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Abstract Let $\mathbf{V}$ be a free module of rank n over a commutative ring $\Bbbk$. We prove that tensor space $\mathbf{V}^{\otimes r}$ satisfies Schur–Weyl duality, regarded as a bimodule for the action of the group algebra of the Weyl group of $\mathrm{GL}(\mathbf{V})$ and the partition algebra $\mathcal{P}_r(n)$ over $\Bbbk$. We also prove a similar result for the half partition algebra.

Introduction

A number of instances of Schur–Weyl duality (a bimodule for which the centraliser of each action equals the image of the other) have been established in positive characteristic over the past forty years, including: [3] (extending [6, 7, 19]) for general linear and symmetric groups; [9] for symplectic groups and Brauer algebras; [15] for orthogonal groups and Brauer algebras (characteristic not 2); [16] for special orthogonal groups and the Brauer–Grood algebra; [8, 10, 11, 12] for general linear groups and walled-Brauer algebras. In all these cases, semisimple versions of Schur–Weyl duality were observed much earlier as an application of Artin–Wedderburn theory, and the extension to positive characteristic (where representations tend to be non-semisimple) is much more difficult to establish. This paper continues the above body of work, by extending the Schur–Weyl duality between symmetric groups and partition algebras to non-semisimple situations.

Let $\Bbbk$ be a commutative ring (always with 1). For G a group, we let $\Bbbk G$ denote its group algebra. Fix a free $\Bbbk$-module $\mathbf{V}$ of rank n, and fix a $\Bbbk$-basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbf{V}$. As explained in Section 1, tensor space $\mathbf{V}^{\otimes r}$ is a $(\Bbbk W_n, \mathcal{P}_r(n))$-bimodule, where W_n is the Weyl group of $\mathrm{GL}(\mathbf{V})$. We identify $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n \subset \mathbf{V}^{\otimes (r+1)}$, which becomes a $(\Bbbk W_{n-1}, \mathcal{P}_{r+1/2}(n))$-bimodule by restriction.

The purpose of this paper is to show that Schur–Weyl duality holds for both bimodule structures on $\mathbf{V}^{\otimes r}$; that is, we have the following result.

Theorem (Schur–Weyl duality). *Let $\Bbbk$ be a commutative ring. Then:*

(a) *The centraliser algebras $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, $\mathrm{End}_{W_n}(\mathbf{V}^{\otimes r})$ coincide with the images of the representations $\Bbbk W_n \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$, $\mathcal{P}_r(n)^{\mathrm{op}} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$, respectively.*

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(b) *Similarly, the centraliser algebras* $\mathrm{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$, $\mathrm{End}_{W_{n-1}}(\mathbf{V}^{\otimes r})$ *coincide with the images of the representations* $\Bbbk W_{n-1} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$, $\mathcal{P}_{r+1/2}(n)^{\mathrm{op}} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$*, respectively.*

This result is well known in case $\Bbbk = \mathbb{C}$; it follows from standard facts in the theory of semisimple algebras (see e.g. [20, Thms 5.4, 3.6]). Our contribution is to extend the result to arbitrary commutative rings $\Bbbk$. After this paper was written, Donkin [14] found a different approach to our main result.

As an application of the main theorem, in [4, Cor. 7.6] we prove that the centraliser algebras $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, $\mathrm{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$ are cellular algebras over a commutative ring, in the sense of [18]. See [13] for the cellularity of $\mathrm{End}_{W_n}(\mathbf{V}^{\otimes r})$. See also [1, 2] for related results over fields of characteristic zero.

The proof of the theorem (for general $\Bbbk$) is obtained as follows. First, the fact that the images of the maps from the partition algebras coincide with the W_n and W_{n-1} centralisers already appears in the proof of [20, Thm. 3.6]; the combinatorial argument there works for any commutative ring $\Bbbk$. So we only need to consider the representations of $\Bbbk W_n$ and $\Bbbk W_{n-1}$. The proof that those representations surject onto the appropriate centraliser algebras is Theorem 4.10 of this paper, our main result. In particular, the centraliser algebras $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, $\mathrm{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$ are spanned over $\Bbbk$ by elements of the form $\mathbf{P}(w)^{\otimes r}$, where $w \in W_n$, W_{n-1} respectively. Here, $\mathbf{P}(w)$ is the permutation matrix corresponding to w.

1. Tensor space

As above, $\Bbbk$ is a fixed commutative ring (with unit). We denote its zero by $0_\Bbbk$ and its unit by $1_\Bbbk$. We identify ordinary integers $m \in \mathbb{Z}$ with elements of $\Bbbk$ by means of the canonical map $\mathbb{Z} \to \Bbbk$ defined by $m \mapsto m1_\Bbbk$. In particular, this identifies $0, 1 \in \mathbb{Z}$ respectively with $0_\Bbbk, 1_\Bbbk \in \Bbbk$.

Throughout this paper, $\mathbf{V}$ denotes a fixed free $\Bbbk$-module of rank n with a given basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, by means of which we identify $\mathbf{V}$ with $\Bbbk^n$. For any positive integer r, the set

$$\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} : i_1, \dots, i_r \in \{1, \dots, n\}\} \tag{1}$$

is a basis of the rth tensor power $\mathbf{V}^{\otimes r}$. The general linear group $\mathrm{GL}(\mathbf{V})$ of $\Bbbk$-linear automorphisms of $\mathbf{V}$ acts naturally on the left on $\mathbf{V}$; this action extends diagonally to an action on $\mathbf{V}^{\otimes r}$. The symmetric group $\mathfrak{S}_r$ acts on the right on $\mathbf{V}^{\otimes r}$ by permuting the tensor positions; this action is known as the *place-permutation* action, defined by

$$(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r})^\sigma = \mathbf{v}_{i_{1\sigma^{-1}}} \otimes \cdots \otimes \mathbf{v}_{i_{r\sigma^{-1}}}, \text{ for } \sigma \in \mathfrak{S}_r \tag{2}$$

extended linearly. (We write maps in $\mathfrak{S}_r$ on the *right* of their arguments.) Thus we have commuting actions of the groups $\mathrm{GL}(\mathbf{V})$, $\mathfrak{S}_r$ on $\mathbf{V}^{\otimes r}$, making $\mathbf{V}^{\otimes r}$ into a $(\Bbbk\,\mathrm{GL}(\mathbf{V}), \Bbbk\mathfrak{S}_r)$-bimodule. Classical Schur–Weyl duality is the statement that the centraliser of each action is generated by the image of the other. It was proved originally over $\Bbbk = \mathbb{C}$ by Schur, extended to infinite fields by J. A. Green and many other authors, and extended to sufficiently large fields in [3].

Let W_n be the Weyl group of $\mathrm{GL}(\mathbf{V})$; that is, the group of elements of $\mathrm{GL}(\mathbf{V})$ permuting the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. We identify W_n with the group of permutation matrices, regarded as matrices with entries from $\Bbbk$. By restricting the action of $\mathrm{GL}(\mathbf{V})$ to W_n, we obtain left actions of W_n on $\mathbf{V}$ and $\mathbf{V}^{\otimes r}$. To be explicit, $w \in W_n$ acts by

$$w(\mathbf{v}_{j_1} \otimes \cdots \otimes \mathbf{v}_{j_r}) = \mathbf{v}_{w(j_1)} \otimes \cdots \otimes \mathbf{v}_{w(j_r)}. \tag{3}$$

(We write maps in W_n on the *left* of their arguments.) Extended linearly, the action of W_n defines a linear representation $\Bbbk W_n \to \mathrm{End}_\Bbbk(\mathbf{V}^{\otimes r})$ of the group algebra $\Bbbk W_n$.

Of course, $W_n \cong \mathfrak{S}_n^{\mathrm{op}}$, the opposite group. We distinguish these symmetric groups notationally throughout this paper, because their actions on tensor space are very different even when $n = r$.

Let $\mathcal{P}_r(\delta) \supset \mathfrak{S}_r$ be the partition algebra (introduced independently by Paul P. Martin [24, 25] and V. F. R. Jones [22] in relation to the Potts model in particle physics) on $2r$ vertices, with parameter $\delta \in \Bbbk$. The algebra $\mathcal{P}_r(\delta)$ has a $\Bbbk$-basis in bijection with the collection of set partitions on

$$\{1, \dots, r\} \cup \{1', \dots, r'\}.$$

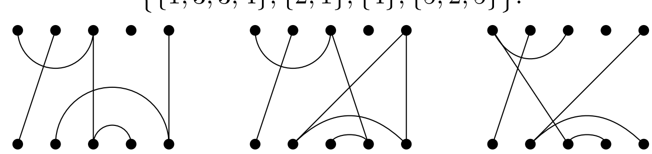
Each basis element d may be regarded as a graph with $2r$ vertices arranged in two rows, with vertices numbered $1, \dots, r$ along the top and $1', \dots r'$ along the bottom. Two vertices in the graph are connected by an edge if and only if they lie in the same subset of the set partition d. Multiplication in the algebra may be defined on the basis elements by stacking diagrams and removing any connected components that contain no vertices from the top and bottom rows of the stack. After removing such interior components, the result of stacking d_1 above d_2 is a new diagram d_3, and the multiplication is defined by

$$d_1 d_2 = \delta^k \, d_3 \tag{4}$$

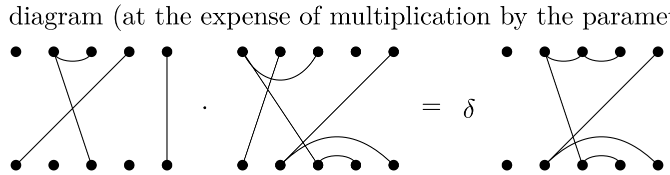
where k is the number of removed interior connected components. It can be checked that this rule, extended linearly, defines an associative multiplication on $\mathcal{P}_r(\delta)$.

Example 1.1. The following 3 diagrams all depict the same set partition:

$$\big\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\big\}.$$



Example 1.2. In the following example, the diagram on the right-hand side of the equality is obtained by stacking the two diagrams on the left of the equality on top of each other (with the leftmost diagram on top) and removing the singleton from the middle of the diagram (at the expense of multiplication by the parameter δ^1).



The half partition algebra $\mathcal{P}_{r+1/2}(\delta)$, introduced in [26], is the subalgebra of $\mathcal{P}_{r+1}(\delta)$ spanned by diagrams such that vertex $r+1$ is connected to vertex $(r+1)'$.

By specialising the parameter δ to n, we obtain a linear representation of $\mathcal{P}_r(n)^{\mathrm{op}}$, defined as follows. Let $I(n, r) = \{1, \dots, n\}^r$ be the set of multi-indices of length r. To simplify the notation, we often write

$$i_1 \cdots i_r \text{ instead of } (i_1, \dots, i_r)$$

for an element of $I(n, r)$. Elements of $I(n, r)$ can also be written as $i_{1'} \cdots i_{r'}$ or $(i_{1'}, \dots, i_{r'})$. Connected components of a diagram are called *blocks*; they correspond to the subsets of the underlying set partition. Following [20], we define a scalar $(d)^{i_1 \cdots i_r}_{i_{1'} \cdots i_{r'}} \in \Bbbk$, for a diagram d and any $i_1 \cdots i_r$, $i_{1'} \cdots i_{r'}$ in $I(n, r)$, by

$$(d)^{i_1 \cdots i_r}_{i_{1'} \cdots i_{r'}} = \begin{cases} 1 & \text{if } i_\alpha = i_\beta \text{ whenever } \alpha \neq \beta \text{ are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases}$$

Here, the indices α, β may be primed or unprimed. The diagram d acts on $\mathbf{V}^{\otimes r}$, on the right, by the rule

$$(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r})^d = \sum_{i_{1'} \cdots i_{r'}} (d)^{i_1 \cdots i_r}_{i_{1'} \cdots i_{r'}} (\mathbf{v}_{i_{1'}} \otimes \cdots \otimes \mathbf{v}_{i_{r'}}) \tag{5}$$

extended linearly. Note that if $d \in \mathfrak{S}_r$ is a permutation diagram, then d acts by the usual place-permutation action. Extended linearly, this action defines a linear representation $\mathcal{P}_r(n)^{\mathrm{op}} \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$. The commuting actions define a $(\Bbbk W_n, \mathcal{P}_r(n))$-bimodule structure on $\mathbf{V}^{\otimes r}$. By identifying $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n \subset \mathbf{V}^{\otimes (r+1)}$, by restriction $\mathbf{V}^{\otimes r}$ may also be regarded as a $(\Bbbk W_{n-1}, \mathcal{P}_{r+1/2}(n))$-bimodule, where W_{n-1} is identified with the subgroup of W_n consisting of the permutations fixing n.

Because the actions commute, the representations $\Bbbk W_n \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$, $\mathcal{P}_r(n) \to \mathrm{End}_{\Bbbk}(\mathbf{V}^{\otimes r})$ induce $\Bbbk$-algebra homomorphisms

$$\begin{aligned} &\Phi_{n,r} : \Bbbk W_n \to \mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r}), \\ &\Psi_{n,r} : \mathcal{P}_r(n)^{\mathrm{op}} \to \mathrm{End}_{W_n}(\mathbf{V}^{\otimes r}) \end{aligned} \tag{6}$$

into the respective centraliser algebras. Similarly, by restriction we have induced $\Bbbk$-algebra homomorphisms

$$\begin{aligned} &\Phi_{n,r+1/2} : \Bbbk W_{n-1} \to \mathrm{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r}), \\ &\Psi_{n,r+1/2} : \mathcal{P}_{r+1/2}(n)^{\mathrm{op}} \to \mathrm{End}_{W_{n-1}}(\mathbf{V}^{\otimes r}). \end{aligned} \tag{7}$$

Schur–Weyl duality is equivalent to the surjectivity of these induced homomorphisms. As noted above, the combinatorial argument given in [20, Thm. 3.6], which works over any commutative ring $\Bbbk$, proves the surjectivity of the maps $\Psi_{n,r}$, $\Psi_{n,r+1/2}$. Indeed, the partition algebra was originally defined with that property in mind.

Thus, we only need to prove the surjectivity of the induced maps $\Phi_{n,r}$, $\Phi_{n,r+1/2}$ defined in (6), (7).

2. Generalised doubly-stochastic matrices

We denote the entry in the ith row and jth column of a matrix $\mathbf{M}$ by m^i_j, and write $\mathbf{M} = (m^i_j)$. In this paper, we will always follow this convention of using bold letters for matrices and lower case letters for their entries. It will be convenient to employ the following terminology; see e.g. [5, 17, 21, 23].

Definition 2.1. *An $n \times n$ matrix $\mathbf{M} = \left(m^i_j\right)_{i,j=1,\dots,n}$ is* generalised doubly-stochastic *(GDS) if there is some $s = s(\mathbf{M})$ in $\Bbbk$ such that both:*

(a) $\sum_{j=1}^n m^i_j = s$, *for all $i = 1, \dots, n$, and*
(b) $\sum_{i=1}^n m^i_j = s$, *for all $j = 1, \dots, n$.*

In other words, $\mathbf{M}$ is GDS if there is a common value for all its row and column sums.

Lemma 2.2. *Assume that $n > 1$. An $n \times n$ matrix $\mathbf{M}$ over the ring $\Bbbk$ is GDS if and only if $\mathbf{M}$ commutes with the matrix $\mathbf{J}_n = (1)_{i,j=1,\dots,n}$ of all ones.*

Proof. The matrix $\mathbf{J}_n\mathbf{M}$ is the $n \times n$ matrix with the column sum $\mathrm{co}_i(\mathbf{M}) := \sum_j m^i_j$ in each entry of the ith column. On the other hand, $\mathbf{M}\mathbf{J}_n$ is the $n \times n$ matrix with the row sum $\mathrm{ro}_j(\mathbf{M}) := \sum_i m^i_j$ in each entry of the jth row. So $\mathbf{M}$ commutes with $\mathbf{J}_n$ if and only if $\mathrm{co}_i(\mathbf{M}) = \mathrm{ro}_j(\mathbf{M})$ for all $i, j = 1, \dots, n$. Since $n > 1$, it follows that this is so if and only if all the row and column sums have a common value. □

Let $\mathbf{M}$ be a generalised doubly-stochastic matrix. If $\mathbf{M}$ is identified with the $\Bbbk$-linear endomorphism of $\mathbf{V}$ defined by $\mathbf{v}_j \mapsto \sum_i m^i_j \mathbf{v}_i$, then $s(\mathbf{M})$ is an eigenvalue for the eigenvector $\mathbf{v}_1 + \cdots + \mathbf{v}_n$. The proof is easy. This observation leads to the following characterisation of GDS operators, where GDS operators on $\mathbf{V}$ are defined to be linear operators whose matrices with respect to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are GDS.

