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Twisted quadrics and α-flocks

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Abstract In this article, we provide a general study of what we call twisted quadrics and consider flocks of the variant of α-conics and α-hyperbolic quadrics. We extend the notion of the Klein quadric to what we call an α-Klein quadric. Blended kernel translation planes are defined and analysed when considering α-conical flocks and α-twisted hyperbolic flocks.

The Thas–Walker constructions of conical flocks and flocks of hyperbolic quadrics are extended to their α-analogues. Using the idea that any derivable net can be embedded into a 3-dimensional projective space over a skewfield, allows us to formulate what might be called a projective version of work previously given in an algebraic framework. The theory of deficiency one flocks is extended to both α-conical flocks and α-twisted hyperbolic flocks. j-planes are used to construct two infinite classes of finite α-hyperbolic flocks.

1. Introduction

This article is something of a culmination of results of flocks of hyperbolic quadrics, flocks of quadratic cones and their connections to translation planes admitting so-called regulus-inducing groups. The study of flocks of finite quadratic cones has generated the most interest. The theory of Thas and Bader–Lunardon [33, 2] completely classifies finite flocks of hyperbolic quadrics. Then there are the deficiency one flocks of quadratic cones and deficiency one flocks of hyperbolic quadrics. These are connected by results of the author to translation planes that admit certain Baer groups, (see [11]). By ingenious arguments, Thas and Payne [29] and, later by other mathematicians, every such deficiency one partial flock of a finite quadratic cone may be extended uniquely to a flock of a quadratic cone–saying something very interesting about the associated Baer group planes being derivable. However, the study of deficiency one hyperbolic flocks is still alive, as there are a few–just a few–examples showing that an extension is not always possible.

Jha and Johnson [10] also analyze deficiency one conical and hyperbolic flocks over an arbitrary field and show these are equivalent to a translation plane admitting certain Baer groups.

There are appearances of flocks of quadratic cones, as in hyperbolic fibrations with constant backhalf, which oddly enough are also connected to translation planes admitting certain collineation groups (cyclic homology groups of order $q + 1$ if the field is $GF(q)$, and analogous groups when the field K is infinite, Johnson et al ([17, 25])). As certain j-planes (translation planes, also) admit such groups, there are

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corresponding flocks of quadratic cones from j-planes, as well. In this way, in the infinite case, all j-planes over the field of real numbers have been determined by the author [17].

In the finite case, all of these planes involve the analysis of translation planes admitting regulus nets. The only other derivable net (in the finite case) is the twisted regulus net (twisted by an automorphism of $GF(q)$).

It might be noted that the twisted regulus net is also a regulus net over a different scalar field, due to the embedding theory of the author [16]. In this general study, any derivable net may be shown to be a "pseudo-regulus" net in some 3-dimensional projective space over a skewfield F. As an aside, the author recently completed a study of all derivable nets over any skewfield (see [21]), where there is quite a variety of interesting derivable nets other than the twisted regulus net. So, at least in the infinite case, the theory of flocks of different colors and corresponding translation planes is far from complete. We shall come back to the classification of derivable nets later in this article.

Concerning twisted regulus nets, which occur both in the finite and infinite cases, when the associated field K admits an automorphism α (they all do, if $\alpha = 1$), we can consider the twisted versions of conical flocks and hyperbolic quadrics over arbitrary fields.

The theory of α-flocks of α-cones has been completed by Cherowitzo, Johnson and Vega [6] and connects translation planes admitting α-regulus inducing elation groups with flocks of α-cones (here the term is α-flokki, coined by Kantor and Pentilla [27], who pointed the way to this theory).

Furthermore, the author [14] shows that the theory of hyperbolic flocks and associated translation planes may be generalized by considering twisted hyperbolic flocks and translation planes admitting twisted regulus inducing homology groups.

It is known that there cannot be finite partial flocks of quadratic cones of deficiency one, and the concept of deficiency one partial flocks of α-cones has been analyzed in Cherowitzo, Johnson and Vega [6]. In that article, it was shown that all finite deficiency one α-flocks (α-flokki) may be extended to a flock, using an ingenious argument of Sziklai [31].

There are, however, deficiency one partial hyperbolic flocks and there is a corresponding equivalent theory of translation planes admitting certain Baer homology groups. In this article, we discuss the known examples of such deficiency one hyperbolic flocks and further consider and complete the analogous deficiency one twisted hyperbolic flock theory.

It might be mentioned that for all of the previous work, the translation planes were always of dimension 2; that is, their spreads lie in $PG(3, K)$, for some field K. For deficiency one twisted hyperbolic flocks, we develop some theory on what we call blended kernels to completely describe the translation planes associated, as they no longer are of dimension 2.

In [6], the study of maximal partial α-flokki was considered and connected with maximal partial spreads that are called "quasifibrations". These occur only in the infinite cases and are strange and wonderful objects. The quasifibrations studied previously admit elation α-regulus inducing groups. Here we consider quasifibrations that admit homology α-regulus-inducing groups. Then there are the associated Baer groups but now acting on quasifibrations that are not of dimension 2 and have blended kernels. So, everything one can say about translation planes and associated flocks and α-flocks, one can say about quasifibrations and maximal partial flocks called quasi α-flocks.

We list the main result of the author, for convenience.

Theorem 1.1 (Johnson [20]). *Let Σ be a translation plane with spread in $PG(3,K)$, for K an arbitrary field. Let α denote an automorphism of K, possibly trivial. Assume that Σ admits an affine homology group one orbit of which, together with the axis and coaxis, is a twisted regulus net. Then all orbits are twisted regulus nets and the spread may be coordinatized in the following form: Let V_4 be the associated 4-dimensional vector space over K. Letting x and y denote 2-vectors, then the spread is:*

$$x=0,\ y=0,\ y=x\begin{bmatrix}u^{\alpha} & 0\\ 0 & u\end{bmatrix},$$
$$\text{and } y=x\begin{bmatrix}F(t) & G(t)\\ 1 & t\end{bmatrix}\begin{bmatrix}v^{\alpha} & 0\\ 0 & v\end{bmatrix};$$
$$\forall u,t,v,uv\neq 0, of\ K,$$

and functions F,G on K. Furthermore, F is bijective. The "α-twisted hyperbolic quadric" has the form:

$$\{(x_1,x_2,x_3,x_4);\ \text{such that } x_1x_4^{\alpha}=x_2^{\alpha}x_3\}.$$

Then there is a flock of the α-twisted hyperbolic quadric with the flock of planes of $PG(3,K)$, as follows:

$$\pi_t:-x_1G(t)^{\alpha}+x_2F(t)-x_3t^{\alpha}+x_4=0, and\ \rho:x_2=x_3,$$

where the intersection with each plane is a non-degenerate α-conic. Conversely, a flock of a twisted hyperbolic flock corresponds to a translation plane admitting an α-regulus inducing homology group.

When $\alpha=1$, we have a flock of a hyperbolic quadric and an associated translation plane admitting a regulus-inducing affine homology group. These two geometries, the hyperbolic flocks and the translation planes are equivalent. There are exactly the following classes; the flocks are the linear flock, where the associated planes of $PG(3,q)$ share a line, and the Thas flocks, with a few exceptions. The Thas flocks correspond to the regular nearfield planes and the exceptional flocks correspond to the irregular nearfield planes and are due to a number of mathematicians from various different points of view, Bader [1], Baker–Ebert [3], Bonisoli [6], Johnson [14]. There are three irregular nearfield planes that correspond to hyperbolic flocks of orders $11^2,23^2,59^2$ which Bader, Bonisoli and Johnson found, independently for all three orders, and by Baker and Ebert for orders $11^2,23^2$. All of these mathematicians determined the flocks/translation planes by using essentially different methods. The main point here is that the associated translation planes are all Bol planes; which has been of considerable interest, and there is a complete classification due to Thas and Bader-Lunardon [2, 32]. There are two (at least) possible formulations for this classification, depending on when it is phrased in the associated translation plane or in the hyperbolic flock.

Theorem 1.2 (Bader–Lunardon [2], Thas [32]). *Classification of finite hyperbolic flocks/translation planes admitting regulus-inducing homology groups in $PG(3,q)$.*

Plane version: The translation planes are nearfield planes; based upon the regular nearfields of order q^2 and the three irregular nearfields of orders $11^2,23^2$ and 59^2.

Flock version: The hyperbolic flocks are exactly the Thas flocks and the flocks of Bader, Baker–Ebert, Bonisoli, Johnson.

In this article, we consider partial flocks of twisted hyperbolic flocks of deficiency one and show that these correspond to a certain type of translation plane admitting a Baer group that we call an α-Baer homology group. We also study partial flocks of twisted hyperbolic type and of α-flocks of deficiency one, and this time, there is an associated translation plane admitting an α-Baer elation group. The translation planes are equivalent to the partial twisted flocks of deficiency one. However, the translation planes do not have spreads in $PG(3,K)$, but have what we call "blended kernels".

We also are interested in the Baer theory for α-conical flocks, and complete that theory as well. We list the main theorem for background. The notation is changed to fit our definition of α-regulus nets.

