficients is 1, and hence the coefficient associated to the graph (9) is 1.

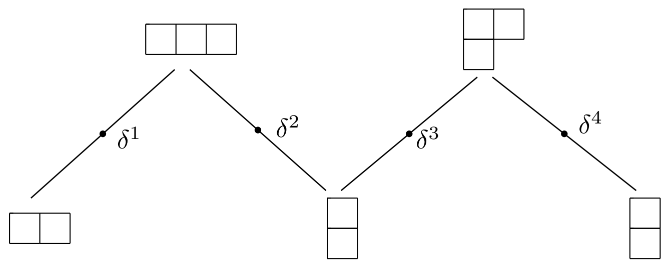
In the general case, we see that the coefficient associated to a good labelling of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ is formed by taking the product, over all partitions γ which are nodes of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ (that is, over all partitions of m and of $m-1$), of the Littlewood–Richardson coefficients $c\big(\gamma; (\delta^1, \ldots, \delta^s)\big)$, where $\delta^1, \ldots, \delta^s$ are the partitions labelling all of the edges which touch γ in $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$. If $\mathcal{L}$ is a good labelling of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$, we denote this coefficient by $\mathcal{M}\big(\mathcal{L}\big)$.

We have now proved the following Specht branching rule, and we note that the multiplicities in this theorem are independent of the field k.

Theorem 4.1. *Let $m > 0$, and as above let r be the number of distinct partitions of m and t the number of distinct partitions of $m-1$. Let $\underline{\lambda}$ be an r-multipartition of n. Then we have a filtration of the $k(S_{m-1} \wr S_n)$-module $S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1)\wr n}$ by the Specht modules $S^{\underline{\nu}}$ for t-multipartitions $\underline{\nu}$ of n, admitting the multiplicities $(D_{\underline{\nu}})_{\underline{\nu}}$, where $D_{\underline{\nu}}$ is the sum over all good labellings $\mathcal{L}$ of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ of the coefficients $\mathcal{M}\big(\mathcal{L}\big)$.*

Let us now extend our example to calculate the multiplicity which $S^{((3),(2,1))}$ has in our filtration of $S^{((2),(1,1),(1,1))}\big\downarrow^{3\wr 6}_{2 \wr 6}$. We have already calculated that the coefficient $\mathcal{M}\big(\mathcal{L}\big)$ is equal to 1 when $\mathcal{L}$ is the labelling (9). We shall show that if $\underline{\lambda} = \big((2),(1,1),(1,1)\big)$ and $\underline{\nu} = \big((3),(2,1)\big)$, then for any good labelling $\mathcal{L}$ of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$ other than (9), we have $\mathcal{M}\big(\mathcal{L}\big) = 0$. Thus the multiplicity which we seek is in fact 1.

Indeed, suppose that we have some good labelling $\mathcal{L}$ of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$. Then $\mathcal{L}$ is equal to



for some integer partitions $\delta^1, \delta^2, \delta^3, \delta^4$. Now by the definition of a good labelling of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$, we see that we must have $|\delta^1| = 2, |\delta^2| = 1, |\delta^3| = 1, |\delta^4| = 2$, so that $\delta^2 = \delta^3 = (1)$. We now see that

$$\mathcal{M}(\mathcal{L}) = c\big((2); \big(\delta^1\big)\big) \cdot c\big((1,1); \big((1),(1)\big)\big) \cdot c\big((1,1); \big(\delta^4\big)\big) \cdot \\ c\big((3); \big(\delta^1, (1)\big)\big) \cdot c\big((2,1); \big((1), \delta^4\big)\big).$$

By our definition of the Littlewood–Richardson coefficient $c(\lambda; \underline{\alpha})$, the only case where this is nonzero is the case where $\delta^1 = (2)$ and $\delta^4 = (1,1)$, as in (9).

5. Tableau combinatorics

We now examine some tableau combinatorics which we shall use to help us understand the double cosets of certain pairs of subgroups in S_n. The material in this section is taken from the account given by Wildon in his unpublished note [12].