3. Description of the invariants $\mathbb{E}_\Bbbk(n, r)$

To ease the notation, we henceforth put $\mathbb{E}_\Bbbk(n, r) = \mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$. Similarly, we set $\mathbb{E}_\Bbbk(n, r+\frac{1}{2}) = \mathrm{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$. The purpose of this section is to describe $\mathbb{E}_\Bbbk(n, r)$; a description of $\mathbb{E}_\Bbbk(n, r+\frac{1}{2})$ will be given in the next section.

We introduce some additional notation. For each multi-index $\boldsymbol{i} = i_1 \cdots i_r$ in $I(n, r)$, σ in $\mathfrak{S}_r$, and w in W_n, we set

$$\boldsymbol{i}^\sigma = (i_1 \cdots i_r)^\sigma = i_{1\sigma^{-1}} \cdots i_{r\sigma^{-1}}, \quad w\boldsymbol{i} = w(i_1 \cdots i_r) = w(i_1) \cdots w(i_r).$$

The assignment $(\boldsymbol{i}, \sigma) \mapsto \boldsymbol{i}^\sigma$ defines a right action (the place-permutation action) of the symmetric group $\mathfrak{S}_r$ on the set $I(n, r)$; the assignment $(w, \boldsymbol{i}) \mapsto w\boldsymbol{i}$ defines a left action of W_n on $I(n, r)$. These left and right actions on $I(n, r)$ commute: $(w\boldsymbol{i})^\sigma = w(\boldsymbol{i}^\sigma)$, for all $w \in W_n$, $\sigma \in \mathfrak{S}_r$. Set:

$$\mathbf{v}_{\boldsymbol{i}} = \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r}. \tag{8}$$

Then the basis of $\mathbf{V}^{\otimes r}$ given in (1) is $\{\mathbf{v}_{\boldsymbol{i}} : \boldsymbol{i} \in I(n, r)\}$, and the commuting actions of W_n and $\mathfrak{S}_r$ on $\mathbf{V}^{\otimes r}$ considered in Section 1 are given by the rules $(w, \mathbf{v}_{\boldsymbol{i}}) \mapsto \mathbf{v}_{w\boldsymbol{i}}$ and $(\mathbf{v}_{\boldsymbol{i}}, \sigma) \mapsto \mathbf{v}_{\boldsymbol{i}^\sigma}$.

Orbits for the left action of W_n on $I(n, r)$ are called *value-types* and may be identified with set partitions of $\{1, \dots, r\}$. The subsets in the value-type $\mathrm{vt}(\boldsymbol{i}) = \Lambda$ of $\boldsymbol{i} = i_1 \cdots i_r$ record the positions at which the distinct values that appear are constant. More precisely, we make the following definition.

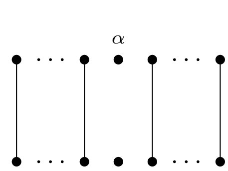
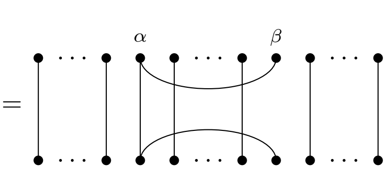
Definition 3.1. *Let $\boldsymbol{i} = i_1 \cdots i_r \in I(n, r)$ be given. For each $v = 1, \dots, n$, let Λ'_v be the set of all positions $\alpha = 1, \dots, r$ for which $i_\alpha = v$. Then $\{1, \dots, r\} = \bigcup_{v=1}^n \Lambda'_v$. Discard any empty Λ'_v to obtain the set partition $\Lambda = \{\Lambda'_v : \Lambda'_v \neq \varnothing\}$ which defines* $\mathrm{vt}(\boldsymbol{i})$. *Let $\ell(\Lambda)$ be the number of non-empty subsets in Λ.*

For example, the multi-index $\boldsymbol{i} = abbcabc$ (for distinct elements $a, b, c \in \{1, \dots, n\}$) has value-type $\mathrm{vt}(\boldsymbol{i}) = \{\{1, 5\}, \{2, 3, 6\}, \{4, 7\}\}$.

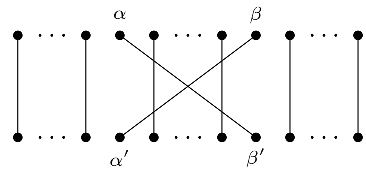
Recall that orbits for the right action of $\mathfrak{S}_r$ on $I(n, r)$ are called *weights* and may be identified with weak compositions of r of length at most n. To be precise, the weight $\mathrm{wt}(\boldsymbol{i})$ of a multi-index $\boldsymbol{i} = i_1 \cdots i_r$ is the composition $\mathrm{wt}(\boldsymbol{i}) = (\mu_1, \dots, \mu_n)$ where for each value $v = 1, \dots, n$ the statistic μ_v counts the number of positions $\alpha = 1, \dots, r$ such that $i_\alpha = v$. In other words, $\mu_v = |\Lambda'_v|$ for each $v = 1, \dots, n$.

We also need to consider the set $\Omega(n, r)$ of orbits for the right action of $\mathfrak{S}_r$ on $I(n, r) \times I(n, r)$ defined by $(\boldsymbol{i}, \boldsymbol{j})^\sigma = (\boldsymbol{i}^\sigma, \boldsymbol{j}^\sigma)$.

In order to describe the invariants $\mathrm{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$ it suffices to consider a set of generators. Halverson and Ram [20] showed that $\mathcal{P}_r(\delta)$ is generated by the diagrams:

$p_\alpha =$  $\qquad p_{\alpha,\beta} =$ 

$s_{\alpha,\beta} =$



where $1 \leqslant \alpha < \beta \leqslant r$. In fact, $\mathcal{P}_r(\delta)$ is generated by the usual Coxeter generators $s_{\alpha,\alpha+1}$ for $\alpha = 1, \dots, r-1$ along with just one p_α and one $p_{\alpha,\beta}$.

Let $\mathcal{G}_r(n)$, $\mathcal{H}_r$, $\Bbbk\mathfrak{S}_r$ be the subalgebras of $\mathcal{P}_r(n)$ respectively generated by the p_α, $p_{\alpha,\beta}$, and $s_{\alpha,\beta}$ pictured above. Note that $\mathcal{H}_r$ is independent of n. We shall separately consider the centraliser algebras of these three subalgebras:

$$G_\Bbbk(n,r) = \mathrm{End}_{\mathcal{G}_r(n)}(\mathbf{V}^{\otimes r}), \quad H_\Bbbk(n,r) = \mathrm{End}_{\mathcal{H}_r}(\mathbf{V}^{\otimes r}), \quad S_\Bbbk(n,r) = \mathrm{End}_{\mathfrak{S}_r}(\mathbf{V}^{\otimes r}).$$

Then we have $\mathbb{E}_\Bbbk(n,r) = G_\Bbbk(n,r) \cap H_\Bbbk(n,r) \cap S_\Bbbk(n,r)$. Note that $S_\Bbbk(n,r)$ is the classical Schur algebra appearing in [19]. Furthermore, we have $\mathcal{H}_1 = \Bbbk\mathfrak{S}_1 = \Bbbk$ and thus $H_\Bbbk(n,1) = S_\Bbbk(n,1) = \mathrm{End}_\Bbbk(\mathbf{V})$, which means that $\mathbb{E}_\Bbbk(n,1) = G_\Bbbk(n,1)$. By Lemma 2.2, the algebra $G_\Bbbk(n,1)$ coincides with the set of $n \times n$ GDS matrices considered in Section 2.

Henceforth, we write $\mathrm{Mat}_{I(n,r)}(\Bbbk)$ for the set of $n^r \times n^r$ matrices over $\Bbbk$, with rows and columns indexed by the set $I(n,r)$ according to the lexicographic ordering. We always identify the matrix $\mathbf{A} = (a^{\boldsymbol{i}}_{\boldsymbol{j}})$ with the $\Bbbk$-linear endomorphism of $\mathbf{V}^{\otimes r}$ defined on basis elements by $\mathbf{v}_{\boldsymbol{j}} \mapsto \sum_{\boldsymbol{i} \in I(n,r)} a^{\boldsymbol{i}}_{\boldsymbol{j}} \mathbf{v}_{\boldsymbol{i}}$.

Proposition 3.2. *Let $G_\Bbbk(n,r)$, $H_\Bbbk(n,r)$, and $S_\Bbbk(n,r)$ be the subalgebras of $\mathrm{End}_\Bbbk(\mathbf{V}^{\otimes r})$ consisting of the endomorphisms commuting with the action of all the p_α, $p_{\alpha,\beta}$, and $s_{\alpha,\beta}$ respectively. Let $\mathbf{A} = (a^{\boldsymbol{i}}_{\boldsymbol{j}}) \in \mathrm{Mat}_{I(n,r)}(\Bbbk)$. Then*

(a) *$\mathbf{A}$ belongs to $G_\Bbbk(n,r)$ if and only if for each place $\alpha = 1, \dots, r$ and for each fixed $\boldsymbol{p} = i_1 \cdots i_{\alpha-1} i_{\alpha+1} \cdots i_r$, $\boldsymbol{q} = j_1 \cdots j_{\alpha-1} j_{\alpha+1} \cdots j_r$ in $I(n, r-1)$ there is some scalar $b^{\boldsymbol{p}}_{\boldsymbol{q}}(\alpha)$ in $\Bbbk$ such that*

$$\sum_{i=1}^{n} a^{i_1 \cdots i_{\alpha-1}\, i\, i_{\alpha+1} \cdots i_r}_{j_1 \cdots j_{\alpha-1}\, j\, j_{\alpha+1} \cdots j_r} = b^{\boldsymbol{p}}_{\boldsymbol{q}}(\alpha), \text{ for any } j = 1, \dots, n$$

and

$$\sum_{j=1}^{n} a^{i_1 \cdots i_{\alpha-1}\, i\, i_{\alpha+1} \cdots i_r}_{j_1 \cdots j_{\alpha-1}\, j\, j_{\alpha+1} \cdots j_r} = b^{\boldsymbol{p}}_{\boldsymbol{q}}(\alpha), \text{ for any } i = 1, \dots, n.$$

(b) *$\mathbf{A}$ belongs to $H_\Bbbk(n,r)$ if and only if $\mathbf{A}$ preserves value-type, in the sense: $a^{\boldsymbol{i}}_{\boldsymbol{j}} = 0$ for all pairs $(\boldsymbol{i}, \boldsymbol{j}) \in I(n,r) \times I(n,r)$ such that $\mathrm{vt}(\boldsymbol{i}) \neq \mathrm{vt}(\boldsymbol{j})$.*

(c) *$\mathbf{A}$ belongs to $S_\Bbbk(n,r)$ if and only if $\mathbf{A}$ is constant on each place-permutation orbit $\mathcal{O}$ in $\Omega(n,r)$, i.e. if $a^{\boldsymbol{i}}_{\boldsymbol{j}} = a^{\boldsymbol{k}}_{\boldsymbol{l}}$ whenever $(\boldsymbol{i}, \boldsymbol{j})$ and $(\boldsymbol{k}, \boldsymbol{l})$ are in the same orbit $\mathcal{O} \in \Omega(n,r)$.*

Proof. (a) Suppose that $1 \leqslant \alpha \leqslant r$. By the definition in equation (5), the matrix $\Psi_{n,r}(p_\alpha)$ representing p_α with respect to the basis $\{\mathbf{v}_{\boldsymbol{i}} : \boldsymbol{i} \in I(n,r)\}$ has the form

$$\left(\delta_{i_1,j_1} \cdots \delta_{i_{\alpha-1},j_{\alpha-1}} \delta_{i_{\alpha+1},j_{\alpha+1}} \cdots \delta_{i_r,j_r}\right)_{\boldsymbol{i},\boldsymbol{j} \in I(n,r) \times I(n,r)}.$$

More succinctly, the matrix can be written as $(\mathbf{I}_n)^{\otimes \alpha-1} \otimes \mathbf{J}_n \otimes (\mathbf{I}_n)^{\otimes r-\alpha}$, where $\mathbf{J}_n$ is the $n \times n$ matrix defined in Section 2. It follows immediately from Lemma 2.2 that the commutant $\mathrm{End}_{p_\alpha}(\mathbf{V}^{\otimes r})$ of p_α is the set of endomorphisms satisfying the displayed condition in part (a) of the proposition. Thus, the centraliser $G_\Bbbk(n,r)$ of all the p_α for $1 \leqslant \alpha \leqslant r$ is the set of endomorphisms satisfying the condition for all α. This proves part (a).

(b) Suppose that $1 \leqslant \alpha < \beta \leqslant r$. By (5), the matrix $\Psi_{n,r}(p_{\alpha,\beta})$ representing $p_{\alpha,\beta}$ with respect to the basis $\{\mathbf{v}_{\boldsymbol{i}} : \boldsymbol{i} \in I(n,r)\}$ is

$$\big(\delta_{i_1,j_1} \cdots \delta_{i_{\alpha-1},j_{\beta-1}} \delta_{i_\alpha,i_\beta,j_\alpha,j_\beta} \delta_{i_{\alpha+1},j_{\beta+1}} \cdots \delta_{i_r,j_r}\big)_{(\boldsymbol{i},\boldsymbol{j}) \in I(n,r)\times I(n,r)}.$$

Here $\delta_{i_\alpha,i_\beta,j_\alpha,j_\beta} = \delta_{i_\alpha,i_\beta}\delta_{i_\alpha,j_\alpha}\delta_{i_\alpha,j_\beta}$ is a generalised Kronecker delta symbol, which is 1 if $i_\alpha = i_\beta = j_\alpha = j_\beta$ and 0 otherwise. So the matrix $\Psi_{n,r}(p_{\alpha,\beta})$ is a diagonal matrix with $(\boldsymbol{i},\boldsymbol{i})$-entry equal to

$$\begin{cases} 1 & \text{if } i_\alpha = i_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

for $\boldsymbol{i} = i_1 \cdots i_r$ in $I(n,r)$. If we reorder $I(n,r)$ so that all the nonzero diagonal entries come before the zero ones, then the matrix $\Psi_{n,r}(p_{\alpha,\beta})$ has the block form

$$\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$$

and an easy calculation with block matrices shows that the commutant $\operatorname{End}_{p_{\alpha,\beta}}(\mathbf{V}^{\otimes r})$ of $p_{\alpha,\beta}$ consists of all block matrices of the form

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B} \end{array}\right]$$

where $\mathbf{A}, \mathbf{B}$ are arbitrary matrices (of the relevant sizes). So $p_{\alpha,\beta}$ sends all $\mathbf{v}_{\boldsymbol{i}}$ satisfying the condition $i_\alpha = i_\beta$ to a linear combination of $\mathbf{v}_{\boldsymbol{j}}$ such that $j_\alpha = j_\beta$ and sends the $\mathbf{v}_{\boldsymbol{i}}$ satisfying $i_\alpha \neq i_\beta$ to a linear combination of $\mathbf{v}_{\boldsymbol{j}}$ such that $j_\alpha \neq j_\beta$.

It follows that the centraliser algebra $H_\Bbbk(n,r)$, which is the intersection of the commutants of the various $p_{\alpha,\beta}$ for $1 \leqslant \alpha < \beta \leqslant r$, is the algebra of all value-type preserving endomorphisms. This proves part (b).

