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Stratified operations on maniplexes

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Abstract There is an increasingly extensive literature on the problem of describing the connection (monodromy) groups and automorphism groups of families of polytopes and maniplexes that are not regular or reflexible. Many such polytopes and maniplexes arise as the result of constructions such as truncations and products. Here we show that for a wide variety of these constructions, the connection group of the output can be described in a nice way in terms of the connection group of the input. We call such operations *stratified*. Moreover, we show that, if F is a maniplex operation in one of two broad subclasses of stratified operations, and if $\mathcal{R}$ is the smallest reflexible cover of some maniplex $\mathcal{M}$, then the connection group of $F(\mathcal{R})$ is equal to the connection group of $F(\mathcal{M})$. In particular, we show that this is true for truncations and medials of maps, for products of polytopes (including pyramids and prisms over polytopes), and for the mix of maniplexes. As an application, we determine the smallest reflexible covers of the pyramids over the equivelar toroidal maps.

1. Introduction

Maniplexes are combinatorial structures that simultaneously generalize maps on surfaces to structures of ranks higher than 3, and abstract polytopes to structures where faces may self-intersect (see [39]). A maniplex is assembled from a set Ω of *flags*, and can either be viewed as an action of a permutation group on Ω, or as a graph with vertex set Ω, subject to some structural restrictions. Symmetry of maniplexes is measured by the number of orbits of flags under the action of the automorphism group. In that sense, the most symmetric maniplexes are the reflexible ones, which are those having only one flag orbit. In the context of maps and polytopes, reflexibility of these objects as maniplexes is equivalent to their being regular.

There are many works discussing regular maps (see for example [25, 30, 35]) and regular polytopes (see for example [4, 5, 12, 17, 36]). On the other hand, only a few classes of non-regular maps and polytopes have been systematically analyzed, highlighting the need for more techniques to study such objects.

Every maniplex is a quotient of some reflexible maniplexes. This connection suggests that part of the theory of reflexible maniplexes can be used to describe non-reflexible maniplexes as well. Among all reflexible maniplexes that cover a given maniplex $\mathcal{M}$, there is one minimal cover $\mathrm{src}(\mathcal{M})$ with respect to the partial order given

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by the quotient relationship (for more details, see Section 2). (Here "src" stands for "smallest reflexible cover".) It is to be expected that much of the information that can be obtained by viewing $\mathcal{M}$ as a quotient of a reflexible maniplex, can be obtained by viewing $\mathcal{M}$ as a quotient of $\mathrm{src}(\mathcal{M})$.

The maniplex $\mathrm{src}(\mathcal{M})$ is often described in terms of specific generators of some permutation group on the flags of $\mathcal{M}$ (called the *connection group* of $\mathcal{M}$). This provides a recipe to construct $\mathrm{src}(\mathcal{M})$ for individual finite maniplexes $\mathcal{M}$ whose set of flags is relatively small. Describing $\mathrm{src}(\mathcal{M})$ for an infinite maniplex $\mathcal{M}$ or for every maniplex $\mathcal{M}$ in an infinite family of maniplexes turns out to be complicated in general.

In the last decade, many publications have studied $\mathrm{src}(\mathcal{M})$ for various families of non-reflexible maniplexes $\mathcal{M}$, providing some ideas on how to work with the smallest reflexible cover $\mathrm{src}(\mathcal{M})$ (see [1, 2, 18, 28, 33, 34]).

The present article is motivated by operations that can be performed on maniplexes in order to obtain other maniplexes. Given such an operation F and a maniplex $\mathcal{M}$, it would be desirable to be able to construct $\mathrm{src}(F(\mathcal{M}))$ from F, $\mathcal{M}$ and $\mathrm{src}(\mathcal{M})$. We introduce the concept of a *stratified operation* that groups together many of the usual operations on maniplexes. We then show that if F is a stratified operation and satisfies an additional mild hypothesis, then $\mathrm{src}(F(\mathcal{M})) \cong \mathrm{src}(F(\mathrm{src}(\mathcal{M})))$ (see Theorems 3.9 and 3.21). These results imply that if F is a suitable stratified operation and we already know how to compute $\mathrm{src}(F(\mathcal{R}))$ for all reflexible maniplexes $\mathcal{R}$, then we have already solved the problem of computing $\mathrm{src}(F(\mathcal{M}))$ as long as we know $\mathrm{src}(\mathcal{M})$.

The paper is organized as follows. In Section 2 we review definitions and basic results on maniplexes. We motivate, state and prove our main result in Section 3, where we also provide a list of stratified operations that have already appeared in published works. In Section 4 we show an application of our main theorem; we choose F to be the operation of taking the pyramid over a maniplex, and show for which equivelar maps on the torus $\mathcal{M}$ the maniplex $\mathrm{src}(F(\mathcal{M}))$ is an abstract polytope. This extends one of the main results in [34], which was stated only for the case where $\mathcal{M}$ is a regular polytope. Another application of our main result is shown in Section 5, where the theorem is used to determine properties of $\mathrm{src}(F(\mathcal{R}))$ for some reflexible maniplex $\mathcal{R}$ in terms of $\mathrm{src}(F(\mathcal{M}))$ for some non-reflexible maniplex $\mathcal{M}$ such that $\mathcal{R} = \mathrm{src}(\mathcal{M})$. Finally, Section 6 contains proofs that some of the maniplex operations from the literature are indeed stratified operations.