Throughout this section we fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ to be a composition of n of length l, and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$ to be a composition of n of length t. For our fixed composition α of n, we let S_n act (from the right) on both the set of tableaux of shape α and type γ, by permuting the entries of a tableau as follows. For a tableau τ, number the boxes of the tableau from 1 to n going from left to right across each row in turn, starting with the top row and working down. Let $\sigma \in S_n$. Then $\tau\sigma$ is defined to be the tableau obtained from τ by moving the number in box number i to box number $(i)\sigma$, for each $i = 1, \ldots, n$. For example, let us take $n = 13$, $\alpha = (5,3,4,1)$, $\gamma = (4,5,4)$, $\sigma = (1,12,3,6)(5,7,13)(8,10) \in S_{13}$, and

$$\tau \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 1 & 3 & 2 \\ \hline 2 & 3 & 2 \\ \cline{1-3} 2 & 3 & 1 & 3 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array} \quad .$$

The reader may verify that we have

$$\tau\sigma \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 3 & 1 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 2 & 2 & 1 & 1 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array} \quad .$$

It is easy to see that this definition does indeed yield a S_n action as claimed, and it is obvious that this S_n action is transitive. It is natural to ask what the stabilizer of a given tableau is under this action, and in order to answer this we now consider certain special tableaux of shape α and type γ. Indeed, for our compositions α and γ, we construct the *standard tableau of shape α and type γ* as follows: we begin with a Young diagram of shape α with the boxes numbered as described above, and then working from box 1 to box n we enter first γ_1 1's, then γ_2 2's, and so on. We denote this tableau by τ_γ^α. For example, if we take $n = 13$, $\alpha = (2, 0, 3, 1, 3, 4)$ and $\gamma = (3, 5, 0, 4, 1)$, then we have

$$\tau_\gamma^\alpha \quad = \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 \\ \cline{1-2} \\ \hline 1 & 2 & 2 \\ \cline{1-3} 2 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 4 & 4 & 4 & 5 \\ \hline \end{array} \quad .$$

Proposition 5.1. *For any $\sigma \in S_n$, we have $\mathrm{Stab}\big(\tau_\gamma^\alpha \sigma\big) = \big(S_\gamma\big)^\sigma$, where we write $\mathrm{Stab}(-)$ to denote a stabilizer.*

Proof. Elementary. □

We say that a tableau of shape α and type γ has *weakly increasing rows* if the entries in its rows are weakly increasing from left to right. Define $\mathcal{W}_\gamma^\alpha$ to be the set of all tableaux of shape α and type γ with weakly increasing rows.

Recall that the *length* of a permutation is defined to be the total number of inversions of the permutation, where an *inversion* of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \leqslant i < j \leqslant n$ and $(i)\sigma > (j)\sigma$. Let us take Ω_γ^α to be a complete system of (S_γ, S_α)-double coset representatives in S_n, where each element σ of Ω_γ^α is of minimal length in its left coset σS_α.

Proposition 5.2. ([12]*, Theorem 4.1 and Corollary 5.1) If $\sigma \in S_n$ is of minimal length in its left S_α-coset σS_α, then $\tau_\gamma^\alpha \sigma$ has weakly increasing rows. Thus we have a map $\Omega_\gamma^\alpha \longrightarrow \mathcal{W}_\gamma^\alpha$, $\sigma \longmapsto \tau_\gamma^\alpha \sigma$, and this map is in fact a bijection.*

Corollary 5.3. *Suppose that we have $\sigma_1, \ldots, \sigma_N \in S_n$ such that if $i \neq j$ then $\tau_\gamma^\alpha \sigma_i \neq \tau_\gamma^\alpha \sigma_j$ and further $\{\tau_\gamma^\alpha \sigma_i \mid 1 \leqslant i \leqslant N\} = \mathcal{W}_\gamma^\alpha$. Then $\sigma_1, \ldots, \sigma_N$ is a complete system of (S_γ, S_α)-double coset representatives in S_n without redundancy.*

