ALGEBRAIC COMBINATORICS

Norihide Tokushige

Application of hypergraph Hoffman's bound to intersecting families

Volume 5, issue 3 (2022), p. 537-557.

https://doi.org/10.5802/alco.222

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing

www.tccpublishing.org www.centre-mersenne.org

Algebraic Combinatorics
Volume 5, issue 3 (2022), p. 537–557
https://doi.org/10.5802/alco.222

Application of hypergraph Hoffman's bound to intersecting families

Norihide Tokushige

Abstract Using the Filmus–Golubev–Lifshitz method [7] to bound the independence number of a hypergraph, we solve some problems concerning multiply intersecting families with biased measures. Among other results we obtain a stability result of a measure version of the Erdős–Ko–Rado theorem for multiply intersecting families.

1. Introduction

It is an important problem to estimate the size of a maximum independent set in a graph, and Hoffman's bound[(1)] is one the most useful algebraic tools for the problem. Recently, Filmus, Golubev, and Lifshitz [7] extended the bound to hypergraphs. In this paper we apply these bounds to some problems concerning multiply intersecting families with biased measures.

We start with the easiest and the most basic result about intersecting families with a biased measure. Let V be a finite set and let $\mathcal{A} \subset 2^V$ be a family of subsets of V. For a fixed real number p with $0 < p < 1$ we define the p-biased measure of the family $\mathcal{A}$ by

$$\mu_p(\mathcal{A}) = \sum_{A \in \mathcal{A}} p^{|A|}(1-p)^{|V|-|A|}.$$

By definition it follows that $\mu_p(2^V) = 1$. We say that $\mathcal{A}$ is *intersecting* if $A \cap A' \neq \varnothing$ for all $A, A' \in \mathcal{A}$. A typical intersecting family is

$$\mathcal{S} = \{A \in 2^V : v \in A\}$$

for some fixed $v \in V$. This family is called a *star* centered at v. The star can be rewritten as $\{\{v\} \cup B : B \in 2^W\}$ where $W = V \setminus \{v\}$, and it follows that

$$\mu_p(\mathcal{S}) = \sum_{A \in \mathcal{S}} p \cdot p^{|A|-1}(1-p)^{|V|-|A|} = p \sum_{B \in 2^W} p^{|B|}(1-p)^{|W|-|B|} = p.$$

Indeed, it is not difficult to show that if $p \leqslant \frac{1}{2}$ and $\mathcal{A} \subset 2^V$ is intersecting, then $\mu_p(\mathcal{A}) \leqslant p$, see e.g. [2, 8], or [10, Chapter 12]. We will extend this result in several ways.

Manuscript received 5th January 2021, revised 18th October 2021, accepted 3rd January 2022.

Keywords. Hoffman's bound, intersecting family, independence number, boolean function.

Acknowledgements. This research was supported by JSPS KAKENHI Grant No. 18K03399.

(1)See [13] for the history of the bound, and some related results including Delsarte's LP bound and Lovász's theta bound.

To state our problems and results we need some more notation and definitions. Let n, r be positive integers and let $[n] \coloneqq \{1, 2, \dots, n\}$. We say that a family of subsets $\mathcal{A} \subset 2^{[n]}$ is *r-wise intersecting* if $A_1 \cap A_2 \cap \cdots \cap A_r \neq \varnothing$ for all $A_1, A_2, \dots, A_r \in \mathcal{A}$. Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ be real numbers, and let $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. We define the $\boldsymbol{p}$-biased measure (or $\mu_{\boldsymbol{p}}$-measure) $\mu_{\boldsymbol{p}} : 2^{[n]} \to (0, 1)$ by

$$\mu_{\boldsymbol{p}}(A) \coloneqq \prod_{i \in A} p_i \prod_{j \in [n] \setminus A} (1 - p_j)$$

for $A \in 2^{[n]}$, and for a family $\mathcal{A} \subset 2^{[n]}$ we define

$$\mu_{\boldsymbol{p}}(\mathcal{A}) \coloneqq \sum_{A \in \mathcal{A}} \mu_{\boldsymbol{p}}(A).$$

The star centered at $i \in [n]$ is an r-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p_i$.

Fishburn et al. [8] studied the maximal $\mu_{\boldsymbol{p}}$-measure for 2-wise intersecting families using combinatorial tools. Then Suda et al. [16] extended their result to cross-intersecting families (see Theorem 4.10 in the last section) by solving a semidefinite programming problem, and posed the following conjecture.

Conjecture 1.1 ([16]). *Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. Let $p_3 \leqslant \frac{1}{2}$. If $\mathcal{A} \subset 2^{[n]}$ is a 2-wise intersecting family, then $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p_1$. Moreover, if $p_1 > p_3$ (except for the case $p_1 = p_2 > p_3 = \frac{1}{2}$), or $p_1 < \frac{1}{2}$, then equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_1 = p_i$.*

This conjecture is true if the condition $p_3 \leqslant \frac{1}{2}$ is replaced with $p_2 \leqslant \frac{1}{2}$, which is proved in [8] and [16]. In this paper we apply Hoffman's bound to show the following result which supports the conjecture.

Theorem 1.2. *Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. Let $p_3 < \frac{1}{2}$. Suppose that $p_1 \leqslant \frac{1}{2}$ or $1 - p_2 > p_3$. If $\mathcal{A} \subset 2^{[n]}$ is a 2-wise intersecting family, then $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p_1$. Moreover equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_1 = p_i$.*

Frankl and the author [9] studied r-wise intersecting families with a $\mu_{\boldsymbol{p}}$-measure where $\boldsymbol{p} = (p, p, \dots, p)$, and proved that if $p < \frac{r-1}{r}$ then $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p$ for every r-wise intersecting family $\mathcal{A} \subset 2^{[n]}$. The proof was combinatorial. Filmus et al. [7] gave a new proof by extending Hoffman's bound to r-uniform hypergraphs (r-graphs). In this paper we further extend their method to obtain the following result.

Theorem 1.3. *Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. If $\frac{2}{3} > p_2$ and $\mathcal{A} \subset 2^{[n]}$ is a 3-wise intersecting family, then $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p_1$. Moreover equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_1 = p_i$.*

Friedgut [11] studied the case $r = 2$ and $\boldsymbol{p} = (p, p, \dots, p)$, and found a stability result. We combine his method with Filmus–Golubev–Lifshitz (FGL) bound to get a stability result for the case $r = 3$.

Theorem 1.4. *Let $0 < p < \frac{2}{3}$ be fixed, and let $\boldsymbol{p} = (p, p, \dots, p)$. Then there exists a positive constant ϵ_p such that the following holds for all $0 < \epsilon < \epsilon_p$. If $\mathcal{A} \subset 2^{[n]}$ is a 3-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p - \epsilon$, then there exists a star $\mathcal{B} \subset 2^{[n]}$ such that*

(i) *if $p \leqslant \frac{1}{2}$ then $\mathcal{A} \subset \mathcal{B}$, and*
(ii) *if $p > \frac{1}{2}$ then $\mu_{\boldsymbol{p}}(\mathcal{A} \triangle \mathcal{B}) < (C_p + o(1))\epsilon$, where $C_p = \frac{16p(1-p)^2}{(2p-1)(3-4p)}$, and the $o(1)$ term vanishes as $\epsilon \to 0$.*

Finally we mention that there are different and more combinatorial approaches to the related problems concerning weighted intersecting families, see, e.g. [3] and [4].

In Section 2 we prepare tools for the proofs, and then we prove Theorems 1.2–1.4 in Section 3. In Section 4 we discuss an easy generalization and some related problems.

2. Preliminaries

In this section we collect some tools used to prove our results. In the first two subsections we reproduce the proof of Hoffman's bound for a hypergraph established by Filmus, Golubev, and Lifshitz, for in the next section we will use not only the bound itself but also some equalities and inequalities appearing in their proof. Our formulation and definitions follow those in [6], which are slightly different from [7] (but of course essentially the same). In the last subsection we present some basic facts about spectral information related to families with the $\mu_{\boldsymbol{p}}$-measure.

Remark for notation: In this paper we often identify a set and its indicator. Let V be a finite set and let $\{0,1\}^V$ denote the set of boolean functions from V to $\{0,1\}$. For $I \in 2^V$ we write $\mathbf{1}_I \in \{0,1\}^V$ to denote the indicator of I, that is,

$$\mathbf{1}_I(v) = \begin{cases} 1 & \text{if } v \in I, \\ 0 & \text{if } v \notin I, \end{cases}$$

where $v \in V$. We identify I and $\mathbf{1}_I$, and we write $v \in \mathbf{1}_I$ to mean $v \in I$. For simplicity we just write $\mathbf{1}$ to mean $\mathbf{1}_V$, so $\mathbf{1}(v) = 1$ for all $v \in V$. For $x, y \in V$ and a matrix T, where the rows and columns are indexed by V, we write $(T)_{x,y}$ for the (x,y)-entry of T. Throughout the paper let $q \coloneqq 1-p$ and $q_i \coloneqq 1-p_i$.

2.1. Hoffman's bound for a graph. Let V be a finite set with $|V| \geqslant 2$. We say that $\mu_2 : V \times V \to \mathbb{R}$ is a symmetric signed measure if

- $\mu_2(x,y) = \mu_2(y,x)$ for all $x, y \in V$, and
- $\sum_{x\in V}\sum_{y\in V}\mu_2(x,y) = 1$.

Note that $\mu_2(x,y)$ can be negative, which is essential for the proof of Theorem 1.2. Let $\mu_1 : V \to \mathbb{R}$ be the marginal of μ_2, that is,

$$\mu_1(x) \coloneqq \sum_{y\in V}\mu_2(x,y). \tag{1}$$

Then $\sum_{x\in V}\mu_1(x) = \sum_{x\in V}\sum_{y\in V}\mu_2(x,y) = 1$.

Definition 2.1. *Let $\mu_2 : V \times V \to \mathbb{R}$ be a symmetric signed measure. We say that $G = (V, \mu_2)$ a weighted graph if*

$$\mu_1(x) > 0 \text{ for all } x \in V. \tag{2}$$

In this paper we only deal with μ_2 whose marginal μ_1 satisfies (2). Here V is the vertex set, and each $(x,y) \in V \times V$ is an ordered edge (or a directed edge) including loops with possibly negative weight $\mu_2(x,y)$. So (x,y) should be considered a non-edge if and only if $\mu_2(x,y) = 0$. We say that $I \subset V$ is an *independent set* if $x, y \in I$ implies $\mu_2(x,y) = 0$. Write $\mu_1(I)$ for $\sum_{x\in I}\mu_1(x)$, and define the independence ratio $\alpha(G)$ by

$$\alpha(G) \coloneqq \max\{\mu_1(I) : I \text{ is an independent set in } G\}.$$

Example 2.2. Let $G = (V, E)$ be a usual simple d-regular graph. Let us construct a symmetric measure μ_2 so that μ_1 becomes a uniform measure $\mu_1(x) \equiv 1/|V|$. To this end we just set

$$\mu_2(x,y) \coloneqq \begin{cases} 0 & \text{if } \{x,y\} \notin E, \\ \frac{1}{d|V|} & \text{if } \{x,y\} \in E. \end{cases}$$

Then (V,μ_2) is a weighted graph, and in this case $\alpha(G)=|I|/|V|$, where $I\subset V$ is a usual maximum independent set in G.

Let $\mathbb{R}^V$ be the set of functions from V to $\mathbb{R}$, and for $f,g\in\mathbb{R}^V$ let

$$\mathbb{E}_{\mu_1}[f]\coloneqq\sum_{x\in V}f(x)\mu_1(x),$$
$$\mathbb{E}_{\mu_2}[f,g]\coloneqq\sum_{x\in V}\sum_{y\in V}f(x)g(y)\mu_2(x,y).$$

Since μ_2 is symmetric it follows $\mathbb{E}_{\mu_2}[f,g]=\mathbb{E}_{\mu_2}[g,f]$.