2. Maniplexes and polytopes

In this section we recall the definitions of maniplex and polytope, together with some previously known results about them. For further details we refer to [27] and [29].

2.1. Maniplexes. The concept of maniplex was first defined in [39] as a combinatorial object generalizing the combinatorial aspects of maps on surfaces. They can be described in several equivalent ways; here we find the following definition convenient.

A *maniplex* of rank n (or n-*maniplex*) is a pair $\mathcal{M} = (\Omega, [r_0, \dots, r_{n-1}])$ where Ω is a set whose elements are called flags, and for every i we have that r_i is a fixed-point-free involution on Ω satisfying the following properties:

- $\langle r_0, \dots, r_{n-1}\rangle$ acts transitively on Ω,
- if $|i-j| \geqslant 2$ then $r_i r_j = r_j r_i$,
- if $i \neq j$ then r_i and r_j have no transpositions in common.

The group $\langle r_0, \dots, r_{n-1}\rangle$ is called the *connection group* of $\mathcal{M}$, it is denoted $\mathrm{Conn}(\mathcal{M})$, and its elements are called *connections*. Elsewhere $\mathrm{Conn}(\mathcal{M})$ is also called the monodromy group of $\mathcal{M}$. By convention, $\mathrm{Conn}(\mathcal{M})$ acts on Ω on the left.

We can also represent an n-maniplex by its *flag graph*, which is an edge-labeled graph whose vertex-set is the set of flags Ω, and where two vertices are connected with an edge of label i if and only if r_i interchanges those flags. Then the restrictions above imply that the flag graph has no loops or multiple edges, that it is connected, and that whenever $|i-j| \geqslant 2$, then the full subgraph consisting of edges of label i and j is a union of disjoint 4-cycles. Given an n-regular graph that satisfies these conditions, we may recover the permutations r_i from the perfect matching consisting of edges of label i, and so we may identify an n-maniplex with its flag graph.

When $\mathcal{M}$ is a map on a surface, the set Ω consists of the triangles in the barycentric subdivision of $\mathcal{M}$, and each triangle represents a flag. The involution r_0 swaps each flag with the neighbouring triangle across the segment between the center of the face and the midpoint of the edge, r_1 is the involution that swaps each flag with the neighbouring triangle across the segment between the vertex and the center of the face, and r_2 is the involution that swaps each flag with the neighbouring triangle across the edge. The flags labeled 0, 1 and 2 in Figure 1 correspond to the images of the gray flag under r_0, r_1 and r_2, respectively.

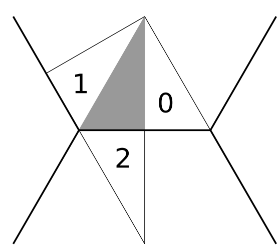

FIGURE 1. The action of the connections r_0, r_1 and r_2.

The connection group is defined as a permutation group on the set of flags. Its structure as an abstract group may be hard to understand from the maniplex itself. This has motivated the study of the connection groups of certain families of maniplexes in the previous decade (see for example [1, 2, 18, 17, 33]).

Given any flag Φ and $i \in \{0, \dots, n-1\}$ we may also denote $r_i\Phi$ by Φ^i, and say that Φ and Φ^i are *adjacent* (or *i-adjacent* if we want to emphasize i).

The faces of a maniplex are defined by extending the notion of the faces of a map. The i-faces are the orbits of $\langle r_j : j \neq i \rangle$. Furthermore, if F_i and F_j are an i-face and a j-face, respectively, then $F_i \leqslant F_j$ if and only if $i \leqslant j$ and $F_i \cap F_j \neq \varnothing$. In this way the faces of a maniplex form a partially ordered set with a rank function. Note, however, that the partially ordered set of faces may not capture all of the information about the maniplex – see Section 3 in [14].

For some purposes we may fix a *base flag* Φ of a maniplex $\mathcal{M}$. In that case we say that $\mathcal{M}$ is *rooted* at Φ. In other words, a rooted maniplex is a pair consisting of a maniplex and one of its flags.

For convenience we denote by M_n the class of all n-maniplexes.

2.2. POLYTOPES. An *abstract polytope* of rank n (or *n-polytope*) is a partially ordered set (*poset*, for short) with a strictly increasing rank function in the range $\{-1, \dots, n\}$ that satisfies some of the main properties of the face-lattices of convex polytopes. Namely, it has a unique maximum F_n of rank n and a unique minimum F_{-1} of rank -1, and it satisfies the diamond condition and strong flag-connectivity defined below. The elements of rank i are called *i-faces*.

A poset having a rank function with range $\{-1,\dots,n\}$ satisfies the *diamond condition* if for every $i \in \{0,\dots,n-1\}$, for every $(i-1)$-face F_{i-1} and for every $(i+1)$-face F_{i+1} such that $F_{i-1} \leqslant F_{i+1}$, there exist precisely two i-faces F and F' such that $F_{i-1} \leqslant F, F' \leqslant F_{i+1}$.