Proof. With our system of (S_γ, S_α)-double coset representatives Ω_γ^α as above, we may by Proposition 5.2 list the distinct elements of Ω_γ^α as $\omega_1, \ldots, \omega_N$ such that $\tau_\gamma^\alpha \sigma_i = \tau_\gamma^\alpha \omega_i$. This implies that $\tau_\gamma^\alpha = \tau_\gamma^\alpha \omega_i \sigma_i^{-1}$, and hence that $\omega_i \sigma_i^{-1} \in \mathrm{Stab}(\tau_\gamma^\alpha)$, so that by Proposition 5.1 we have $\omega_i \sigma_i^{-1} \in S_\gamma$. Hence $S_\gamma \sigma_i S_\alpha = S_\gamma (\omega_i \sigma_i^{-1}) \sigma_i S_\alpha = S_\gamma \omega_i S_\alpha$, and so $\sigma_1, \ldots, \sigma_N$ is a complete system of (S_γ, S_α)-double coset representatives in S_n without redundancy. □

6. Second Specht branching rule for wreath products

For $n > 0$, we can embed $S_m \wr S_{n-1}$ into $S_m \wr S_n$ by mapping $(\sigma; \alpha_1, \ldots, \alpha_{n-1})$, where $\sigma \in S_{n-1}$, $\alpha_i \in S_m$, to $(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)$, making use of the canonical embedding of S_{n-1} into S_n. Hence for $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ an r-multipartition of n, we can consider the $k(S_m \wr S_{n-1})$-module

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{m\wr(n-1)} \cong T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{m\wr(n-1)}$$

obtained by restricting $S^{\underline{\lambda}}$ from $k(S_m \wr S_n)$ to $k(S_m \wr S_{n-1})$. By Mackey's Theorem we have

$$T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{m\wr(n-1)} \cong \bigoplus_{u \in \mathcal{U}} \big(T^{\underline{\lambda}}\big)^u \big\downarrow^{(m\wr|\underline{\lambda}|)^u}_{(m\wr|\underline{\lambda}|)^u \cap m\wr(n-1)} \big\uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^u \cap m\wr(n-1)} \tag{10}$$

with minor notational abuses as in the argument for the first branching rule, and where $\mathcal{U}$ represents a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$. We thus want to find such a set of double coset representatives. For $\sigma \in S_n$, let us write $\hat{\sigma}$ for the element $(\sigma; e, \dots, e)$ of $S_m \wr S_n$. Let $\sigma_1, \dots, \sigma_N$ be a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n. We claim that $\hat{\sigma}_1, \dots, \hat{\sigma}_N$ is then a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$. Indeed, if $(\theta; \alpha_1, \dots, \alpha_n) \in S_m \wr S_n$, then we have $\theta = \epsilon\sigma_i\delta$ for some $i \in \{1, \dots, N\}$, $\epsilon \in S_{|\underline{\lambda}|}$ and $\delta \in S_{n-1}$, and it follows that

$$(\theta; \alpha_1, \dots, \alpha_n) = \underbrace{(\epsilon; \alpha_{(1)\sigma_i}, \dots, \alpha_{(n)\sigma_i})}_{\in S_m \wr S_{|\underline{\lambda}|}} \underbrace{(\sigma_i; e, \dots, e)}_{= \hat{\sigma}_i} \underbrace{(\delta; e, \dots, e)}_{\in S_m \wr S_{n-1}}$$

which establishes completeness. For non-redundancy, suppose that we have some i, j such that

$$\big(S_m \wr S_{|\underline{\lambda}|}\big)\hat{\sigma}_i\big(S_m \wr S_{n-1}\big) = \big(S_m \wr S_{|\underline{\lambda}|}\big)\hat{\sigma}_j\big(S_m \wr S_{n-1}\big).$$