Fact 2.3. *Let $\varphi\coloneqq\mathbf{1}_I\in\{0,1\}^V$ be the indicator of an independent set I. Then we have $\mathbb{E}_{\mu_1}[\varphi]=\mu_1(I)$ and $\mathbb{E}_{\mu_2}[\varphi,\varphi]=0$.*

Proof. Indeed we have

$$\mathbb{E}_{\mu_1}[\varphi]=\sum_{x\in V}\varphi(x)\mu_1(x)=\sum_{x\in I}\mu_1(x)=\mu_1(I).$$

Since $\mu_2(x,y)=0$ for $x,y\in I$ we also have

$$\mathbb{E}_{\mu_2}[\varphi,\varphi]=\sum_{x\in V}\sum_{y\in V}\varphi(x)\varphi(y)\mu_2(x,y)=\sum_{x\in I}\sum_{y\in I}\mu_2(x,y)=0. \qquad \square$$

We define a measure version of the adjacency matrix, which is an extension of the usual adjacency matrix, and we simply call it an adjacency matrix in this paper.

Definition 2.4. *Let $G=(V,\mu_2)$ be a weighted graph. We define the adjacency matrix $T=T(G)$. This is a $|V|\times|V|$ matrix, and for $x,y\in V$ the (x,y)-entry of T is given by*

$$(T)_{x,y}=\frac{\mu_2(x,y)}{\mu_1(x)}. \tag{3}$$

We introduce an inner product $\langle\cdot,\cdot\rangle_{\mu_1}:\mathbb{R}^V\times\mathbb{R}^V\to\mathbb{R}$ by

$$\langle f,g\rangle_{\mu_1}\coloneqq\mathbb{E}_{\mu_1}[fg]=\sum_{x\in V}f(x)g(x)\mu_1(x). \tag{4}$$

Note that condition (2) is necessary to define the above inner product properly. Clearly $\langle f,g\rangle_{\mu_1}=\langle g,f\rangle_{\mu_1}$. We list some easy facts. (We include the proof in Appendix.)

Fact 2.5. *Let $G=(V,\mu_2)$ be a weighted graph with adjacency matrix T. Let $f,g\in\mathbb{R}^V$ and $\varphi\in\{0,1\}^V$.*

(i) $\langle f,Tg\rangle_{\mu_1}=\mathbb{E}_{\mu_2}[f,g]$.
(ii) $\langle f,Tg\rangle_{\mu_1}=\langle Tf,g\rangle_{\mu_1}$*, that is, T is self-adjoint.*
(iii) $T\mathbf{1}=\mathbf{1}$.
(iv) $\langle\mathbf{1},\mathbf{1}\rangle_{\mu_1}=1$.
(v) $\langle\varphi,\mathbf{1}\rangle_{\mu_1}=\mathbb{E}_{\mu_1}[\varphi]$.
(vi) $\langle\varphi,\varphi\rangle_{\mu_1}=\mathbb{E}_{\mu_1}[\varphi]$.

Setup 2.6. *Let $G=(V,\mu_2)$ be a weighted graph with adjacency matrix T. By Fact 2.5(iii) the matrix T has eigenvector $\mathbf{1}$ with eigenvalue 1. Since T is self-adjoint, T has $|V|$ real eigenvalues $l_0=1,l_1,l_2,\dots,l_{|V|-1}$ with corresponding eigenvectors $\boldsymbol{v}_0=\mathbf{1},\boldsymbol{v}_1,\boldsymbol{v}_2\dots,\boldsymbol{v}_{|V|-1}$, that is, $T\boldsymbol{v}_i=l_i\boldsymbol{v}_i$. We may assume that these vectors* *consist of an orthonormal basis of* $\mathbb{R}^V$ *with respect to the inner product* $\langle\cdot,\cdot\rangle_{\mu_1}$*. Then, for* $\varphi\in\{0,1\}^V$*, we can expand* φ *using the basis:*

$$\varphi=\widehat{\varphi}_0\mathbf{1}+\sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i, \tag{5}$$

where $\widehat{\varphi}_i=\langle\varphi,\boldsymbol{v}_i\rangle_{\mu_1}$*. Let* $\lambda_{\min}(T)$ *denote the minimum eigenvalue of* T*.*

Fact 2.7. *Let* $\varphi\in\{0,1\}^V$*.*

(i) $\mathbb{E}_{\mu_1}[\varphi]=\widehat{\varphi}_0$ *and* $\mathbb{E}_{\mu_1}[\varphi]=\widehat{\varphi}_0^2+\sum_{i\geqslant 1}\widehat{\varphi}_i^2$*.*
(ii) $\mathbb{E}_{\mu_2}[\varphi,\varphi]=\widehat{\varphi}_0^2+\sum_{i\geqslant 1}\widehat{\varphi}_i^2 l_i$*.*

Lemma 2.8. *For* $\varphi\in\{0,1\}^V$ *we have* $\mathbb{E}_{\mu_2}[\varphi,\varphi]\geqslant\mathbb{E}_{\mu_1}[\varphi]\big(1-(1-\lambda_{\min}(T))(1-\mathbb{E}_{\mu_1}[\varphi])\big)$*.*

Proof. By Fact 2.7(ii)

$$\begin{aligned}\mathbb{E}_{\mu_2}[\varphi,\varphi]&=\widehat{\varphi}_0^2+\sum_{i\geqslant 1}\widehat{\varphi}_i^2 l_i\\&\geqslant\widehat{\varphi}_0^2+\lambda_{\min}(T)\sum_{i\geqslant 1}\widehat{\varphi}_i^2\\&=\mathbb{E}_{\mu_1}[\varphi]^2+\lambda_{\min}(T)(\mathbb{E}_{\mu_1}[\varphi]-\mathbb{E}_{\mu_1}[\varphi]^2)\quad\text{(by Fact 2.7(i))}\\&=\mathbb{E}_{\mu_1}[\varphi]\big(1-(1-\lambda_{\min}(T))(1-\mathbb{E}_{\mu_1}[\varphi])\big).\end{aligned}$$

□

Theorem 2.9 (Hoffman's bound, see [13]). *Let* $G=(V,\mu_2)$ *be a weighted graph with adjacency matrix* T*. Let* φ *be the indicator of an independent set of* G*. Suppose that* $\lambda_{\min}(T)<1$*. Then we have*

$$1-\mathbb{E}_{\mu_1}[\varphi]\geqslant\frac{1}{1-\lambda_{\min}(T)},$$

and

$$\alpha(G)\leqslant\frac{-\lambda_{\min}(T)}{1-\lambda_{\min}(T)}.$$

Proof. By Lemma 2.8 with $\mathbb{E}_{\mu_2}[\varphi,\varphi]=0$ it follows

$$0\geqslant\mathbb{E}_{\mu_1}[\varphi]\big(1-(1-\lambda_{\min}(T))(1-\mathbb{E}_{\mu_1}[\varphi])\big)=\mathbb{E}_{\mu_1}[\varphi]\big(\lambda_{\min}(T)+(1-\lambda_{\min}(T))\mathbb{E}_{\mu_1}[\varphi]\big).$$

Since $\mathbb{E}_{\mu_1}(\varphi)>0$ and $\lambda_{\min}(T)<1$ we get the desired inequality. □

2.2. Hoffman's bound for a 3-graph.

Let V be a finite set with $|V|\geqslant 2$, and let $\mu_3:V^3\to\mathbb{R}$ be a symmetric signed measure, that is,

- $\mu_3(x,y,z)=\mu_3(p,q,r)$ whenever (p,q,r) is a permutation of (x,y,z), and
- $\sum_{x\in V}\sum_{y\in V}\sum_{z\in V}\mu_3(x,y,z)=1$.

Define the marginals $\mu_2\in\mathbb{R}^{V^2}$ and $\mu_1\in\mathbb{R}^V$ as follows:

$$\begin{aligned}\mu_2(x,y)&\coloneqq\sum_{z\in V}\mu_3(x,y,z).\\\mu_1(x)&\coloneqq\sum_{y\in V}\mu_2(x,y)=\sum_{y\in V}\sum_{z\in V}\mu_3(x,y,z).\end{aligned}$$

Definition 2.10. *Let* $\mu_3:V^3\to\mathbb{R}$ *be a symmetric signed measure. We say that* $H=(V,\mu_3)$ *is a weighted* 3*-graph if* $\mu_2(x,y)>0$ *for all* $x,y\in V$*.*

Note that $\mu_1(x)>0$ for all x follows from (1). We will consider two inner products, one is with respect to $\mu_1(x)$, and the other is with respect to $\mu_2(x,y)/\mu_1(x)$ for fixed x. We need the conditions in the above definition to ensure that these inner products are defined properly.

We say that a subset $I \subset V$ is an independent set in H if $\mu_3(x,y,z) = 0$ for all $x,y,z \in I$, and we define the independence ratio $\alpha(H)$ by

$$\alpha(H) \coloneqq \max\{\mu_1(I) : I \text{ is an independent set in } H\}.$$

Suppose that $H = (V, \mu_3)$ is a weighted 3-graph. Then $G = (V, \mu_2)$ is a weighted 2-graph because μ_2 is a symmetric measure and μ_1 satisfies (2). The adjacency matrix $T = T(G)$ is defined by (3). Now we define a link graph relative to x which will be denoted by H_x. To this end, for $x, y, z \in V$, we define $\mu_{2,x} \in \mathbb{R}^{V^2}$ and its marginal $\mu_{1,x} \in \mathbb{R}^V$ by

$$\begin{aligned}\mu_{2,x}(y,z) &\coloneqq \frac{\mu_3(x,y,z)}{\mu_1(x)},\\ \mu_{1,x}(y) &\coloneqq \sum_{z\in V} \mu_{2,x}(y,z) = \frac{\mu_2(x,y)}{\mu_1(x)}.\end{aligned}$$

Then $H_x \coloneqq (V, \mu_{2,x})$ is a weighted 2-graph because $\mu_{2,x}$ is a symmetric measure and $\mu_{1,x}$ satisfies (2). The adjacency matrix $T_x = T_x(H_x)$ is also defined by (3), so the (y,z)-entry of T_x is

$$(T_x)_{y,z} = \frac{\mu_{2,x}(y,z)}{\mu_{1,x}(y)} = \frac{\mu_3(x,y,z)}{\mu_2(x,y)}. \tag{6}$$

By definition both T and T_x are self-adjoint, and they have $|V|$ real eigenvalues.

We can relate $\mathbb{E}_{\mu_2}$ and $\mathbb{E}_{\mu_{1,x}}$ as follows. Here we write $x \in \varphi$ to mean $\varphi(x) = 1$.

Lemma 2.11. *For $\varphi \in \{0,1\}^V$ we have $\mathbb{E}_{\mu_2}[\varphi,\varphi] \leqslant \mathbb{E}_{\mu_1}[\varphi] \max_{x\in\varphi} \mathbb{E}_{\mu_{1,x}}[\varphi]$.*

Proof. Note that if $x \notin \varphi$ then $\varphi(x) = 0$ and the term having $\varphi(x)$ does not contribute in the sum below. Thus we have

$$\begin{aligned}\mathbb{E}_{\mu_2}[\varphi,\varphi] &= \sum_{x\in V}\sum_{y\in V} \varphi(x)\varphi(y)\mu_2(x,y)\\ &= \sum_{x\in\varphi} \varphi(x)\mu_1(x) \sum_{y\in V} \varphi(y)\frac{\mu_2(x,y)}{\mu_1(x)}\\ &= \sum_{x\in\varphi} \varphi(x)\mu_1(x) \sum_{y\in V} \varphi(y)\mu_{1,x}(y)\\ &\leqslant \sum_{x\in V} \varphi(x)\mu_1(x) \max_{x\in\varphi} \sum_{y\in V} \varphi(y)\mu_{1,x}(y)\\ &= \mathbb{E}_{\mu_1}[\varphi] \max_{x\in\varphi} \mathbb{E}_{\mu_{1,x}}[\varphi].\end{aligned}$$

□

Theorem 2.12 (Hoffman's bound for a 3-graph [7]). *Let $H = (V, \mu_3)$ be a weighted 3-graph. Let φ be the indicator of an independent set. Suppose that $\lambda_{\min}(T) < 1$ and $\lambda_{\min}(T_x) < 1$ for $x \in \varphi$. Then*

$$1 - \mathbb{E}_{\mu_1}[\varphi] \geqslant \frac{1}{(1-\lambda_{\min}(T)) \max_{x\in\varphi}(1-\lambda_{\min}(T_x))}.$$