The maximal totally ordered subsets of a poset $(\mathcal{P}, \leqslant)$ are called *flags*. If $(\mathcal{P}, \leqslant)$ has a strictly increasing rank function then all flags have the same size. Furthermore, if $(\mathcal{P}, \leqslant)$ satisfies the diamond condition then for every flag Φ and every $i \in \{0,\dots,n-1\}$ there exists a unique flag Φ^i that differs from Φ only in its i-face. We say that Φ^i and Φ are *adjacent*, or *i-adjacent* if we want to stress the rank i.

A poset satisfying the diamond condition is said to be *strongly flag-connected* if for every two flags Φ and Ψ, there exists a sequence $\Phi = \Phi_0, \dots, \Phi_k = \Psi$ of flags such that any two consecutive flags are adjacent and such that $\Phi \cap \Psi \subseteq \Phi_j$ for every $j \in \{1,\dots,k-1\}$.

The face-lattices of convex polytopes are abstract polytopes, but there are abstract polytopes that are not the face-lattices of convex polytopes. Examples of this are maps on surfaces of genera at least 1 such that the closure of every face is homeomorphic to a disk.

Since this paper is devoted to maniplexes and abstract polytopes, unless explicitly clarified, in what follows we refer to abstract polytopes simply by "polytopes".

Every polytope $\mathcal{P}$ may be interpreted as a maniplex via its *flag graph* $FG(\mathcal{P})$, that is, an edge-labelled graph whose vertices are the flags of $\mathcal{P}$, two of which are adjacent by an edge with label i if and only if the corresponding flags are i-adjacent. The diamond condition ensures that the edges of each color form a perfect matching, and hence we may define the involution r_i as the permutation that swaps every flag with its i-adjacent flag. The partially ordered set of faces of $\mathcal{P}$ can be recovered naturally from its flag graph, and hence we may abuse notation and use $\mathcal{P}$ to denote both the partially ordered set and the corresponding maniplex.

Not every maniplex is a polytope. For example, a 3-maniplex may have a face incident to only one edge and one vertex, which violates the diamond condition. We say that a maniplex is *polytopal* if the corresponding poset is a polytope.

2.3. Symmetry. Most of the study on maniplexes and abstract polytopes has been devoted to those that possess a high degree of symmetry.

An *automorphism* of a maniplex $\mathcal{M} = (\Omega, [r_0,\dots,r_{n-1}])$ is a permutation of Ω whose action commutes with that of $\langle r_0,\dots,r_{n-1}\rangle$. If we represent $\mathcal{M}$ as its flag graph then the automorphisms of $\mathcal{M}$ are the automorphisms of the graph that fix the sets of edges corresponding to each label. If $\mathcal{M}$ is a polytope then the automorphisms are precisely the order-preserving permutations of the faces.

We denote the automorphism group of $\mathcal{M}$ by $\Gamma(\mathcal{M})$, and follow the convention that it acts on the right. This has the convenient advantage that $(w\Phi)\gamma = w(\Phi\gamma)$ for every flag Φ, every $w \in \mathrm{Conn}(\mathcal{M})$ and every $\gamma \in \Gamma(\mathcal{M})$.

The transitivity of the action of $\mathrm{Conn}(\mathcal{M})$ implies that every automorphism is completely determined by the image of any flag. As a consequence, $\Gamma(\mathcal{M})$ acts semi-regularly on the flags of $\mathcal{M}$.

A maniplex $\mathcal{M}$ is said to be *reflexible* if $\Gamma(\mathcal{M})$ acts transitively (and hence, regularly) on the set of flags. If $\Gamma(\mathcal{M})$ induces k orbits on the set of flags we say that $\mathcal{M}$ is a k-orbit maniplex. Polytopes for which $\Gamma(\mathcal{M})$ acts transitively on its flags are *regular*.

A *string group generated by involutions* (sggi for short) is a group generated by a list of involutions where every two involutions that are not consecutive in the list commute. A *string C-group* is an sggi $\langle r_0,\dots,r_{n-1}\rangle$ where for every $I, J \subseteq \{0,\dots,n-1\}$

we have that

$$\langle r_k : k \in I\rangle \cap \langle r_k : k \in J\rangle = \langle r_k : k \in I \cap J\rangle.$$

The automorphism group of every reflexible n-maniplex $\mathcal{M}$ is an sggi $\Gamma(\mathcal{M}) = \langle \rho_0, \dots, \rho_{n-1}\rangle$, where ρ_i is the automorphism mapping a fixed base flag Φ to Φ^i. If $\mathcal{M}$ is a regular polytope then $\Gamma(\mathcal{M})$ is a string C-group. Conversely, given a string C-group $\Gamma = \langle \rho_0, \dots, \rho_{n-1}\rangle$ there exists a unique regular abstract n-polytope $\mathcal{P}$ such that $\Gamma(\mathcal{P}) = \Gamma$ and with the property that there exists a flag Φ such that $\Phi\rho_i = \Phi^i$ for every i. The base i-face F_i is taken to be the parabolic subgroup $\langle \rho_j : j \neq i\rangle$ and the remaining i-faces are the cosets $F_i\gamma$ with $\gamma \in \Gamma$. Face incidence is then defined as non-trivial intersection of the cosets.