(c) The proof of part (c) is well known and can be found, for instance, in [19]. $\square$

Definition 3.3. *Let $\boldsymbol{i} = i_1 \cdots i_r$, $\boldsymbol{j} = j_1 \cdots j_r$ be given multi-indices. Suppose that $\alpha \in \{1, \ldots, r\}$ is a place. The row and column α-*slices *of $\mathbf{A}$ determined by $(\boldsymbol{i},\boldsymbol{j})$ are respectively the n-vectors*

$$\big(a^{i_1\cdots i_{\alpha-1}\, i\, i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1}\, j_\alpha\, j_{\alpha+1}\cdots j_r}\big)_{i=1,\ldots,n} \quad \text{and} \quad \big(a^{i_1\cdots i_{\alpha-1}\, i_\alpha\, i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1}\, j\, j_{\alpha+1}\cdots j_r}\big)_{j=1,\ldots,n}.$$

We denote these vectors respectively by the suggestive shorthand notations

$$a^{i_1\cdots i_{\alpha-1} * i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1}\, j_\alpha\, j_{\alpha+1}\cdots j_r} \quad \text{and} \quad a^{i_1\cdots i_{\alpha-1}\, i_\alpha\, i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1} * j_{\alpha+1}\cdots j_r}.$$

Putting a $\sum$ symbol in front of a slice implies a sum over the elements of the slice. We also extend this notation in the obvious way to allow two or more $*$s to appear; we call them double slices, *etc.*

Here are some basic properties of invariants in $\mathbb{E}_\Bbbk(n,r)$, for $r \geqslant 2$. (If $r = 1$, invariants in $\mathbb{E}_\Bbbk(n,r)$ are just $n \times n$ GDS matrices; see Section 2.)

Proposition 3.4. *Suppose that $r \geqslant 2$ and that $\mathbf{A}$ is an element of $\mathbb{E}_\Bbbk(n,r)$. Then:*

(a) *For any $\boldsymbol{p}$, $\boldsymbol{q} \in I(n, r-1)$, the scalars $b^{\boldsymbol{p}}_{\boldsymbol{q}}(\alpha)$ appearing in Proposition* 3.2(a) *are independent of α. That is, all the slice sums determined by $\boldsymbol{p}, \boldsymbol{q}$ have the same value $b^{\boldsymbol{p}}_{\boldsymbol{q}}$.*

(b) *For any $\boldsymbol{p}$, $\boldsymbol{q} \in I(n, r-1)$, let $\mathbf{A}^{\boldsymbol{p}}_{\boldsymbol{q}}$ be the $n \times n$ block $\mathbf{A}^{\boldsymbol{p}}_{\boldsymbol{q}} = (a^{\boldsymbol{p}\,i}_{\boldsymbol{q}\,j})_{i,j=1,\ldots,n}$. Then $\mathbf{A}^{\boldsymbol{p}}_{\boldsymbol{q}}$ is GDS, with common row and column sums equal to $b^{\boldsymbol{p}}_{\boldsymbol{q}}$.*

(c) *If $\boldsymbol{i} = i_1 \cdots i_r$, $\boldsymbol{j} = j_1 \cdots j_r$ are in $I(n,r)$ and $\operatorname{vt}(\boldsymbol{i}) \neq \operatorname{vt}(\boldsymbol{j})$ then $a^{\boldsymbol{i}}_{\boldsymbol{j}} = 0$.*

(d) *If $\boldsymbol{i} = i_1 \cdots i_r$, $\boldsymbol{j} = j_1 \cdots j_r$ are in $I(n,r)$, $\operatorname{vt}(\boldsymbol{i}) = \operatorname{vt}(\boldsymbol{j})$, and i_r appears in $\boldsymbol{p} = i_1 \cdots i_{r-1}$, then j_r also appears in the same place in $\boldsymbol{q} = j_1 \cdots j_{r-1}$, and $a^{\boldsymbol{i}}_{\boldsymbol{j}} = b^{\boldsymbol{p}}_{\boldsymbol{q}}$.*

Proof. (a) Since $\mathbb{E}_{\Bbbk}(n,r)$ is the intersection of $G_{\Bbbk}(n,r)$, $H_{\Bbbk}(n,r)$, and $S_{\Bbbk}(n,r)$ it is clear that $\mathbf{A}$ must be constant on place-permutation orbits, by Proposition 3.2(c). This immediately implies that its slice sums are independent of α.

(b) This follows from part (a). Explicitly, for any $\alpha = 1,\dots,r$ and any given $\boldsymbol{p} = i_1\cdots i_{\alpha-1}i_{\alpha+1}\cdots i_r$, $\boldsymbol{q} = j_1\cdots j_{\alpha-1}\, j_{\alpha+1}\cdots j_r$ in $I(n,r-1)$, part (a) says that there exists a scalar $b^{\boldsymbol{p}}_{\boldsymbol{q}}$ in $\Bbbk$ such that

$$\sum a^{i_1\cdots i_{\alpha-1}\,*\,i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1}\,j\,j_{\alpha+1}\cdots j_r} = b^{\boldsymbol{p}}_{\boldsymbol{q}} \quad\text{and}\quad \sum a^{i_1\cdots i_{\alpha-1}\,i\,i_{\alpha+1}\cdots i_r}_{j_1\cdots j_{\alpha-1}\,*\,j_{\alpha+1}\cdots j_r} = b^{\boldsymbol{p}}_{\boldsymbol{q}}$$

for any $i,j = 1,\dots,n$. Part (b) is the particular case where $\boldsymbol{p} = i_1\cdots i_{r-1}$, $\boldsymbol{q} = j_1\cdots j_{r-1}$.

(c) This is the same as part (b) of Proposition 3.2, repeated here for the sake of convenience.

(d) This follows from parts (b) and (c). By part (c), under these hypotheses, all terms except $a^{\boldsymbol{p}\,i_r}_{\boldsymbol{q}\,j_r}$ of the slice sum $\sum a^{\boldsymbol{p}\,i_r}_{\boldsymbol{q}\,*} = b^{\boldsymbol{p}}_{\boldsymbol{q}}$ must be zero. □

Definition 3.5. *Assume that $r \geqslant 1$. Let $\mathbf{A} \in \mathbb{E}_{\Bbbk}(n,r)$ be given. For any $\boldsymbol{p}$, $\boldsymbol{q} \in I(n,r-1)$ let $b^{\boldsymbol{p}}_{\boldsymbol{q}}$ be the common value of the slice sums in $\mathbf{A}$ indexed by $\boldsymbol{p},\boldsymbol{q}$. (This is a single scalar if $r=1$.) The matrix $\mathbf{B} = (b^{\boldsymbol{p}}_{\boldsymbol{q}})_{\boldsymbol{p},\boldsymbol{q}\in I(n,r-1)}$ in $\mathrm{Mat}_{I(n,r-1)}(\Bbbk)$ is the* restriction *of $\mathbf{A}$. We write $\rho(\mathbf{A}) = \mathbf{B}$ for the restriction.*

Proposition 3.4(b) says that the invariant $\mathbf{A}$ is obtained by "blowing up" its restriction $\mathbf{B} = \rho(\mathbf{A})$ by a process which replaces each matrix entry $b^{\boldsymbol{p}}_{\boldsymbol{q}}$ of $\mathbf{B}$ by an $n\times n$ GDS matrix with row and column sums equal to $b^{\boldsymbol{p}}_{\boldsymbol{q}}$. Of course, $\mathbf{A}$ must also be invariant under place-permutations, so blowing up is not the only consideration.

Proposition 3.6. *Let $r \geqslant 1$, and suppose that $\mathbf{A} \in \mathbb{E}_{\Bbbk}(n,r)$. Then the restriction $\rho(\mathbf{A})$ belongs to $\mathbb{E}_{\Bbbk}(n,r-1)$. Thus, ρ is a $\Bbbk$-linear map from $\mathbb{E}_{\Bbbk}(n,r)$ into $\mathbb{E}_{\Bbbk}(n,r-1)$.*

Proof. Set $\mathbf{B} = \rho(\mathbf{A})$. If $r=1$ then $\mathbf{B}$ is just a scalar and there is nothing to prove, since $\mathbb{E}_{\Bbbk}(n,0) = \Bbbk$. For the moment, we assume that $r=2$. Given any $i_1i_2 \in I(n,2)$, we compute the double-slice sum $\sum a^{i_1i_2}_{**}$ two ways, by applying the independence in Proposition 3.4 and changing the order of summation:

$$\begin{aligned}\sum a^{i_1i_2}_{**} &= \sum_{j=1}^{n}\sum_{j'=1}^{n} a^{i_1i_2}_{jj'} = \sum_{j=1}^{n} b^{i_1}_{j} = \sum b^{i_1}_{*}\\ &= \sum_{j'=1}^{n}\sum_{j=1}^{n} a^{i_1i_2}_{jj'} = \sum_{j=1}^{n} b^{i_2}_{j'} = \sum b^{i_2}_{*}.\end{aligned}$$

Thus all the row sums in $\mathbf{B}$ have a common value. Similarly, by considering the double-slice sum $\sum a^{**}_{j_1j_2}$ we see that $\sum b^{*}_{j_1} = \sum b^{*}_{j_2}$, for arbitrary j_1j_2 in $I(n,2)$, so all the column sums in $\mathbf{B}$ have a common value. Finally, for arbitrary $i_1, j_2 = 1,\dots,n$ we compute the mixed-double-slice sum $\sum a^{i_1*}_{*j_2}$ two ways:

$$\begin{aligned}\sum a^{i_1*}_{*j_2} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a^{i_1i}_{jj_2} = \sum_{i=1}^{n} b^{i}_{j_2} = \sum b^{*}_{j_2}\\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} a^{i_1i}_{jj_2} = \sum_{j=1}^{n} b^{i_1}_{j} = \sum b^{i_1}_{*}.\end{aligned}$$

Thus $\sum b^{*}_{j_2} = \sum b^{i_1}_{*}$. This shows that all the row sums are equal to all the column sums in $\mathbf{B}$. In other words, $\mathbf{B}$ is GDS, and thus belongs to $\mathbb{E}_{\Bbbk}(n,1)$. This proves the result in case $r=2$.

Now assume that $r>2$. We need to show that $\mathbf{B}$ satisfies the conditions of Proposition 3.2(a)–(c). It is easy to see that conditions (b), (c) hold for $\mathbf{B}$ since they hold for

$\mathbf{A}$. Then calculations similar to the above (and place-permutation invariance) show that $\mathbf{B}$ satisfies the slice-sum equations in part (a). Thus $\mathbf{B}$ belongs to $\mathbb{E}_{\Bbbk}(n, r-1)$ as claimed. ☐

Going forward, we will often regard a given $\mathbf{A} = (a^{i_1\cdots i_r}_{j_1\cdots j_r})_{i_1\cdots i_r, j_1\cdots j_r \in I(n,r)}$ in $\mathbb{E}_{\Bbbk}(n,r)$ as the $n \times n$ block matrix

$$\mathbf{A} = (\mathbf{A}^i_j)_{i,j=1,\dots,n} = \begin{bmatrix} \mathbf{A}^1_1 & \cdots & \mathbf{A}^1_n \\ \vdots & \ddots & \vdots \\ \mathbf{A}^n_1 & \cdots & \mathbf{A}^n_n \end{bmatrix} \tag{9}$$

where each block $\mathbf{A}^i_j$ is defined by $\mathbf{A}^i_j = (a^{i\, i_2\cdots i_r}_{j\, j_2\cdots j_r})_{i_2\cdots i_r, j_2\cdots j_r \in I(n,r-1)}$.

Remark 3.7. The block notation just introduced provides a convenient description of the restriction map. Given $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$, we have

$$\rho(\mathbf{A}) = \sum \mathbf{A}^i_* = \sum \mathbf{A}^*_j$$

for any i, j. That is, the restriction $\mathbf{B} = \rho(\mathbf{A})$ is the common value of the block row and column sums in the block matrix (9).

Example 3.8. We now illustrate how, using only the value-type condition of Proposition 3.2(b) we can obtain general forms for the invariants $\mathbb{E}_{\Bbbk}(n,r)$ for small values of n and r (we will use blank entries to denote entries which are zero due to value-type mismatches). The general form of invariants in $\mathbb{E}_{\Bbbk}(2,2)$, $\mathbb{E}_{\Bbbk}(3,2)$ are displayed below, respectively:

	11	12	21	22
11	*			*
12		*	*	
21		*	*	
22	*			*

,

	11	12	13	21	22	23	31	32	33
11	*				*				*
12		*	*	*		*	*	*	
13		*	*	*		*	*	*	
21		*	*	*		*	*	*	
22	*				*				*
23		*	*	*		*	*	*	
31		*	*	*		*	*	*	
32		*	*	*		*	*	*	
33	*				*				*

.

The general form of invariants in $\mathbb{E}_{\Bbbk}(4,2)$ is displayed below:

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	*					*					*					*
12		*	*	*	*		*	*	*	*		*	*	*	*	
13		*	*	*	*		*	*	*	*		*	*	*	*	
14		*	*	*	*		*	*	*	*		*	*	*	*	
21		*	*	*	*		*	*	*	*		*	*	*	*	
22	*					*					*					*
23		*	*	*	*		*	*	*	*		*	*	*	*	
24		*	*	*	*		*	*	*	*		*	*	*	*	
31		*	*	*	*		*	*	*	*		*	*	*	*	
32		*	*	*	*		*	*	*	*		*	*	*	*	
33	*					*					*					*
34		*	*	*	*		*	*	*	*		*	*	*	*	
41		*	*	*	*		*	*	*	*		*	*	*	*	
42		*	*	*	*		*	*	*	*		*	*	*	*	
43		*	*	*	*		*	*	*	*		*	*	*	*	
44	*					*					*					*

.

In these depictions, starred entries can be non-zero, but they are not arbitrary, because they must be invariant under place-permutations and their slice sums must satisfy the GDS conditions in Proposition 3.4(b).

Let $\mathbf{P}(w) = \Phi_{n,1}(w)$ be the permutation matrix representing $w \in W_n$. As a linear endomorphism of $\mathbf{V}$, $\mathbf{P}(w)$ is the linear map sending $\mathbf{v}_j$ to $\mathbf{v}_{w(j)}$, for $j = 1, \dots, n$. In terms of matrix coordinates, $\mathbf{P}(w) = (\delta_{i,w(j)})_{i,j=1,\dots,n}$. It follows that

$$\Phi_{n,r}(w) = \mathbf{P}(w)^{\otimes r} = \left(\delta_{i_1,w(j_1)} \cdots \delta_{i_r,w(j_r)}\right)^{j_1\cdots j_r}_{i_1\cdots i_r}. \tag{10}$$

It is easy to compute the restriction of such matrices.

Lemma 3.9. *For any $w \in W_n$ and any $r \geqslant 1$, $\rho(\mathbf{P}(w)^{\otimes r}) = \mathbf{P}(w)^{\otimes (r-1)}$. In particular, $\rho(\mathbf{P}(w)) = 1$.*

Proof. Write $\mathbf{A} = \mathbf{P}(w)^{\otimes r}$. Express $\mathbf{A}$ as an $n \times n$ block matrix $\mathbf{A} = (\mathbf{A}^i_j)$ as in (9). Then by (10), $\mathbf{A}^i_j$ is given by

$$\mathbf{A}^i_j = \delta_{i,w(j)}\, \mathbf{P}(w)^{\otimes (r-1)}, \quad \text{all } i,j = 1, \dots, n.$$

The result now follows by Remark 3.7. □

Since the actions of W_n and $\mathcal{P}_r(n)$ on $\mathbf{V}^{\otimes r}$ commute, we have induced left and right actions of W_n on $\mathbb{E}_{\Bbbk}(n,r) = \operatorname{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$, which are given by left and right multiplication of the corresponding permutation matrices. Explicitly,

$$(w, \mathbf{A}) \mapsto \Phi(w)\mathbf{A} = \mathbf{P}(w)^{\otimes r}\mathbf{A}, \quad (\mathbf{A}, w) \mapsto \mathbf{A}\Phi(w) = \mathbf{A}\mathbf{P}(w)^{\otimes r} \tag{11}$$

defines the left and right actions, respectively. In other words, the algebra $\mathbb{E}_{\Bbbk}(n,r)$ is stable under row and column permutations by rth Kronecker powers of $\mathbf{P}(w)$ for $w \in W_n$.