In particular, if φ is the indicator of a maximum independent set, then

$$\alpha(H) \leqslant 1 - \frac{1}{(1-\lambda_{\min}(T)) \max_{x\in\varphi}(1-\lambda_{\min}(T_x))}.$$

Proof. Let $I \subset V$ be the independent set in H such that $\mathbf{1}_I = \varphi$. By Lemma 2.8 we have

$$\mathbb{E}_{\mu_2}[\varphi,\varphi] \geqslant \mathbb{E}_{\mu_1}[\varphi]\big(1 - (1-\lambda_{\min}(T))(1-\mathbb{E}_{\mu_1}[\varphi])\big).$$

This together with Lemma 2.11 yields

$$\mathbb{E}_{\mu_1}[\varphi] \max_{x\in\varphi} \mathbb{E}_{\mu_{1,x}}[\varphi] \geqslant \mathbb{E}_{\mu_1}[\varphi]\big(1-(1-\lambda_{\min}(T))(1-\mathbb{E}_{\mu_1}[\varphi])\big),$$

that is,

$$1-\mathbb{E}_{\mu_1}[\varphi] \geqslant \frac{1-\max_{x\in\varphi}\mathbb{E}_{\mu_{1,x}}[\varphi]}{1-\lambda_{\min}(T)}. \tag{7}$$

Next we bound $\mathbb{E}_{\mu_{1,x}}[\varphi]$ by using the link graph $H_x := (V, \mu_{2,x})$ relative to $x \in I$. Note that I is an independent set in H_x as well. Indeed if $y, z \in I$ then $\mu_{2,x}(y,z) = 0$ because $\mu_3(x,y,z) = 0$. By applying Theorem 2.9 to the adjacency matrix T_x of H_x we get

$$1-\mathbb{E}_{\mu_{1,x}}[\varphi] \geqslant \frac{1}{1-\lambda_{\min}(T_x)},$$

and

$$1-\max_{x\in\varphi}\mathbb{E}_{\mu_{1,x}}[\varphi] \geqslant \frac{1}{\max_{x\in\varphi}(1-\lambda_{\min}(T_x))}. \tag{8}$$

By (7) and (8) we obtain the desired inequality. □

2.3. Tools for uniqueness. In Setup 2.6 any $\varphi \in \{0,1\}^V$ can be expanded in the form in (5). We first show that if $\mathbb{E}_{\mu_2}[\varphi,\varphi]$ is small, then we only need the eigenvectors corresponding to the largest and the smallest eigenvalues for the expansion.

Lemma 2.13. *We assume Setup* 2.6*. If*

$$\mathbb{E}_{\mu_2}[\varphi,\varphi] \leqslant \widehat{\varphi}_0^2 + \lambda_{\min}(T)(\widehat{\varphi}_0 - \widehat{\varphi}_0^2), \tag{9}$$

then $\varphi = \widehat{\varphi}_0\mathbf{1} + \sum_{i\in J}\widehat{\varphi}_i\boldsymbol{v}_i$*, where* $J = \{i : 1 \leqslant i < |V|,\, l_i = \lambda_{\min}(T)\}$*.*

Proof. By Fact 2.7 and (9) we have

$$\mathbb{E}_{\mu_2}[\varphi,\varphi] = \widehat{\varphi}_0^2 + \sum_{i\geqslant 1}\widehat{\varphi}_i^2 l_i \leqslant \widehat{\varphi}_0^2 + \lambda_{\min}(T)(\widehat{\varphi}_0 - \widehat{\varphi}_0^2) = \widehat{\varphi}_0^2 + \lambda_{\min}(T)\sum_{i\geqslant 1}\widehat{\varphi}_i^2,$$

and

$$\sum_{i\geqslant 1}(l_i - \lambda_{\min}(T))\widehat{\varphi}_i^2 \leqslant 0.$$

This yields $l_i = \lambda_{\min}(T)$ or $\widehat{\varphi}_i = 0$, and the result follows. □

Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ be given. Let $V = 2^{[n]}$ and let $\mu_1 : V \to (0,1)$ be the measure defined by $\mu_1(S) = \prod_{i\in S} p_i \prod_{j\in[n]\setminus S} q_j$ for $S \in V$. Then we can view $\mathbb{R}^V$ as a 2^n-dimensional inner space over $\mathbb{R}$, where the inner product is defined by (4). We will construct an orthonormal basis that suits our purpose.

Fact 2.14. *Let* $p_i > \frac{r-2}{r-1}$ *and let*

$$T^{(i)} = \begin{bmatrix} 1-\frac{p_i}{(r-1)q_i} & \frac{p_i}{(r-1)q_i} \\ \frac{1}{r-1} & 1-\frac{1}{r-1} \end{bmatrix}.$$

Then $T^{(i)}$ *has eigenvalues 1 and* $\lambda_i := 1 - \frac{1}{(r-1)q_i} < 0$ *with the corresponding eigenvectors* $\boldsymbol{v}_{\varnothing}^{(i)} = \binom{1}{1}$ *and* $\boldsymbol{v}_{\{i\}}^{(i)} = \begin{pmatrix} c_i \\ -\frac{1}{c_i} \end{pmatrix}$*, where* $c_i = \sqrt{p_i/q_i}$*.*

Let $T = T^{(n)} \otimes T^{(n-1)} \otimes \cdots \otimes T^{(1)}$. Then the rows and columns of T are indexed by the order $\varnothing, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \ldots$. For each $S \in V$ the corresponding indicator is given by the column vector of the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

One can construct the eigenvectors of T by routine computation, and we have the following, see e.g. [11, 16],

Fact 2.15. *Let $S \in V$ and let $\boldsymbol{v}_S$ be the column vector (indexed by S) of the matrix*

$$C_n := \begin{bmatrix} 1 & c_n \\ 1 & -\frac{1}{c_n} \end{bmatrix} \otimes \begin{bmatrix} 1 & c_{n-1} \\ 1 & -\frac{1}{c_{n-1}} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & c_1 \\ 1 & -\frac{1}{c_1} \end{bmatrix}.$$

(i) *$\mathbb{R}^V$ with the inner product defined by* (4) *is spanned by the orthonormal basis $\{\boldsymbol{v}_S : S \in V\}$.*
(ii) *The vector $\boldsymbol{v}_S$ is an eigenvector of T with the corresponding eigenvalue $\lambda_S := \prod_{j \in S} \lambda_j$. In particular, $\boldsymbol{v}_\varnothing = \mathbf{1}$ and $\lambda_\varnothing = 1$.*
(iii) *The entry of $\boldsymbol{v}_{\{i\}}$ corresponding to $S \in V$ is c_i if $i \notin S$ and $-1/c_i$ if $i \in S$, and $p_i\mathbf{1} - \sqrt{p_i q_i}\boldsymbol{v}_{\{i\}}$ is the indicator of the star centered at i, i.e. $\{S \in V : i \in S\}$.*
(iv) *We have $\boldsymbol{v}_S = \prod_{i \in S} \boldsymbol{v}_{\{i\}}$, where the product is taken componentwise.*

For example, the matrix C_3 is as follows, where the columns are in the order $\boldsymbol{v}_\varnothing$, $\boldsymbol{v}_{\{1\}}$, $\boldsymbol{v}_{\{2\}}$, $\boldsymbol{v}_{\{1,2\}}$, $\boldsymbol{v}_{\{3\}}$, $\boldsymbol{v}_{\{1,3\}}$, $\boldsymbol{v}_{\{2,3\}}$, $\boldsymbol{v}_{\{1,2,3\}}$.

$$C_3 = \begin{bmatrix} 1 & c_1 & c_2 & c_1c_2 & c_3 & c_1c_3 & c_2c_3 & c_1c_2c_3 \\ 1 & -\frac{1}{c_1} & c_2 & -\frac{c_2}{c_1} & c_3 & -\frac{c_3}{c_1} & c_2c_3 & -\frac{c_2c_3}{c_1} \\ 1 & c_1 & -\frac{1}{c_2} & -\frac{c_1}{c_2} & c_3 & c_1c_3 & -\frac{c_3}{c_2} & -\frac{c_1c_3}{c_2} \\ 1 & -\frac{1}{c_1} & -\frac{1}{c_2} & \frac{1}{c_1c_2} & c_3 & -\frac{c_3}{c_1} & -\frac{c_3}{c_2} & \frac{c_3}{c_1c_2} \\ 1 & c_1 & c_2 & c_1c_2 & -\frac{1}{c_3} & -\frac{c_1}{c_3} & -\frac{c_2}{c_3} & -\frac{c_1c_2}{c_3} \\ 1 & -\frac{1}{c_1} & c_2 & -\frac{c_2}{c_1} & -\frac{1}{c_3} & \frac{1}{c_1c_3} & -\frac{c_2}{c_3} & \frac{c_2}{c_1c_3} \\ 1 & c_1 & -\frac{1}{c_2} & -\frac{c_1}{c_2} & -\frac{1}{c_3} & -\frac{c_1}{c_3} & \frac{1}{c_2c_3} & \frac{c_1}{c_2c_3} \\ 1 & -\frac{1}{c_1} & -\frac{1}{c_2} & \frac{1}{c_1c_2} & -\frac{1}{c_3} & \frac{1}{c_1c_3} & \frac{1}{c_2c_3} & -\frac{1}{c_1c_2c_3} \end{bmatrix}.$$

Lemma 2.16. *Let $L = \{i \in [n] : p_i = p_1\}$ and $\varphi \in \{0,1\}^V$. Suppose that $\lambda_{\min}(T) < \lambda_S$ for all $S \in V \setminus \binom{L}{1}$. If $\varphi(\varnothing) = 0$ and $\varphi([n]) = 1$, and φ is expanded as*

$$\varphi = p_1\mathbf{1} + \sum_{k \in L} \widehat{\varphi}_{\{k\}}\boldsymbol{v}_{\{k\}}, \tag{10}$$

then φ is the indicator of a star centered at some $i \in L$.

Proof. We first show that there is only one $i \in [n]$ such that $\varphi = p_1\mathbf{1} - \sqrt{p_i q_i}\,\boldsymbol{v}_{\{i\}}$. Suppose, to the contrary, that there are distinct i, j such that both $\widehat{\varphi}_{\{i\}}$ and $\widehat{\varphi}_{\{j\}}$ are non-zero. Let $\varphi^2 \in \{0,1\}^V$ be such that $\varphi^2(x) = \varphi(x)^2$. Then, by Fact 2.15(iv), $\varphi^2 = (p_1\mathbf{1} + \sum_{k \in L} \widehat{\varphi}_{\{k\}}\boldsymbol{v}_{\{k\}})^2$ must contain the term

$$\widehat{\varphi}_{\{i\}}\widehat{\varphi}_{\{j\}}\boldsymbol{v}_{\{i\}}\boldsymbol{v}_{\{j\}} = \widehat{\varphi}_{\{i\}}\widehat{\varphi}_{\{j\}}\boldsymbol{v}_{\{i,j\}}$$

whose coefficient $\widehat{\varphi}_{\{i\}}\widehat{\varphi}_{\{j\}}$ is non-zero. But this contradicts the fact that the expansion (10) is unique and $\varphi = \varphi^2$.

Therefore we can write $\varphi = p_1\mathbf{1} + \widehat{\varphi}_{\{i\}}\boldsymbol{v}_{\{i\}}$ for some $i \in [n]$. By Fact 2.15(iii) we have $\boldsymbol{v}_{\{i\}}(\varnothing) = c_i$ and $\boldsymbol{v}_{\{i\}}([n]) = -1/c_i$, and so

$$\varphi(\varnothing) = 0 = p_1 + \widehat{\varphi}_{\{i\}}c_i \text{ and } \varphi([n]) = 1 = p_1 - \widehat{\varphi}_{\{i\}}/c_i.$$

Solving the equations we get $\widehat{\varphi}_{\{i\}} = -\sqrt{p_1 q_1}$ and $c_i = c_1$. This means that $i \in L$. Consequently $\varphi = p_i\mathbf{1} - \sqrt{p_i q_i}\boldsymbol{v}_{\{i\}}$, and by Fact 2.15(iii) we complete the proof. ☐

3. Application

Recall that a family of subsets $\mathcal{A} \subset 2^{[n]}$ is r-wise intersecting if $A_1 \cap A_2 \cap \cdots \cap A_r \neq \varnothing$ for all $A_1, \ldots, A_r \in \mathcal{A}$. Let $\boldsymbol{p} = (p_1, \ldots, p_n) \in (0,1)^n$ be a fixed real vector. The

$\mu_{\boldsymbol{p}}$-measure of a family $\mathcal{A} \subset 2^{[n]}$ is defined by

$$\mu_{\boldsymbol{p}}(\mathcal{A}) \coloneqq \sum_{A\in\mathcal{A}} \prod_{i\in A} p_i \prod_{j\in[n]\smallsetminus A} q_j.$$

3.1. 2-wise case: Proof of Theorem 1.2.