Hence $\hat{\sigma}_i \in \big(S_m \wr S_{|\underline{\lambda}|}\big)\hat{\sigma}_j\big(S_m \wr S_{n-1}\big)$, so that we have $\epsilon \in S_{|\underline{\lambda}|}$, $\delta \in S_{n-1}$ and elements α_i, β_i of S_m such that

$$(\sigma_i; e, \dots, e) = (\epsilon; \alpha_1, \dots, \alpha_n)(\sigma_j; e, \dots, e)(\delta; \beta_1, \dots, \beta_{n-1}, e)$$

from which it follows that $\sigma_i = \epsilon\sigma_j\delta$ and hence that $i = j$. Thus we now seek such $\sigma_1, \dots, \sigma_N$, and to do this we shall make use of our work on tableaux.

Now recall that if α, γ are compositions of n, then we have defined the tableau τ^α_γ to be the tableau of shape α whose entries, read from left to right across each row in turn starting with the top row, consist of γ_1 1's, then γ_2 2's, then γ_3 3's, and so on. So for example if $n = 9$, $\alpha = (8, 1)$ and $\gamma = (3, 1, 0, 2, 3)$, then

$$\tau^\alpha_\gamma = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 4 & 5 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

Further, we know by Corollary 5.3 that if we have $\sigma_1, \dots, \sigma_N \in S_n$ such that $\tau^\alpha_\gamma\sigma_1, \dots, \tau^\alpha_\gamma\sigma_N$ is a complete list, with no repetition, of the tableaux of shape α and type γ with weakly increasing rows, then $\sigma_1, \dots, \sigma_N$ is in fact a complete system of (S_γ, S_α)-double coset representatives without redundancy. We now apply this in the case where $\alpha = (n-1, 1)$ and $\gamma = |\underline{\lambda}|$ to obtain our desired system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n, noting that the subgroup S_{n-1} of S_n is exactly the Young subgroup $S_{(n-1,1)}$. The following example should serve to illustrate the general argument which we shall give below.

Keep $n = 9$, and suppose that $|\underline{\lambda}| = (3, 1, 0, 2, 3)$ as above. Then the possible tableaux of shape $(n-1, 1)$ and type $|\underline{\lambda}|$ with weakly increasing rows are

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 4 & 5 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 5 & 5 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 4 & 4 & 5 & 5 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 5 & 5 & 5 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Thus, a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{(n-1,1)})$-double coset representatives is $e, (6, 9, 8, 7), (4, 9, 8, 7, 6, 5), (3, 9, 8, 7, 6, 5, 4)$, recalling that in our action of S_n on tableaux, $\sigma \in S_n$ acts by moving the contents of the i^{th} box to the $(i)\sigma^{\text{th}}$ box, where the boxes of a tableau are numbered with the numbers $1, \ldots, n$ from left to right across each row, working from the top row to the bottom row.

The general case works in exactly the same way as the example. Indeed, recall that $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$. For $i = 1, \ldots, r$ we let $b_i = |\lambda^1| + \cdots + |\lambda^i|$, so that we have a sequence $0 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_r = n$. Then for each $i = 1, \ldots, r$ such that $b_i \neq 0$ we define an element ρ_i of S_n by letting

$$\rho_i = \begin{cases} (b_i, n, n-1, \ldots, b_i + 1) & \text{if } b_i < n \\ e \quad \text{if } b_i = n \end{cases}$$

(where e is the identity element). By letting i run through all $1, \ldots, r$ such that $|\lambda^i| > 0$, we obtain a complete list of all the distinct ρ_i without repetition. As in the above example, we see that the set of all tableaux $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ for i such that $|\lambda^i| > 0$ forms a complete list of all of the tableaux of shape $(n-1, 1)$ and type $|\underline{\lambda}|$ with weakly increasing rows. Hence by Corollary 5.3 we see that the collection of all ρ_i for i such that $|\lambda^i| > 0$ forms a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n, and hence the collection of all $\hat{\rho}_i$ for i such that $|\lambda^i| > 0$ forms a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$.