In this article, we define an α-regulus net as follows:

$$x=0, y=x\begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix}; u\in K,\ \alpha \text{ an automorphism of } K.$$

Where in Cherowitzo, Johnson, and Vega [6], this would have been called an α^{-1}-regulus net.

Theorem 1.3 (Cherowitzo–Johnson–Vega [6]). *Let K be any field and α an automorphism of K. Let (x_1,x_2,x_3,x_4) denote homogeneous coordinates of $PG(3,K)$. Define C_α an α-cone as $x_1^{\alpha}x_2 = x_3^{\alpha+1}$, with vertex $(0,0,0,1)$. A set of planes which partition the non-vertex points of C_α will be called an α-flokki (also an α-conical flock). The plane intersections are called α-conics. Assume that π is a translation plane that admits an elation group E^{α}, one component of which together with the axis is an α-regulus net has the following form*

$$x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u^{\alpha} \end{bmatrix}; t,u\in K, f,g \text{ functions of } K,$$

when writing the α-regulus in the form

$$x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u^{\alpha} \end{bmatrix} = \begin{bmatrix} v^{\alpha^{-1}} & 0 \\ 0 & v \end{bmatrix}; t,u,v\in K.$$

Then there is a corresponding α-flokki with planes, corresponding to

$$\rho_t : x_1 t - x_2 f(t)^{\alpha} + x_3 g(t)^{\alpha} - x_4; t\in K.$$

Conversely, an α-flokki may be written in the form $\{\rho_t; t\in K\}$, for some functions $f(t)$ and $g(t)$, which constructs a translation plane with the previous form. Hence, translation planes with spreads in $PG(3,K)$ that admit α-regulus inducing elation groups are equivalent to flocks of an α-quadratic cone.

We also give a Thas–Walker style construction-proof of the Cherowitzo–Johnson–Vega result, which will allow a more complete way to deal with deficiency one α-conical flocks.

Concerning deficiency one partial flocks of quadratic cones and deficiency one partial flocks of hyperbolic quadrics, we list for convenience the following theorem of Jha and the author. The approach by Jha and Johnson using the Klein quadric will be generalized using what we shall call blended kernel translation planes. The two theorems corresponding to the elation groups and homology groups shall be separated for clarity.

Theorem 1.4 (Jha-Johnson [10]). *Let K be a field and let π be a translation plane with spread in $PG(3,K)$ that admits a full point-Baer elation group. Then there is a corresponding partial conical flock of deficiency one in $PG(3,K)$. Conversely, any partial conical flock of deficiency one constructs a translation plane with spread in $PG(3,K)$ that admits a full point-Baer elation group. The partial conical flock of deficiency one may be extended to a flock of a quadratic cone if and only if the net containing the point-Baer axis is a regulus net sharing the axis.*

Theorem 1.5 (Jha–Johnson [10]). *Let K be a field and let Σ be a translation plane with spread in $PG(3,K)$ that admits a full point-Baer homology group. Then there is a corresponding partial hyperbolic flock of deficiency one in $PG(3,K)$. Conversely, any partial hyperbolic flock of deficiency one constructs a translation plane with spread in $PG(3,K)$ that admits a full point-Baer homology group. The partial hyperbolic flock may be extended to a hyperbolic flock if and only if the net containing the point-Baer axis and coaxis is a regulus net.*

Remark 1.6. Note that in the above two theorems there is the term "point-Baer" in the hypotheses. A Baer subplane in an affine plane must have two properties: Every point of the plane must be on a line of the subplane ("point-Baer") and every line of the plane must be on a line of the subplane ("line-Baer"). These conditions are equivalent in the finite case but, in the infinite case, Barlotti [4] shows that they are not! The author [15] constructs dual translation planes that admit a derivable net which are not derivable! So, when dealing with the infinite case, care must be taken (the interested reader might also look at (25.3) of [16] for more explanation). When we give our main extension results, we will show that we may remove the point-Baer hypothesis in the setting under consideration.

We also point out that the $j = \frac{p^s-1}{2}-$planes of Johnson, Pomareda, Wilke [12] provide two infinite classes of finite p^s-twisted hyperbolic flocks.

2. Blended kernels

In the author's work [16], it is shown that every derivable net may be embedded into a 3-dimensional vector space over a skewfield K, $PG(3,K)$, such that points, lines of the derivable net correspond to lines and points not incident with a fixed line N of $PG(3,K)$. The Baer subplanes of the net correspond to planes that do not contain N, and parallel classes of the derivable net correspond to the planes of $PG(3,K)$ that contain N. It then follows that the collineation group of the derivable net may be determined as $P\Gamma L(4,K)_N$.

The main result of this work shows that every derivable net is a classical pseudo-regulus net, which is then a classical regulus net when K is a field. Using certain natural subgroups of the derivable net, we may realize an embedding in an associated 4-dimensional vector space over K, with left kernel mappings $(x_1,x_2,x_3,x_4) \to (\delta x_1, \delta x_2, \delta x_3, \delta x_4)$, for $\delta \in K^*$. And thus it is possible to obtain the form for the derivable net, which shall here be called the "classical form" is as follows: Assume that we are considering V a left K-space.

$$x = 0, y = \delta x; \forall \delta \in K; \text{components are right spaces,}$$
$$P(a,b) = \{(ca, cb, da, db); \forall c, d \in K\};\ \text{Baer subspaces are left spaces.}$$

The derived net is

$$x = 0, y = x\delta; \forall \delta \in K;\ \text{components are left spaces;}$$
$$P(a,b) = \{(ac, bc, ad, bd); \forall c, d \in K\};\ \text{Baer subspaces are right spaces.}$$

So, the form is "classical" due to the choice of the subgroup of $P\Gamma L(4,K))_N$, that we use to represent the translation group back into the constructed 4-dimensional vector space. Suppose that K has an automorphism α, and consider, for the moment, that K is a field. Then a classical regulus net R has the form:

$$R\ :\ x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K;\ \text{the components and}$$
$$P(a,b) = \{(ac, bc, ad, bd); \forall c, d \in K\}; \textit{Baer subspaces.}$$

Now it is known that there are also “twisted regulus nets”, twisted by an automorphism α of K, and the classical ones have the form:

$$R^{\alpha} \; : \; x = 0, y = x \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \forall u \in K; \text{ the components and}$$
$$P(a,b) = \{(ac^{\alpha}, bc, ad^{\alpha}, bd); \forall c, d \in K\}; Baer \text{ components.}$$

Now note that the Baer subplanes are no longer subspaces over the original kernel mappings. Since every derivable net is a classical regulus net, we seem to have a contradiction. The catch is that using the embedding theory, a different “kernel” group could be used to turn the derivable net into a classical regulus net. In the twisted version, we use the new kernel with mappings

$$(x_1, x_2, x_3, x_4) \to (\delta^{\alpha} x_1, \delta x_2, \delta^{\alpha} x_3, \delta x_4); \; for \; \delta \in K^{*}.$$

Call this group $K^{*\alpha}$. If we then define $x \cdot u = x \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix}$, where x is a 2-vector over K, we see that that the components and the Baer subplanes are then both K^{α}-subspaces, so we have the classical derivable net but over a different but isomorphic field K^{α}. Note that R^{α} has components that are K and K^{α}-subspaces, where the Baer subplanes are only K^{α}-subspaces, when $\alpha \neq 1$.

Now to switch ideas again, consider the twisted regulus net over K, and form the derived net. Now this net is also embeddable and becomes a classical regulus net under a suitable field L. So, that we can appreciate what the form must be, if we wish to transform the Baer “components” to standard form components, we make a transformation of the Baer subspaces as follows:

$P(a,b) = \{(ac^{\alpha}, bc, ad^{\alpha}, bd); \forall c, d \in K\}; Baer$ components, and form the basis change transformation

$$\theta : (x_1, x_2, x_3, x_4) \to (x_1, x_3, x_2, x_4).$$

If $x^{\alpha^{-1}} = (x_1^{\alpha^{-1}}, x_2^{\alpha^{-1}})$, where $x = (x_1, x_2)$, we see that $P(a,b)$ under θ consists of the following vectors: $(ac^{\alpha}, ad^{\alpha}, bc, bd)$, for $a \neq 0$, is incident with $y = x^{\alpha^{-1}} \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$, for $x = (c^{\alpha}, d^{\alpha})$ and $u = a^{-1}b$. Note that c and d vary over K. When $a = 0$, we see we have the form of the derived net as:

$$x = 0, y = x^{\alpha^{-1}} \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K;$$

But, now we are using a different kernel group:

$$^{\alpha}K^{*} : (x_1, x_2, x_3, x_4) \to (\delta^{\alpha} x_1, \delta^{\alpha} x_2, \delta x_3, \delta x_4); \; for \; \delta \in K^{*}.$$

Again, this group may be used in the embedded 3-dimensional projective space $PG(3,K)$, to create a different 4-dimensional vector space over $^{\alpha}K$, again creating a classical regulus net with a different scalar mapping and, hence, a different but isomorphic kernel.