Proof of Theorem 1.2. The case $n = 1$. In this case the only intersecting family is $\mathcal{A} = \{\{1\}\}$, and $\mu_{\boldsymbol{p}}(\mathcal{A}) = p_1$, where $\boldsymbol{p} = (p_1)$. But we will get this result by using Hoffman's bound because the spectral information in this case will be used later to get the spectral information for the general $n \geqslant 2$ case.

Let $V^{(1)} = 2^{\{1\}} = \{\varnothing, \{1\}\}$, and define the symmetric signed measure $\mu_2^{(1)} : V^{(1)} \times V^{(1)} \to \mathbb{R}$ by

$$\mu_2^{(1)}(\varnothing, \{1\}) = \mu_2^{(1)}(\{1\}, \varnothing) = p_1, \quad \mu_2^{(1)}(\varnothing, \varnothing) = 1 - 2p_1, \quad \mu_2^{(1)}(\{1\}, \{1\}) = 0.$$

This induces the marginal

$$\mu_1^{(1)}(\{1\}) = p_1, \quad \mu_1^{(1)}(\varnothing) = q_1.$$

Then we obtain a weighted 2-graph $G = (V, \mu_2^{(1)})$. Note that $\mu_{\boldsymbol{p}} = \mu_1^{(1)}$. (Indeed this μ_2 is the only symmetric signed measure which satisfies $\mu_2^{(1)}(\{1\}, \{1\}) = 0$ and $\mu_{\boldsymbol{p}} = \mu_1^{(1)}$.) The adjacency matrix $T^{(1)}$ is given by

$$T^{(1)} = \begin{bmatrix} 1 - \frac{p_1}{q_1} & \frac{p_1}{q_1} \\ 1 & 0 \end{bmatrix},$$

where the rows and columns are indexed in the order $\varnothing, \{1\}$. This matrix has eigenvalues 1 and $-\frac{p_1}{q_1}$. Thus $\lambda_{\min}(T^{(1)}) = -\frac{p_1}{q_1}$. Then by Theorem 2.9 we have

$$1 - \alpha(G) \geqslant \frac{1}{1 - \lambda_{\min}(T^{(1)})} = \frac{1}{1 + \frac{p_1}{q_1}} = q_1,$$

and $\alpha(G) \leqslant 1 - q_1 = p_1$. Now it follows from the definition of $\mu_2^{(1)}$ that a 2-wise intersecting family $\mathcal{A} \subset V^{(1)}$ is an independent set in G. Thus we have shown that $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p_1$ in this case $n = 1$.

The general case $n \geqslant 2$. For $i = 1, 2, \ldots, n$, let $V_i = 2^{\{i\}}$ and let $\mu_2^{(i)}$ be defined as in the previous $n = 1$ case. Let $G^{(i)} = (V^{(i)}, \mu_2^{(i)})$ with the adjacency matrix $T^{(i)}$, where

$$T^{(i)} = \begin{bmatrix} 1 - \frac{p_i}{q_i} & \frac{p_i}{q_i} \\ 1 & 0 \end{bmatrix}.$$

Now we define $G = (V, \mu_2)$ to be a product of $G^{(1)}, \ldots, G^{(n)}$. To this end let $V = V^{(1)} \times \cdots \times V^{(n)} \cong 2^{[n]}$, and define $\mu_2 : V^2 \to \mathbb{R}$ by $\mu_2 = \mu_2^{(1)} \times \cdots \times \mu_2^{(n)}$, that is, for $S, S' \in V$, let

$$\mu_2(S, S') \coloneqq \mu_2^{(1)}(s_1, s_1') \times \cdots \times \mu_2^{(n)}(s_n, s_n'),$$

where $s_i = S \cap \{i\}$ and $s_i' = S' \cap \{i\}$ for $1 \leqslant i \leqslant n$.

Claim 3.1. *The marginal μ_1 satisfies $\mu_1 = \mu_1^{(1)} \times \cdots \times \mu_1^{(n)}$, and $\mu_1 = \mu_{\boldsymbol{p}}$.*

Proof. Indeed, by (1), we have

$$\begin{aligned}\mu_1(S) &= \sum_{S'\in V} \mu_2(S,S')\\ &= \sum_{S'\in V} \mu_2^{(1)}(s_1,s_1')\times\cdots\times\mu_2^{(n)}(s_n,s_n')\\ &= \sum_{s_1'\in V^{(1)}} \mu_2^{(1)}(s_1,s_1')\times\cdots\times\sum_{s_n'\in V^{(n)}}\mu_2^{(n)}(s_n,s_n')\\ &= \mu_1^{(1)}(s_1)\times\cdots\times\mu_1^{(n)}(s_n)=\mu_{\boldsymbol{p}}(S). \end{aligned}$$ □

Thus $0<\mu_1(S)<1$ for all $S\in V$, and G is a weighted graph. By construction we see that the adjacency matrix is $T=T^{(n)}\otimes\cdots\otimes T^{(1)}$. We can apply Fact 2.14 with $r=2$ and Fact 2.15. Then, for each $S\in V$, T has an eigenvalue $\lambda_S\coloneqq\prod_{j\in S}\left(-\frac{p_j}{q_j}\right)$ with the corresponding eigenvector $\boldsymbol{v}_S$. Now we determine $\lambda_{\min}(T)=\min_S\lambda_S$.

Claim 3.2. *We have $\lambda_{\min}(T)=\lambda_{\{1\}}=-\frac{p_1}{q_1}$, and if $\lambda_S=\lambda_{\min}(T)$ then $S=\{i\}$ with $p_i=p_1$.*

Proof. Since $p_i\geqslant p_{i+1}$ we have $\lambda_{\{i\}}=-\frac{p_i}{q_i}\leqslant-\frac{p_{i+1}}{q_{i+1}}=\lambda_{\{i+1\}}<0$, and $\min_i\lambda_{\{i\}}=\lambda_{\{1\}}$. The assumption $p_3<\frac12$ means $-1<\lambda_{\{3\}}$, and so $-1<\lambda_{\{i\}}<0$ for all $3\leqslant i\leqslant n$. Thus if $\lambda_{\min}(T)=\lambda_S$ then, using Fact 2.15(ii), S contains at most one i with $i\geqslant 3$. In particular if $\lambda_S=\lambda_{\{1\}}$ then $S=\{i\}$ with $p_i=p_1$.

If $p_1\leqslant\frac12$ then $-1\leqslant\lambda_{\{1\}}$, and $-1\leqslant\lambda_{\{1\}}\leqslant\lambda_{\{2\}}<0$. Therefore if $i\in S\subset[n]$ then $\lambda_{\{i\}}\leqslant\lambda_S$ with equality holding if and only if $S=\{j\}$ with $p_j=p_i$. Thus we get the statement of the claim in this case.

If $p_1>\frac12$ then $\lambda_{\{1\}}<-1$. Thus we have $\lambda_{\min}(T)=\min\{\lambda_{\{1\}},\lambda_{\{1,2,3\}}\}$. By simple computation we see that $\lambda_{\{1\}}<\lambda_{\{1,2,3\}}$ is equivalent to $p_3<q_2$, which is our assumption. Thus we get the statement of the claim again. □

Thus by Theorem 2.9 we have $\alpha(G)\leqslant p_1$. Now let $\mathcal{A}\subset 2^{[n]}$ be a 2-wise intersecting family. If $A,B\in\mathcal{A}$ then there is some $i\in A\cap B$. Then $\mu_2(A,B)=0$ follows from the fact that $\mu_2^{(i)}(\{i\},\{i\})=0$ with the definition of μ_2. This means that $\mathcal{A}\subset V$ is an independent set of G. Since $\mu_1=\mu_{\boldsymbol{p}}$ we see that $\mu_{\boldsymbol{p}}(\mathcal{A})=\mu_1(\mathcal{A})\leqslant\alpha(G)\leqslant p_1$, which completes the proof of inequality.

Finally we prove the uniqueness. Suppose that $\alpha(G)=p_1$ and let φ be the indicator of a maximum independent set. Then $\widehat{\varphi}_\varnothing=\langle\varphi,\mathbf{1}\rangle_{\mu_1}=\mathbb{E}_{\mu_1}[\varphi]=p_1$. We also have $\mathbb{E}_{\mu_2}[\varphi,\varphi]=0$ by Fact 2.3. Thus, using $\widehat{\varphi}_\varnothing=p_1$ and $\lambda_{\min}(T)=-\frac{p_1}{q_1}$, we can verify (9), and by Lemma 2.13 we have $\varphi=p_1\mathbf{1}+\sum_{S\in W}\widehat{\varphi}_S\boldsymbol{v}_S$, where $W=\{S\in V:\lambda_S=\lambda_{\min}(T)\}$. Since $\lambda_S=\lambda_{\min}(T)$ is equivalent to $S=\{i\}$ with $p_i=p_1$, we can rewrite $\varphi=p_1\mathbf{1}+\sum_{k\in L}\widehat{\varphi}_{\{k\}}\boldsymbol{v}_{\{k\}}$, where $L=\{i\in[n]:p_i=p_1\}$. Consequently it follows from Lemma 2.16 that φ is the indicator of a star centered at some $i\in L$. This completes the proof of Theorem 1.2. □

Example 3.3. Define a 2-wise intersecting family $\mathcal{A}\subset 2^{[n]}$ by $\mathcal{A}=\{A\in 2^{[n]}:|A\cap[3]|\geqslant 2\}$, and let $\boldsymbol{p}=(p_1,p_2,\ldots,p_n)$. Then

$$\mu_{\boldsymbol{p}}(\mathcal{A})=p_1p_2q_3+p_1q_2p_3+q_1p_2p_3+p_1p_2p_3=p_1p_2+p_1p_3+p_2p_3-2p_1p_2p_3.$$

If $p_1=p_2=p_3$ then $\mu_{\boldsymbol{p}}(\mathcal{A})=p_1^2(3-2p_1)$ and $\mu_{\boldsymbol{p}}(\mathcal{A})>p_1$ iff $\frac12<p_1<1$. If $p_1=p_2$ and $p_3=\frac12$ then $\mu_{\boldsymbol{p}}(\mathcal{A})=p_1$.

These examples show the sharpness of the condition $p_3<\frac12$ in Conjecture 1.1 (if true) in the following sense. First, for the inequality ($\mu_{\boldsymbol{p}}(\mathcal{A})\leqslant p_1$) we cannot replace the condition with $p_4 < \frac{1}{2}$. Second, to ensure the uniqueness we cannot replace the condition with $p_3 \leqslant \frac{1}{2}$.

3.2. 3-wise case: Proof of Theorem 1.3.

Proposition 3.4. *Let $\frac{2}{3} > p_1 \geqslant \frac{1}{2}$ and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$. Let $\mathcal{A} \subset 2^{[n]}$ be a 3-wise intersecting family. Then*
$$\mu(\mathcal{A}) \leqslant p_1.$$
Moreover equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_i = p_1$.