Looking back to (10), we see that we want to understand the module

$$\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \Big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)} \Big\uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)}$$

for i such that $|\lambda^i| > 0$. Our first step in doing so will be to understand the subgroup $\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i} \cap \left(S_m \wr S_{n-1}\right)$ of $S_m \wr S_n$ and its action on the module $\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}$.

So choose i such that $|\lambda^i| > 0$. It is easy to show directly that $\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i}$ is equal to $S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i}$. Thus we have

$$\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i} \cap \left(S_m \wr S_{n-1}\right) = S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap \left(S_m \wr S_{n-1}\right)$$

and it is easy to show directly that $S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap \left(S_m \wr S_{n-1}\right)$ is equal to the subgroup of $S_m \wr S_n$ consisting of all elements of the form

$$(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e) \tag{11}$$

where σ is an element of the subgroup $\left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap S_{n-1}$ of S_n and $\alpha_i \in S_m$. We thus wish to understand the subgroup $\left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap S_{n-1}$ of S_n. By Proposition 5.1, $\left(S_{|\underline{\lambda}|}\right)^{\rho_i}$ is the stabilizer (under the action of S_n) of the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$. It is easy to see that the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ is the unique tableau of shape $(n-1, 1)$ and type $|\underline{\lambda}|$ with weakly increasing rows which has an i in the box on the second row; such tableaux are illustrated in the above example. For any subset Ω of $\{1, \ldots, n\}$, let us write $S(\Omega)$ to denote the subgroup of S_n consisting of all permutations which fix any number not lying in Ω. We easily see that the stabilizer of the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ is the subgroup $X_{|\underline{\lambda}|}^i$ of S_n, where we define (recalling that $|\lambda^i| > 0$ and hence $b_i > b_{i-1}$, where b_0 is taken to be 0)

$$\begin{aligned} X_{|\underline{\lambda}|}^i = {} & S\big(\{1, \ldots, b_1\}\big) \times S\big(\{b_1 + 1, \ldots, b_2\}\big) \times \cdots \\ & \times S\big(\{b_{i-1} + 1, \ldots, b_i - 1, n\}\big) \times S\big(\{b_i, \ldots, b_{i+1} - 1\}\big) \times S\big(\{b_{i+1}, \ldots, b_{i+2} - 1\}\big) \times \\ & \qquad \cdots \times S\big(\{b_{r-1}, \ldots, b_r - 1 = n - 1\}\big) \end{aligned}$$

(note that here we are using the $\times$ symbol to denote an *internal* direct product of subgroups, and that if $b_i = b_{i+1}$ then $\{b_i, \ldots, b_{i+1} - 1\}$ represents the empty set, and that if $b_i = b_{i-1}+1$ then $\{b_{i-1}+1,\dots,b_i-1,n\} = \{n\}$), and hence $\big(S_{|\underline{\lambda}|}\big)^{\rho_i} = X^i_{|\underline{\lambda}|}$. We now introduce a small piece of notation. Indeed, if $\gamma = (\gamma_1,\dots,\gamma_r)$ is a composition of n, and $i \in \{1,\dots,r\}$ such that $\gamma_i > 0$, then we write $[\gamma]_i$ for the composition $(\gamma_1,\dots,\gamma_{i-1},\gamma_i - 1,\gamma_{i+1},\gamma_r)$ of $n-1$. We see that $X^i_{|\underline{\lambda}|} \cap S_{n-1}$ is the subgroup

$$\begin{gathered}S\big(\{1,\dots,b_1\}\big) \times S\big(\{b_1+1,\dots,b_2\}\big) \times \cdots \\ \times S\big(\{b_{i-1}+1,\dots,b_i-1\}\big) \times S\big(\{b_i,\dots,b_{i+1}-1\}\big) \times S\big(\{b_{i+1},\dots,b_{i+2}-1\}\big)\times \\ \cdots \times S\big(\{b_{r-1},\dots,b_r-1=n-1\}\big)\end{gathered}$$