The proof of the following theorem is essentially the same as that of [29, Theorem 3.9].

Theorem 2.1. *Let $\mathcal{M}$ be a reflexible n-maniplex with $\Gamma(\mathcal{M}) = \langle \rho_0, \dots, \rho_{n-1}\rangle$ and connection group* $\mathrm{Conn}(\mathcal{M}) = \langle r_0, \dots, r_{n-1}\rangle$. *Then there is an isomorphism between* $\Gamma(\mathcal{M})$ *and* $\mathrm{Conn}(\mathcal{M})$ *mapping each ρ_i to r_i.*

We shall denote by W_n the universal string Coxeter group of rank n, that is, the group with presentation

$$\langle s_0, \dots, s_{n-1} : (s_i s_j)^{p_{i,j}} = id\rangle, \tag{1}$$

where $p_{i,i} = 1$, $p_{i,j} = 2$ if $|i - j| \geqslant 2$, and $p_{i,j} = \infty$ if $|i - j| = 1$. It is well-known that W_n is a string C-group and hence it is the automorphism group of the universal polytope $\mathcal{P}(W_n)$ of rank n. Note that $\mathcal{P}(W_n)$ is regular.

2.4. Smallest reflexible covers. A *covering* from a maniplex $\mathcal{M}_1 = (\Omega_1, [r_0, \dots, r_{n-1}])$ to a maniplex $\mathcal{M}_2 = (\Omega_2, [r'_0, \dots, r'_{n-1}])$ is a function $\eta : \Omega_1 \to \Omega_2$ such that $(r_i\Phi)\eta = r'_i(\Phi\eta)$ for every $\Phi \in \Omega_1$ and every i. Such a function is necessarily surjective. Coverings are also called rap-maps (rank and adjacency preserving maps). If η is a covering from $\mathcal{M}_1$ to $\mathcal{M}_2$ we say that $\mathcal{M}_1$ is a *cover* of $\mathcal{M}_2$, that $\mathcal{M}_1$ *covers* $\mathcal{M}_2$, or that $\mathcal{M}_2$ is a *quotient* of $\mathcal{M}_1$.

Every maniplex is covered by at least one reflexible maniplex. For instance, every n-maniplex $\mathcal{M}$ is covered by the universal n-polytope $\mathcal{P}(W_n)$. This is a consequence of Theorem 2.1, since the connection group of every n-maniplex satisfies the relations in the presentation (1).

Coverings by reflexible maniplexes provide us with useful tools to better understand connection groups of non-reflexible maniplexes. The following proposition is the version for maniplexes of [29, Proposition 3.11], and it can be proven in the same way as the proposition in that article.

Proposition 2.2. *Let $\mathcal{L} = (\Omega_{\mathcal{L}}, [r'_0, \dots, r'_{n-1}])$ and $\mathcal{M} = (\Omega_{\mathcal{M}}, [r_0, \dots, r_{n-1}])$ be maniplexes such that $\mathcal{M}$ covers $\mathcal{L}$. Then there is a group epimorphism μ from* $\mathrm{Conn}(\mathcal{M})$ *to* $\mathrm{Conn}(\mathcal{L})$ *that maps r_i to r'_i for all i.*

This proposition essentially says that we may think of $\mathrm{Conn}(\mathcal{M})$ as acting on the flags of $\mathcal{L}$. In particular, $W_n = \langle s_0, \dots, s_{n-1}\rangle$ acts on the flags of any n-maniplex $\mathcal{L}$, and we can identify the action of each s_i with a generator of $\mathrm{Conn}(\mathcal{L})$.

Among all reflexible covers of $\mathcal{M} = (\Omega, [r_0, \dots, r_{n-1}])$ the following cover has special relevance. Let $\mathrm{src}(\mathcal{M}) := (\overline{\Omega}, [\bar{r}_0, \dots, \bar{r}_{n-1}])$, where $\overline{\Omega} = \mathrm{Conn}(\mathcal{M})$ and for $w \in \overline{\Omega}$ and $i \in \{0, \dots, n-1\}$, $\bar{r}_i w := r_i w$. (We will describe the reason for the name "src" in a moment.) Then a covering $\eta : \overline{\Omega} \to \Omega$ is defined by specifying some $\Phi \in \Omega$ as the image of the identity element id, and then mapping each $w \in \overline{\Omega}$ to $w\Phi$.

The following remark is immediate from the definition of $\mathrm{src}(\mathcal{M})$.

Remark 2.3. For every maniplex $\mathcal{M}$ there is an isomorphism from $\mathrm{Conn}(\mathcal{M})$ to $\mathrm{Conn}(\mathrm{src}(\mathcal{M}))$ mapping r_i to $\bar{r}_i$.

Furthermore, the cover $\mathrm{src}(\mathcal{M})$ is the *smallest reflexible cover* of $\mathcal{M}$ in the sense that any other reflexible cover of $\mathcal{M}$ is also a cover of $\mathrm{src}(\mathcal{M})$. (A straightforward adaptation of Proposition 3.16 from [29] can be used to prove this fact.)