4. Description of the invariants $\mathbb{E}_{\Bbbk}(n, r+\frac{1}{2})$

We now study how the algebra $\mathbb{E}_{\Bbbk}(n, r+\frac{1}{2}) = \operatorname{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$ is related to $\mathbb{E}_{\Bbbk}(n,r) = \operatorname{End}_{\mathcal{P}_r(n)}(\mathbf{V}^{\otimes r})$. Recall that in this context we identify $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$. The following terminology is useful for our study.

Definition 4.1.

(a) *If $i_1 \cdots i_r \in I(n,r)$, and if $j \in \{1, \dots, n\}$, let $\Lambda_j(i_1 \cdots i_r)$ be the set of places in which the value j appears:*

$$\Lambda_j(i_1 \cdots i_r) = \{\alpha \in \{1, \dots, r\} : i_\alpha = j\}.$$

We say that $i_1 \cdots i_r$ contains *j if $\Lambda_j(i_1 \cdots i_r)$ is non-empty.*

(b) *Fix i,j such that $1 \leqslant i,j \leqslant n$. Let $\mathbb{E}_{\Bbbk}(n,r)^i_j$ be the set of invariants $\mathbf{A} = (a^{i_1 \cdots i_r}_{j_1 \cdots j_r})$ in $\mathbb{E}_{\Bbbk}(n,r)$ satisfying the following condition:*

$$\text{if } a^{i_1 \cdots i_r}_{j_1 \cdots j_r} \neq 0 \text{ then } \Lambda_i(i_1 \cdots i_r) = \Lambda_j(j_1 \cdots j_r).$$

We call elements of any $\mathbb{E}_{\Bbbk}(n,r)^i_j$ special invariants*. Note that $\mathbb{E}_{\Bbbk}(n,r)^i_j$ is a $\Bbbk$-module, for any i,j. For any $i = 1, \dots, n$, $\mathbb{E}_{\Bbbk}(n,r)^i_i$ is an algebra over $\Bbbk$, the algebra of $\mathbf{v}_i$-fixing invariants.*

Lemma 4.2. *Suppose that $r \geqslant 1$ and $\mathbf{A}$ is in $\mathbb{E}_{\Bbbk}(n,r)$. If all blocks except for $\mathbf{A}^i_j$ in the ith block row and jth block column are zero blocks, then $\mathbf{A}$ must be a special invariant in $\mathbb{E}_{\Bbbk}(n,r)^i_j$.*

Proof. Suppose $a^{i_1 \cdots i_r}_{j_1 \cdots j_r} \neq 0$. We must show that $\Lambda_i(i_1 \cdots i_r) = \Lambda_j(j_1 \cdots j_r)$.

Case 1. Assume that $i_1 \cdots i_r$ contains i or $j_1 \cdots j_r$ contains j. Since $\mathbf{A}$ is invariant under place-permutations we can assume that $i_1 = i$ or $j_1 = j$. Then the hypothesis implies that *both* $i_1 = i$ and $j_1 = j$. Since $\mathbf{A}$ preserves value-type, it follows that the places in $i_1 \cdots i_r$ containing i must agree with the places in $j_1 \cdots j_r$ containing j, so $\Lambda_i(i_1 \cdots i_r) = \Lambda_j(j_1 \cdots j_r)$.

Case 2. Otherwise, $i_1 \cdots i_r$ does not contain i and $j_1 \cdots j_r$ does not contain j. Again $\Lambda_i(i_1 \cdots i_r) = \Lambda_j(j_1 \cdots j_r)$, as both sets are empty. □

The next result says in particular that the blocks of any invariant are always special invariants in the previous degree.

Lemma 4.3. *Suppose that $r \geqslant 1$ and that $\mathbf{A}$ is in $\mathbb{E}_{\Bbbk}(n,r)$. For any fixed $1 \leqslant i,j \leqslant n$, the block matrix $\mathbf{A}^i_j = (a^{i\,i_2\cdots i_r}_{j\,j_2\cdots j_r})_{i_2\cdots i_r,\, j_2\cdots j_r \in I(n,r-1)}$ belongs to $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$, after re-indexing its rows and columns via the forgetful map that omits the initial term of each multi-index.*

Proof. Clearly $\mathbf{A}^i_j$ belongs to $\mathbb{E}_{\Bbbk}(n,r-1)$, since it satisfies the conditions of Proposition 3.2. Furthermore, the fact that $\mathbf{A}$ preserves value-type implies that

$$\text{if } a^{i\,i_2\cdots i_r}_{j\,j_2\cdots j_r} \neq 0 \text{ then } \Lambda_i(i_2\cdots i_r) = \Lambda_j(j_2\cdots j_r)$$

since $\Lambda_i(i\,i_2\cdots i_r) = \Lambda_j(j\,j_2\cdots j_r)$ by hypothesis. So $\mathbf{A}^i_j \in \mathbb{E}_{\Bbbk}(n,r-1)^i_j$, as required. $\square$

Lemma 4.4. *$\mathbb{E}_{\Bbbk}(n, r+\frac{1}{2})$ is isomorphic to $\mathbb{E}_{\Bbbk}(n,r)^n_n$ (as algebras). In particular, $\mathbb{E}_{\Bbbk}(n, r+\frac{1}{2})$ embeds in $\mathbb{E}_{\Bbbk}(n,r)$.*

Proof. This is more or less immediate from the definitions. Thanks to the identification of $\mathbf{V}^{\otimes r}$ with $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$, an invariant in

$$\mathbb{E}_{\Bbbk}\left(n, r+\frac{1}{2}\right) = \operatorname{End}_{\mathcal{P}_{r+1/2}(n)}(\mathbf{V}^{\otimes r})$$

is a $\Bbbk$-linear endomorphism of $\mathbf{V}^{\otimes r} \otimes \mathbf{v}_n$ commuting with the action of $\mathcal{P}_{r+1/2}(n)$. So it must preserve value-type, which means that it must fix $\mathbf{v}_n$ in all tensor places, since it does so in the last place. Also, it is constant on place-permutation orbits for $\mathfrak{S}_r$ acting on the first r places, and satisfies the slice equations in Proposition 3.2(a) in all places. If we index rows and columns of the invariant by elements of $I(n,r)$, by forgetting the last tensor factor (of $\mathbf{v}_n$), then we get an invariant in $\mathbb{E}_{\Bbbk}(n,r)$. $\square$

Example 4.5. A special invariant in $\mathbb{E}_{\Bbbk}(4,2)^4_4 \cong \mathbb{E}_{\Bbbk}(4, 2+\frac{1}{2})$ is of the form

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	*					*					*					
12		*	*		*		*		*	*						
13		*	*		*		*		*	*						
14				*				*				*				
21		*	*		*		*		*	*						
22	*					*					*					
23		*	*		*		*		*	*						
24				*				*				*				
31		*	*		*		*		*	*						
32		*	*		*		*		*	*						
33	*					*					*					
34				*				*				*				
41													*	*	*	
42													*	*	*	
43													*	*	*	
44																*

where all blank positions must be zero, and the starred positions can be non-zero. This should be compared with Example 3.8. All special invariants in $\mathbb{E}_{\Bbbk}(4,2)$ look like this, up to a reordering of rows and columns.

Notice that deleting the rows and columns indexed by labels containing 4 yields the general form of an invariant in $\mathbb{E}_{\Bbbk}(3,2)$; see Example 3.8. This observation motivates Proposition 4.6 below.

The reader may wish to refer to Example 4.5 when working through the proof of the next result.

Proposition 4.6. *Suppose that $n \geqslant 2$. For any $1 \leqslant p,q \leqslant n$, there is a $\Bbbk$-linear isomorphism*

$$\mathbb{E}_{\Bbbk}(n,r)^p_q \xrightarrow{\approx} \mathbb{E}_{\Bbbk}(n-1,r)$$

given by respectively excising all rows, columns labeled by a multi-index containing p, q respectively and re-indexing the sets $\{1,\dots,p-1,p+1,\dots,n\}$ and $\{1,\dots,q-1,q+1\dots,n\}$ to match $\{1,\dots,n-1\}$. In particular, we have a $\Bbbk$-linear isomorphism

$$\mathbb{E}_{\Bbbk}\left(n, r+\frac{1}{2}\right) \xrightarrow{\approx} \mathbb{E}_{\Bbbk}(n-1,r)$$

given by excising all rows and columns labeled by a multi-index containing n (with no re-indexing needed).

Proof. We first prove the special case in which $p=q=n$. Suppose that $\mathbf{A}\in\mathbb{E}_{\Bbbk}(n,r)^n_n$. We obtain a corresponding invariant $\eta(\mathbf{A})\in\mathbb{E}_{\Bbbk}(n-1,r)$ by excising all rows and columns of $\mathbf{A}$ indexed by a label containing n. This defines a $\Bbbk$-linear map $\eta:\mathbb{E}_{\Bbbk}(n,r)^n_n\to\mathbb{E}_{\Bbbk}(n-1,r)$.

For the opposite direction, suppose that $\mathbf{C}\in\mathbb{E}_{\Bbbk}(n-1,r)$ is given. We define a $\Bbbk$-linear map $\theta_r:\mathbb{E}_{\Bbbk}(n-1,r)\to\mathbb{E}_{\Bbbk}(n,r)^n_n$ by induction on r, holding n fixed. If $r=1$, we set $\theta_1(\mathbf{C})=\mathbf{A}=(a^i_j)_{i,j=1,\dots,n}$, where

$$a^i_j=\begin{cases} c^i_j & \text{if } i\neq n \text{ and } j\neq n,\\ s & \text{if } i=n \text{ and } j=n,\\ 0 & \text{otherwise.}\end{cases}$$

Here, s is the common value of the row and column sums in the given GDS matrix $\mathbf{C}$. If $r>1$, we regard $\mathbf{C}=(\mathbf{C}^i_j)_{i,j=1,\dots,n-1}$ as an $(n-1)\times(n-1)$ block matrix, where each $\mathbf{C}^i_j=(b^{i\,i_2\cdots i_r}_{j\,j_2\cdots j_r})_{i_2\cdots i_r,j_2\cdots j_r\in I(n-1,r)}$, and then we set $\theta_r(\mathbf{C})=\mathbf{A}=(\mathbf{A}^i_j)_{i,j=1,\dots,n}$, again as a block matrix, where the blocks $\mathbf{A}^i_j=(a^{i\,i_2\cdots i_r}_{j\,j_2\cdots j_r})_{i_2\cdots i_r,j_2\cdots j_r\in I(n,r-1)}$ are given by

$$\mathbf{A}^i_j=\begin{cases} \theta_{r-1}(\mathbf{C}^i_j) & \text{if } i\neq n \text{ and } j\neq n,\\ \mathbf{S} & \text{if } i=n \text{ and } j=n,\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Here, $\mathbf{S}$ is the common value of the block sum of the first $n-1$ block rows and columns in $\mathbf{A}$; that is, $\mathbf{S}=\sum\mathbf{A}^*_j=\sum\mathbf{A}^i_*$, for any $i,j=1,\dots,n-1$. Alternatively, $\mathbf{S}=\theta_{r-1}(\mathbf{S}')$, where $\mathbf{S}'=\sum\mathbf{C}^i_*=\sum\mathbf{C}^*_j$ for any $i,j=1,\dots,n-1$.

Having defined $\theta=\theta_r$, we claim that θ is a two-sided inverse of η, so η is the desired isomorphism (and $\eta^{-1}=\theta$). Details are left to the reader. This proves the special case. The general case, for arbitrary $1\leqslant p,q\leqslant n$, follows from the special case by re-indexing (interchange n with p and q in row and column indices, respectively). $\square$

Remark 4.7. For general $1\leqslant p,q\leqslant n$, whenever necessary we will denote the $\Bbbk$-linear isomorphisms η, θ in the above proof by η^p_q, θ^p_q respectively.

Lemma 4.3 tells us that blocks of any invariant are always special invariants. If the given invariant is itself special, then we can be more precise about the nature of its blocks. Let $\overline{p}$ be the image of p under the renumbering bijection $\{1,\dots,i-1,i+1,\dots,n\}\cong\{1,\dots,n-1\}$; similarly $\overline{q}$ is the image of q under $\{1,\dots,j-1,j+1,\dots,n\}\cong\{1,\dots,n-1\}$.

Proposition 4.8. *If $\mathbf{A}\in\mathbb{E}_{\Bbbk}(n,r)^i_j$ is a special invariant then with $\eta=\eta^i_j$ we have:*

(a) *$\mathbf{A}^i_q=\mathbf{0}$ for any $q\neq j$ and $\mathbf{A}^p_j=\mathbf{0}$ for any $p\neq i$.*

(b) *If $p\neq i$ and $q\neq j$ then $\eta(\mathbf{A}^p_q)\in\mathbb{E}_{\Bbbk}(n-1,r-1)^{\overline{p}}_{\overline{q}}$.*

(c) *$\rho(\mathbf{A})=\mathbf{A}^i_j$. Thus $\rho(\mathbf{A})\in\mathbb{E}_{\Bbbk}(n,r-1)^i_j$.*

Proof. (a) Clear from the definition of $\mathbb{E}_{\Bbbk}(n,r)^i_j$.

(b) This follows from Proposition 4.6 and Lemma 4.3, applied to $\eta(\mathbf{A})$.

(c) Set $\mathbf{B} = \rho(\mathbf{A})$. By Remark 3.7, all the block row and column sums of $\mathbf{A}$ are equal to $\mathbf{B}$. Thus, by part (a), we have $\mathbf{B} = \mathbf{A}^i_j$. The last statement in (c) then follows by Lemma 4.3. $\square$

Note that Proposition 4.8 is well illustrated by Example 4.5.

We finish this section with the following observation.

Lemma 4.9. *With $\theta = \theta^i_j$ and $\eta = \eta^i_j$ the following diagram commutes:*

$$\begin{tikzcd} \mathbb{E}_{\Bbbk}(n-1,r) \arrow[r, "\theta"] \arrow[d, "\rho"] & \mathbb{E}_{\Bbbk}(n,r)^i_j \arrow[d, "\rho"] \\ \mathbb{E}_{\Bbbk}(n-1,r-1) \arrow[r, "\theta"] & \mathbb{E}_{\Bbbk}(n,r-1)^i_j \end{tikzcd}.$$

In other words, the restriction map ρ commutes with θ. (It also commutes with $\eta = \theta^{-1}$.)

Proof. Left to the reader. $\square$

Restriction $\rho : \mathbb{E}_{\Bbbk}(n,r) \to \mathbb{E}_{\Bbbk}(n,r-1)$ gives a way of obtaining invariants in degree $r-1$ from invariants in degree r (for $r \geqslant 1$). The opposite problem is the *extension problem:*

Given $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)$, find some $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$ such that $\rho(\mathbf{A}) = \mathbf{B}$.

A closely related problem is the *decomposition problem:*

Given $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$, write $\mathbf{A}$ as a sum of special invariants.

We will prove in Theorem 6.16 ahead that both problems can always be solved. We show now that this implies the main result of this paper.