of S_n, and under our embedding of S_{n-1} into S_n this is exactly the subgroup $S_{[|\underline{\lambda}|]_i}$ of S_{n-1}. Hence, recalling that we are viewing $S_m \wr S_{n-1}$ as a subgroup of $S_m \wr S_n$ via the embedding $(\sigma;\alpha_1,\dots,\alpha_{n-1}) \longmapsto (\sigma;\alpha_1,\dots,\alpha_{n-1},e)$, we see that the subgroup $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$ of $S_m \wr S_n$ is equal to the subgroup $S_m \wr S_{[|\underline{\lambda}|]_i}$ of the subgroup $S_m \wr S_{n-1}$ of $S_m \wr S_n$.

We now turn our attention to the action of $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$ on the $k\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i}$-module $\big(T^{\underline{\lambda}}\big)^{\hat\rho_i}$. We know by the definition of conjugate modules that $\big(T^{\underline{\lambda}}\big)^{\hat\rho_i}$ is the module formed by equipping $T^{\underline{\lambda}}$ with the $k\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i}$-action $*$ given for $x \in T^{\underline{\lambda}}$ and $y \in \big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i}$ by $x * y = x(\hat\rho_i y \hat\rho_i^{-1})$ (where the action on the right-hand side is the action of $S_m \wr S_{|\underline{\lambda}|}$ on $T^{\underline{\lambda}}$, noting that $\hat\rho_i y \hat\rho_i^{-1}$ does indeed lie in $S_m \wr S_{|\underline{\lambda}|}$). Thus to calculate the action of an element

$$(\sigma;\alpha_1,\dots,\alpha_{n-1},e) \in \big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$$

on the module $\big(T^{\underline{\lambda}}\big)^{\hat\rho_i}$, we need to calculate $\hat\rho_i(\sigma;\alpha_1,\dots,\alpha_{n-1},e)\hat\rho_i^{-1}$. We have

$$\begin{aligned}\hat\rho_i(\sigma;\alpha_1,\dots,\alpha_{n-1},e)\hat\rho_i^{-1} &= (\rho_i;e,\dots,e)(\sigma;\alpha_1,\dots,\alpha_{n-1},e)(\rho_i^{-1};e,\dots,e) \\ &= (\rho_i\sigma\rho_i^{-1};\alpha_{(1)\rho_i},\dots,\alpha_{(n)\rho_i}) \qquad (\text{taking } \alpha_n = e) \\ &= (\rho_i\sigma\rho_i^{-1};\alpha_1,\alpha_2,\dots,\alpha_{b_i-1},e,\alpha_{b_i},\alpha_{b_i+1}, \\ &\qquad\qquad \dots,\alpha_{n-2},\alpha_{n-1}).\end{aligned}$$

But by our description (11) of the elements of $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$, we see that $\sigma \in \big(S_{|\underline{\lambda}|}\big)^{\rho_i} \cap S_{n-1}$, which implies that $\rho_i\sigma\rho_i^{-1} \in S_{|\underline{\lambda}|} \cap \big(S_{n-1}\big)^{\rho_i^{-1}}$. By direct calculation, any element of $\big(S_{n-1}\big)^{\rho_i^{-1}}$ fixes b_i, and hence we see that $\rho_i\sigma\rho_i^{-1}$ is an element of $S_{|\underline{\lambda}|}$ which fixes b_i. Now we know that the subgroup $S_{|\underline{\lambda}|}$ of S_n has an internal direct product factorisation

$$\begin{gathered}S\big(\{1,\dots,b_1\}\big) \times S\big(\{b_1+1,\dots,b_2\}\big) \times \cdots \\ \cdots \times S\big(\{b_{i-1}+1,\dots,b_i\}\big) \times S\big(\{b_i+1,\dots,b_{i+1}\}\big) \times \cdots \\ \cdots \times S\big(\{b_{r-1}+1,\dots,b_r=n\}\big).\end{gathered}$$