Note that the definition of $\mathrm{src}(\mathcal{M})$ is actually a definition of a covering of rooted maniplexes and in principle it depends on the image Φ of $id \in \overline{\Omega}$. However, since $\mathrm{src}(\mathcal{M})$ is reflexible, different choices of such a flag Φ yield isomorphic covers, and so $\mathrm{src}(\mathcal{M})$ is a well-defined construction on the category of unrooted maniplexes.

If $\mathrm{src}(\mathcal{M})$ is polytopal then it has been denoted elsewhere as the *minimal regular cover*, which is indeed minimal even when restricting only to polytopal covers. However, the smallest reflexible cover of $\mathcal{M}$ may not be polytopal, even if $\mathcal{M}$ is a polytope. If a polytope $\mathcal{P}$ has a non-polytopal smallest reflexible cover then $\mathcal{P}$ has rank at least 4 ([29, Proposition 6.1]). To date, relatively few examples of non-polytopal smallest reflexible covers of polytopes are known, which motivates the search for more such examples.

To conclude this section we state a simple result about the action of the connection group of $\mathrm{src}(\mathcal{M})$ on the flags of $\mathcal{M}$.

Proposition 2.4. *If $\mathcal{R} = \mathrm{src}(\mathcal{M})$, then the identity element is the only element in $\mathrm{Conn}(\mathcal{R})$ that fixes every flag of $\mathcal{M}$.*

Proof. First, recall that $\mathrm{Conn}(\mathcal{M})$ is a concrete permutation group on Ω, the set of flags of $\mathcal{M}$. Remark 2.3 says that $\mathrm{Conn}(\mathcal{R})$ is isomorphic to $\mathrm{Conn}(\mathcal{M})$. It follows that if an element of $\mathrm{Conn}(\mathcal{R})$ fixes every flag of $\mathcal{M}$, then it corresponds to the identity permutation and thus the identity element of $\mathrm{Conn}(\mathcal{R})$. □

3. Stratified operations

If we want to find 4-polytopes whose smallest reflexible cover is not polytopal, where do we start? One appealing way is to apply well-known operations to small polytopes and analyze the result. Most of the literature so far on the connection group of polytopes has used this approach, considering truncations of polytopes as well as pyramids and prisms over polytopes; see [1], [2], [18], [19], [28] and [33]. In [34], two of the authors analyzed the connection groups of pyramids over the regular toroidal maps, determining which pyramids have a polytopal smallest reflexible cover. What can we say about pyramids over non-regular equivelar toroidal maps?

We will show that if $\mathcal{M}$ is a maniplex with smallest reflexible cover $\mathcal{R}$, then the smallest reflexible cover of the pyramid over $\mathcal{M}$ is the same as the smallest reflexible cover of the pyramid over $\mathcal{R}$. Moreover, it turns out that the properties of the pyramid operation that lead to this result are in fact shared by most common maniplex operations. Let us start by considering pyramids over polytopes, and then generalize to other operations.

3.1. Smallest reflexible covers of pyramids.

The notion of a pyramid over an n-polytope $\mathcal{P}$ comes from convex geometry, where it is defined as the ($(n+1)$-dimensional) convex hull of $\mathcal{P} \cup \{a_0\}$, for some point a_0 outside the affine span of $\mathcal{P}$ (see for example [16]). The point a_0 is denoted the *apex*, and $\mathcal{P}$ the *base* of the pyramid. This concept has been extended to abstract polytopes. In [15] it is a particular instance of a product of polytopes.

Here we find it convenient to define the pyramid over an arbitrary maniplex as follows.

DEFINITION 3.1. *Given an n-maniplex $\mathcal{M} = (\Omega, [r_0, \ldots, r_{n-1}])$, the* pyramid over $\mathcal{M}$ *is the $(n+1)$-maniplex $Pyr(\mathcal{M}) = (\{0, \ldots, n+1\} \times \Omega, [s_0, \ldots, s_n])$, where*

$$s_i(k, \Phi) = \begin{cases} (k, r_i\Phi) & \text{if } k < n-i, \\ (k+1, \Phi) & \text{if } k = n-i, \\ (k-1, \Phi) & \text{if } k = n-i+1, \\ (k, r_{i-1}\Phi) & \text{if } k > n-i+1. \end{cases}$$

The elements of the set $\{0, \ldots, n+1\}$ are called the layers *of* $\mathrm{Pyr}(\mathcal{M})$.

In Definition 3.1, the base of the pyramid corresponds to the n-face consisting of the flags $(0, \Phi)$ and the apex corresponds to the vertex consisting of the flags $(n+1, \Phi)$ (see Figure 2).