Theorem 4.10. *Let $\Bbbk$ be a commutative ring. For any $n \geqslant 2$, $r \geqslant 1$ the maps $\Phi_{n,r} : \Bbbk W_n \to \mathbb{E}_{\Bbbk}(n,r)$ and $\Phi_{n,r+1/2} : \Bbbk W_{n-1} \to \mathbb{E}_{\Bbbk}(n, r+\frac{1}{2})$ are surjective.*

Proof. By Proposition 4.6, the surjectivity of $\Phi_{n,r+1/2}$ follows from the surjectivity of $\Phi_{n-1,r}$, so it suffices to prove the surjectivity of $\Phi_{n,r}$. This surjectivity is trivial if $r = 0$ since $\mathbb{E}_{\Bbbk}(n,r) \cong \Bbbk$. We proceed by induction on n. Let $\mathbf{A} \in \mathbb{E}_{\Bbbk}(n,r)$. By Theorem 6.16, we can write

$$\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$$

where $\mathbf{A}(j)$ is in $\mathbb{E}_{\Bbbk}(n,r)^n_j$ for each $j = 1, \dots, n$. By Proposition 4.6 and the inductive hypothesis, each $\eta^n_j \mathbf{A}(j)$ belongs to the image of $\Phi_{n-1,r}$, for $j = 1, \dots, n$. This implies that each $\mathbf{A}(j) = \theta^n_j \eta^n_j \mathbf{A}(j)$ belongs to the image of $\Phi_{n,r}$. Hence so does $\mathbf{A}$, and the proof is complete. $\square$

Remark 4.11. Another proof of Theorem 4.10 is based on the existence of extensions (also proved in Theorem 6.16). First note that it is easy to prove Theorem 4.10 if $n = r$, because after all we just need to solve the equation

$$\mathbf{A} = \textstyle\sum_{w \in W_n} x_w \Phi_{n,r}(w).$$

When $n = r$ the equation has at most one solution, given by setting

$$x_w = a^{w(1)w(2)\cdots w(n)}_{12\cdots n} \quad \text{for each } w \in W_n.$$

This works because only one permutation w can contribute to that entry in the matrix $\mathbf{A}$. It is not difficult to check that this is actually a solution. It follows immediately (given the existence of extensions) that the "same" linear combination

$$\mathbf{A}' = \sum_{w \in W_n} x_w \Phi_{n,r+1}(w)$$

is the unique extension of $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,n+1)$. Repeating the argument inductively, we see that Theorem 4.10 holds for all $r \geqslant n$.

Assume now that $r < n$, which is the difficult case. The existence of extensions implies in particular that the restriction map ρ is always surjective. Thus the commutative diagram

$$\begin{tikzcd} \mathbb{E}_{\Bbbk}(n,n) \arrow[r, "\rho^{n-r}"] & \mathbb{E}_{\Bbbk}(n,r) \\ \Bbbk W_n \arrow[u, "\Phi_{n,n}"] \arrow[ur, "\Phi_{n,r}"'] & \end{tikzcd}$$

expresses the map $\Phi_{n,r}$ as the composite of two surjections, hence it is a surjection, and the theorem is proved. The idea behind this proof is: given $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$ we first extend up to degree n, where we can read off a solution, and then restrict it back down to degree r.

Corollary 4.12. *The kernel of restriction* $\rho : \mathbb{E}_{\Bbbk}(n,r) \to \mathbb{E}_{\Bbbk}(n,r-1)$ *is isomorphic to* $\Phi_{n,r}(\ker \Phi_{n,r-1}) \cong (\ker \Phi_{n,r-1})/(\ker \Phi_{n,r})$.

Proof. Since $\Phi_{n,r}$ is surjective (by Theorem 4.10) and ρ is surjective (Remark 4.11), the commutativity of the diagram

$$\begin{tikzcd} \mathbb{E}_{\Bbbk}(n,r) \arrow[r, "\rho"] & \mathbb{E}_{\Bbbk}(n,r-1) \\ \Bbbk W_n \arrow[u, "\Phi_{n,r}"] \arrow[ur, "\Phi_{n,r-1}"'] & \end{tikzcd}$$

implies that $\ker \rho \cong \Phi_{n,r}(\ker \Phi_{n,r-1})$. $\square$

In [4], the authors find an explicit cellular basis for the kernel of $\Phi_{n,r}$ for all n, r. This means that we have an explicit basis for the kernel of ρ in Corollary 4.12.

5. Extensions and decompositions

It remains to prove the existence of extensions and decompositions. This is the purpose of Sections 5 and 6.

First we discuss the extension problem. Let $\mathbf{B}$ be a given fixed invariant in $\mathbb{E}_{\Bbbk}(n,r-1)$. Suppose that $\mathbf{A} \in \mathbb{E}_{\Bbbk}(n,r)$ is an extension of the given $\mathbf{B}$, and write $\mathbf{A} = (a^{\boldsymbol{i}}_{\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j} \in I(n,r)}$. Then the matrix coordinates $a^{\boldsymbol{i}}_{\boldsymbol{j}}$ of $\mathbf{A}$ must satisfy the conditions:

$$a^{(i_1\cdots i_{r-1} i_{r-1})^\sigma}_{(j_1\cdots j_{r-1} j_{r-1})^\sigma} = b^{i_1\cdots i_{r-1}}_{j_1\cdots j_{r-1}} \text{ for all } \sigma \in \mathfrak{S}_r\,, \tag{I-1}$$

$$a^{i_1\cdots i_r}_{j_1\cdots j_r} = 0 \text{ whenever } \operatorname{vt}(i_1\cdots i_r) \neq \operatorname{vt}(j_1\cdots j_r) \tag{I-2}$$

for all $i_1\cdots i_{r-1}$, $j_1\cdots j_{r-1}$ in $I(n,r-1)$ and all $i_1\cdots i_r$, $j_1\cdots j_r$ in $I(n,r)$.

Condition (I-1) comes from Proposition 3.4(d), while condition (I-2) restates value-type preservation, from Proposition 3.2(b) (also Proposition 3.4(c)). We call (I-1), (I-2) *initialisation conditions*. They determine the value of all entries $a^{\boldsymbol{i}}_{\boldsymbol{j}} = a^{i_1\cdots i_r}_{j_1\cdots j_r}$ for which either $\#(\boldsymbol{i}) < r$ or $\#(\boldsymbol{j}) < r$, where we define $\#(\boldsymbol{i})$ to be the number of distinct values appearing in the multi-index $\boldsymbol{i}$.

Thus, to find an extension $\mathbf{A}$ of the given $\mathbf{B}$, we start by initialising $\mathbf{A} \in \operatorname{Mat}_{I(n,r)}(\Bbbk)$ to satisfy (I-1), (I-2). Then it only remains to assign values to the $a^{i_1\cdots i_r}_{j_1\cdots j_r}$ for which $\#(i_1\cdots i_r) = r = \#(j_1\cdots j_r)$. In other words, if we set

$$I'(n,r) = \{\boldsymbol{i} \in I(n,r) : \#(\boldsymbol{i}) = r\}$$

then we can focus just on how to assign entries $a_{\boldsymbol{j}}^{\boldsymbol{i}}$ such that $\boldsymbol{i}, \boldsymbol{j} \in I'(n,r)$. By Proposition 3.2, those entries of $\mathbf{A}$ must satisfy the following conditions. For any $\boldsymbol{p} = i_1 \cdots i_{\alpha-1} i_{\alpha+1} \cdots i_r$, $\boldsymbol{q} = j_1 \cdots j_{\alpha-1} j_{\alpha+1} \cdots j_r$ in $I'(n, r-1)$,

$$\sum a_{j_1 \cdots j_{\alpha-1} * j_{\alpha+1} \cdots j_r}^{i_1 \cdots i_{\alpha-1}\, i\, i_{\alpha+1} \cdots i_r} = b_{\boldsymbol{q}}^{\boldsymbol{p}} \text{ for all } i = 1, \dots, n \tag{12}$$

$$\sum a_{j_1 \cdots j_{\alpha-1}\, j\, j_{\alpha+1} \cdots j_r}^{i_1 \cdots i_{\alpha-1} * i_{\alpha+1} \cdots i_r} = b_{\boldsymbol{q}}^{\boldsymbol{p}} \text{ for all } j = 1, \dots, n \tag{13}$$

and for all $\boldsymbol{i} = i_1 \cdots i_r$, $\boldsymbol{j} = j_1 \cdots j_r$ in $I'(n, r)$,

$$a_{j_1 \cdots j_r}^{i_1 \cdots i_r} - a_{(j_1 \cdots j_r)^\sigma}^{(i_1 \cdots i_r)^\sigma} = 0 \text{ all } \sigma \in \mathfrak{S}_r. \tag{14}$$

So finding an extension $\mathbf{A}$ of the given $\mathbf{B}$ is (after initialisation) equivalent to solving the linear system given by equations (12)–(14).

Any special invariant $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)_j^i$ is necessarily an extension of its block $\mathbf{A}_j^i$. This follows from Proposition 4.8(a) and Remark 3.7. So we sometimes refer to such special invariants as *special extensions*.

Our goal now is to establish the following four interrelated properties, the first of which is about the existence of extensions. They will be established by an interleaved double induction on n, r. Note that that each property is based on some fixed (but arbitrary) block row or column.

Property 1 (extension property). *For any* $\mathbf{B}$ *in* $\mathbb{E}_{\Bbbk}(n, r-1)$*, there exists some* $\mathbf{A}$ *in* $\mathbb{E}_{\Bbbk}(n,r)$ *such that* $\rho(\mathbf{A}) = \mathbf{B}$*. More precisely, for any fixed* $1 \leqslant i \leqslant n$ *(respectively,* $1 \leqslant j \leqslant n$*), there exist* $\mathbf{A}(j)$ *(resp.* $\mathbf{A}(i)$*) in* $\mathbb{E}_{\Bbbk}(n,r)_j^i$ *such that* $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$*. Call such an* $\mathbf{A}$ *an* i*th block row (resp.* j*th block column) extension of* $\mathbf{B}$*.*

A priori, it is not clear that every extension can be constructed as a block row or column extension. However, the next property guarantees that every extension so arises.

Property 2 (decomposition property). *For any given* $\mathbf{A}$ *in* $\mathbb{E}_{\Bbbk}(n,r)$ *and any fixed* $1 \leqslant i \leqslant n$ *(respectively,* $1 \leqslant j \leqslant n$*), there exist* $\mathbf{A}(j)$ *(resp.* $\mathbf{A}(i)$*) in* $\mathbb{E}_{\Bbbk}(n,r)_j^i$ *such that* $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$*. Call such a decomposition an* i*th block row (resp.* j*th block column) decomposition.*

Property 1 for (n,r) immediately implies Property 2 for $(n, r-1)$; this follows from Remark 3.7, because if $\mathbf{A}$ is in $\mathbb{E}_{\Bbbk}(n,r)$ and $\rho(\mathbf{A}) = \mathbf{B}$ then the sum of any chosen block row or column of $\mathbf{A}$ is equal to $\mathbf{B}$. It should be noted however that we work in the opposite direction: we need Property 2 for $(n, r-1)$ as an inductive hypothesis in order to prove Property 1 for (n, r).

Now we introduce free patterns, which index a set of entries in a matrix which can be freely assigned to arbitary values in the ring $\Bbbk$. Free patterns correspond to a choice of free variables in the solution of a consistent linear system. Once the free pattern entries have been assigned, the system has a unique solution.

Property 3 (free patterns for extensions). *For any fixed index* $1 \leqslant i \leqslant n$ *(respectively,* $1 \leqslant j \leqslant n$*), there exists a subset* $F(n,r)$ *of* $I'(n,r) \times I'(n,r)$*, possibly empty and depending on the index, such that for any* $\mathbf{B}$ *in* $\mathbb{E}_{\Bbbk}(n, r-1)$ *and any assignment* $f : F(n,r) \to \Bbbk$ *there exists a unique* $\mathbf{A}$ *in* $\mathbb{E}_{\Bbbk}(n,r)$ *with* $\rho(\mathbf{A}) = \mathbf{B}$*.*

Remark 5.1. We often identify the elements of $F(n,r)$ with the matrix entries they index.

It follows from the symmetry in equations (12)–(14) or from the existence of the actions in (11) that if $F(n,r)$ is a given free pattern, then by applying an arbitrary permutation $y \in W_n$ to all its row or column indices, we obtain another free pattern.

Furthermore, by interchanging row and column indices in $F(n,r)$ (transposing) we obtain another free pattern.

Property 4 (free patterns for decompositions). *For any given* $\mathbf{A}$ *in* $\mathbb{E}_{\Bbbk}(n,r)$ *and any fixed* $1 \leqslant i \leqslant n$ *(respectively,* $1 \leqslant j \leqslant n$*), there exists a subset* $D(n,r)$ *of* $\{1,\dots,n\} \times I'(n,r) \times I'(n,r)$ *such that for any given assignment* $f : D(n,r) \to \Bbbk$ *there exist unique* $\mathbf{A}(j)$ *(resp.* $\mathbf{A}(i)$*) in* $\mathbb{E}_{\Bbbk}(n,r)^i_j$ *such that* $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$*.*

Properties 3, 4 are refinements of Properties 1, 2 (respectively) that precisely quantify the amount of freedom in constructing solutions to their underlying linear systems.

Suppose that $\mathbf{A}$ is in $\mathbb{E}_{\Bbbk}(n,r)$. Fix i and consider its ith block row $\mathbf{A}^i_* = (\mathbf{A}^i_j)^n_{j=1}$. By Lemma 4.3, each block $\mathbf{A}^i_j$ of $\mathbf{A}^i_*$ gives an invariant $\mathbf{B}(j)$ in $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$, and the correspondence is given by

$$a^{i\, i_2\cdots i_r}_{j\, j_2\cdots j_r} = b(j)^{i_2\cdots i_r}_{j_2\cdots j_r} \tag{15}$$

as j runs from 1 to n. We let π^i be the bijection between the set of pairs $(i\, i_2\cdots i_r, j\, j_2\cdots j_r)$ indexing entries of $\mathbf{A}^i_*$ on the left hand side of (15) and the set of triples $(j, i_2\cdots i_r, j_2\cdots j_r)$ indexing entries on the right hand side. Similarly, transposing rows and columns and i,j we obtain a bijection π_j between the set of pairs indexing entries of the jth block column $\mathbf{A}^*_j$ and a corresponding set of triples. The following lemma is immediate from Remark 3.7.

Lemma 5.2. *Let* $\mathbf{A} \in \mathbb{E}_{\Bbbk}(n,r)$ *and set* $\mathbf{B} = \rho(\mathbf{A}) \in \mathbb{E}_{\Bbbk}(n,r-1)$*. The map* π^i *(resp.* π_j*) is a bijection between a set of labels for the entries of* $\mathbf{A}^i_*$ *(resp.* $\mathbf{A}^*_j$*) and a set of labels for a solution* $\mathbf{B} = \mathbf{B}(1) + \cdots + \mathbf{B}(n)$ *to the decomposition problem in degree* $r-1$*.*

Now suppose that $F(n,r)$ exists, in other words, that the free extension pattern for constructing $\mathbf{A}$ (from $\mathbf{B}$) in Lemma 5.2 exists (note that $F(n,r)$ is necessarily based on a designated block row or column). If $F(n,r)$ is based on the ith block row (resp. jth block column) then we define

$$F'(n,r) = \{(i_1\cdots i_r, j_1\cdots j_r) \in F(n,r) : i_1 = i \text{ (resp. } \; j_1 = j)\}. \tag{16}$$

Furthermore, we define $F''(n,r) = F(n,r) \smallsetminus F'(n,r)$, so that

$$F(n,r) = F'(n,r) \sqcup F''(n,r). \tag{17}$$

This disjoint decomposition is the crux of the interleaved induction that will prove Properties 1–4. Of great importance for our results is the fact that the map π^i (resp. π_j) in Lemma 5.2 restricts to a bijection

$$F'(n,r) \cong D(n,r-1),$$

where the right-hand side is the free decomposition pattern for writing $\mathbf{B} = \mathbf{B}(1) + \cdots + \mathbf{B}(n)$ in Lemma 5.2. In particular, this means that Property 3 for (n,r) immediately implies Property 4 for $(n,r-1)$. However, our induction proceeds in the reverse direction, using the inverse of the above bijection to construct $F'(n,r)$ from $D(n,r-1)$. Then we construct $F''(n,r)$ and glue them together according to (17) in order to obtain $F(n,r)$. In this manner, we explicitly construct and interrelate the free extension patterns and free decomposition patterns.