These ideas come into play when we have a translation plane with spread in $PG(3,K)$, with the standard kernel group K^{*}, but we have a twisted regulus net, whose components are both K and K^{α} subspaces. When we derive this twisted regulus net, the derived plane does not have kernel K and it does not have kernel K^{α}, the kernel might be said to be “blended”.

Definition 2.1. *Given two fields K and R. If a translation plane π is a union of subspaces $P \cup L$, where all subspaces in P are 2-dimensional K-subspaces and all subspaces of L are 2-dimensional R-subspaces, we shall say that π is a translation plane with blended kernel (K,R) of dimension 2.*

In the situation under consideration, the kernel would be $Fix(\alpha)$, and the blended kernel would be (K, K^{α}).

Definition 2.2. *Clearly the idea of blended kernel extends to a set of distinct fields K_i for $i \in \lambda$ and $\cup P_i$ subspaces making up the spread for π, where P_i are K_i-subspaces. Furthermore, the dimensions need not always be equal. We use the dimension 2 only as this is the situation in the current article.*

Translation planes with blended kernels have nice properties since the collineation group must leave all P_i invariant. When a translation plane with blended kernel arises from net replacement, there are then a set of interesting and mutually non-isomorphic translation planes obtained.

Consider the following translation planes:

$$x = 0, y = x \begin{bmatrix} u^{\alpha} & bt^{\alpha} \\ t & u \end{bmatrix}; u, t \in K,\ K \text{ a field.}$$

K could be finite or infinite. In fact, the matrix set could also define a quaternion division ring, a finite Hughes–Kleinfeld semifield plane or a (generalized) Hughes–Kleinfeld semifield plane over an infinite field. We note that this plane corresponds to an α-conical flock and to an α-twisted hyperbolic flock, (see Cherowitzo, Johnson, Vega [6], Johnson [20], [21]). Consider the finite case, where the field K is $GF(q)$. Then there are q twisted regulus nets corresponding to the α-conical flock sharing $x = 0$ and $q+1$ twisted regulus nets sharing $x = 0$ and $y = 0$. Similarly, there are the same/analogous twisted regulus nets when K is an infinite field.

When one of the nets is derived, we obtain a blended translation plane with respect to (K, K^{α}) or with respect to $(K,^{\alpha} K)$. Hence, the full collineation group of the translation plane will leave the derived net invariant. Furthermore, it follows that given any two of these $2q + 1$ possible translation planes or infinitely many possible translation planes in the infinite field case, any isomorphism Γ must map the derived net of one plane to the derived net of the other plane, and then must act as a collineation group of the original translation plane. In these cases, since the translation plane is always a semifield plane, we have that the q possible blended translation planes obtained from deriving a twisted regulus net sharing $x = 0$ are all isomorphic, whereas, in the finite case, we would obtain a set of $q + 1$ mutually non-isomorphic blended translation planes. In the infinite case, the translation planes obtained from deriving a twisted regulus net sharing $x = 0$ and $y = 0$ are isomorphic if and only if the translation plane is a quaternion division ring plane.

Thus we have:

Theorem 2.3. *The translation planes with blended kernel (K, K^{α}) in the finite case, where K is $GF(q)$, produce exactly $q+2$ mutually non-isomorphic blended translation planes. When K is infinite either there are infinitely many mutually non-isomorphic blended translation planes or α has order 2 and the associated translation plane is a quaternion division ring plane, and there are exactly two non-isomorphic blended translation planes.*

2.1. Lifting quasifibrations.

In this section, we give a short reminder of the concept of lifting a translation plane of dimension 2. The term "quasifibration" might not be well known, so we provide a more general definition for any finite dimension n.

Definition 2.4. *Let Q be any partial spread over a field K, where the associated vector space is a $2k-$dimensional vector space over K and there is a matrix representation of Q as a set of mutually disjoint k-dimensional K-subspaces, where k is* *a positive integer (the components). If there is a row among the matrix spread set of the form $[e_1, e_2, ..., e_k]$, for all $e_i \in K$, Q shall be called a quasifibration of dimension k. Therefore, clearly Q is a maximal partial spread. If there is a vector that is not incident with a component of Q, then Q is said to be a proper quasifibration. If the context is clear, we shall often just use the term "quasifibration" to mean proper quasifibration.*

For this subsection, the reader is referred to Biliotti, Jha, Johnson [5], and Johnson, Jha [24] for any additional reference material. The main idea of lifting is that from any translation plane or quasifibration with spread in $PG(3, K)$, where K is a field that admits a quadratic extension with non-trivial automorphism σ of order 2 of $F = K(\theta)$ fixing K pointwise, then there is a translation plane admitting a σ-twisted derivable net in $PG(3, K(\theta))$ as follows:

If the quasifibration (there may not be a complete cover) is represented in the form:

$$x = 0, y = x \begin{bmatrix} f(t,u) & g(t,u) \\ t & u \end{bmatrix}; t, u \in K$$

then the "lifted quasifibration has the following form

$$x = 0, y = x \begin{bmatrix} w^\sigma & (\theta f(t,u) + g(t,u))^\sigma \\ \theta t + u & w \end{bmatrix}; t, u \in K,\ w \in F,$$

and is a spread or a quasifibration equivalent to a σ-flock of a σ-cone (or a maximal partial structure). Notice that we have a σ-twisted derivable net

$$x = 0, y = x \begin{bmatrix} w^\sigma & 0 \\ 0 & w \end{bmatrix}; t, u \in K,\ w \in F.$$

Therefore, the derived plane is a translation plane/quasifibration with blended kernel (K, K^σ). For a translation plane that corresponds to an α-flock of an α-cone, but is not a lifted plane, we consider the following:

2.2. The Kantor–Pentilla translation planes.

Kantor and Pentilla [27] construct the following translation planes:

$$x = 0, y = x \begin{bmatrix} u^2 + t^5 & t^{14} \\ t & u \end{bmatrix}; t, u \in GF(2^e), \text{where 3 does not divide } e.$$

We note here that again we have a 2-twisted derivable net. The derived translation plane has blended kernel (K, K^2), for $K = GF(2^e)$.

2.3. Blended quasifibrations.

There are examples of quasifibrations in $PG(3, K)$ that may be lifted to quasifibrations or correspond to partial α-conical flocks in Cherowitzo, Johnson, Vega [6], and Biliotti, Jha, Johnson [5]. All of these quasifibrations correspond to quasifibrations that admit an α-twisted regulus net and hence may be derived. All of these derived quasifibrations or translation planes are with blended kernel (K, K^α).

2.4. Twisted quadrics.

- In preparation for the statement of the main results, we mention the concept of "twisted quadrics" of $PG(3, K)$. Recall the idea of a twisted regulus net:

$$x = 0, y = x \begin{bmatrix} w^\alpha & 0 \\ 0 & w \end{bmatrix}; t, u \in K,\ w \in K,\ K \text{ a field},$$

$$\alpha \text{ an automorphism of } K.$$

$$P^\alpha(a,b) = \left\{ ((a,b) \begin{bmatrix} u^\alpha & 0 \\ 0 & u \end{bmatrix}, (a,b) \begin{bmatrix} v^\alpha & 0 \\ 0 & v \end{bmatrix}); \forall u, v \in K \right\};$$

Baer components.

We note that although the twisted regulus net is not a regulus over K, it is a regulus over K^α.

- Consider a hyperbolic quadric Γ over K with homogeneous coordinates (x_1, x_2, x_3, x_4) such that

$$x_1 x_4 = x_2 x_3.$$

Now write (x_1, x_2, x_3, x_4) over K^α, so that the same vector with the same vector basis but writing over K^α, becomes $(x_1^{\alpha^{-1}}, x_2, x_3^{\alpha^{-1}}, x_4)$, that we have the associated α-twisted regulus as

$$x_1^\alpha x_4 = x_2 x_3^\alpha.$$

(In the author's work, (see [20]) on twisted hyperbolic flocks, the form,

$$x_1 x_4^\alpha = x_2^\alpha x_3$$

is used, as this fits better with the algebraic form).

- Consider a quadratic cone over K, with homogeneous coordinates (x_1, x_2, x_3, x_4) such that

$$x_1 x_4 = x_2^2.$$

Note that there is a conic in the plane in $x_2 = x_3$. Now consider α-twisted regulus net over K^α as a regulus and then consider the associated α-quadric over K^α has the form

$$x_1^\alpha x_4 = x_2^{\alpha+1}.$$

3. Foundations and examples

As mentioned, there are α-conical flocks associated with translation planes that admit certain elation groups and there are α-hyperbolic flocks associated with translation planes that admit certain homology groups. All of these planes may be derived to produce translation planes with blended kernel (K, K^α). Now the elation and homology groups become Baer collineation groups under derivation.

It is possible to study Baer collineation groups in translation planes with blended kernel (K, K^α), without the assumption that the associated net containing the Baer axis (Baer axis and coaxis) is derivable. It will become apparent that such a translation plane will be missing exactly one twisted regulus net (regulus net if $\alpha = 1$) and then will correspond to a partial flock of a α-conical or α-twisted hyperbolic flock of deficiency one. In fact, we show that any such deficiency one partial flock corresponds to such a translation plane with blended kernel.