Proof. The case $n = 1$. Let $V^{(1)} = 2^{\{1\}} = \{\varnothing, \{1\}\}$, and we will define a symmetric signed measure $\mu_3^{(1)} : V^{(1)} \times V^{(1)} \times V^{(1)} \to \mathbb{R}$. Here, for simplicity, we write 0 and 1 to mean $\varnothing$ and $\{1\}$, e.g. we write $\mu_3^{(1)}(0,1,1)$ to mean $\mu_3^{(1)}(\varnothing, \{1\}, \{1\})$. Now $\mu_3^{(1)}$ is defined by
$$\mu_3^{(1)}(0,1,1) = \mu_3^{(1)}(1,0,1) = \mu_3^{(1)}(1,1,0) = \frac{1}{2}p_1, \quad \mu_3^{(1)}(0,0,0) = 1 - \frac{3}{2}p_1,$$
$$\mu_3^{(1)}(1,0,0) = \mu_3^{(1)}(0,1,0) = \mu_3^{(1)}(0,0,1) = \mu_3^{(1)}(1,1,1) = 0.$$
Then
$$\mu_2^{(1)}(1,1) = \frac{1}{2}p_1, \quad \mu_2^{(1)}(1,0) = \mu_2^{(1)}(0,1) = \frac{1}{2}p_1, \quad \mu_2^{(1)}(0,0) = 1 - \frac{3}{2}p_1,$$
and
$$\mu_1^{(1)}(1) = p_1, \quad \mu_1^{(1)}(0) = q_1.$$
It follows from $0 < p_1 < \frac{2}{3}$ that $\mu_1^{(1)}$ and $\mu_2^{(1)}/\mu_1^{(1)}$ take values in $(0,1)$. So we can define a weighted 3-graph $H = (V^{(1)}, \mu_3^{(1)})$. Then, from (3) and (6), we have the following matrices.
$$T^{(1)} = \begin{bmatrix} 1 - \frac{p_1}{2q_1} & \frac{p_1}{2q_1} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad T_{\varnothing}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_{\{1\}}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
By direct computation we get the following table concerning spectral information.

	$T^{(1)}$	$T_{\varnothing}^{(1)}$	$T_{\{1\}}^{(1)}$
eigenvalues λ	$1, 1 - \frac{1}{2q_1}$	$1, 1$	$1, -1$
$\lambda_{\min}$	$1 - \frac{1}{2q_1}$	1	-1

Let φ be the indicator of a maximum independent set in H. Then $\alpha(H) = \mathbb{E}_{\mu_1^{(1)}}[\varphi]$ and $\varnothing \notin \varphi$. So, by Theorem 2.12, we have
$$\begin{aligned} 1 - \alpha(H) &\geqslant \frac{1}{(1 - \lambda_{\min}(T^{(1)})) \max_{x \in \varphi}(1 - \lambda_{\min}(T_x^{(1)}))} \\ &= \frac{1}{(1 - 1 + \frac{1}{2q_1})(1+1)} = q_1, \end{aligned}$$
and $\alpha(H) \leqslant 1 - q_1 = p_1$.

The general case $n \geqslant 2$. For $i = 1, 2, \ldots, n$ let $V_i = 2^{\{i\}}$ and let $\mu_3^{(i)}$ be defined as in the previous $n = 1$ case. Let $H^{(i)} = (V^{(i)}, \mu_3^{(i)})$ be the weighted 3-graph. This induces the weighted 2-graph and the link graphs with the adjacency matrices $T^{(i)}, T_{\varnothing}^{(i)}, T_{\{i\}}^{(i)}$, where
$$T^{(i)} = \begin{bmatrix} 1 - \frac{p_i}{2q_i} & \frac{p_i}{2q_i} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad T_{\varnothing}^{(i)} = T_{\varnothing}^{(1)}, \quad T_{\{i\}}^{(i)} = T_{\{1\}}^{(1)}.$$

We will construct a weighted 3-graph $H = (V, \mu_3)$ from $H^{(1)}, \dots, H^{(n)}$. Let $V = V^{(1)} \times \dots \times V^{(n)} \cong 2^{[n]}$. Define $\mu_3 : V^3 \to \mathbb{R}$ by $\mu_3 = \mu_3^{(1)} \times \dots \times \mu_3^{(n)}$. Let $\mu_2 \in \mathbb{R}^{V^2}$ and $\mu_1 \in \mathbb{R}^V$ be the marginals. Then, as in Claim 3.1, we see that $\mu_i = \mu_i^{(1)} \times \dots \times \mu_i^{(n)}$ for $i = 1, 2$, as well, in particular, $\mu_1 = \mu_{\boldsymbol{p}}$. Note also that both μ_1 and μ_2/μ_1 take values in $(0, 1)$, and we need the condition $p_1 < \frac{2}{3}$ here. Consequently H is a weighted 3-graph with adjacency matrix $T = T^{(n)} \otimes \dots \otimes T^{(1)}$.

We apply Fact 2.14 with $r = 3$ and Fact 2.15. Then, for each $S \in V$, the matrix T has an eigenvalue $\lambda_S := \prod_{j \in S} \left(1 - \frac{1}{2q_j}\right)$ with the corresponding eigenvector $\boldsymbol{v}_S$ from Fact 2.15. Since $\frac{1}{2} \leqslant p_1 < \frac{2}{3}$ and $\lambda_{\{1\}} = 1 - \frac{1}{2q_1}$ we have $-\frac{1}{2} < \lambda_{\{1\}} \leqslant 0$ and $\lambda_{\min}(T) = \lambda_{\{1\}}$. The adjacency matrix of the link graph $H_S = (V, \mu_{2,S})$ is $T_S = T_{s_n}^{(n)} \otimes \dots \otimes T_{s_1}^{(1)}$, where $s_i = S \cap \{i\}$. If $S \neq \varnothing$ then T_S has eigenvalues $\{1, -1\}$ and

$$\lambda_{\min}(T_S) = -1. \tag{11}$$

Let φ be the indicator of a maximum independent set in H. We have $\varnothing \notin \varphi$ because

$$\mu_3(\varnothing, \varnothing, \varnothing) = \prod_{i=1}^{n} \mu_3^{(i)}(0, 0, 0) = \prod_{i=1}^{n} (1 - \tfrac{3}{2} p_i) \neq 0.$$

Thus, by Theorem 2.12, we have

$$1 - \alpha(H) \geqslant \frac{1}{(1 - \lambda_{\min}(T)) \max_{S \in \varphi}(1 - \lambda_{\min}(T_S))} = \frac{1}{\left(1 - (1 - \frac{1}{2q_1})\right)(1 - (-1))} = q_1,$$

and $\alpha(H) \leqslant 1 - q_1 = p_1$.

Now let $\mathcal{A} \subset 2^{[n]}$ be a 3-wise intersecting family. If $A, B, C \in \mathcal{A}$ then there is some $i \in A \cap B \cap C$. Then $\mu_3(A, B, C) = 0$ follows from the fact that $\mu_3^{(i)}(\{i\}, \{i\}, \{i\}) = 0$ with the definition of μ_3. This means that $\mathcal{A} \subset V$ is an independent set in H. Since $\mu_1 = \mu_1^{(1)} \times \dots \times \mu_n^{(n)} = \mu_{\boldsymbol{p}}$ we have $\mu_{\boldsymbol{p}}(\mathcal{A}) = \mu_1(\mathcal{A}) \leqslant \alpha(H) \leqslant p_1$, which completes the proof of inequality.

Finally we show the uniqueness of equality case. Suppose that $\alpha(H) = p_1$ and let φ be the indicator of a maximum independent set I in H. Then $\varnothing \notin I$ and I is also an independent set in the link graph $H_S = (V, \mu_{2,S})$ if $S \neq \varnothing$. Thus by applying Theorem 2.9 to H_S with (11) we have

$$\max_{S \in \varphi} \mathbb{E}_{\mu_1, S}[\varphi] \leqslant \max_{S \neq \varnothing} \frac{-\lambda_{\min}(T_S)}{1 - \lambda_{\min}(T_S)} = \frac{1}{2}. \tag{12}$$

(We note that (12) holds for the indicator of *any* independent set, not necessarily a maximum one, and we will use this fact in the next subsection.) Then by Lemma 2.11 we have $\mathbb{E}_{\mu_2}[\varphi, \varphi] \leqslant \frac{p_1}{2}$. This together with $\widehat{\varphi}_0 = p_1$ and $\lambda_{\min}(T) = 1 - \frac{1}{2q_1}$ verifies (9), and we can apply Lemma 2.13. Since $\lambda_{\min}(T)$ is attained only by $\lambda_{\{i\}}$ with $i \in J := \{j \in [n] : p_j = p_1\}$ we have $\varphi = p_1 \mathbf{1} + \sum_{j \in J} \widehat{\varphi}_{\{j\}} \boldsymbol{v}_{\{j\}}$. Finally by Lemma 2.16 it follows that φ is the indicator of a star centered at some $i \in J$. $\square$

Proof of Theorem 1.3. We note that the μ_1 in Theorem 1.2 and the μ_1 in Proposition 3.4 are the same, and moreover $\mu_{\boldsymbol{p}} = \mu_1$. Then Theorem 1.3 for the case $p_1 \leqslant \frac{1}{2}$ follows from Theorem 1.2, and the case $\frac{1}{2} \leqslant p_1 < \frac{2}{3}$ follows from Proposition 3.4. Thus we may assume that $p_1 \geqslant \frac{2}{3}$ and $p_2 < \frac{2}{3}$. Now we follow the argument in [8]. Let $\boldsymbol{p} = (p_1, p_2, p_3, \dots, p_n)$ and $\boldsymbol{p}' = (p_2, p_2, p_3, \dots, p_n)$, that is, $\boldsymbol{p}'$ is obtained from $\boldsymbol{p}$ by replacing p_1 with p_2. Let $\mathcal{A}$ be the star centered at 1, and let $\mathcal{B}$ be an inclusion maximal intersecting family. Suppose that $\mathcal{B} \neq \mathcal{A}$, and we will show that $\mu_{\boldsymbol{p}}(\mathcal{A}) > \mu_{\boldsymbol{p}}(\mathcal{B})$.

By the construction we have $\mu_{\boldsymbol{p}}(\mathcal{A}) = p_1$ and $\mu_{\boldsymbol{p}'}(\mathcal{A}) = p_2$. Thus $\mu_{\boldsymbol{p}}(\mathcal{A}) = \frac{p_1}{p_2}\mu_{\boldsymbol{p}'}(\mathcal{A})$.

On the other hand, by Proposition 3.4, we have $\mu_{\boldsymbol{p}'}(\mathcal{B}) \leqslant p_2$. Let $B \in \mathcal{B}$. If $1 \in B$ then $\mu_{\boldsymbol{p}}(B) = \frac{p_1}{p_2}\mu_{\boldsymbol{p}'}(B)$. If $1 \notin B$ then $\mu_{\boldsymbol{p}}(B) = \frac{q_1}{q_2}\mu_{\boldsymbol{p}'}(B) < \frac{p_1}{p_2}\mu_{\boldsymbol{p}'}(B)$, where we used $p_1 > p_2$. Since $\mathcal{B} \neq \mathcal{A}$ and $\mathcal{B}$ is inclusion maximal there is some $B \in \mathcal{B}$ such that $1 \notin B$, e.g. $\{2, 3, \dots, n\} \in \mathcal{B}$. Thus we have $\mu_{\boldsymbol{p}}(\mathcal{B}) < \frac{p_1}{p_2}\mu_{\boldsymbol{p}'}(\mathcal{B}) \leqslant p_1 = \mu_{\boldsymbol{p}}(\mathcal{A})$, as needed. This means that $\mathcal{A}$ is the only intersecting family which attains the maximum $\mu_{\boldsymbol{p}}$-measure. ☐

3.3. Stability: Proof of Theorem 1.4. Friedgut [11] obtained a stability result for 2-wise t-intersecting families. The special case $t = 1$, which is a stability version of a result by Ahlswede–Katona [1], reads as follows.

Proposition 3.5 ([11]). *Let $0 < p < \frac{1}{2}$ be fixed. Then there exists a constant $\epsilon_p > 0$ such that the following holds for all $0 < \epsilon < \epsilon_p$. If $\mathcal{A} \subset 2^{[n]}$ is a 2-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p - \epsilon$, then there exists a star $\mathcal{B} \subset 2^{[n]}$ such that $\mu_{\boldsymbol{p}}(\mathcal{A} \triangle \mathcal{B}) < (C_p + o(1))\epsilon$, where $\boldsymbol{p} = (p, p, \dots, p)$ and $C_p = \frac{4q^2}{1-2p}$.*

We include the proof for convenience in Appendix. (The C_p is not explicitly computed in [11].)