Definition 5.3. *We order* $I(n,r)$ *lexicographically, and we do the same for the row and column indices of any matrix in* $\mathrm{Mat}_{I(n,r)}(\Bbbk)$*. A free pattern* $F(n,r)$ *is* row-initial *(resp.* column-initial*) if, after identifying free-pattern elements with their corresponding matrix entries, free-pattern entries precede all other entries in each row (resp. column) slice. Similarly, it is* row-terminal *(resp.* column-terminal*) if, after identifying free-pattern elements with matrix entries, free-pattern entries come after all other entries in each row (resp. column) slice.*

It is evident that free patterns $F(n,1)$ exist. Indeed, it is easily checked that

$$F(n,1) = \{(i,j) : 2 \leqslant i,j \leqslant n\}$$

is a row- and column-terminal pattern for $(n,1)$. To see this, notice that once an assignment $f : F(n,1) \to \Bbbk$ has been chosen, there is a unique extension $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,1)$ of any given $b \in \mathbb{E}_{\Bbbk}(n,0) = \Bbbk$ satisfying $a^i_j = f(i,j)$ for all $(i,j) \in F(n,1)$. Note that a^1_i and a^i_1 are then uniquely forced for $i > 1$. The first entry a^1_1 is forced in two ways, but easily seen to be well-defined.

By applying appropriate permutations to the rows and/or columns of $F(n,1)$ above, one gets row- and column-terminal, row-initial and column-terminal, and row-terminal and column-initial free patterns for $(n,1)$. Our proof of the four basic properties in the next section will show that these variations of free patterns always exist, for any (n,r). Note that the distinguished element $w_0 \in W_n$ given by $w_0(j) = n+1-j$ for $j = 1,\dots,n$ interchanges initial and terminal patterns.

We conclude this section with the following algorithm, which (as we show in Corollary 6.6) determines a terminal (respectively, initial) free pattern in a randomly chosen row or column of an extension $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$ of a given $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)$, assuming that it is the first such row or column to be completed. The algorithm is independent of Properties 1–4, and independent of the inductive proof of those properties.

Algorithm 5.4. *The α-slices of $I'(n,r) = \{\boldsymbol{i} \in I(n,r) : \#(\boldsymbol{i}) = r\}$ are the subsets*

$$\{i_1 \cdots i_{\alpha-1}\, i\, i_{\alpha+1} \cdots i_r : i = 1,\dots,n\}$$

for $\alpha = 1,\dots,r$. We start by listing the elements of $I'(n,r)$ in lexicographical order. We recursively assign a colour $(0$ *or* $1)$ *to each element of $I'(n,r)$ as follows.*

As long as uncoloured elements exist, we find the largest (*respectively, smallest*) *uncoloured element, colour it, and repeat. Colouring an element consists of the following two steps:*

(a) *Examine all the element's slices. If the element is the only uncoloured element in one of its slices, we say it is forced, and colour it* 0. *Otherwise, colour the element* 1 *to indicate that it is free.*

(b) *If colouring the current element forces any additional elements in any of its slices, then colour those elements as well.* (*This happens when there is just one remaining uncoloured element in the slice, after the current element is coloured.*)

We note that colouring is a recursive process, because of (b). *Let $I'_1(n,r)$ be the set of elements of $I'(n,r)$ coloured* 1 *by the above procedure.*

Example 5.5. By always choosing the largest uncoloured element, for the case of $I'_1(5,2)$ the algorithm produces the following colouring:

$I'_1(5,2)$:

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0	0	0	0	0	1	1	0	1	1	1	0	1	1	1	0	1	1	1

Thus we have $I'_1(5,2) = \{54,53,52,45,43,42,35,34,32,25,24\}$.

Let $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)$ be given, and fix some index $\boldsymbol{i}$ (respectively, $\boldsymbol{j}$) in $I'(n,r)$. Then any assignment

$$f : \{a^{\boldsymbol{i}}_{\boldsymbol{j}} : \boldsymbol{j} \in I'_1(n,r)\} \to \Bbbk \qquad (\text{resp. } f : \{a^{\boldsymbol{i}}_{\boldsymbol{j}} : \boldsymbol{i} \in I'_1(n,r)\} \to \Bbbk)$$

determines the $\boldsymbol{i}$th row $a^{\boldsymbol{i}}_*$ (resp. $\boldsymbol{j}$th column $a^*_{\boldsymbol{j}}$) of a matrix $\mathbf{A}$ satisfying equations (12) (resp. (13)) with respect to the given $\mathbf{B}$.

Remark 5.6.

(i) In Corollary 6.6 we will show that that, under suitable inductive hypotheses, there exists an extension $\mathbf{A}$ of the given $\mathbf{B}$ which agrees with the row or column determined by the above algorithm.

(ii) We prefer to work with terminal free patterns, because they are compatible with restriction, in the sense that by excising all indices containing an n we obtain a free pattern for $n-1$. This preference pervades all of the examples and some of the proofs in the next section.

6. Proof of Properties 1–4

We remind the reader that $\Bbbk$ is an arbitrary commutative ring. Now we are ready to start the inductive proof of the four properties. For each property, we need to assume an earlier instance of one or more properties. We begin with Property 1.

Proposition 6.1. *Assume Property* 1 *for* $(n-1,r)$. *Then:*

(a) *For any i, j with $1 \leqslant i,j \leqslant n$ and any given $\mathbf{B}$ in $\mathbb{E}_\Bbbk(n,r-1)^i_j$, there exists some $\mathbf{A}$ in $\mathbb{E}_\Bbbk(n,r)^i_j$ such that $\rho(\mathbf{A}) = \mathbf{B}$.*

(b) (i) *Fix some j in $\{1,\dots,n\}$. Suppose given the data*

$$\mathbf{B}(1),\dots,\mathbf{B}(n)$$

with $\mathbf{B}(i)$ in $\mathbb{E}_\Bbbk(n,r-1)^i_j$ for $i=1,\dots,n$. Then there exist corresponding $\mathbf{A}(i)$ in $\mathbb{E}_\Bbbk(n,r)^i_j$ for $i=1,\dots,n$ such that $\rho(\mathbf{A}(i)) = \mathbf{B}(i)$ for all i.

(ii) *Similarly, fix some i in $\{1,\dots,n\}$. Suppose given the data*

$$\mathbf{B}(1),\dots,\mathbf{B}(n)$$

with $\mathbf{B}(j)$ in $\mathbb{E}_\Bbbk(n,r-1)^i_j$ for $j=1,\dots,n$. Then there exist corresponding $\mathbf{A}(j)$ in $\mathbb{E}_\Bbbk(n,r)^i_j$ for $j=1,\dots,n$ such that $\rho(\mathbf{A}(j)) = \mathbf{B}(j)$ for all j.

(iii) *In either case* (i) *or* (ii)*, the sum $\mathbf{A}(1)+\cdots+\mathbf{A}(n)$ extends $\mathbf{B}(1)+\cdots+\mathbf{B}(n)$.*

(c) *If in addition Property* 2 *holds for* $(n,r-1)$ *then Property* 1 *holds for* (n,r).

Proof. (a) This follows from Proposition 4.6, which reduces the question to the problem of extending from $\mathbb{E}_\Bbbk(n-1,r-1)$ to $\mathbb{E}_\Bbbk(n-1,r)$, which is solved by the hypothesis.

(b) is immediate from part (a) and the linearity of ρ.

(c) Let $\mathbf{B}$ in $\mathbb{E}_\Bbbk(n,r-1)$ be given. By the decomposition property for $(n,r-1)$, we can find $\mathbf{B}(j)$ in $\mathbb{E}_\Bbbk(n,r-1)^n_j$ for $j=1,\dots,n$ such that $\mathbf{B} = \mathbf{B}(1)+\cdots+\mathbf{B}(n)$. By part (b)(ii), there exist corresponding $\mathbf{A}(j)$ in $\mathbb{E}_\Bbbk(n,r)^n_j$ such that $\rho(\mathbf{A}(j)) = \mathbf{B}(j)$ for all $j=1,\dots,n$. Put $\mathbf{A} = \mathbf{A}(1)+\cdots+\mathbf{A}(n)$. Then by linearity of ρ it follows that $\rho(\mathbf{A}) = \mathbf{B}$, as required. This shows that $\mathbf{A}$ is a last block row extension of $\mathbf{B}$. The proof for any other block row or column is similar. ☐

Having dealt with the existence of extensions, we now consider the question of their uniqueness.

Lemma 6.2. *Suppose that $n \leqslant r$. Then*

(a) *Extensions* (*if any*) *from $\mathbb{E}_\Bbbk(n,r-1)$ to $\mathbb{E}_\Bbbk(n,r)$ are unique.*

(b) *Restriction $\rho : \mathbb{E}_\Bbbk(n,r) \to \mathbb{E}_\Bbbk(n,r-1)$ is injective.*

(c) *$\mathbf{0}$ is the only extension in $\mathbb{E}_\Bbbk(n,r)$ of $\mathbf{0}$.*

Proof. (a) Suppose that $\mathbf{A}$ in $\mathbb{E}_\Bbbk(n,r)$ extends $\mathbf{B}$ in $\mathbb{E}_\Bbbk(n,r-1)$. We use Proposition 3.4. Let $\boldsymbol{i} = i_1\cdots i_r$, $\boldsymbol{j} = j_1\cdots j_r$ be in $I(n,r)$. If $\mathrm{vt}(\boldsymbol{i}) \neq \mathrm{vt}(\boldsymbol{j})$ then by Proposition 3.4(c), $a^{\boldsymbol{i}}_{\boldsymbol{j}} = 0$.

So assume for the rest of the proof that $\mathrm{vt}(\boldsymbol{i}) = \mathrm{vt}(\boldsymbol{j})$. If $\boldsymbol{i}$ has a repeated value, then by Proposition 3.4(d), $a^{\boldsymbol{i}}_{\boldsymbol{j}} = b^{\boldsymbol{p}}_{\boldsymbol{q}}$, where $\boldsymbol{p}, \boldsymbol{q}$ are obtained from $\boldsymbol{i}, \boldsymbol{j}$ by removing one of the duplicate values (from the same place). So such entries of $\mathbf{A}$ are determined by $\mathbf{B}$. If $r > n$, then all multi-indices in $I(n,r)$ must have at least one duplicate value, so we are done in that case.

We are left with the case $r = n$ and $\#(\boldsymbol{i}) = n$ ($\boldsymbol{i}$ has no repeated values). Then the same is true of $\boldsymbol{j}$, and by Proposition 3.4(b), with $\boldsymbol{p} = i_1 \cdots i_{n-1}$ and $\boldsymbol{q} = j_1 \cdots j_{n-1}$ we have

$$\sum a^{\boldsymbol{p}\,*}_{\boldsymbol{q}\,j_r} = b^{\boldsymbol{p}}_{\boldsymbol{q}}.$$

Exactly $n-1$ of the possible n values from $\{1, \dots, n\}$ appear in $\boldsymbol{i}$; similarly for $\boldsymbol{j}$. Hence at most one term in the above sum can be non-zero, because of value-type, so $a^{\boldsymbol{i}}_{\boldsymbol{j}} = b^{\boldsymbol{p}}_{\boldsymbol{q}}$. So in this case, $\mathbf{A}$ is also determined by $\mathbf{B}$. This proves part (a).

(b) This follows from part (a). If $\mathbf{A} \in \ker \rho$, then $\mathbf{A} = \mathbf{0}$ by uniqueness.

(c) This follows from part (b). □

Remark 6.3. Suppose that $n \leqslant r+1$. If $\mathbf{C}$ is any invariant in $\mathbb{E}_{\Bbbk}(n,r)$ with zero last block row, then $\mathbf{C} = \mathbf{0}$. The same holds for any other block row or column. This follows from the proof of Lemma 6.2(a); the assumption of a zero block row removes one degree of freedom from column slice sums, so the uniqueness conclusion is still valid, so $\mathbf{C} = \mathbf{0}$.

The next result gives conditions under which we can construct extensions with prescribed partial information in a chosen block row or column. This is a crucial technical result needed to prove Property 2. The chosen block row or column is controlled by its free-pattern $F'(n,r) \cong D(n,r-1)$. We say that the prescribed information is *compatible* with the extension problem for a given $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)$ if there is some partial assignment from a subset of some $F'(n,r)$ to $\Bbbk$ which gives the prescribed information.

Lemma 6.4. *Assume Property* 1 *for* $(n-1,r)$ *and Property* 4 *for* $(n,r-1)$. *Suppose that* $\mathbf{B}$ *in* $\mathbb{E}_{\Bbbk}(n,r-1)$ *is given. Fix a choice of block row* (*respectively, block column*) *and a choice of any number of compatible prescribed rows* (*resp. columns*) *in the chosen block row* (*resp. column*). *Then there exists an* $\mathbf{A}$ *in* $\mathbb{E}_{\Bbbk}(n,r)$ *satisfying* $\rho(\mathbf{A}) = \mathbf{B}$ *which agrees with the prescribed rows* (*resp. columns*).

Proof. Suppose we have fixed on an ith block row extension (the argument for a block column extension is similar and left to the reader). By hypothesis, a set $D(n,r-1)$ exists satisfying Property 4 for $(n,r-1)$. Any assignment to this set determines a decomposition of the given $\mathbf{B}$, say $\mathbf{B} = \mathbf{B}(1) + \cdots + \mathbf{B}(n)$ where each $\mathbf{B}(j) \in \mathbb{E}_{\Bbbk}(n,r-1)^i_j$. We define $F'(n,r)$ to correspond to $D(n,r-1)$ under the bijection π^i of Lemma 5.2. Assignments to $D(n,r-1)$ correspond to assignments to $F'(n,r)$, which complete the ith block row of $\mathbf{A}$ in a way that satisfies all relevant extension equations. Under this correspondence, $\mathbf{A}^i_j = \mathbf{B}(j)$ for each $j = 1, \dots, n$. Once the ith block row of $\mathbf{A}$ is complete, we apply Proposition 6.1(b) to find special extensions $\mathbf{A}(j)$ in $\mathbb{E}_{\Bbbk}(n,r)^i_j$ for all j such that $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$ extends $\mathbf{B}$.

To finish, we simply note that by hypothesis the prescribed rows in the ith block row are compatible with the extension problem for the given $\mathbf{B}$, so they are specified by a partial assignment f' to some subset of some $F'(n,r)$. We can extend f' to an assignment $f : F'(n,r) \to \Bbbk$ which then determines the ith block row of $\mathbf{A}$ as above, in a way that coincides with the prescribed information. □

Example 6.5. We illustrate the above proof. Take $(n,r)=(4,2)$, and suppose that row 42 of an extension has been prescribed, for a given $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(4,1)$. The prescribed row is part of the final block row, so we construct a final block row extension. The following

$F'(4,2)$:

	12	13	14	21	23	24	31	32	34	41	42	43
42						✓		✓	✓		✓	✓
43						✓		✓	✓		✓	✓

depicts a row- and column-terminal free-pattern $F'(4,2)$ for the last block row of our desired extension $\mathbf{A}$, where the checkmarked entries correspond to elements of $F'(4,2)$, which can be freely assigned in order to determine the last block row of a general extension. This free pattern is calculated in Example 6.14 below.

The prescribed 42-row merely determines the values of the five free entries in that row. By assigning arbitrary values to the remaining five entries in $F'(4,2)$ we complete the last block row of $\mathbf{A}$, and then complete $\mathbf{A}$ by choosing special extensions of each block in the last block row, and summing, as in the first paragraph of the proof of Lemma 6.4.

We note the following immediate consequence of Lemma 6.4, which was promised in Remark 5.6.

Corollary 6.6. *Assume the same hypotheses as in Lemma* 6.4. *Suppose that* $(a^{\boldsymbol{i}}_{*})$ *or* $(a^{*}_{\boldsymbol{j}})$ *is a row or column* (*where* $\boldsymbol{i}$ *or* $\boldsymbol{j} \in I'(n,r)$) *determined by an assignment to the variables in that row or column labelled by the set* $I_1(n,r)$ *in Algorithm* 5.4. *Then that row or column is a row or column of some extension* $\mathbf{A}$ *of any given* $\mathbf{B}$ *in* $\mathbb{E}_{\Bbbk}(n,r-1)$.

Now we are ready to prove Property 2.

Proposition 6.7. *Assume Property* 1 *for* $(n-1,r)$ *and Property* 4 *for* $(n-1,r-1)$. *Then Property* 2 *holds for* (n,r).