When K is finite, the deficiency one conical flock may be extended to a flock and this result has been proved by Payne and Thas [29]. Considering deficiency 1 α-conical flocks in the finite case also may be extended to α-conical flocks by result of Cherowitzo, Johnson, and Vega [6]. We also extend the ideas of deficiency one in this article.

3.1. An unusual twisted hyperbolic flock. We note that any deficiency $q-1$ partial spread in $PG(3,K)$, may be a maximal partial spread but still be embeddable in an affine plane. By a result of Jungnickel [26], if the maximal partial spread is embeddable then the affine plane is the unique extension and is a translation plane. The catch is that the translation plane will not have dimension 2. That is, the spread will not be in $PG(3,K)$, it will be in a translation plane with blended kernel (K,K^{α}), or a similar variant.

Consider the following putative spread with blended kernel $(GF(4),GF(4)^2)$

$$x=0, y=(x_1^2,x_1^2)\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$
$$y=x\begin{bmatrix} v^2 & b+1 \\ 1 & v \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; v,u,s\in GF(4), \text{ and } b+1 \text{ and } b \text{ both } \neq 0.$$

Assume, for the moment, that this is a translation plane. Recall that a classical K-regulus net has the form

$$x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u\in K$$

where all vectors are over a 4-dimensional vector space V_4 with standard scalar multiplication K^*. Since $\begin{bmatrix} v^2 & b+1 \\ 1 & v \end{bmatrix}$ are non-singular matrices, a transformation by

$$\begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} v^2 & b+1 \\ 1 & v \end{bmatrix}^{-1} \end{bmatrix}$$

will transform

$$x=0, y=x\begin{bmatrix} v^2 & b+1 \\ 1 & v \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; \forall s\in GF(4)^*$$

to the classical form. Hence, we have a set of 4 regulus nets sharing $x=0, y=0$.

Now while the remaining

$$x=0, y=(x_1^2,x_1^2)\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u\in GF(4)$$

admits the "regulus-inducing" group

$$\begin{bmatrix} I_2 & 0_2 \\ 0_2 & \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \end{bmatrix}; u\in GF(4)^*.$$

of order $4-1$, this set of matrices is no longer invariant under the mappings $GF(4)^*$; this is not a spread in $PG(3,GF(4))$, but is a spread with a blended kernel.

Hence, this set is not a regulus. It is the derived version of a 2-twisted regulus net.

Now note that since we have $q=4$ regulus nets sharing $x=0$ and $y=0$, there is a corresponding partial hyperbolic flock of deficiency one which cannot be extended to a hyperbolic flock. That is, if it could be so extended, it would correspond to a translation plane with spread in $PG(3,16)$. Then this would be the unique affine plane extension of the corresponding partial spread in $PG(3,16)$ of deficiency $q-1=3$. Now to see that we indeed have a spread, we derive the structure given. We leave the details to the reader to see that the derived structure has the following form:

$$x=0, y=x\begin{bmatrix} u^2 & bt^2 \\ t & u \end{bmatrix}; t,u\in GF(4).$$

The regulus-inducing group now has the following form:

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} & 0_2 \\ 0_2 & \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} \end{bmatrix}; v \in GF(4)^*.$$

Here the group is what we call a full Baer homology group with axis $(x_1, 0, x_3, 0)$ and coaxis $(0, x_2, 0, x_4)$, where x_i vary over K, $i = 1, 2, 3, 4$. We see that using the images of $y = x \begin{bmatrix} s^2 & b \\ 1 & s \end{bmatrix}$ the group will map these elements to $y = x \begin{bmatrix} s^2 & bv \\ v^{-1} & s \end{bmatrix}, \forall s, v \neq 0$. Letting $v^{-1} = t$, we see that $t^2 = v$, since $t \in GF(4)$.

So, we obtain a semifield translation plane of order 16, and to check that we have a spread we need only check that the non-zero matrices are non-singular. The determinant is $s^3 + bv^3$, which is either $1 + b, 1$ or b. Taking b and $b + 1$ nonzero, we have proved that we have a spread.

- This spread is unusual in that it provides a 2-flock of an $2-$cone, a 2-twisted hyperbolic flock and a partial hyperbolic flock of deficiency one. Most unusual.

3.2. The examples of deficiency one partial hyperbolic flocks. In addition to the spread of the previous subsection, there are exactly five other examples.

There is a similar derivable translation plane of order 81, due to Johnson and Pomareda [13], also of order p^4, for p a prime, with blended kernel also containing

$$x = 0, y = (x_1^p, x_1^p) \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; u \in GF(p^2),\ p \text{ a prime}.$$

There is the associated derived spread has spread

$$x = 0, y = x \begin{bmatrix} u^p & bt^{-1} \\ t, & u \end{bmatrix}; u, t \in GF(p^2),\ p = 3, b \text{ an appropriate constant}.$$

And the net listed above is the derived net of the twisted regulus net, given when $t = 0$ $(0^{-1} = 0)$. Consider the derivates of these two translation planes. This translation plane and the previous one, when derived are derivable by a p-twisted regulus net and are also transitive on the components other than $x = 0$. In Johnson and Cordero [22], there is a classification of all such translation planes. Note that the previous homology group of order $p^2 - 1$ now appears as a Baer group in the plane with spread in $PG(3, p^2)$.

These two planes of orders 16 and 81 of order p^2, are extraordinarily exceptions, as seen in the following result.

Theorem 3.1 (Johnson-Cordero [22]). *Let π be a translation plane in $PG(3, p^2)$, for p a prime that is derivable by a p-twisted regulus net and is also transitive on the components distinct from $x = 0$. If the plane admits a Baer group of order $p^2 - 1$, then the plane is the translation plane corresponding to either the Johnson partial hyperbolic flock of deficiency one or the Johnson-Pomareda partial hyperbolic flock of deficiency one, of order p^4 for $p = 2$ and 3, respectively.*

The four other partial hyperbolic flocks are of orders 25 or 49 and are found by using a computer by Royle [30].

Since translation planes of order p^2, for p a prime, do not admit twisted regulus nets, the associated translation planes cannot be derivable. Hence, we have:

Theorem 3.2. *The four Royle deficiency one partial hyperbolic flocks, two of order 5^2 and two of order 7^2, correspond to translation planes in $PG(3, GF(p))$, for $p = 5, 7$ that admit Baer groups of order $p - 1$. There are p orbits $(p^2 + 1 - (p + 1))/(p - 1)$*

such that the Baer axis and coaxis together with the p orbits form a partial hyperbolic flock of deficiency one.

- All translation planes of orders 25 and 49 are known by computer searches by Mathon and Royle [28] and Czerwinski and Oakden [7].
- The two translation planes of order 25 in question are identified as A_2 and B_5. These identifications might not be correct. This is due to the fact that A_2 is identified further in Czerwinski and Oakden [7] as a Dickson nearfield plane. But this plane corresponds not to a partial hyperbolic flock but to an extended one, which would imply that the Baer net is derivable. Hence, the partial hyperbolic flock is not maximal or the plane was misidentified.
- B_5 is identified as a Walker plane by Czerwinski and Oakden [7]. However, if it is meant that this is the Walker plane that admits a group of order $5(5-1)$, this cannot correspond to a translation plane that admits a Baer group (see the last section). We must have a Baer group of order 4, which does not occur in the Walker plane. Also, another problem is that if B_5 is a Walker plane, the Walker plane is derivable, by Jha–Johnson [9], and the translation plane in question does not contain a regulus, according to the computer program. But, even if it did, the partial hyperbolic flock would be extendable.
- The translation planes order 49 are identified as $S771$ and $S773$. Since the associated translation planes must admit a Baer group of order $7-1$, there is probably a Baer net of degree $7+1=8$ that is invariant (this is just a speculation, at this point). In both of these translation planes there is an orbit of length 8. This would be very interesting, as it would say that the components of the Baer net are permuted in one orbit.

4. Quasifibrations as T-copies

Previously, we mentioned quasifibrations of dimension 2. Therefore, we have a quasifibration of the following form in $PG(3,K)$, for K a field:

$$x=0, y=x\begin{bmatrix} f(t,u) & g(t,u) \\ t & u \end{bmatrix}; \forall t,u\in K, \text{ and } f, \text{ and } g \text{ functions on } K\times K.$$

This is a maximal partial spread and when K is finite, it is clearly a spread. When this maximal partial spread is not a spread, it is a proper quasifibration of dimension 2.

At the time of the writing of this article, all of the known quasifibrations were of dimension 2 and all but one were constructed by what is called "transcendental copying", or T-copying, by which a spread over K is copied in $K(z)$, the rational function field over K.

The interesting part of T-copying is that the properties of the spread from which it is copied are preserved. If there is a regulus-inducing group, either an elation or homology group acting on a translation plane, and if the T-copy exists, there will be a quasifibration with this same property.