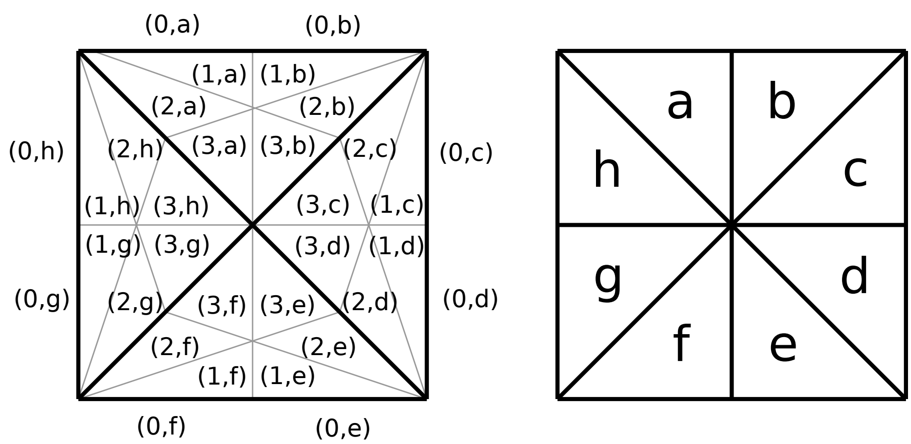

FIGURE 2. The flags of the pyramid over a square (left) and its base square (right).

The motivation of Definition 3.1 is the following. The pyramid $\mathrm{Pyr}(\mathcal{M})$ has one facet isomorphic to $\mathcal{M}$, namely that consisting of the flags whose first coordinate equals 0. Then, for every facet $\mathcal{L}$ of $\mathcal{M}$ there is a facet of $Pyr(\mathcal{M})$ consisting of the flags (k, Ψ) with Ψ a flag containing $\mathcal{L}$ and $k > 1$. A careful inspection shows that this facet of $\mathcal{M}$ satisfies the definition to be the pyramid over $\mathcal{L}$ (with first coordinates shifted by 1). The equivalence between Definition 3.1 and the ones in [16] and [15] (applied to the appropriate class of objects) can be easily shown by induction, with base given by the triangle (pyramid over a 1-maniplex).

Pyramids are well-behaved maniplexes in several respects. One of them is with their interaction with quotients, as shown by the following result.

LEMMA 3.2. *If $\mathcal{M}$ covers $\mathcal{L}$ then* $\mathrm{Pyr}(\mathcal{M})$ *covers* $\mathrm{Pyr}(\mathcal{L})$.

Proof. Let π be the covering from $\mathcal{M} = (\Omega_{\mathcal{M}}, [r_0, \ldots, r_{n-1}])$ to $\mathcal{L} = (\Omega_{\mathcal{L}}, [r'_0, \ldots, r'_{n-1}])$. Then it follows from Definition 3.1 that the mapping $\tilde{\pi} : \{0, \ldots, n+1\} \times \Omega_{\mathcal{M}} \to \{0, \ldots, n+1\} \times \Omega_{\mathcal{L}}$ given by $(k, \Phi)\tilde{\pi} = (k, \Phi\pi)$ is the desired covering. □

There is also a nice interaction between the layers of the pyramid and the action of the connection group. Observe that the generator s_i acts as the transposition $\sigma_i := (n-i, n-i+1)$ on the first coordinates of all flags. More generally, if $x \in W_{n+1}$ with $x = s_{i_1} \cdots s_{i_t}$, then we will use σ_x to denote $\sigma_{i_1} \cdots \sigma_{i_t}$, so that x acts on the first coordinates of all flags as σ_x.

The action on the second coordinate is also independent of the choice of flag Φ and the choice of maniplex. In other words, for each k and i, there is an element $w_{k,i} \in W_n$ such that $s_i(k, \Phi) = (\sigma_i k, w_{k,i}\Phi)$. Indeed, we have:

$$w_{k,i} = \begin{cases} r_i & \text{if } k < n-i, \\ id & \text{if } n-i \leqslant k \leqslant n-i+1, \\ r_{i-1} & \text{if } k > n-i+1. \end{cases}$$

We extend this notation recursively as follows. First, we understand w_{k,s_i} to mean the same thing as $w_{k,i}$. Now, if x is the product of some t generators of W_{n+1} with $t > 1$, then we may write $x = y\sigma_i$, where y is the product of $t-1$ generators. Then we define

$$w_{k,x} = w_{\sigma_i k, y} w_{k,i},$$

which makes it so that $x(k, \Phi) = (\sigma_x k, w_{k,x}\Phi)$.

Summarizing, we have shown:

Proposition 3.3.

(1) *W_{n+1} acts on $\{0, \ldots, n+1\}$ via the homomorphism that sends each s_i to $\sigma_i = (n-i, n-i+1)$, and where we denote the image of x by σ_x. This action does not depend on the choice of base maniplex.*
(2) *For every $k \in \{0, \ldots, n+1\}$ and every $x \in W_{n+1}$, there is an element $w_{k,x} \in W_n$ such that, for all Φ,*

$$x(k, \Phi) = (\sigma_x k, w_{k,x}\Phi).$$

Furthermore, $w_{k,x}$ does not depend on the choice of base maniplex.

We are now ready for the main theorem of this section.