In this subsection we adapt his proof to 3-wise intersecting families to show the following.

Proposition 3.6. *Let $\frac{1}{2} < p < \frac{2}{3}$ be fixed. Then there exists a constant $\epsilon_p > 0$ such that the following holds for all $0 < \epsilon < \epsilon_p$. If $\mathcal{A} \subset 2^{[n]}$ is a 3-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p - \epsilon$, then there exists a star $\mathcal{B} \subset 2^{[n]}$ such that $\mu_{\boldsymbol{p}}(\mathcal{A} \triangle \mathcal{B}) < (C_p + o(1))\epsilon$, where $\boldsymbol{p} = (p, p, \dots, p)$ and*

$$C_p = \frac{16pq^2}{(2p-1)(3-4p)}.$$

For the proof we use the Kindler–Safra theorem, which extends the Friedgut–Kalai–Naor theorem [12]. To state the result we need a definition. Let $V = 2^{[n]}$ and $g \in \{0,1\}^V$. We say that a boolean function $g \in \{0,1\}^V$ *depends on at most one coordinate* if g is one of the following:

(G1) there is some $i \in [n]$ such that $g = \mathbf{1}_{\{i\}}$, or
(G2) there is some $i \in [n]$ such that $g = \mathbf{1} - \mathbf{1}_{\{i\}}$, or
(G3) g is a constant function, that is, $g = \mathbf{0}$, or $g = \mathbf{1}$.

(G1) means that g is the indicator of the star centered at i, and (G2) means that g is the indicator of the complement of the star.

We can expand any boolean function $f \in \{0,1\}^V$ as $f = \sum_{S \in V} \widehat{f}_S \boldsymbol{v}_S$, where $\boldsymbol{v}_s$ is defined in Subsection 2.3 (see Fact 2.15). Let $f^{>1} := \sum_{|S|>1} \widehat{f}_S \boldsymbol{v}_S$, and let $\|f\|$ denote the square root of $\langle f, f \rangle_{\mu_{\boldsymbol{p}}}$, where $\boldsymbol{p} = (p, \dots, p)$. For example, if $f = \mathbf{1}_{\{i\}}$ then $f = p\boldsymbol{v}_{\varnothing} - \sqrt{pq}\boldsymbol{v}_{\{i\}}$, and $\|f\| = p$, $\|f^{>1}\| = 0$.

Theorem 3.7 (Kindler–Safra, Corollary 15.2 in [14], see also [15]). *Let $p \in (0,1)$ be fixed and let $V = 2^{[n]}$. Let $f \in \{0,1\}^V$ and $\|f^{>1}\|^2 \leqslant \delta \ll p$. Then there exists $g \in \{0,1\}^V$ which depends on at most one coordinate and $\|f - g\|^2 < (4 + o(1))\delta$.*

The $o(1)$ term is actually smaller than $c_1 \exp(-\frac{c_2}{\delta})$, where c_1, c_2 are positive constants depending only on p, see Corollary 6.1 in [15] for more details.

Proof of Proposition 3.6. Let $V = 2^{[n]}$ and let $\mu_3 \in \mathbb{R}^{V^3}$ be the measure defined in the proof of Proposition 3.4. By definition of μ_3 with $\frac{1}{2} < p < \frac{2}{3}$ we have that

$0 < \mu_2(x,y) < 1$ for all $x, y \in V$, and $\mu_1 = \mu_{\boldsymbol{p}}$. Thus we can define a weighted 3-graph (V, μ_3).

Let φ be the indicator of $\mathcal{A}$ and we write

$$\varphi = \widehat{\varphi}_\varnothing \mathbf{1} + \sum_{S \neq \varnothing} \widehat{\varphi}_S \boldsymbol{v}_S.$$

To apply Theorem 3.7 we need to show that $\|\varphi^{>1}\|$ is small. By Fact 2.7(ii) we have

$$\mathbb{E}_{\mu_2}[\varphi, \varphi] = \widehat{\varphi}_\varnothing^2 + \sum_{|S|=1} \widehat{\varphi}_S^2 \lambda_S + \sum_{|S|>1} \widehat{\varphi}_S^2 \lambda_S,$$

where $\lambda_S = (1 - \frac{1}{2q})^{|S|}$. Since $-\frac{1}{2} < 1 - \frac{1}{2q} < 0$ the minimum and the second minimum eigenvalues come from the cases $|S| = 1$ and $|S| = 3$, respectively. So let $\lambda_1 \coloneqq 1 - \frac{1}{2q}$ and $\lambda_3 \coloneqq (1 - \frac{1}{2q})^3$. Then we have

$$\mathbb{E}_{\mu_2}[\varphi, \varphi] \geqslant \widehat{\varphi}_\varnothing^2 + \lambda_1 \sum_{|S|=1} \widehat{\varphi}_S^2 + \lambda_3 \sum_{|S|>1} \widehat{\varphi}_S^2. \tag{13}$$

Define τ by $\sum_{|S|>1} \widehat{\varphi}_S^2 = \tau \widehat{\varphi}_\varnothing$. Then, by Fact 2.7(i), $\sum_{|S|=1} \widehat{\varphi}_S^2 = \widehat{\varphi}_\varnothing - \widehat{\varphi}_\varnothing^2 - \tau \widehat{\varphi}_\varnothing$. Thus we have

$$\mathbb{E}_{\mu_2}[\varphi, \varphi] \geqslant \widehat{\varphi}_\varnothing^2 + \lambda_1 (\widehat{\varphi}_\varnothing - \widehat{\varphi}_\varnothing^2 - \tau \widehat{\varphi}_\varnothing) + \lambda_3 \tau \widehat{\varphi}_\varnothing.$$

On the other hand we have $\mathbb{E}_{\mu_1}(\varphi) = \widehat{\varphi}_\varnothing$ and $\max_{x \in \varphi} \mathbb{E}_{\mu_1, x}[\varphi] \leqslant \frac{1}{2}$ by (12). Thus it follows from Lemma 2.11 that $\mathbb{E}_{\mu_2}[\varphi, \varphi] \leqslant \frac{1}{2} \widehat{\varphi}_\varnothing$. So estimating $\mathbb{E}_{\mu_2}[\varphi, \varphi] / \widehat{\varphi}_\varnothing$ we get

$$\frac{1}{2} \geqslant \widehat{\varphi}_\varnothing + \lambda_1 (1 - \widehat{\varphi}_\varnothing - \tau) + \lambda_3 \tau,$$

which yields

$$\tau \leqslant \frac{1}{\lambda_3 - \lambda_1} \left(\frac{1}{2} - \widehat{\varphi}_\varnothing - \lambda_1 (1 - \widehat{\varphi}_\varnothing) \right) = \frac{1}{\lambda_3 - \lambda_1} \cdot \frac{\epsilon}{2q} = \frac{4q^2}{(2p-1)(3-4p)} \epsilon,$$

where we used $\widehat{\varphi}_\varnothing = p - \epsilon$ for the first equality. Since

$$\|\varphi^{>1}\|^2 = \sum_{|S|>1} \widehat{\varphi}_S^2 = \tau \widehat{\varphi}_\varnothing < \tau p$$

we obtain

$$\|\varphi^{>1}\|^2 < \frac{4pq^2}{(2p-1)(3-4p)} \epsilon.$$

By applying Theorem 3.7 with $\delta = \frac{4pq^2}{(2p-1)(3-4p)} \epsilon$ we can find $g \in \{0,1\}^V$ which depends at most one coordinate and $\|\varphi - g\|^2 < (4 + o(1))\delta$.

We claim that g is the indicator of a star, that is, (G1) happens. Note that $\frac{1}{2} < p < \frac{2}{3}$ and $p \gg \epsilon + \delta$ by the choice of ϵ. So we have $\|\varphi - \mathbf{0}\|^2 = \|\varphi\|^2 = p - \epsilon \gg \delta$ and $\|\varphi - \mathbf{1}\|^2 = 1 - (p - \epsilon) \gg \delta$. Thus (G3) cannot happen. If g is the indicator of the complement of a star, then $\|g\|^2 = 1 - p$ and

$$\|\varphi - g\|^2 \geqslant (\|\varphi\| - \|g\|)^2 = (\sqrt{p - \epsilon} - \sqrt{1 - p})^2 > \frac{16pq^2}{(2p-1)(3-4p)} \epsilon = 4\delta$$

by choosing $\epsilon \ll p$ small enough (we need to choose ϵ quite small when p is close to 1/2). This shows that (G2) cannot happen. So the only possibility is (G1), as needed. $\square$

Proof of Theorem 1.4. Let $\mathcal{A} \subset 2^{[n]}$ be a 3-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p-\epsilon$. First let $\frac{1}{2} < p < \frac{2}{3}$. Then (ii) of the theorem follows from Proposition 3.6.

Next let $p = \frac{1}{2}$. It follows from the Brace–Daykin Theorem [5] that if $\mathcal{F} \subset 2^{[n]}$ is a 3-wise intersecting family which is not a subfamily of a star, then

$$|\mathcal{F}| \leqslant |\{F \subset [n] : |F \cap [4]| \geqslant 3\}|,$$

or equivalently $\mu_{\boldsymbol{p}}(\mathcal{F}) \leqslant \frac{5}{16}$, where $\boldsymbol{p} = (\frac{1}{2}, \dots, \frac{1}{2})$. Thus if $p - \epsilon > \frac{5}{16}$, that is, $0 < \epsilon < \frac{3}{16}$, then $\mathcal{A}$ is a subfamily of a star, which shows (i) of the theorem in this case.

Finally let $0 < p < \frac{1}{2}$. We say that $\mathcal{A}$ is 2-wise 2-intersecting if $|A \cap A'| \geqslant 2$ for all $A, A' \in \mathcal{A}$. If $\mathcal{A}$ is *not* 2-wise 2-intersecting, then there exist $A, A' \in \mathcal{A}$ such that $|A\cap A'| = 1$, say, $A\cap A' = \{i\}$. In this case every $A \in \mathcal{A}$ must contain i due to the 3-wise intersecting condition. Thus $\mathcal{A}$ is contained in a star $\mathcal{B}$ centered at i, and we get (i) of the theorem in this case. The only remaining case is that $\mathcal{A}$ is 2-wise 2-intersecting, and we show that this cannot happen. For $i = 0, 1, \dots$, let $\frac{i}{2i+1} \leqslant p \leqslant \frac{i+1}{2i+3}$. Then it follows from the Ahlswede–Khachatrian theorem [2] that $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant \mu_{\boldsymbol{p}}(\mathcal{G}_i)$, where

$$\mathcal{G}_i = \{G \subset [n] : |G \cap [2i + 2]| \geqslant i + 2\}.$$

A direct computation shows that $\mu_{\boldsymbol{p}}(\mathcal{G}_i) = \sum_{j=0}^{i} \binom{2i+2}{j} p^{2i+2-j}(1-p)^j < p$. So by choosing $\epsilon < \epsilon_p \coloneqq p - \mu_{\boldsymbol{p}}(\mathcal{G}_i)$ we see that $\mu_{\boldsymbol{p}}(\mathcal{A}) = p - \epsilon > p - \epsilon_p = \mu(\mathcal{G}_i)$, a contradiction. ☐

4. Concluding remarks

4.1. Generalization to r-graphs. Filmus et al. extended Hoffman's bound to an r-graph in [7]. We briefly explain how to extend Theorem 2.12 to an r-graph by induction on r. Let V be a finite set with $|V| \geqslant 2$ and we define a weighted r-graph on V as follows.

Definition 4.1. *Let $\mu_r : V^r \to \mathbb{R}$ be a symmetric signed measure. We say that $H = (V, \mu_r)$ is a weighted r-graph if $\mu_{r-1}(x_1, \dots, x_{r-1}) > 0$ for all $x_1, \dots, x_{r-1} \in V$, where*

$$\mu_{r-1}(x_1, \dots, x_{r-1}) \coloneqq \sum_{y \in V} \mu_r(x_1, \dots, x_{r-1}, y).$$

For $i = r-2, r-3, \dots, 1$ we define a measure $\mu_i \in \mathbb{R}^{V^i}$ inductively by

$$\mu_i(x_1, \dots, x_i) \coloneqq \sum_{y \in V} \mu_{i+1}(x_1, \dots, x_i, y).$$

Note that $\mu_i(x_1, \dots, x_i) > 0$ for all $x_1, \dots, x_i \in V$.