Definition 4.1. *A quasifibration of dimension 2 and is not a spread, which admits α-regulus inducing elation or homology group shall be called a quasi α-flock, either of an α-cone or of an α-twisted quadric. If a matrix spread of n dimensions over K can be considered a partial spread over $K(z)$, the rational function field over K, it is called a T-copy. This also can work over infinite fields K, wherever the exponents in the spread are meaningful in the rational function field extension. If there are automorphisms in a group G within the entries of the spread sets, then extend these automorphisms to $K(z)$, so that the automorphisms leave $Fix(G(z))$ fixed pointwise. In this setting, if this is also a T-copy, we use the term "twisted T-copy".*

The following are examples of T-copies as quasifibrations:

$$x=0,y=x\begin{bmatrix} u^{\sigma} & bt^{\rho} \\ t & u \end{bmatrix};\forall t,u\in K=GF(q)(\theta),$$

$\sigma\neq 1$, ρ automorphisms of $GF(q)$,

K a transcendental field extension of $GF(q)$,

b a non-square in $GF(q), q$ odd.

This is an example of a quasi σ-flock of a σ-cone.

- If $\sigma=\rho$ then this maximal partial spread, is also a quasi σ-hyperbolic partial spread;

$$x=0,y=x\begin{bmatrix} u^{\sigma} & bt^{\sigma} \\ t & u \end{bmatrix}=(y=x\begin{bmatrix} s^{\sigma} & b \\ 1 & s \end{bmatrix}\begin{bmatrix} v^{\sigma} & 0 \\ 0 & v \end{bmatrix}).$$

  Also, a twisted T-copy of this form becomes a spread by results of Johnson and Jha [23].
- If we try the same analysis with the Johnson partial hyperbolic flock of deficiency one, it is possible to show that we have a maximal partial hyperbolic quasi flock of deficiency one.

$$x=0,y=x\begin{bmatrix} u^{2} & bt^{-1} \\ t & u \end{bmatrix};\forall t,u\in GF(4)(\theta),\ b,b+1 \text{ not } 0,$$

where $GF(4)(\theta)$ is a transcendental field extension of $GF(4)$.

Noting that $t^{-1}=t^{2}$, we see now we have a T-copy of the Johnson partial hyperbolic flock of deficiency one. Note that $u\rightarrow u^{2}$ is not bijective, thus we have a proper quasifibration.

Furthermore, if we take a twisted T-copy, we have a spread, and contains a subplane that admits a partial hyperbolic flock of deficiency one.

Hence, we have the following two results Biliotti, Jha and Johnson [5], for the first result, (slightly generalized). More details and the complete proof are in Johnson and Jha [23].

(1) The T-copy of a Klein flock of a quadratic cone and flock of a hyperbolic quadric is a proper quasifibration (not a spread).

(2) The T-copy of the Johnson partial flock of deficiency one of a hyperbolic quadric over $GF(4)$ is a proper quasifibration and a twisted T-copy is a spread that contains a subplane producing a deficiency one hyperbolic flock.

(3) Consider also

$$x=0,y=x\begin{bmatrix} u+nt^{3} & nt^{9}+n^{3}t \\ t & u \end{bmatrix};$$

n non-square in K a field of characteristic 3

(Biliotti–Jha–Johnson [5, 29.3.5]).

These quasifibrations are called the "generalized Ganley" additive quasifibrations. When K is infinite, there are fields that determine proper quasifibrations.

(4) The Cherowitzo–Johnson–Vega [6, (2.6)] quasifibrations:

$$x=0,y=x\begin{bmatrix} u^{\alpha} & -t^{3\alpha^{-1}} \\ t & u \end{bmatrix};\ K \text{ an ordered field}$$

and α an automorphism of K.

Then this is a quasifibration providing a proper quasi α-conical flock, but is not a T-copy.

Recently, the author considers Galois chains of quasifibrations [18], where given any quasifibration of dimension 2, over a field K and a Galois tower of quadratic extensions with base K, it is possible to form an associated chain of quasifibrations. The basic question if the quasifibration is proper, that is, not a spread, is there any way to determine if there is a quasifibration at link k that becomes a spread? There is a method developed for this question and it is shown that there can never be a spread in a Galois chain of arbitrary length, with one of the known proper quasifibrations as base.

Therefore, although there are just a few proper quasifibrations to serve as bases of Galois chains of quasifibrations, every link in the chain is a proper quasifibration, as is each of the derived planes with blended kernel. In this way, we have infinitely many proper quasifibrations both of dimension 2 and infinitely many proper quasifibrations of dimension 4.

4.1. Baer theory of twisted flocks; Part I. Our main results in this article extend the above result for partial α-conical flocks and partial α-twisted hyperbolic flocks of deficiency one, over any field K that admits an automorphism α.

In Jha and Johnson [10], the analysis focused on understanding the structure of the hyperbolic flocks and of flocks of quadratic cones. The analysis is that from a translation plane with spread in $PG(3,K)$ admitting a regulus-inducing elation group E or a regulus-inducing homology group H actually involves partitioning the hyperbolic quadric or quadratic cone by the invariant Baer subplanes of the associated group H or E, respectively.

We shall give the pertinent theory providing the structure of the set of invariant 2-dimensional K-subspaces in the associated 4-dimensional K-vector spaces. But, also, we have noted that instead of a partition of V_4 over K by regulus-inducing groups, we may also consider analogous theory by twisted regulus-inducing groups, either elation or homology groups. We now consider a partition of the twisted quadrics; the twisted conical flocks and the twisted hyperbolic flocks. After the preliminary material below, the extension of the theorem of Jha–Johnson [10] above may be directly extended by using α-reguli instead of reguli. The idea is to work over K^{a}, when dealing with the Baer subplanes, then all of the arguments extend directly.

Our main results for the hyperbolic situation are as follows. However, “translation plane” may be replaced by “quasifibration” in the most general version of the theorem.

Theorem 4.2. *Let Σ be a translation plane with blended kernel (K,K^{α}) that admits a full Baer α-homology group. Then there is a corresponding twisted hyperbolic flock of deficiency one in $PG(3,K)$. Conversely, any partial twisted hyperbolic flock of deficiency one constructs a translation plane of blended kernel (K,K^{α}) that admits a full Baer α-homology group. The partial α-hyperbolic flock may be extended to an α-hyperbolic flock if and only if the Baer net is a derived α-regulus net.*

The theorem for Baer α-elation groups is then as follows:

Theorem 4.3. *Let Σ be a translation plane with blended kernel (K,K^{α}) that admits a full Baer α-elation group. Then there is a corresponding partial α-flock of an α-cone of deficiency one in $PG(3,K)$. Conversely, any partial flock of deficiency one of an α-cone constructs a translation plane of blended kernel (K,K^{α}) that admits a full Baer α-elation group. The partial α-flock may be extended to an α-flock if and only if the Baer net is a derived α-regulus net.*

The proofs will follow after some results on group orbits. We shall state the elation and homology Baer theorems separately, but will tend to combine the proofs, as there are definite similarities. First the Baer elation group situation:

Theorem 4.4. *Let K be a field and let α be an automorphism of K. Let V_4 be a vector space over K and let $x = 0$ and $y = 0$ be disjoint 2-dimensional K and K^{α} subspaces, where x and y are 2−vectors. We note that V_4 may also be considered a 4-dimensional vector space over K^{α}. Consider the following group in $GL(4, K)$:*

$$E^{\alpha} = \left\{ \begin{bmatrix} I_2 & \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \\ 0_2 & I_2 \end{bmatrix} ; u \in K \right\}.$$

We note that E^a fixes $x = 0$ pointwise. (1)(a) Then the elements of the set of E^{α} invariant 2-dimensional subspaces not equal to $x = 0$ are each generated from a 1-dimensional K^{α} subspace of $x = 0$ and another 1-dimensional space K^{α}. Two of these 2-dimensional subspaces over K^{α} are disjoint if and only if they are disjoint on $x = 0$. (1)(b) Let $\{x_i; i \in \lambda\}$ denote the set of 1-dimensional K^{α} subspaces of $x = 0$ and for each x_i, choose any other 1-dimensional subspace w_i not incident with $x = 0$ such that the generated 2-dimensional subspace is E^{α}-invariant. Then the set of all 2-dimensional K^{α}-subspaces of

$$\{\langle x_i, w_i \rangle, i \in \lambda\}$$

is a partial spread net covering $x = 0$. If we can form these subspaces into a derivable net, then the derived net is an α-regulus net over K.