Proof. Let $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$ be given. We choose to decompose $\mathbf{A}$ based on its last block row. (The argument for any other block row or column is similar.) By Proposition 4.6, the problem of extending from $\mathbb{E}_{\Bbbk}(n,r-1)^n_j$ to $\mathbb{E}_{\Bbbk}(n,r)^n_j$ is equivalent to the problem of extending from $\mathbb{E}_{\Bbbk}(n-1,r-1)$ to $\mathbb{E}_{\Bbbk}(n-1,r)$. So by the first hypothesis and Proposition 6.1(a),

$$\text{there exists } \mathbf{A}(j) \text{ in } \mathbb{E}_{\Bbbk}(n,r)^n_j \text{ such that } \rho(\mathbf{A}(j)) = \mathbf{A}^n_j \tag{18}$$

for all $j=1,\dots,n$. Any choice of $\mathbf{A}(j)$ satisfying (18) makes the last block row of $\mathbf{A}$ agree with that of the sum $\mathbf{A}(1)+\cdots+\mathbf{A}(n)$, so that the last block row of the difference $\mathbf{C}=\mathbf{A}-\mathbf{A}(1)-\cdots-\mathbf{A}(n)$ is zero. We need to show that it is always possible to choose the $\mathbf{A}(j)$ satisfying (18) in such a way that $\mathbf{C}=\mathbf{0}$.

Case 1. *If* $n \leqslant r+1$ *then* $\mathbf{C}$ *as above is an invariant whose last block row is zero. The zero invariant* $\mathbf{0}$ *is another such invariant. By Remark* 6.3, *it follows that* $\mathbf{C}=\mathbf{0}$. *This completes Case 1.*

Case 2. *Assume for the rest of the proof that* $n>r+1$ *(so* $n-r\geqslant 2$*). We aim to show that* $\mathbf{A}(j)$ *for* $j=2,\dots,n$ *can be chosen satisfying* (18) *in such a way that all but the last block of the first block column of* $\mathbf{A}-\mathbf{A}(2)-\cdots-\mathbf{A}(n)$ *is zero. Working in* reverse *order from right to left along the last block row of* $\mathbf{A}$, *we choose* $\mathbf{A}(j)$ *for* $j=n,n-1,\dots,2$ *satisfying* (18) *by the following process. For each* j, *assuming that* $\mathbf{A}(n),\dots,\mathbf{A}(j)$ *have already been chosen, we set* $\mathbf{C}(j)=\mathbf{A}-\mathbf{A}(n)-\cdots-\mathbf{A}(j)$.

Step 1. First we choose arbitrary $\mathbf{A}(n),\dots,\mathbf{A}(r+2)$ subject only to condition (18). This step is not inductive.

STEP 2. We proceed by reverse induction on j running from $r+1$ down to 2. At each stage, we claim that $\mathbf{A}(j)$ satisfying (18) can be chosen so that:

(19) $\eta_j^n(\mathbf{A}(j))$ agrees with $\eta_j^n(\mathbf{C}(j+1))$ on all columns indexed by a label in $\mathscr{L}_{j-1}$

where $\mathscr{L}_j = \{j_1 \cdots j_r \in I'(n,r) : j_\alpha = \alpha \text{ for all } \alpha = 1, \dots, j\}$. To see this, we apply Lemma 6.4 in the case $(n-1,r)$, since $\eta_j^n(\mathbf{C}(j+1))$ belongs to $\mathbb{E}_{\Bbbk}(n-1,r)$. (Property 1 for $(n-1,r)$ implies the existence of special extensions for $(n-1,r)$, which is equivalent to Property 1 for $(n-2,r)$, so the hypotheses of Lemma 6.4 for the case $(n-1,r)$ are satisfied.) By the lemma, we can find $\mathbf{A}'(j)$ in $\mathbb{E}_{\Bbbk}(n-1,r)$ which agrees on the image under η_j^n of the columns of $\mathbf{C}(j)$ indexed by $\mathscr{L}_{j-1}$. Then we set $\mathbf{A}(j) = \theta_j^n \mathbf{A}'(j)$. This matrix satisfies (19), so the claim is proved.

At this point we have inductively chosen $\mathbf{A}(r+1)$ down to $\mathbf{A}(2)$, and we are ready for the final step, choosing $\mathbf{A}(1)$. Since $\mathscr{L}_1$ indexes all the non-initialised columns in the first block column, the only nonzero block in the first block column of $\mathbf{C}(2)$ is $\mathbf{C}(2)_1^n$. By construction, the same is true of the blocks in the last block row. Thus, we may apply Lemma 4.2 to conclude that $\mathbf{C}(2)$ belongs to $\mathbb{E}_{\Bbbk}(n,r)_1^n$, and hence by setting $\mathbf{A}(1) = \mathbf{C}(2)$ we are done. $\square$

Step 2 above starts with the unique column in $\mathscr{L}_r = \{1 \cdots r\}$, which isn't affected by any special invariant in $\mathbb{E}_{\Bbbk}(n,r)_j^n$ for $j < r+1$. So the last opportunity to zero that column is when we choose $\mathbf{A}(r+1)$. Similarly, as the induction in Step 2 proceeds, controlled by the nested sequence

$$\mathscr{L}_r \subset \mathscr{L}_{r-1} \subset \cdots \subset \mathscr{L}_1,$$

the new columns that are zeroed in the running difference are precisely those columns that cannot be affected in subsequent steps.

EXAMPLE 6.8. Assume that $\mathbf{A} \in \mathbb{E}_{\Bbbk}(4,2)$. We illustrate Case 2 of the above proof for a last block row decomposition, working from right to left through the last block row. Step 1 consists of subtracting an arbitrary $\mathbf{A}(4)$ in $\mathbb{E}_{\Bbbk}(4,2)_4^4$ such that $\rho(\mathbf{A}(4)) = \mathbf{A}_4^4$. Referring to the matrix forms depicted in Examples 3.8, 4.5 we see that after Step 1,

$$\mathbf{C}(4) = \mathbf{A} - \mathbf{A}(4) = \begin{array}{c|cccc|cccc|cccc|cccc|}
 & 11 & 12 & 13 & 14 & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\
\hline
11 & \ast & & & & & \ast & & & & & \ast & & & & & \ast \\
12 & & t_1 & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
13 & & t_2 & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
14 & & \ast & s & 0 & \ast & & \ast & 0 & \ast & \ast & & 0 & \ast & \ast & \ast & \\
\hline
21 & & t_1' & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
22 & \ast & & & & & \ast & & & & & \ast & & & & & \ast \\
23 & & t_2' & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
24 & & \ast & s' & 0 & \ast & & \ast & 0 & \ast & \ast & & 0 & \ast & \ast & \ast & \\
\hline
31 & & t_1'' & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
32 & & t_2'' & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & & \ast & \ast & \ast & \ast & \\
33 & \ast & & & & & \ast & & & & & \ast & & & & & \ast \\
34 & & \ast & s'' & 0 & \ast & & \ast & 0 & \ast & \ast & & 0 & \ast & \ast & \ast & \\
\hline
\end{array}$$

where blank entries represent zero as usual. We omit showing the last block row, where there are no choices. The nine explicit zeros shown above are place-permutation symmetric to entries of $\mathbf{C}(4)_4^4$, which is $\mathbf{0}$ by construction. We must have $t_1 + t_2 = s$, $t_1' + t_2' = s'$, and $t_1'' + t_2'' = s''$ thanks to local GDS conditions in the blocks.

The above equations imply that the image of column 12 under η_3^4 is consistent with that column of an extension of $\eta_3^4(\mathbf{A}_3^4)$. By Lemma 6.4, we can find an $\mathbf{A}(3)$ in $\mathbb{E}_{\Bbbk}(4,2)_3^4$ such that $\rho(\mathbf{A}(3)) = \mathbf{C}(4)_3^4$ and $\eta_3^4(\mathbf{A}(3))$ agrees with $\eta_3^4(\mathbf{C}(4))$ in column 12.

(Note that $\mathscr{L}_2 = \{12\}$.) This implies that $\mathbf{C}(3)$ has the form

$$\mathbf{C}(3) = \mathbf{C}(4) - \mathbf{A}(3) = \begin{array}{c|cccc|cccc|cccc|cccc|}
 & 11 & 12 & 13 & 14 & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\ \hline
11 & * & & & & & * & & & & & * & & & & & * \\
12 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
13 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
14 & & * & 0 & 0 & * & & 0 & 0 & * & * & & 0 & * & * & 0 & \\ \hline
21 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
22 & * & & & & & * & & & & & * & & & & & * \\
23 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
24 & & * & 0 & 0 & * & & 0 & 0 & * & * & & 0 & * & * & 0 & \\ \hline
31 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
32 & & 0 & * & * & 0 & & * & * & * & * & & * & * & * & * & \\
33 & * & & & & & * & & & & & * & & & & & * \\
34 & & * & 0 & 0 & * & & 0 & 0 & * & * & & 0 & * & * & 0 & \\ \hline
\end{array}.$$

The explicit zeros in column 21 must be zero because invariants are place-permutation invariant, and column 21 is place-permutation symmetric to column 12.

By Lemma 6.4 there exists an $\mathbf{A}(2)$ in $\mathbb{E}_{\Bbbk}(4,2)^4_2$ such that $\rho(\mathbf{A}(2)) = \mathbf{A}^4_2$ and $\eta^4_2(\mathbf{A}(2))$ agrees with $\eta^4_2(\mathbf{C}(3))$ in columns $\mathscr{L}_1 = \{1*\} = \{12, 13, 14\}$. Hence the difference $\mathbf{C}(2) = \mathbf{C}(3) - \mathbf{A}(2)$ satisfies the property

$$\mathbf{C}(2)^i_4 = \mathbf{0} \text{ and } \mathbf{C}(2)^4_j = \mathbf{0} \text{ for all } i, j < 4.$$

By Lemma 4.2 it follows that $\mathbf{C}(2)$ is in $\mathbb{E}_{\Bbbk}(4,2)^4_1$, so by setting $\mathbf{A}(1) = \mathbf{C}(2)$ we obtain the desired decomposition $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(4)$. This completes Example 6.8.

To prove Property 3 we need the following result, which describes how to construct a free pattern for the special extension problem, for a given pair i, j of indices in the set $\{1, \dots, n\}$. We remind the reader that, by Proposition 4.8(c), the restriction of any $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)^i_j$ belongs to $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$. Let θ^i_j be the isomorphism in Proposition 4.6.

Lemma 6.9. *Assume Property 3 for $(n-1,r)$. Then $F(n,r)^i_j = \theta^i_j F(n-1,r)$ is a free pattern for the problem of extending invariants from $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$ to $\mathbb{E}_{\Bbbk}(n,r)^i_j$, in the sense that for any given $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$ and any assignment $f^i_j : F(n,r)^i_j \to \Bbbk$, there is a unique special invariant $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)^i_j$ such that $\rho(\mathbf{A}) = \mathbf{B}$.*

Proof. Suppose some $\mathbf{B}$ in $\mathbb{E}_{\Bbbk}(n,r-1)^i_j$ is given. To construct a special extension $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)^i_j$ we must set $\mathbf{A}^i_j = \mathbf{B}$ and all other blocks in the ith block row and jth block column to $\mathbf{0}$. The rest of $\mathbf{A}$ is determined by place-permutation symmetry and the isomorphism $\theta^i_j : \mathbb{E}_{\Bbbk}(n-1,r) \to \mathbb{E}_{\Bbbk}(n,r)^i_j$ of Proposition 4.6, thus determined by assigning values to images of the free variables in the free pattern $F(n-1,r)$. $\square$

Now we are ready for the proof of Property 3.

Proposition 6.10. *Property 4 for $(n, r-1)$ and Property 3 for $(n-1,r)$ imply Property 3 for (n,r).*

Proof. We choose to base the construction of $F(n,r)$ on the last block row. The argument for any other block row or column is similar. Let $\pi = \pi^n$ be the map in Lemma 5.2. Set $F'(n,r) = \pi^{-1} D(n, r-1)$. Then $F'(n,r)$ is a free pattern for completing the last block row of an extension $\mathbf{A}$ of $\mathbf{B}$, because doing so is equivalent to decomposing $\mathbf{B}$ along its last block row. Let $\{\mathbf{A}^n_j : j = 1, \dots, n\}$ be the last block row determined by some chosen assignment $f' : F'(n,r) \to \Bbbk$. Fix this choice of last block row of $\mathbf{A}$ for the rest of the argument.

To find the desired extension $\mathbf{A}$, we apply Proposition 6.1(a) to choose arbitrary special extensions $\mathbf{A}(j)$ in $\mathbb{E}_{\Bbbk}(n,r)^n_j$ such that $\rho(\mathbf{A}(j)) = \mathbf{A}^n_j$ for all j, and set $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$. By Lemma 6.9, there exists a free pattern $F(n,r)^n_j$ for each j and an assignment $f(j) : F(n,r)^n_j \to \Bbbk$ that determines $\mathbf{A}^n_j$. Let $F''(n,r) = \bigcup_{j=1}^n F(n,r)^n_j$.

To continue, we introduce the notation

$$\mathbf{A}_{F''} := (a^{\boldsymbol{i}}_{\boldsymbol{j}})_{(\boldsymbol{i},\boldsymbol{j})\in F''(n,r)}$$

for the restriction of $\mathbf{A}$ to $F''(n,r)$, and similarly for each $\mathbf{A}(j)$. We have $\mathbf{A}_{F''} = \sum_j \mathbf{A}(j)_{F''}$, so the given assignments $f(j)$ induce a corresponding assignment $f'' : F''(n,r) \to \Bbbk$. For the given fixed last block row of $\mathbf{A}$, it is clear that there is an extension $\mathbf{A}$ whose restriction to $F''(n,r)$ induces f'', for an arbitrary assignment $f'' : F''(n,r) \to \Bbbk$, because we can choose the $f(j)$ entries arbitrarily.

Every extension $\mathbf{A}$ of $\mathbf{B}$ with the specified last block row must be of the form $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$, where $\mathbf{A}(j)$ is in $\mathbb{E}_\Bbbk(n,r)^n_j$ and $\rho(\mathbf{A}(j)) = \mathbf{A}^n_j$ for all j. This follows from the hypotheses, which by Proposition 6.7 imply that the decomposition property holds for (n,r).

To finish, we need to argue that $\mathbf{A}$ (with the fixed last block row) is uniquely determined by its assignment f''. To see this, suppose that $\mathbf{A}'$ is another extension of $\mathbf{B}$, with the same last block row, given by the same assignment f''. Then $\mathbf{A}_{F''} = \mathbf{A}'_{F''}$. Thus the equations $\mathbf{A} = \sum_j \mathbf{A}(j)$, $\mathbf{A}' = \sum_j \mathbf{A}'(j)$ imply by restriction that

$$\textstyle\sum_j \mathbf{A}(j)_{F''} = \sum_j \mathbf{A}'(j)_{F''}\,.$$

Each entry of $\mathbf{A}(j)$, $\mathbf{A}'(j)$ is uniquely expressible by the same linear combination of the free pattern variables $a(j)^{\boldsymbol{i}}_{\boldsymbol{j}}$, $a'(j)^{\boldsymbol{i}}_{\boldsymbol{j}}$ indexed by the free pattern $F(n,r)^n_j$. Hence, any linear relations among the $\{\mathbf{A}(j)\}$, $\{\mathbf{A}'(j)\}$ are determined by their restriction to $F''(n,r)$, which contains $F(n,r)^n_j$. So the above displayed equality implies that

$$\textstyle\sum_j \mathbf{A}(j) = \sum_j \mathbf{A}'(j)\,.$$

Hence $\mathbf{A} = \mathbf{A}'$, as required. This proves the desired uniqueness statement. We have now shown that $F''(n,r)$ is a free pattern for constructing an extension $\mathbf{A}$ having the specified last block row. Thus, the disjoint union $F(n,r) = F'(n,r) \sqcup F''(n,r)$ is a free pattern for the extension problem (for the given $\mathbf{B}$). ☐

We note the following immediate consequence of the above proof.

Corollary 6.11. *Under the same hypotheses as the preceeding result based on an ith block row* (*respectively, jth block column*) *construction,*

$$F(n,r) = F'(n,r) \sqcup F''(n,r)$$

where $F''(n,r) = \bigcup_{j=1}^n F(n,r)^i_j$ (resp. $F''(n,r) = \bigcup_{i=1}^n F(n,r)^i_j$) and $F'(n,r)$ is determined by the condition $\pi F'(n,r) = D(n,r-1)$.