Let φ be the indicator of an independent set I in the weighted r-graph $H = (V, \mu_r)$. Then Lemma 2.8 and Lemma 2.11 work for H as well. Here we define $\mu_{r-1,x}(y_2, \dots, y_r) \coloneqq \mu_r(x, y_2, \dots, y_r)/\mu_1(x)$. Then $\mathbb{E}_{\mu_{1,x}}[\varphi]$ is bounded from above by $\alpha(H_x)$, where $H_x = (V, \mu_{r-1,x})$ is the link $(r-1)$-graph of H relative to x. By induction hypothesis we can bound $\alpha(H_x)$, and we eventually bound $\mathbb{E}_{\mu_1}[\varphi]$ using Lemma 2.8 and Lemma 2.11. To state the bound, for $s = 1, \dots, r-2$, and $S = \{v_1, \dots, v_s\}$, where $v_1, \dots, v_s \in V$, let T_S be the adjacency matrix of the link $(r-s)$-graph relative to S, defined by

$$(T_S)_{x,y} = \frac{\mu_{s+2}(v_1, \dots, v_s, x, y)}{\mu_{s+1}(v_1, \dots, v_s, x)}. \tag{14}$$

Let $\lambda_s \coloneqq \min_S \lambda_{\min}(T_S)$ and $\min_S$ is taken over all s-element (multi)subset S of I. Also let $\lambda_0 \coloneqq \lambda_{\min}(T)$ where T is the adjacency matrix of H defined by (3). Then the FGL bound is stated as follows.

$$\mathbb{E}_{\mu_1}[\varphi] \leqslant 1 - \prod_{s=0}^{r-2} \frac{1}{1-\lambda_s}. \tag{15}$$

With this bound it is not difficult to extend Theorem 1.3 to an r-graph.

Theorem 4.2. *Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. Let $r \geqslant 3$. If $\frac{r-1}{r} > p_2$ and $\mathcal{A} \subset 2^{[n]}$ is an r-wise intersecting family, then $\mu_{\boldsymbol{p}}(\mathcal{A}) \leqslant p_1$. Moreover equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_1 = p_i$.*

The proof of Theorem 4.2 goes exactly the same as that of Theorem 1.3, and the main part is the proof of the following result which corresponds to Proposition 3.4.

Proposition 4.3. *Let $\frac{r-1}{r} > p_1 \geqslant \frac{r-2}{r-1}$ and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$. Let $\mathcal{A} \subset 2^{[n]}$ be an r-wise intersecting family. Then*

$$\mu(\mathcal{A}) \leqslant p_1.$$

Moreover equality holds if and only if $\mathcal{A}$ is a star centered at some $i \in [n]$ with $p_i = p_1$.

Proof. The matrices for the proof are different from the ones used in the proof of Theorem 7.1 in [7], because we will introduce a parameter $\epsilon > 0$ so that all the measures μ_i take positive values and (14) is well defined. Here we only record the matrices and the corresponding eigenvalues. Otherwise the proof is the same as the one for Proposition 3.4.

Let $n = 1$ and $V = \{\varnothing, \{1\}\}$. We need to define a symmetric measure $\mu_r^{(1)}$. For this we start with a symmetric function $\mu_r^{(1)} : V^r \to \mathbb{R}$ defined by

$$\mu_r^{(1)}(0^i, 1^{r-i}) = \begin{cases} 0 & \text{if } i = 0, \\ \frac{p_1}{r-1} - \delta_1 & \text{if } i = 1, \\ \epsilon & \text{if } 2 \leqslant i \leqslant r-1, \\ 1 - \frac{rp_1}{r-1} - \delta_2 & \text{if } i = r, \end{cases}$$

where $\epsilon, \delta_1, \delta_2$ are small positive constants, and $(0^i, 1^{r-i})$ in the LHS means i repeated $\varnothing$ and $r-i$ repeated $\{1\}$. Since $\mu_r^{(1)}$ is symmetric, any permutation of $(0^i, 1^{r-i})$ takes the same value. We require $\sum_{x \in V^r} \mu_r^{(1)}(x) = 1$ for $\mu_r^{(1)}$ to be a measure, that is,

$$\sum_{x \in V^r} \mu_r^{(1)}(x) = \binom{r}{1}\left(\frac{p_1}{r-1} - \delta_1\right) + \sum_{i=2}^{r-1} \binom{r}{i} \epsilon + \binom{r}{r}\left(1 - \frac{rp_1}{r-1} - \delta_2\right) = 1.$$

We also require that the induced measure $\mu_1^{(1)}$ is the p_1-biased one, that is, $\mu_1^{(1)}(1) = p_1$, and so

$$\begin{aligned} \mu_1^{(1)}(1) &= \sum_{x_2 \in V} \mu_2^{(1)}(1, x_2) = \cdots = \sum_{(x_2, \dots, x_r) \in V^{r-1}} \mu_r^{(1)}(1, x_2, \dots, x_r) \\ &= \binom{r-1}{1}\left(\frac{p_1}{r-1} - \delta_1\right) + \sum_{i=2}^{r-1} \binom{r-1}{i} \epsilon = p_1. \end{aligned}$$

These two requirements yield that

$$\delta_1 = \frac{2^{r-1} - r}{r-1}\epsilon, \quad \delta_2 = \frac{(2^{r-1}-1)(r-2)}{r-1}\epsilon.$$

Then $H = (V, \mu_r^{(1)})$ is a weighted r-graph by choosing $\epsilon > 0$ sufficiently small. For each $s = 1, \dots, r-2$, and $S \in V^s$, the link $(r-s)$-graph $H_S = (V, \mu_{r-s,S}^{(1)})$ is induced from H. Let $T^{(1)}(\epsilon)$ and $T_S^{(1)}(\epsilon)$ be the adjacency matrices corresponding to H and H_S, respectively, and let $T^{(1)} = \lim_{\epsilon\to 0} T^{(1)}(\epsilon)$ and $T_S^{(1)} = \lim_{\epsilon\to 0} T_S^{(1)}(\epsilon)$. After a somewhat tedious but direct computation one can verify that, for $i \geqslant 2$ and $j \geqslant 1$,

$$T^{(1)} = \begin{bmatrix} 1 - \frac{p_1}{(r-1)q_1} & \frac{p_1}{(r-1)q_1} \\ \frac{1}{r-1} & 1 - \frac{1}{r-1} \end{bmatrix}, \qquad T_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad T_{1^j}^{(1)} = \begin{bmatrix} 0 & 1 \\ \frac{1}{r-j-1} & \frac{r-j-2}{r-j-1} \end{bmatrix},$$
$$T_{0^i1^j}^{(1)} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad T_{0^i}^{(1)} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad T_{01^j}^{(1)} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

The above six matrices have the corresponding eigenvalues below:

$$\begin{array}{lll} \{1 - \frac{1}{(r-1)q_1}, 1\}, & \{1, 1\}, & \{-\frac{1}{r-j-1}, 1\}, \\ \{0, 1\}, & \{\frac{1}{2}, 1\}, & \{\frac{1}{2}, 1\}. \end{array}$$

Thus we have $\lambda_0^{(1)} \coloneqq \lambda_{\min}(T) = 1 - \frac{1}{(r-1)q_1} < 0$, and $\lambda_s^{(1)} \coloneqq \min_{S\in V^s} \lambda_{\min}(T_S) = -\frac{1}{r-s-1}$ for $s = 1, \dots, r-2$. Let φ be the indicator of an independent set in H. Then by (15) we have $\mathbb{E}_{\mu_1}[\varphi] \leqslant 1 - \prod_{s=0}^{r-2} \frac{1}{1-\lambda_s^{(1)}} = p_1$.

For the general case let $n \geqslant 2$ and $V = 2^{[n]}$. We define the measure $\mu_r : V^r \to \mathbb{R}$ by $\mu_r \coloneqq \mu_r^{(1)} \times \cdots \times \mu_r^{(n)}$. Then the corresponding adjacency matrices are obtained by taking tensor product of the ones in the $n = 1$ case. So $T = T^{(1)} \otimes \cdots \otimes T^{(n)}$ with eigenvalues

$$\lambda_v(T) \coloneqq \prod_{i\in v} \left(1 - \frac{1}{(r-1)q_i}\right) \tag{16}$$

for $v \in V$, and $\lambda_0 \coloneqq \min_{v\in V} \lambda_v(T) = \lambda_{\{1\}} = 1 - \frac{1}{(r-1)q_1}$. For $1 \leqslant s \leqslant r-2$ and $S \in V^s$ we have $T_S = T_S^{(1)} \otimes \cdots \otimes T_S^{(n)}$ with

$$\lambda_s \coloneqq \min_{S\in V^s} \lambda_{\min}(T_S) = \lambda_{\min}(T_{1^s}) = -\frac{1}{r-s-1}. \tag{17}$$

Finally it follows from (15) that $\mathbb{E}_{\mu_1}[\varphi] \leqslant p_1$. ☐

Conjecture 4.4. *The condition $\frac{r-1}{r} > p_2$ in Theorem 4.2 can be replaced with $\frac{r-1}{r} > p_{r+1}$. In particular, Theorem 1.3 holds if $p_4 < \frac{2}{3}$ instead of $p_3 < \frac{2}{3}$.*

On the other hand, the condition above cannot be replaced with $\frac{r-1}{r} > p_{r+2}$. To see this let $\mathcal{A} = \{A \in 2^{[n]} : |A \cap [r+1]| \geqslant r\}$, and $p_1 = \cdots = p_{r+1} =: p$. Then $\mathcal{A}$ is an r-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = (r+1)p^r q + p^{r+1}$. A computation shows $\mu_{\boldsymbol{p}}(\mathcal{A})$ is greater than p provided, e.g. $p \geqslant 1 - \frac{1}{r^2}$. More generally we can ask the following.

Problem 4.5. *Let $1 > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ and $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$. Determine the maximum of $\mu_{\boldsymbol{p}}(\mathcal{A})$, where $\mathcal{A} \subset 2^{[n]}$ is an r-wise intersecting family.*

Proposition 3.6 can be extended to r-wise intersecting families as follows.

Proposition 4.6. *Let $r \geqslant 3$ and $\frac{r-2}{r-1} < p < \frac{r-1}{r}$ be fixed. Then there exists a constant $\epsilon_{r,p} > 0$ such that the following holds for all $0 < \epsilon < \epsilon_{r,p}$. If $\mathcal{A} \subset 2^{[n]}$ is an r-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}) = p - \epsilon$, then there exists a star $\mathcal{B} \subset 2^{[n]}$ such that $\mu_{\boldsymbol{p}}(\mathcal{A} \triangle \mathcal{B}) < (C_p + o(1))\epsilon$, where $\boldsymbol{p} = (p, p, \dots, p)$ and*

$$C_p = \frac{4(r-1)^2 pq^2}{((r-1)p - (r-2))\,((2r-3) - 2(r-1)p)}.$$

Proof. The proof is the same as the proof of Proposition 3.6. We estimate (13) from both sides. For the RHS we see from (16) that the minimum and the second minimum eigenvalues come from the cases $|v| = 1$ and $|v| = 3$, so

$$\lambda_1 = 1 - \frac{1}{(r-1)q}, \quad \lambda_3 = \lambda_1^3.$$

For the LHS we use Lemma 2.11 with (17), and we have

$$\mathbb{E}_{\mu_2}[\varphi, \varphi] \leqslant \varphi_\varnothing \left(1 - \prod_{s=1}^{r-2} \frac{1}{1-\lambda_s}\right) = 1 - \frac{1}{r-1}.$$

Then we get the C_p exactly in the same way as in the proof of Proposition 3.6. $\square$

4.2. More about stability. Let $0 < p < 1$ and $\boldsymbol{p} = (p, \dots, p) \in (0,1)^n$. In Theorem 1.4 the statement of stability differs between the two cases (i) and (ii). In (ii) (the case $p > \frac{1}{2}$) we have the following example:

$$\mathcal{A}_n \coloneqq \left(\{A \in 2^{[n]} : 1 \in A, \, |A| > \tfrac{n}{2}\} \setminus \{[1]\}\right) \sqcup \{[n] \setminus [1]\}.$$

Then $\mathcal{A}_n$ is a 3-wise intersecting family with $\mu_{\boldsymbol{p}}(A_n) \to p$ as $n \to \infty$, but $\mathcal{A}_n$ is not contained in any star. It is worth noting that the stability in (i) (the case $p \leqslant \frac{1}{2}$) also differs from the situation in 2-wise intersecting case in Proposition 3.5. Indeed let

$$\mathcal{A}'_n \coloneqq \left(\{A \in 2^{[n]} : 1 \in A \setminus \{\{1\}\}\right) \cup \{[n] \setminus \{1\}\},$$

then $\mathcal{A}'_n$ is a 2-wise intersecting family with $\mu_{\boldsymbol{p}}(\mathcal{A}'_n) \to p$, but no star can contain $\mathcal{A}'_n$.