For the Baer homology situation, we have:

Theorem 4.5. *Let K be a field and let α be an automorphism of K. Let V_4 be a vector space over K and let $x = 0$ and $y = 0$ be disjoint 2-dimensional K and K^{α} subspaces, where x and y are 2−vectors. We note that V_4 may also be considered a 4-dimensional vector space over K^{α}. Consider the following group in $GL(4, K)$,*

$$H^{\alpha} = \left\{ \begin{bmatrix} I_2 & 0_2 \\ 0_2 & \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \end{bmatrix} ; u \in K^* \right\}.$$

Let $x = 0$ and $y = 0$ be as assumed. (2)(a) Then all H^{α}-invariant 2-dimensional subspaces are generated by a 1-dimensional K^{α}−space from $x = 0$ and a 1-dimensional K^{α}−space of $y = 0$. (2) (b) Let $\{x_i; i \in \lambda\}$ denote the set of 1-dimensional K^{α} subspaces of $x = 0$ and let $\{y_i; i \in \lambda\}$ be the set of all 1-dimensional K^{α}-subspaces of $y = 0$. Let Γ be any bijective mapping from $\{x_i; i \in \lambda\}$ onto $\{y_i; i \in \lambda\}$. Then the set of 2-dimensional K^{α}-subspaces

$$\{(x_i, \Gamma(x_i)); i \in \lambda\}$$

defines a partial spread net. If these subspaces can be formed into a derivable net then the derived net is an α-regulus net whose components are K-subspaces and K^{α}-subspaces, and whose 1-dimensional K^{α}-subspaces on $x = 0$ and $y = 0$ are completely covered.

Proof. (1)(a). $E^{\alpha} = \left\{ \begin{bmatrix} I_2 & \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \\ 0_2 & I_2 \end{bmatrix} ; u \in K \right\}$. Consider any vector (x_1, x_2, x_3, x_4) and consider the E^{α} orbit of this vector. This is

$$\langle (x_1, x_2, x_1 u^{\alpha} + x_3, x_2 u + x_4); u \in K \rangle.$$

This subtracting (x_1, x_2, x_3, x_4) from a general term, we see that we have the vectors $(0, 0, x_1u^\alpha, x_2u)$. Recall that V_4 is a K^α-vector space as well, and we have the scalar multiplication $\cdot$ given by $(x_1, x_2, x_3, x_4) \cdot u = (x_1u^\alpha, x_2u, x_3u^\alpha, x_4u)$, we see that we must have $(0, 0, x_1, x_2)$ and the 1-dimensional K^α space within this subspace, in any two dimensional subspace that is invariant E^α. This subspace is 2-dimensional over K if and only if $x_1 = x_2 = 0$, for $\alpha \neq 1$. We consider Ω^α as the set of "points" that are either E^α or H^α 2-dimensional K^α orbits: The set of points has the following form

$$\{(x_1^\alpha x_4, x_1x_3^\alpha, x_2^\alpha x_4, x_2x_3^\alpha); x_i \in L.\}$$

Abstractly, this forms the basis for the 3-dimensional projective intersection with the K^α-Klein quadric, which will be discussed in the subsequent material. We note the following: The E^α and H^α-orbits define "points" and furthermore, if a vector $(x_1, x_2, x_3, x_4) \to (x_1^\alpha x_4, x_1x_3^\alpha, x_2^\alpha x_4, x_2x_3^\alpha)$ then every K-space generated by this vector $(x_1\beta, x_2\beta, x_3\beta, x_4\beta)$ maps to the same "point"/modulo K $\to (x_1^\alpha x_4\beta^{\alpha+1}, x_1x_3^\alpha\beta^{\alpha+1}, x_2^\alpha x_4\beta^{\alpha+1}, x_2x_3^\alpha\beta^{\alpha+1}) \equiv (x_1^\alpha x_4, x_1x_3^\alpha, x_2^\alpha x_4, x_2x_3^\alpha)$. So, in the associated flocks, the plane intersections could be visualized as K-vectors whose corresponding mapped K^α-Klein quadric objects that arise from either E^α or H^α-2-dimensional or K^α-subspace/orbits. In this manner, we may consider plane intersections as either covering Ω^α in the H^α-case or by covering an associated α-flock of an α-quadratic cone. Now assume that two of the E^α-subspaces are not disjoint

$$\langle (x_1, x_2, x_3, x_4)E^\alpha \rangle \cap \langle (y_1, y_2, y_3, y_4)E^\alpha \rangle .$$

Assume that

$$(x_1, x_2, x_1u^\alpha + x_3, x_2u + x_4) \cdot \delta = (y_1, y_2, y_1v^\alpha + y_3, y_2v + y_4),$$

for $u, v, \delta \in K$. Then $\langle (x_1, x_2) \rangle = \langle (y_1, y_2) \rangle$, and hence these subspaces share a 1-dimensional subspace on $x = 0$. It now follows directly that the two subspaces are equal. This proves (1)(a). Now it is clear that $\{\langle x_i, w_i \rangle, i \in \lambda\}$ is a set of E^α-invariant K^α-subspaces that are mutually disjoint and completely cover $x = 0$. Let $w_i = (x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i})$, where the 2-vector over K^α on $x = 0$ is $x_i = (x_{1,i}, x_{2,i})$. The rest of the theorem will now follow directly once we have worked out the form of the derived nets. (2)(a) Now consider the group

$$H^\alpha = \left\{ \begin{bmatrix} I_2 & 0_2 \\ 0_2 & \begin{bmatrix} u^\alpha & 0 \\ 0 & u \end{bmatrix} \end{bmatrix}; u \in K^* \right\}.$$

Consider any vector (x_1, x_2, x_3, x_4) not in either $x = 0$ or $y = 0$ and consider the H^α-orbit, $\langle (x_1, x_2, x_3u^\alpha, x_4u); u \in K^* \rangle$. By subtracting the generating vector, and then realizing that we have a K^α-subspace, it follows that $(0, 0, x_3, x_4)$ is a non-zero vector in the generated K^α-subspace. Now subtracting the original vector from this preceding vector, we also have $(x_1, x_2, 0, 0)$ a non-zero vector in the generated K^α-space. This says that the only H^α-invariant 2-dimensional subspaces are generated from two non-zero vectors on each of the two invariant K-subspaces $x = 0$ and $y = 0$. Moreover, it now follows that the only way that two of these invariant 2-dimensional K^α-subspaces can non-trivially intersect is if they intersect on either $x = 0$ or $y = 0$. Therefore, $\{(x_i, \Gamma(x_i); i \in \lambda\}$ defines a partial spread net of K^α-invariant 2-dimensional subspaces that cover $x = 0$ and $y = 0$. This completes the proofs of both (2)(a) and (b). □

5. The K^α-Klein quadric

The reader will notice that we are essentially using ideas of Thas and Walker, Thas [32], coupled with insights about derivable nets, for these extension notions.

Consider a 6-dimensional vector space V_6 over K in the standard manner and over K^α with special scalar multiplication $\cdot$ as follows:

$$(x_1, x_2, x_3, x_4, x_5, x_6) \cdot \delta = (x_1\delta^\alpha, x_2\delta, x_3\delta^\alpha, x_4\delta, x_5\delta^\alpha, x_6\delta),$$

and written over the associated standard basis, we obtain vectors as

$$(x_1^{\alpha^{-1}}, x_2, x_3^{\alpha^{-1}}, x_4, x_5^{\alpha^{-1}}, x_6),$$

then the standard hyperbolic form Ω_5 becomes

$$x_1^\alpha x_6 + x_2 x_5^\alpha + x_3^\alpha x_4 = 0.$$

Now embed $PG(2, K)$ as $x_3 = x_4 = 0$. We note that the group E^α now acts on $PG(3, K)$.

We consider the associated α-hyperbolic form, which can be given by $x_1^\alpha x_6 = x_2 x_5^\alpha$, without loss of generality. Now consider the set of points within $x_2 = x_5$, to obtain $x_1^\alpha x_6 = x_2^{\alpha+1}$, an α-conic. We realize that E^α leaves the associated homogeneous points on the α-regulus of Baer subspaces pointwise fixed (as “points”). Identifying the α-regulus projectively as the α-twisted hyperbolic quadric, we see that we may consider the α-conical flock as a set of points within the α-twisted quadric in $PG(3, K)$ as follows: Consider a point of $PG(3, K)$ say $v_0 = (0, 0, 0, 0, 0, 1)$ as the vertex of the cone, then a set of planes of $PG(3, K)$ that partition the points of the lines from v_0 to the α-conic $x_1^\alpha x_6 = x_2^{\alpha+1}$, then the planes correspond to the K-components of a translation plane with spread in $PG(3, K)$ that admit an α-regulus-inducing elation group E^α. The converse that from a translation plane with spread in $PG(3, K)$ with group E^α produces an α-flock of the α-quadratic cone is now immediate.

Hence, this provides an alternative construction of the main theorem of Cherowitzo–Johnson–Vega [6]. We shall use this approach when considering the Baer groups and the deficiency one α-conical flocks.

- This shows that α-flocks and the set of Baer components of the translation plane have a bijective correspondence.
- The results of Johnson [20] show that we may identity the K^α-subspaces of the invariant H^α-2-dimensional subspaces with the elements Ω_3^α, the twisted hyperbolic quadric in $PG(3, K)$ and the Baer components of the associated α-regulus nets associated with the translation plane admitting H^α.
- This also shows that twisted hyperbolic flocks and the set of Baer components of the translation plane have a bijective correspondence.