Theorem 3.4. *Let $\mathcal{M}$ be a maniplex and $\mathcal{R}$ be its smallest reflexible cover. Then $\mathrm{Conn}(\mathrm{Pyr}(\mathcal{M})) = \mathrm{Conn}(\mathrm{Pyr}(\mathcal{R}))$.*

Proof. We know from Lemma 3.2 that $\mathrm{Pyr}(\mathcal{R})$ covers $\mathrm{Pyr}(\mathcal{M})$. Then Proposition 2.2 tells us that there is a surjective group homomorphism $\mu : \mathrm{Conn}(\mathrm{Pyr}(\mathcal{R})) \to \mathrm{Conn}(\mathrm{Pyr}(\mathcal{M}))$. It only remains to show that it is in fact an isomorphism.

Let n be the rank of $\mathcal{R}$. Consider an element $x \in W_{n+1}$ such that the image of x in $\mathrm{Conn}(\mathrm{Pyr}(\mathcal{R}))$ is in the kernel of μ. In other words, x fixes every flag of $\mathrm{Pyr}(\mathcal{M})$, so $x(k, \Phi) = (k, \Phi)$ for every flag Φ of $\mathcal{M}$ and every layer k. By Proposition 3.3, x acts on the layers as some permutation σ_x that is independent of the choice of base maniplex. Thus, since x fixes every layer of $\mathrm{Pyr}(\mathcal{M})$, it also fixes every layer of $\mathrm{Pyr}(\mathcal{R})$. Now, Proposition 3.3 also says that for each k, there is an element $w_{k,x}$ of W_n such that $x(k, \Phi) = (k, w_{k,x}\Phi)$ for every Φ. So for each k, the element $w_{k,x}$ fixes every flag Φ of $\mathcal{M}$. By Proposition 2.4, $w_{k,x}$ is trivial and hence $\mathrm{Conn}(\mathrm{Pyr}(\mathcal{M})) = \mathrm{Conn}(\mathrm{Pyr}(\mathcal{R}))$. □

Having shown that the pyramid operation interacts nicely with connection groups, our attention naturally turns to other constructions. Do prisms over polytopes work the same way? What about truncations of polyhedra? The essential properties of the pyramid construction that led to Theorem 3.4 are that:

(1) If $\mathcal{M}$ covers $\mathcal{L}$, then $\mathrm{Pyr}(\mathcal{M})$ covers $\mathrm{Pyr}(\mathcal{L})$.
(2) In the definition of pyramids in Definition 3.1, each s_i acts on the layers independently of the base maniplex, and each s_i acts on the flags Φ in a way that only depends on i and the layer k.

Roughly speaking, the second condition will be satisfied by any maniplex operation whose definition looks like:

$$s_i(k,\Phi) = \begin{cases} (k', u\Phi) & \text{if (some condition on } i \text{ and } k), \\ \cdots & \\ (k'', v\Phi) & \text{if (some condition on } i \text{ and } k). \end{cases}$$

Example 3.5. Consider a map (3-maniplex) $\mathcal{M}$. Its *medial* is a new map $\mathrm{Me}(\mathcal{M})$ with flags $(0,\Phi)$ and $(2,\Phi)$ for each flag Φ of $\mathcal{M}$. Then we have:

$$s_i(k,\Phi) = \begin{cases} (k, r_1\Phi) & \text{if } i = 0, \\ (k, r_{2-k}\Phi) & \text{if } i = 1, \\ (2-k, \Phi) & \text{if } i = 2. \end{cases}$$

(See [22].) So the medial operation satisfies condition (2) above.

Let us describe condition (2) a little more formally, and then we can see which other maniplex operations satisfy that condition.

3.2. Stratified operations and connection groups.

Definition 3.6. *Consider a maniplex operation $F : \mathsf{M}_n \to \mathsf{M}_m$. We say that F is a* stratified *maniplex operation if there is a set A (called the set of* strata *or* layers*) such that:*

1. *If Ω is the set of flags of a maniplex $\mathcal{M}$, then the set of flags of $F(\mathcal{M})$ is a subset of $A \times \Omega$ such that the canonical projections into A and Ω are both surjective.*
2. *Let $S = \{s_0, \ldots, s_{m-1}\}$ be the set of generators of W_m; then W_m has a well-defined action on A, where we denote by σ_i the permutation of A induced by $s_i \in S$.*
3. *There is a function $\varphi : A \times S \to W_n$ such that, for every maniplex $\mathcal{M}$ and flag Φ, the action of s_i on $A \times \Omega$ is described by*

$$s_i(a,\Phi) = (\sigma_i a, \varphi(a, s_i)\Phi).$$

Additionally, if the set of flags of $F(\mathcal{M})$ is all of $A \times \Omega$, then we say that F is fully stratified. *For brevity, we will typically denote $\varphi(a, s_i)$ by $w_{a,i}$, so that*

$$s_i(a,\Phi) = (\sigma_i a, w_{a,i}\Phi).$$

Proposition 3.7. *Suppose that F is a stratified maniplex operation, with function $\varphi : A \times S \to W_n$ as in Definition 3.6. Then φ can be extended to a function $\tilde{\varphi} : A \times W_m \to W_n$ such that, for every $x \in W_m$ we have $x(a,\Phi) = (\sigma_x a, \tilde{\varphi}(a,x)\Phi)$. We will typically denote $\tilde{\varphi}(a,x)$ by $w_{a,x}$.*