The constant C_p in Theorem 1.4 becomes very large when p is slightly more than $\frac{1}{2}$. This is because our proof relies on Theorem 3.7 and we need to distinguish our indicator from the indicator of (G2). But the family corresponding to (G2) is not 3-wise intersecting at all. This suggests that the C_p could be far from the best possible value especially when p is close to $\frac{1}{2}$, or even more bravely, we conjecture the following.

Conjecture 4.7. *There exists C'_p such that the inequality in* (ii) *of Theorem* 1.4 *can be replaced with $\mu_{\boldsymbol{p}}(\mathcal{A} \triangle \mathcal{B}) < (C'_p + o(1))\epsilon$, where $C'_p \leqslant C_p$ and moreover C'_p is increasing in p for $\frac{1}{2} \leqslant p < \frac{2}{3}$.*

The item (ii) of Theorem 1.4 can be extended to r-wise intersecting case as in Proposition 4.6. So maybe the item (i) could be extended to r-wise intersecting case as well.

Problem 4.8. *Let $r \geqslant 4$ and $p \leqslant \frac{r-2}{r-1}$. Is it true that the item* (i) *of Theorem* 1.4 *holds as well for r-wise intersecting families?*

It is also interesting to see whether or not Theorem 1.4 (and/or Theorem 3.7) can be extended to a general $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$.

Problem 4.9. *What happens if we replace $\boldsymbol{p} = (p, p, \dots, p)$ in Theorem* 1.4 *with $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ where $\frac{2}{3} > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$?*

4.3. Multiply cross intersecting families. We say that r families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_r \subset 2^{[n]}$ are *r-cross intersecting* if $A_1 \cap A_2 \cap \cdots \cap A_r \neq \varnothing$ for all $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, \dots, A_r \in \mathcal{A}_r$. Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_r \in (0,1)^n$ be given vectors. Then one can ask the maximum of $\prod_{i=1}^r \mu_{\boldsymbol{p}_i}(\mathcal{A}_i)$ for r-cross intersecting families. For the case $r = 2$, Suda et al. obtained the following result.

Theorem 4.10 ([16]). *For $i = 1, 2$ let $\boldsymbol{p}_i = (p_i^{(1)}, \dots, p_i^{(n)})$, and $p_i = \max\{p_i^{(\ell)} : \ell \in [n]\}$. Suppose that $p_1^{(\ell)}, p_2^{(\ell)} \leqslant 1/2$ for $\ell \geqslant 2$. If $\mathcal{A}_1, \mathcal{A}_2 \subset 2^{[n]}$ are 2-cross intersecting, then*

$$\mu_{\boldsymbol{p}_1}(\mathcal{A}_1)\mu_{\boldsymbol{p}_2}(\mathcal{A}_2) \leqslant p_1 p_2.$$

Moreover, unless $p_1 = p_2 = 1/2$ and $|w| \geqslant 3$, equality holds if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same star centered at some $\ell \in w$, where $w := \big\{\ell \in [n] : (p_1^{(\ell)}, p_2^{(\ell)}) = (p_1, p_2)\big\}$.

Almost nothing is known for the cases $r \geqslant 3$. Perhaps the easiest open problem is the case when $r = 3$ and $\boldsymbol{p}_i = (p, p, \dots, p)$ for all $1 \leqslant i \leqslant 3$.

Conjecture 4.11. *Let $p \leqslant \frac{2}{3}$ and $\boldsymbol{p} = (p, p, \dots, p)$. If $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \subset 2^{[n]}$ are 3-cross intersecting, then $\mu_{\boldsymbol{p}}(\mathcal{A}_1)\mu_{\boldsymbol{p}}(\mathcal{A}_2)\mu_{\boldsymbol{p}}(\mathcal{A}_3) \leqslant p^3$.*

Appendix A. Proof of Fact 2.5

(i):

$$\begin{aligned}
\langle f, Tg\rangle_{\mu_1} &= \sum_x f(x)(Tg)(x)\mu_1(x) \\
&= \sum_x f(x)\left(\sum_y \frac{\mu_2(x,y)}{\mu_1(x)} g(y)\right)\mu_1(x) \\
&= \sum_x \sum_y f(x)g(y)\mu_2(x,y) \\
&= \mathbb{E}_{\mu_2}[f, g].
\end{aligned}$$

(ii):

$$\langle f, Tg\rangle_{\mu_1} = \mathbb{E}_{\mu_2}[f, g] = \mathbb{E}_{\mu_2}[g, f] = \langle g, Tf\rangle_{\mu_1} = \langle Tf, g\rangle_{\mu_1}.$$

(iii):

$$(T\mathbf{1})(x) = \sum_y \frac{\mu_2(x,y)}{\mu_1(x)}\mathbf{1}(y) = \frac{1}{\mu_1(x)}\sum_y \mu_2(x,y) = 1 = \mathbf{1}(x).$$

(iv):

$$\langle \mathbf{1}, \mathbf{1}\rangle_{\mu_1} = \sum_x \mathbf{1}(x)\mathbf{1}(x)\mu_1(x) = \sum_x \mu_1(x) = 1.$$

(v):

$$\langle \varphi, \mathbf{1}\rangle_{\mu_1} = \sum_x \varphi(x)\mathbf{1}(x)\mu_1(x) = \sum_x \varphi(x)\mu_1(x) = \mathbb{E}_{\mu_1}[\varphi].$$

(vi):

$$\langle \varphi, \varphi\rangle_{\mu_1} = \sum_x \varphi(x)^2\mu_1(x) = \sum_x \varphi(x)\mu_1(x) = \mathbb{E}_{\mu_1}[\varphi].$$

Appendix B. Proof of Fact 2.7

(i): We have $\mathbb{E}_{\mu_1}[\varphi] = \langle \varphi, \mathbf{1}\rangle_{\mu_1} = \widehat{\varphi}_0$ and

$$\mathbb{E}_{\mu_1}[\varphi] = \langle \varphi, \varphi\rangle_{\mu_1} = \langle \widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i, \widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i\rangle_{\mu_1} = \widehat{\varphi}_0^2 + \sum_{i\geqslant 1}\widehat{\varphi}_i^2.$$

Thus $\sum_{i\geqslant 1}\widehat{\varphi}_i^2 = \mathbb{E}_{\mu_1}[\varphi] - \widehat{\varphi}_0^2 = \mathbb{E}_{\mu_1}[\varphi] - \mathbb{E}_{\mu_1}[\varphi]^2$, which we will use to show (ii).

(ii):

$$\begin{aligned}\mathbb{E}_{\mu_2}[\varphi,\varphi] &= \langle \varphi, T\varphi\rangle_{\mu_1} \\ &= \langle \widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i, T(\widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i)\rangle_{\mu_1} \\ &= \langle \widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i\boldsymbol{v}_i, \widehat{\varphi}_0\mathbf{1} + \sum_{i\geqslant 1}\widehat{\varphi}_i l_i\boldsymbol{v}_i\rangle_{\mu_1} \\ &= \widehat{\varphi}_0^2 + \sum_{i\geqslant 1}\widehat{\varphi}_i^2 l_i.\end{aligned}$$

Appendix C. Proof of Proposition 3.5

The proof is almost identical to the proof of Proposition 3.6, and we only give a sketch. The only difference is that in this case we have $\mathbb{E}_{\mu_2}[\varphi,\varphi]=0$, which makes things easier. Then by (13) we have

$$0 \geqslant \widehat{\varphi}_\varnothing + \lambda_1(1-\widehat{\varphi}_\varnothing-\tau)+\lambda_3\tau,$$

where $\widehat{\varphi}_\varnothing = p-\epsilon$, $\lambda_1 = -\frac{p_1}{q_1}$, and $\lambda_3 = (-\frac{p_1}{q_1})^3$. Rearranging we have

$$\tau \leqslant \frac{q^2\epsilon}{p(1-2p)},$$

and $\|\varphi^{>1}\|^2 < \frac{q^2\epsilon}{1-2p} =: \delta$. Then using Theorem 3.7 with this δ we get the result.

Acknowledgements. The author thanks Tsuyoshi Miezaki for valuable discussions. He also thanks the referees for their very careful reading and helpful suggestions.

References

[1] Rudolf Ahlswede and Gyula O. H. Katona, *Contributions to the geometry of Hamming spaces*, Discrete Math. **17** (1977), no. 1, 1–22.

[2] Rudolf Ahlswede and Levon H. Khachatrian, *The diametric theorem in Hamming spaces—optimal anticodes*, Adv. in Appl. Math. **20** (1998), no. 4, 429–449.

[3] Christian Bey and Konrad Engel, *Old and new results for the weighted t-intersection problem via AK-methods*, in Numbers, information and complexity (Bielefeld, 1998), Kluwer Acad. Publ., Boston, MA, 2000, pp. 45–74.

[4] Peter Borg, *Cross-intersecting integer sequences*, `https://arxiv.org/abs/1212.6955`, 2012.

[5] Alan Brace and David E. Daykin, *A finite set covering theorem*, Bull. Austral. Math. Soc. **5** (1971), 197–202.

[6] Yuval Filmus, Konstantin Golubev, and Noam Lifshitz, *High dimensional Hoffman bound*, `https://yuvalfilmus.cs.technion.ac.il/Talks/Hoffman.pdf`, 2020.

[7] Yuval Filmus, Konstantin Golubev, and Noam Lifshitz, *High dimensional Hoffman bound and applications in extremal combinatorics*, Algebr. Comb. **4** (2021), no. 6, 1005–1026.

[8] Peter C. Fishburn, Peter Frankl, Daniel Freed, Jeffrey C. Lagarias, and Andrew M. Odlyzko, *Probabilities for intersecting systems and random subsets of finite sets*, SIAM J. Algebraic Discrete Methods **7** (1986), no. 1, 73–79.

[9] Peter Frankl and Norihide Tokushige, *Weighted multiply intersecting families*, Studia Sci. Math. Hungar. **40** (2003), no. 3, 287–291.

[10] ———, *Extremal problems for finite sets*, Student Mathematical Library, vol. 86, American Mathematical Society, Providence, RI, 2018.

[11] Ehud Friedgut, *On the measure of intersecting families, uniqueness and stability*, Combinatorica **28** (2008), no. 5, 503–528.

[12] Ehud Friedgut, Gil Kalai, and Assaf Naor, *Boolean functions whose Fourier transform is concentrated on the first two levels*, Adv. in Appl. Math. **29** (2002), no. 3, 427–437.

[13] Willem H. Haemers, *Hoffman's ratio bound*, `https://arxiv.org/abs/2102.05529`, 2021.

[14] Guy Kindler, *Property testing, PCP, and juntas*, Ph.D. thesis, Tel Aviv University, 2002.

[15] Guy Kindler and Shmuel Safra, *Noise-resistant boolean-functions are juntas*, `https://www.cs.huji.ac.il/w~gkindler/`, 2004.

[16] Sho Suda, Hajime Tanaka, and Norihide Tokushige, *A semidefinite programming approach to a cross-intersection problem with measures*, Math. Program. **166** (2017), no. 1-2, Ser. A, 113–130.

Norihide Tokushige, University of the Ryukyus, College of Education, 1 Senbaru Nishihara, Okinawa, 903-0213, Japan
E-mail : hide@edu.u-ryukyu.ac.jp