Example 6.12.

(a) We now construct $F(4,2)$, under the assumption that $F(3,2)$ and $D(4,1) \cong F'(4,2)$ are known. It is easy to check that $F(3,2) = \{(32,32)\}$. Hence

$$F''(4,2) = \bigcup_{1\leqslant j\leqslant 4} \theta^j_4(F(3,2)) = \bigcup_{1\leqslant j\leqslant 4} \theta^j_4\{(32),(32)\}$$

where $\theta^1_4\{32,32\} = \{32,43\}$, $\theta^2_4\{32,32\} = \{32,43\}$, $\theta^3_4\{32,32\} = \{32,42\}$, and $\theta^4_4\{32,32\} = \{32,32\}$. We refer to Example 6.14 for $F'(4,2)$. It follows that $F(4,2)$ is as depicted below:

$F(4,2)$:

	12	13	14	21	23	24	31	32	34	41	42	43
32								✓			✓	✓
42						✓		✓	✓		✓	✓
43						✓		✓	✓		✓	✓

(b) Now we consider $F(5,3)$, under the assumption that $F(4,3)$ and $D(5,2)$ are both known. It is easy to check that $F(4,3) = \{(432,432)\}$. It follows that

$F''(5,3)$:

	254	325	352	354	425	432	435	452	453	524	532	534	542	543
432						✓					✓		✓	✓

since $\theta_5^5(432,432) = (432,432)$, $\theta_5^4(432,432) = (432,532)$, $\theta_5^3(432,432) = (432,542)$, and $\theta_5^2(432,432) = (432,543)$. See Example 6.15 for $F'(5,3)$. Taking the union of $F''(5,3)$ with $F'(5,3)$ gives $F(5.3)$ as depicted below:

$F(5,3)$:

	254	325	352	354	425	432	435	452	453	524	532	534	542	543
432						✓					✓		✓	✓
532	✓		✓	✓		✓		✓	✓		✓		✓	✓
542	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
543	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

This ends Example 6.12.

It remains to prove Property 4. The proof given below closely follows the proof of Proposition 6.7.

Proposition 6.13. *Property* 3 *for* $(n-1,r)$ *implies Property* 4 *for* (n,r).

Proof. We need to prove the existence of the set $D(n,r)$ satisfying Property 4. Property 3 for $(n-1,r)$ implies Property 4 for $(n-1,r-1)$ and also implies Property1 for $(n-1,r)$, so the hypotheses of Proposition 6.7 are satisfied, and hence its conclusion holds. In other words, any given $\mathbf{A}$ in $\mathbb{E}_{\Bbbk}(n,r)$ has a decomposition based on a chosen block row or column. We assume for concreteness that it is based on the last block row, as in the proof of Proposition 6.7. The argument closely follows the proof of that proposition.

Case 1. If $n \leqslant r+1$ then by Case 1 of the proof of Proposition 6.7, the desired decomposition $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$ is unique. Hence $D(n,r)$ is empty in this case.

Case 2. Assume henceforth that $n > r+1$. By Lemma 6.9 and the hypothesis, free patterns

$$F(n,r)_j^n = \theta_j^n F(n-1,r)$$

are available for any $j = 1,\dots,n$. We may assume that $F(n-1,r)$ is row- and column-terminal.

Step 1. For each $j = r+2,\dots,n$ we choose arbitrary assignments $F(n,r)_j^n \to \Bbbk$. Each assignment uniquely determines a matrix $\mathbf{A}(j)$ in $\mathbb{E}_{\Bbbk}(n,r)_j^n$. Therefore we define

$$D'(n,r) = \bigcup_{r+2 \leqslant j \leqslant n} \{j\} \times F(n,r)_j^n \,.$$

This set parametrises a set of free entries that uniquely determines matrices $\mathbf{A}(r+2),\dots,\mathbf{A}(n)$ in Case 2 of the proof of Proposition 6.7. Set $\mathbf{C}(j{+}2) = \mathbf{A} - \sum_{j=r+2}^{n} \mathbf{A}(j)$.

Step 2. Now we proceed by reverse induction on j running from $r+1$ down to 2, in order to identify a set $D''(n,r)$ of free entries parametrising the choice of $\mathbf{A}(j+1),\dots,\mathbf{A}(2)$ satisfying condition (19) in the proof of Proposition 6.7. Clearly the union

$$\bigcup_{2 \leqslant j \leqslant r+1} \{j\} \times F(n,r)_j^n$$

is an upper bound on $D''(n,r)$. This bound is not tight due to linear dependencies caused by the prescription of columns labelled by elements of ${\mathscr{L}_{j-1}}^{\mathfrak{S}_r}$ in the proof of Step 2 of Proposition 6.7. We have to remove those additional dependencies in order to obtain $D''(n,r)$.

We do this by applying a modified version of Algorithm 5.4. Namely, for each fixed $j = r+1,\dots,2$ we initialise each entry $j_1 \cdots j_r$ of $I'(n,r)$ containing j to colour 0, and (following Step 2 in Case 2 of the proof of Proposition 6.7) do the same for each entry belonging to ${\mathscr{L}_{j-1}}^{\mathfrak{S}_r}$. Then we run Algorithm 5.4 to determine the independent (free) columns in a generic row of an extension. Let $I'_j(n,r)$ be the set of elements coloured 1 in that algorithm. We define

$$\overline{F}(n,r)^n_j = \{(i_1\cdots i_r, j_1\cdots j_r) \in F(n,r)^n_j : j_1\cdots j_r \in I'_j(n,r)\}.$$

and we accordingly set

$$D''(n,r) = \bigcup_{2\leqslant j\leqslant r+1} \{j\} \times \overline{F}(n,r)^n_j \,.$$

We claim that the desired pattern $D(n,r)$ is equal to the union

$$D(n,r) = D'(n,r) \cup D''(n,r).$$

To see this, observe that every decomposition $\mathbf{A} = \mathbf{A}(1)+\cdots+\mathbf{A}(n)$ in Proposition 6.7 determines a unique assignment

$$f : D(n,r) \to \Bbbk \tag{20}$$

by setting $f(\{j\}\times(\boldsymbol{p},\boldsymbol{q})) = a(j)^{\boldsymbol{p}}_{\boldsymbol{q}}$. Conversely, for each $r+1 \geqslant j \geqslant 2$, each assignment to $\overline{F}(n,r)^n_j$ along with the values in the prescribed columns indexed by ${\mathscr{L}_{j-1}}^{\mathfrak{S}_r}$ forces a corresponding assignment to $F(n,r)^n_j$. (Indeed, the algorithm was designed with that purpose in mind.) It thus follows from the proof of Case 2 of Proposition 6.7 that each assignment as in (20) determines a unique decomposition of the form $\mathbf{A} = \mathbf{A}(1) + \cdots + \mathbf{A}(n)$ with the required properties. $\square$

Example 6.14. To illustrate the above proof, we construct $D(n,1)$, assuming that $F(n-1,1) = \{(i,j) : 2 \leqslant i,j \leqslant n-1\}$. For each $j = 1,\ldots,n$ we have

$$F(n,1)^n_j = \{(p,q) : 2 \leqslant p \leqslant n-1,\ 2 \leqslant q \leqslant n,\ q \neq j\}.$$

Furthermore, $\overline{F}(n,1)^n_2$ is obtained from $F(n,1)^n_2$ by excising all entries in its leftmost column; that is,

$$\overline{F}(n,1)^n_2 = \{(p,q) : 2 \leqslant p \leqslant n-1,\ 4 \leqslant q \leqslant n\}.$$

Thus, as in the proof of Proposition 6.13, we obtain

$$D(n,1) = \{2\} \times \overline{F}(n,r)^n_2 \quad \cup \bigcup_{3\leqslant j\leqslant n} \{j\} \times F(n,r)^n_j \,.$$

For instance, when $n = 4$ this can be depicted by the following table:

$D(4,1)$:

$j=$	1				2				3				4			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1																
2								✓		✓		✓		✓	✓	
3								✓		✓		✓		✓	✓	
4																

in which the indices labelling checkmarked positions correspond to elements of $D(4,1)$. By Lemma 5.2, it follows that $F'(4,2)$ can be depicted by

$F'(4,2)$:

	12	13	14	21	23	24	31	32	34	41	42	43
42						✓		✓	✓		✓	✓
43						✓		✓	✓		✓	✓

.

Example 6.15. We now compute $D(5,2)$, which is in bijection with $F'(5,3)$. We first run Algorithm 5.4 to calculate $I'_j(5,2)$ for $j = 3,2$:

$I'_3(5,2)$:

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	0	1

$I'_2(5,2)$:

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	1	1

.

To justify this, we go through the algorithm for $I'_3(5,2)$ a little more slowly. The initial colouring is

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0			0	0			0	0	0	0			0				0	

.

We now list the colourings obtained after three complete iterations (doing both Steps 1 and 2) of the algorithm:

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0			0	0			0	0	0	0			0				0	1

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0			0	0			0	0	0	0	0	0	0	0	0	1	0	1

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1

respectively. Thus we obtain

$\overline{F}(5,2)^5_3$:

	12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
32																		✓		✓
42								✓										✓		✓
43								✓										✓		✓

.

Now let $j=2$. The initial colouring is

12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
0	0	0	0	0	0	0	0	0	0			0	0			0	0		

.

In Step 1 of the algorithm, we 1-colour 54. In Step 2 this forces us to 0-colour 53; this in turn forces us to 0-colour 43; this in turn forces us to 0-colour 45; this in turn forces us to 0-colour 35; this in turn forces us to 0-colour 34. Thus $I'_2(5,2)=\{54\}$. Therefore we have

$\overline{F}(5,2)^5_2$:

	12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
32																				✓
42																				✓
43																				✓

.

We summarise this in the following table, where the last two blocks are determined by $F(4,2)$ in Example 6.12(a):

$D(5,2)$:

$j=$	2	3			4					5				
	54	25	52	54	25	32	35	52	53	24	32	34	42	43
32	✓		✓	✓		✓		✓	✓		✓		✓	✓
42	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
43	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

in which we omit columns that have no information, in order to save space. We hence obtain

$F'(5,3)$:

	254	325	352	354	425	432	435	452	453	524	532	534	542	543
532	✓		✓	✓		✓		✓	✓		✓		✓	✓
542	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
543	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

.

Theorem 6.16. *All the Properties* 1–4 *hold for any* $n \geqslant 2$, $r \geqslant 1$.

Proof. This follows from the results of this section by a double induction on n, r. To be precise, we summarize the results proved in Propositions 6.1, 6.7, 6.10, and 6.13 in the table below:

Result	Hypotheses	Conclusion
6.1	Prop. $1(n-1,r)$, Prop. $2(n,r-1)$	Prop. $1(n,r)$
6.7	Prop. $1(n-1,r)$, Prop. $4(n-1,r-1)$	Prop. $2(n,r)$
6.10	Prop. $3(n-1,r)$, Prop. $4(n,r-1)$	Prop. $3(n,r)$
6.13	Prop. $3(n-1,r)$, Prop. $4(n-1,r-1)$	Prop. $4(n,r)$

.

Properties 1 and 3 for $(n,1)$ are evident, for any n. Properties 2 and 4 hold vacuously for $r=0$, for any n, as there is nothing to do since $\mathbb{E}_{\Bbbk}(n,0)\cong\Bbbk$. These serve as base cases for the induction. □

Appendix A. Gibson's theorem

We explain the connection to earlier work of P.M. Gibson [17]. Assume in this appendix that $n>1$ and that $\Bbbk$ is a (not necessarily commutative) unital ring. Gibson observed that the algebra $\mathbb{E}_{\Bbbk}(n,1)$ of $n\times n$ GDS matrices is free over the ring $\Bbbk$ and spanned by permutation matrices, and he constructed an explicit basis of $\mathbb{E}_{\Bbbk}(n,1)$ of permutation matrices.

In order to describe Gibson's basis, we define $i+1 \bmod n$ to be the unique element t of $\{1,\dots,n\}$ such that $i\equiv t$ modulo n. Let

$$\mathbf{Q}_n=\big(\delta_{i,\,j+1 \bmod n}\big)_{i,j=1,\dots,n}$$

be the $n\times n$ circulant permutation matrix representing the descending n-cycle $(n,n-1,\dots,1)\in W_n$. For example, if $n=4$ we have

$$\mathbf{Q}_4=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}.$$

Let m^i_j be the (i,j)-entry of $\mathbf{Q}_n+\mathbf{I}_n$, where $\mathbf{I}_n=\big(\delta_{i,j}\big)_{i,j=1,\dots,n}$ is the $n\times n$ identity matrix. Clearly

$$m^i_j=\begin{cases}1 & \text{if } i=j \text{ or } i+1 \bmod n=j,\\ 0 & \text{otherwise.}\end{cases}$$

There are two zero entries in each column of $\mathbf{Q}_n+\mathbf{I}_n=\big(m^i_j\big)$, one for each pair $(r,c)\in\Gamma_n$, where we define Γ_n to be the set of (r,c) such that $r=c$ or $r+1 \bmod n=c$. So there are $n(n-2)$ zero entries in $\mathbf{Q}_n+\mathbf{I}_n$. If $m^r_c=0$, there is a unique $n\times n$ permutation matrix $\mathbf{G}_{r,c}=\big(g^i_j\big)$ such that

$$g^r_c=1, \text{ and } g^i_j\leqslant m^i_j \text{ for all } (i,j)\neq(r,c).$$

In fact, one can show that the $(n-1)\times(n-1)$ submatrix obtained by removing row r and column c of $\mathbf{G}_{r,c}$ is equal to

$$\begin{cases}\mathbf{Q}_{n-1} & \text{if } r<c,\\ \mathbf{I}_{n-1} & \text{if } c<r.\end{cases}$$

This provides a recursive description of $\mathbf{G}_{r,c}$. We obtain $n(n-2)$ linearly independent permutation matrices $\mathbf{G}_{r,c}$ in this way, one for each $(r,c)\in\Gamma_n$. Gibson proved the following result.

Theorem A.1 ([17, Thm. 2.1]). *Let $\Bbbk$ be any unital ring. Then the set of permutation matrices*

$$\{\mathbf{G}_{r,c}:1\leqslant r,c\leqslant n \text{ and } (r,c)\in\Gamma_n\}\cup\{\mathbf{Q}_n,\mathbf{I}_n\}$$

is a basis over $\Bbbk$ of $\mathbb{E}_{\Bbbk}(n,1)$. In particular, the algebra of all $n\times n$ GDS matrices is free over $\Bbbk$ of rank $(n-1)^2+1$.

To express a given $n\times n$ GDS matrix $\mathbf{A}=\big(a^i_j\big)$ as a linear combination of permutation matrices, one sets

$$\mathbf{B}=\big(b^i_j\big)=\mathbf{A}-\textstyle\sum_{(r,c)\in\Gamma_n}a^r_c\,\mathbf{G}_{r,c}. \tag{21}$$

Then it is easy to see that $\mathbf{B} - b_1^n\mathbf{Q}_n - b_n^n\mathbf{I}_n = \mathbf{0}$, so

$$\mathbf{A} = b_1^n\mathbf{Q}_n + b_n^n\mathbf{I}_n + \textstyle\sum_{(r,c)\in\Gamma_n} a_c^r\,\mathbf{G}_{r,c} \tag{22}$$

is the desired linear combination. This shows that the proposed basis spans $\mathbb{E}_{\Bbbk}(n,1)$. One easily checks that it is linearly independent.

Johnsen [21] found a different basis of permutation matrices for $\mathbb{E}_{\Bbbk}(n,1)$ under the assumption that $\Bbbk$ is a field. In that case it suffices to focus on the doubly stochastic matrices (with row and column sums equal to 1). See also [23] for related work.

Remark A.2. It follows from Gibson's theorem that $\mathbb{E}_{\Bbbk}(n,1)$ is spanned by the set of $n\times n$ permutation matrices. This is the $r=1$ case of Theorem 4.10.

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