5.1. Baer theory; Part II. We now complete the proof of our Baer theorems. We begin with the translation plane admitting a Baer elation group. The reader should note that the group

$$E^\alpha = \left\{ \begin{bmatrix} I_2 & \begin{bmatrix} u^\alpha & 0 \\ 0 & u \end{bmatrix} \\ 0_2 & I_2 \end{bmatrix}; u \in K \right\},$$

has this form when considering the axis as $x = 0$. It might be expected that any associated translation plane admitting E^α has $x = 0$ as a component. However, this may not be the case, and will not be the case when E^α is considered a Baer group and, in this case, $x = 0$ becomes a Baer subplane.

If the translation plane admits E^α as a Baer group then the components that lie on the Baer axis (i.e. $x = 0$) are E^α subspaces, so we have the situation of a translation plane with blended kernel (K, K^α). Now, we still have the orbits of K-components as before, we just have one less α-regulus derivable net than in the case when E^α is an elation group. For clarity, we now speak of a Baer elation group. Since the other

α-regulus nets still correspond to sets of E^{α} orbits of K^{α}-subspaces, we clearly have a deficiency one α-conical flock.

Now suppose that we have a deficiency one α-conical flock. Then on each line of the α-cone, we are missing exactly one point. This point corresponds to an E^{α}-orbit and since no two of these are on the same line of the α-cone, then the points are mutually disjoint. Let Λ denote the set of corresponding 2-dimensional K^{α}-subspaces. Let Ω denote the partial spread consisting of the α-regulus nets corresponding to the deficiency one α-conical flock. Since now $\Lambda \cup \{\Omega - (x=0)\}$, covers all of the E^{α}-orbits in the associated vector space over K^{α}, we have a spread, but now with blended kernel (K, K^{α}), admitting E^{α} as a Baer elation group.

Now the question of when the deficiency one α-conical flock situation occurs depends on whether the set Λ is a derivable net or not. This is valid, as this is the only method that can produce a set of K-components to return to a translation plane with kernel K admitting E^{α} as an elation group. Note that it is possible that the net in question is derivable without the associated Baer subplanes being K-subspaces, but it is only when this occurs that the deficiency one partial α-conical flock may be extended to a flock.

This completes the proof of the deficiency one α-conical flock theorem. Now assume that we have a translation plane with blended kernel (K, K^{α}) admitting

$$H^{\alpha} = \left\{ \begin{bmatrix} I_2 & 0_2 \\ 0_2 & \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \end{bmatrix} ; u \in K^{*} \right\}$$

as a Baer homology group, where now $x=0$ and $y=0$ are K-Baer subplanes. In this setting, we are missing one α-regulus net of H^{α}-invariant subplanes and hence the Baer subplanes of the other α-regulus nets produce a deficiency one twisted hyperbolic flock.

Now assume that we have a deficiency one twisted hyperbolic flock. From the analysis of the author in [20], it is clear that we may construct a partial spread Π which consists of a set of K-2-dimensional subspaces that admit H^{α} and is a set of α-regulus nets sharing $x=0$ and $y=0$. Recall that our α-hyperbolic quadric is a hyperbolic quadric with respect to K^{α} (since the α-regulus components and Baer components are all K^{α}-subspaces). Hence, there are two sets of ruling lines with respect to K^{α}. It follows that on each K^{α}-ruling line of each of the two sets of ruling lines, there is exactly one missing point. These points correspond to a set of mutually disjoint K^{α}-subspaces that are disjoint H^{α}-orbits and thus completely cover $x=0$ and $y=0$. Let W denote the set of these K^{α}-subspaces in the associated vector space. Consider $W \cup \Pi$. This partial spread is a spread, since it covers all of the H^{α}-invariant 2-dimensional K^{α}-subspaces. This translation plane has blended kernel (K, K^{α}) and admits H^{α} as a Baer homology group.

For the question of when the deficiency one partial twisted hyperbolic flock may be extended, it depends on whether W is a derivable net with Baer components being K-subspaces, similar to the argument for the Baer elation situation. This completes the proofs of the Baer theory for α-twisted flocks–except for that question about "point Baer" that is in the hypotheses of the corresponding Baer theory for flocks of quadratic cones and for hyperbolic quadrics. So, we have shown that with an α-flock of an α-quadratic cone, there is an associated translation plane admitting an α-regulus-inducing elation group E^{α} and, conversely, such a translation plane constructs the α-conical flock. And, we have shown that flocks of an α-twisted hyperbolic quadric and translation planes admitting α-regulus-inducing groups are equivalent. The real problem seems to come in the deficiency one theory.

Suppose that we have a translation plane of either type. Choose any of the α-regulus nets and suppose that the sublanes involved are "point Baer" but not "line Baer". Now derive the net, what is obtained? This cannot be a translation plane any longer, just a Sperner space with blended kernel (K, K^{α}) (which could be K, if $\alpha = 1$). But, this structure still admits a full "point Baer" elation or full "point Baer" homology group. Therefore, we obtain a deficiency one α-flock, quadratic or hyperbolic, now with a point missing on each of the relative lines of the cone or of the α-rulings. Our proof still shows that we can recover a spread-a "translation plane" –not a Sperner space. So, all of the point Baer subplanes must be Baer subplanes, since all of the α-regulus nets can be derived to affine planes. This completes the proofs of the theorems on Baer theory.

6. Quasi-flocks

Quasifibrations have been mentioned previously. Much of the work on flocks and associated translation planes has been done algebraically. We have a certain form of a translation plane and using this form, we create an associated flock, using that algebraic representation. More generally, we may use the forms defining partial spreads to create what appear to be flocks but are actually quasifibration/maximal partial flocks. If we define a "quasi-flock" as a partial flock that has the basic form of a flock but does not satisfy the covering criterion, we have necessarily a maximal partial flock. Proper quasi-flocks like proper quasifibrations are infinite.

In any case, all our results may be phrased more generally in terms of quasifibrations. Without listing all of various theorems, here is an omnibus theorem:

Theorem 6.1. *Quasifibrations with α-regulus inducing groups are equivalent to quasi α-flocks.*

7. The Baer forms

Here we indicate what the translation planes with blended kernel (K, K^{α}) look like that admit Baer groups.

- Baer elation translation planes would have the following form:
$$x = 0, y = (x_1^{\alpha^{-1}}, x_2^{\alpha^{-1}}) \begin{bmatrix} v & z(v) \\ 0 & v \end{bmatrix}; \forall v \in K,$$
$$y = x \begin{bmatrix} 1 & -u^{\alpha} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F(t) & G(t) \\ t & 0 \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$$
for all $u, t \neq 0 \in K$. z a function on K so that $z(0) = 0$, $z(1) = 1$, and functions F and G on K.
- A partial α-conical flock may be extended if and only if $z(v) = 0$ for all $v \in K$.
- Baer Homology translation planes would have the following form:
$$x = 0, y = (x_1^{\alpha^{-1}}, x_2^{\alpha^{-1}}) \begin{bmatrix} n(v) & 0 \\ 0 & v \end{bmatrix}; \forall v \in K,$$
n a function on K so that $n(0) = 0$, $n(1) = 1$ and
$$y = x \begin{bmatrix} 1 & 0 \\ 0 & u^{-\alpha} \end{bmatrix} \begin{bmatrix} g(t) & f(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}; \forall u \neq 0, t \neq 0 \text{ in } K,$$
for functions g and f on K.
- The deficiency one partial twisted hyperbolic flock may be extended if and only if $n(v) = v$ $\forall v \in K$.

Proof. The elation group changes form when considering $x = 0$ as $(0, x_2, 0, x_4)$ in the formulation of the axis of the elation group. The components incident with the Baer axis are K^α-subspaces that are left fixed by the Baer elation group. We leave, as an exercise, to check out the remaining parts of the form. Similarly, the homology group changes form when considering $x = 0$ as $(0, x_2, 0, x_4)$ and $y = 0$ as $(x_1, 0, x_3, 0)$. Again the components incident with the Baer axis and coaxis are K^α-subspaces that are left fixed by the Baer homology group. Here is another exercise to check out the remaining parts of the form. Note that there are two distinct uses of the notation of $x = 0$ and $y = 0$; one use is when considering the group axis/coaxis and the other use is when considering the form of the translation plane, when $x = 0$ and $y = 0$ are Baer subplanes. □

8. The algebraic and α-Klein methods; α and α^{-1}-flocks

When the ideas of flocks of quadratic cone and flocks of hyperbolic quadrics were introduced, there were a variety of new studies, in the infinite case, and also later with what we are calling α-flocks of α-quadratic cones and α-twisted hyperbolic flocks. Many of these were algebraic in nature. When this was done, there was essentially no connection between α-regulus-inducing groups, elation and homology, that really becomes the essence of understanding the α-Klein methods. Moreover, there is no uniformity with the notation of the α-cones of α-twisted quadrics.