Thus any element π of $S_{|\underline{\lambda}|}$ has a unique factorisation $\pi = \theta_1\cdots\theta_r$ where $\theta_j \in S\big(\{b_{j-1}+1,\dots,b_j\}\big)$ (with b_0 taken to be 0). We thus see that $\rho_i\sigma\rho_i^{-1}$ has such a factorisation $\rho_i\sigma\rho_i^{-1} = \theta_1\cdots\theta_r$, where θ_i fixes b_i. Thus we see that our element $(\sigma;\alpha_1,\dots,\alpha_{n-1},e)$ of $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$ acts on the module $\big(T^{\underline{\lambda}}\big)^{\hat\rho_i}$ as the element

$$\big(\theta_1\cdots\theta_r;\alpha_1,\alpha_2,\dots,\alpha_{b_i-1},e,\alpha_{b_i},\alpha_{b_i+1},\dots,\alpha_{n-2},\alpha_{n-1}\big)$$

of $S_m \wr S_{|\underline{\lambda}|}$ acts on $T^{\underline{\lambda}}$ (recalling that $\big(T^{\underline{\lambda}}\big)^{\hat\rho_i}$ and $T^{\underline{\lambda}}$ are equal as k-vector spaces). But we know that $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat\rho_i} \cap \big(S_m \wr S_{n-1}\big)$ is equal to the subgroup $S_m \wr S_{[|\underline{\lambda}|]_i}$ of the subgroup $S_m \wr S_{n-1}$ of $S_m \wr S_n$, and we now see that if we identify $S_m \wr S_{[|\underline{\lambda}|]_i}$ with

$$(S_m \wr S_{|\lambda^1|}) \times (S_m \wr S_{|\lambda^2|}) \times \cdots$$
$$\cdots \times (S_m \wr S_{|\lambda^{i-1}|}) \times (S_m \wr S_{|\lambda^i|-1}) \times (S_m \wr S_{|\lambda^{i+1}|}) \times \cdots \times (S_m \wr S_{|\lambda^r|})$$

in the canonical way, then by the definition of the $k(S_m \wr S_{|\underline{\lambda}|})$-module $T^{\underline{\lambda}}$, the $k(S_m \wr S_{[|\underline{\lambda}|]_i})$-module

$$\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)} = \left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i}$$

is isomorphic to

$$\left(\left(S^{\mu^1}\right)^{\widetilde{\boxtimes}|\lambda^1|} \oslash S^{\lambda^1}\right) \boxtimes \cdots \boxtimes \left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right) \Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)} \boxtimes \cdots$$
$$\cdots \boxtimes \left(\left(S^{\mu^r}\right)^{\widetilde{\boxtimes}|\lambda^r|} \oslash S^{\lambda^r}\right). \tag{12}$$

Thus, we want to investigate the $k(S_m \wr S_{|\lambda^i|-1})$-module

$$\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right) \Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)}.$$

Now the restriction operation $\Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)}$ may be expressed as

$$\Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} \Big\downarrow^{m\wr(|\lambda^i|-1,1)}_{m\wr(|\lambda^i|-1)},$$

where, we recall, $m \wr (|\lambda^i|-1,1)$ represents the subgroup $S_m \wr S_{(|\lambda^i|-1,1)}$ of $S_m \wr S_{|\lambda^i|}$ consisting of all elements of the form $(\sigma;\alpha_1,\ldots,\alpha_n)$ for $\alpha_i \in S_m$ and $\sigma \in S_{(|\lambda^i|-1,1)}$, while $m \wr (|\lambda^i|-1)$ represents the subgroup $S_m \wr S_{(|\lambda^i|-1)}$ of $S_m \wr S_{|\lambda^i|}$ consisting of all elements of the form $(\sigma;\alpha_1,\ldots,\alpha_{n-1},e)$ for $\alpha_i \in S_m$ and $\sigma \in S_{(|\lambda^i|-1,1)}$. Now we have by Proposition 2.4 that

$$\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right) \Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} = \left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} \oslash S^{\lambda^i} \Big\downarrow^{|\lambda^i|}_{(|\lambda^i|-1,1)}.$$