- In Cherowitzo, Johnson, and Vega [6], it was pointed out that whenever an α-conical flock is constructed, there is also an α^{-1}-conical flock which may be constructed and is isomorphic to the original. How these two examples may be understood using the α-Klein method is by a translation of vectors $(x_1, x_2, x_3, x_4) \to (x_4, x_3, x_2, x_1)$, which changes the α-conic that is used to the associated α^{-1}-conic. To see this, just note that

$$\left\{x = 0, y = x \begin{bmatrix} u^\alpha & 0 \\ 0 & u \end{bmatrix}; u \in K\right\} \to \left\{x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^{\alpha^{-1}} \end{bmatrix}; u \in K\right\}.$$

  This mapping works for the α-twisted hyperbolic flocks. There is also a corresponding change of functions defining the α or α^{-1}-flocks. This is the mapping $(x_1, x_2, x_3, x_4) \to (x_2, x_1, x_4, x_3)$, which does the same thing, changing the α-twisted regulus net to the α^{-1}-twisted regulus net.
- In the author's work on α-twisted hyperbolic quadrics (see [20]), the process of translating components of α-regulus-inducing group H^α to the associated α-twisted hyperbolic quadrics does not use the understanding of the "points" being the invariant 2-dimensional subspaces over K^α (not equal to the two components $x = 0$ and $y = 0$, the axis and coaxis of H^α). In fact, in the algebraic method, it may be seen that the projective connection is accomplished with the invariant 2-dimensional subspaces over $K^{\alpha^{-1}}$. This does not cause any difficulty as, similar to the elation case, given an α-twisted hyperbolic flock, there is always an isomorphic α^{-1}-twisted hyperbolic flock.

9. Quaternion division ring planes are flock planes

To appreciate how the subject of division rings comes into the discussion, we recall the main theorem of the classification of subplane covered nets, Johnson [16], which we have previously discussed. Here we take a more in-depth view, in order to bring in quaternion division rings.

We mention only the classification of derivable nets:

Theorem 9.1 (Johnson [16]). *Let D be a derivable net. Then there is an embedding of D into a 3-dimensional projective space Σ over a skewfield K, as follows: Let N be a fixed line of Σ, then the embedding maps the points, lines, Baer subplanes, parallel classes of D into $PG(3,K)$ as lines, points, planes that do not contain N and planes that do contains N of Σ, respectively. The full collineation group of D is $P\Gamma L(4,K)_N$. Using the collineation group, then there is a contraction method that shows that D may be represented in a 4-dimensional vector space over K as a pseudo-regulus net. If K is a field then the derivable net is a regulus-net.*

The author's article on the classification of derivable nets, [21], tries to understand the nature of other possible derivable nets that can sit in the same vector space as the embedded and contracted pseudo-regulus net.

This is where quaternion division rings come in. There are four classes of such derivable nets that can sit in the same 4-dimensional vector space as does the contracted pseudo-regulus; we look at the classification of a derivable net as a comparison to an existing derivable net. There is exactly one type of derivable net that is a pseudo-regulus net but sits in a embedded/contracted 4-dimensional vector space over a field; the comparison derivable net is a regulus net. This type is the set of quaternion division rings realized as derivable nets.

It is not really necessary to have the non-commutative algebra definition of a quaternion division ring for this discussion, but we will require the matrix definition of a quaternion division ring spread, Johnson [21]:

Definition 9.2. *The matrix construction of a quaternion division ring plane in any dimension 2 matrix spread set is as follows:*

$$x = 0, y = x\begin{bmatrix} u^\sigma & bt^\sigma \\ t & u \end{bmatrix}; t, u \in F(\theta), \text{ a Galois quadratic extension of a field } F,$$

$$b \in F,\ \sigma \text{ the induced automorphism.}$$

This example, in the finite case, has been seen earlier in the article. But in the previous finite version, or quasifibration version, required b to be a non-square in $F(\theta)$, where the skewfield version requires b to be a non-square in F.

But, the interesting fact about the quaternion division ring planes is that they are equivalent to σ-flocks of a σ-cone, and equivalent to also σ-hyperbolic flocks. The flocks, in both cases, are linear, although not the same linear structure, as they are two completely different linear flocks, [20].

10. Lifting skewfield planes

The quaternion division ring planes almost always have what are called "central extensions" of skewfields S, which are quadratic in this case. Considering the associated translation planes, the central extensions are (analogous) to dimension 2 translation planes, as they can be represented over the 3-dimensional projective space with respect to the original skewfield S. Whenever a skewfield S, quaternion or not, has a quadratic central extension (meaning that the generating quadratic polynomial is irreducible over the center of S), the corresponding translation plane can be lifted, just as in the commutative dimension 2 situation, Hiramine, Matsumoto, Oyama [8]. Lifted spreads of dimension two provide a wealth of examples of σ-flocks of σ-quadratic cones.

By lifting non-commutative skewfield planes that admit central quadratic extensions, the constructed spreads are semifield spreads in non-commutative 3-dimensional projective spaces, Johnson, Jha [24]. This is interesting also as the quaternion division ring spreads, as dimension 2 spreads over fields, can also be lifted by the standard procedure. Thus, we have two mutually non-isomorphic spreads, one in $PG(3,K)$, for K a field, and another in $PG(3,L)$, for L a non-commutative skewfield, constructed from the same spread, by essentially the same method.

11. Examples of twisted hyperbolic flocks

In [19], the author shows how to use the Kantor–Knuth flocks of the quadratic cone in $PG(3,p^r)$, p odd, to construct the $j=(p^s-1)/2$-planes of Johnson, Pomareda, and Wilke [12], which provide several infinite classes of p^s-twisted hyperbolic quadrics. In a j-plane, from Johnson, Pomareda and Wilke [12], there is always a cyclic homology group of the following form:

$$H^{\alpha}=\left\{\begin{bmatrix}\begin{bmatrix}u^{2j+1} & 0\\ 0 & u\end{bmatrix} & 0_2\\ 0_2 & I_2\end{bmatrix}; u\in GF(p^r=q)^*\right\},$$

so when $j=(p^s-1)/2$, $2j+1=p^s$, which provides the necessary $\alpha=p^s-$twisted hyperbolic flocks. To obtain the form of the group used in this article, and to see the form that the twisted hyperbolic flocks take, a basis change is required, basically by taking the inverses of all of the matrices, which would make this into a "right" α-twisted regulus-inducing group instead of a "left" α-twisted regulus-inducing group (we need to invert $x=0$ and $y=0$).

- There is also a related $j=(p^s-1)/2+(q-1)/2-$plane that also provides an infinite class of p^s-twisted hyperbolic flocks, obtained by a derivation replacement of the set of regulus nets of "odd" determinant type.

11.1. $x=0,y=x\begin{bmatrix}u^{\alpha}+gt & ft^{\alpha^{-1}}\\ t & u\end{bmatrix}; u,t\in K$, K a field.

In this subsection, we look at translation planes that provide both α-flocks of α-conics and α-twisted hyperbolic flocks. That this form above exactly describes the translation planes that accomplish this is shown in Johnson [20], and are the Hughes–Kleinfeld semifield planes and their infinite generalizations.

- However, in that work the Hughes–Kleinfield planes and their α^{-1}-flocks and α-twisted hyperbolic flocks were all labeled linear. However, this would only be valid in the α^{-1}-flock case, when $\alpha^2=1$ and $g=0$, so that correction should be noted.
- We have noticed this form previously when $\alpha^2=1$, and $\alpha\neq 1$ and $g=0$. In this setting, we have both an α-flock of an α-conic and an α-twisted hyperbolic flock. These flocks are linear and occur also for the quaternion division rings.
- In this more general setting, by noting that

$$x=0,y=x\begin{bmatrix}u^{\alpha}+gt & ft^{\alpha^{-1}}\\ t & u\end{bmatrix}=\begin{bmatrix}v+gt & ft^{\alpha^{-1}}\\ t & v\alpha^{-1}\end{bmatrix}$$

then put in the expression for α^{-1}-flokki (α^{-1}-flock of an α^{-1}-cone, as there is a notation change here) to obtain:

$$\rho_t: x_1t-x_2f(t)^{\alpha^{-1}}+x_3g(t)^{\alpha^{-1}}-x_4; t\in K.$$

and for $f(t)=ft^{\alpha^{-1}}$, and $g(t)=gt$, we have:

$$\rho_t: x_1t-x_2f^{\alpha^{-1}}t^{\alpha^{-2}}+x_3g^{\alpha^{-1}}t^{\alpha^{-1}}-x_4; t\in K.$$

Then, we note that this same spread is

$$= \left\{ \begin{array}{c} x = 0, y = x \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix}; \\ y = x \begin{bmatrix} t^{\alpha} + g = F(t) & f = G(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} v^{\alpha} & 0 \\ 0 & v \end{bmatrix}; u, t, v \neq 0 \text{ in } GF(q) \end{array} \right\},$$

we see that we obtain a α-twisted linear hyperbolic flock,

$$\pi_t : -x_1 G(t)^{\alpha} + x_2 F(t) - x_3 t^{\alpha} + x_4 = 0, and\ \rho : x_2 = x_3,$$

which is given by:

$$\pi_t : -x_1 f^{\alpha} + x_2 (t^{\alpha} + g) - x_3 t^{\alpha} + x_4 = 0, and\ \rho : x_2 = x_3.$$

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