Upon further restriction to $S_m \wr S_{(|\lambda^i|-1)}$, we see that this is isomorphic to the direct sum of $\dim_k(S^{\mu^i})$ copies of the module

$$\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|-1} \oslash S^{\lambda^i} \Big\downarrow^{|\lambda^i|}_{|\lambda^i|-1}. \tag{13}$$

It now follows by Theorem 2.2 and the fact that $- \oslash -$ preserves submodule series (see above) that, if we let $\delta^t \rhd \delta^{t-1} \rhd \cdots \rhd \delta^1$ be all the (distinct) partitions of $|\lambda^i|-1$ which can be obtained by removing a single box from λ^i, then (13) has a submodule series whose quotients are, from top to bottom,

$$\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|-1} \oslash S^{\delta^t}, \ldots, \left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|-1} \oslash S^{\delta^1}.$$

Using (12), it now follows that the $k(S_m \wr S_{[|\underline{\lambda}|]_i})$-module

$$\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)} = \left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i}$$

is the direct sum of $\dim_k(S^{\mu^i})$ copies of a module V_i which has a submodule series whose quotients are, from top to bottom $T^{\underline{\delta}^t}, \ldots, T^{\underline{\delta}^1}$, where $\underline{\delta}^l$ is the multipartition of $n-1$ obtained by replacing λ^i with δ^l in $\underline{\lambda}$. By exactness of the functor $\uparrow^{m\wr(n-1)}_{m\wr[|\underline{\lambda}|]_i}$, it now follows that the $k(S_m \wr S_{n-1})$-module

$$(T^{\underline{\lambda}})^{\hat{\rho}_i}\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}\cap m\wr(n-1)}\uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}\cap m\wr(n-1)} = (T^{\underline{\lambda}})^{\hat{\rho}_i}\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i}\uparrow^{m\wr(n-1)}_{m\wr[|\underline{\lambda}|]_i}$$

is the direct sum of $\dim_k(S^{\mu^i})$ copies of the module $X_i = V_i\uparrow^{m\wr(n-1)}_{m\wr[|\underline{\lambda}|]_i}$, and that X_i has a submodule series whose factors are, from top to bottom, $S^{\underline{\delta}^t}, \dots, S^{\underline{\delta}^1}$. Referring back to the decomposition (10), we now see that we have proved the following result, which is our desired Specht branching rule.

Theorem 6.1. *Let $n > 0$, and let $\underline{\lambda}$ be an r-multipartition of n. Then we have a direct sum decomposition*

$$S^{\underline{\lambda}}\downarrow^{m\wr n}_{m\wr(n-1)} \cong \bigoplus_i \dim_k(S^{\mu^i})X_i$$

where the summation runs over all $i \in \{1, \dots, r\}$ such that $|\lambda^i| > 0$, and where the $k(S_m \wr S_{n-1})$-module X_i has a submodule series as follows. Let $\delta^{i,t} \rhd \delta^{i,t-1} \rhd \cdots \rhd \delta^{i,1}$ be all the (distinct) partitions which can be obtained by removing a single box from λ^i, and let $\underline{\delta}^{i,l}$ be the multipartition of $n-1$ obtained by replacing λ^i with $\delta^{i,l}$ in $\underline{\lambda}$. Then X_i has a series of submodules

$$X_i = M^i_t \supseteq M^i_{t-1} \supseteq M^i_{t-2} \supseteq \cdots \supseteq M^i_1 \supseteq M^i_0 = 0$$

such that the quotient of M^i_l by M^i_{l-1} is isomorphic to $S^{\underline{\delta}^{i,l}}$.

We note the similarity of this result to Theorem 2.2. In particular, we note that the multiplicities $\dim_k(S^{\mu^i})$ occurring in this decomposition have a simple and elegant combinatorial interpretation via the *hook length formula* (see for example [7, Chapter 20]), from which we see that they are in fact independent of the field k (as is also the case for the multiplicities in Theorem 2.2 and Theorem 4.1).

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