Proof. Let $x = s_{i_1} \cdots s_{i_t} \in W_m$. Then

$$\begin{aligned} x(a,\Phi) &= (s_{i_1} \cdots s_{i_{t-1}}) s_{i_t}(a,\Phi) \\ &= s_{i_1} \cdots s_{i_{t-1}} (\sigma_{i_t} a, w_{a,i_t}\Phi). \end{aligned}$$

Continuing in this way, we find that $x(a,\Phi) = (\sigma_x a, w_{a,x}\Phi)$ for some permutation σ_x of A and some element $w_{a,x} \in W_n$ that depends only on a and x. □

Let us offer one cautionary note – if we want to define a new maniplex operation by just picking a set A and a function φ, there are several restrictions. For example, we need

$$uv(a,\Phi) = (\sigma_{uv} a, \tilde{\varphi}(a, uv)\Phi)$$

and also

$$uv(a,\Phi) = u(\sigma_v a, \tilde{\varphi}(a,v)\Phi) = (\sigma_{uv} a, \tilde{\varphi}(va,u)\tilde{\varphi}(a,v)\Phi),$$

implying that $\tilde{\varphi}(a,uv) = \tilde{\varphi}(va,u)\tilde{\varphi}(a,v)$ for all a, u, and v. It is not so easy to pick φ so that it satisfies this unless it does not really depend on a (see Section 3.4). For now, we restrict our attention to merely describing what φ is for well-known operations that already exist.

A maniplex operation is *cover-preserving* if, whenever $\mathcal{M}$ covers $\mathcal{L}$, then $F(\mathcal{M})$ covers $F(\mathcal{L})$. (In fact, in this case usually F will define a functor on the category of rooted n-maniplexes.) Recall that one of the features of the operation Pyr that led to Theorem 3.4 is that Pyr is cover-preserving. It turns out that we get that for free when working with fully stratified operations.

Proposition 3.8. *If F is a fully stratified maniplex operation, then F is cover-preserving.*

Proof. Suppose that $\mathcal{M}$ and $\mathcal{L}$ are n-maniplexes such that $\pi : \mathcal{M} \to \mathcal{L}$ is a covering. We want to define a covering $\tilde{\pi} : F(\mathcal{M}) \to F(\mathcal{L})$, and the obvious possibility is $(a,\Phi)\tilde{\pi} = (a, \Phi\pi)$. (Since F is fully stratified, we can be sure that $(a,\Phi\pi)$ is a flag of $F(\mathcal{L})$.) To show that this is a covering, it suffices to show that $\tilde{\pi}$ commutes with the action of W_m. That is, we want to show that for each j, $(s_j(a,\Phi))\tilde{\pi} = s_j((a,\Phi)\tilde{\pi})$. We find:

$$(s_j(a,\Phi))\tilde{\pi} = (\sigma_j a, \varphi(a,s_j)\Phi)\tilde{\pi} = (\sigma_j a, (\varphi(a,s_j)\Phi)\pi),$$

whereas

$$s_j((a,\Phi)\tilde{\pi}) = s_j(a,\Phi\pi) = (\sigma_j a, \varphi(a,s_j)(\Phi\pi)),$$

and the two expressions on the right are equal since the action of π commutes with the action of connections. □

Now we can prove a generalization of Theorem 3.4.

Theorem 3.9. *Suppose that F is a fully stratified maniplex operation. Let $\mathcal{M}$ be a maniplex with smallest reflexible cover $\mathcal{R}$. Then* $\mathrm{Conn}(F(\mathcal{R})) = \mathrm{Conn}(F(\mathcal{M}))$.

Proof. By Proposition 3.8, F is cover-preserving. So $F(\mathcal{R})$ covers $F(\mathcal{M})$, which implies that there is a surjective group homomorphism π from $\mathrm{Conn}(F(\mathcal{R}))$ to $\mathrm{Conn}(F(\mathcal{M}))$ that sends each generator of the former to the corresponding generator of the latter. It remains to show that this is an isomorphism.

Suppose that $x \in W_m$ is such that the image of x in $\mathrm{Conn}(F(\mathcal{R}))$ is in the kernel of π. In other words, x fixes every flag of $F(\mathcal{M})$. Since F is fully stratified, the flags of $F(\mathcal{M})$ include all pairs (a,Φ) with $a \in A$ and Φ a flag of $\mathcal{M}$. Then it follows that σ_x fixes every $a \in A$. Furthermore, for each a, the element $\varphi(a,x)$ fixes every flag of $\mathcal{M}$. Then Proposition 2.4 says that $\varphi(a,x)$ fixes every flag of $\mathcal{R}$ as well. Thus we have shown that for every flag (a,Φ) of $F(\mathcal{R})$, we have $x(a,\Phi) = (a,\Phi)$, which implies that x corresponds to the identity of $\mathrm{Conn}(F(\mathcal{R}))$. Therefore, the map from $\mathrm{Conn}(F(\mathcal{R}))$ to $\mathrm{Conn}(F(\mathcal{M}))$ is an isomorphism. □

3.3. Examples of fully stratified operations. In light of Theorem 3.9, it is worthwhile to know which maniplex operations are fully stratified. To demonstrate that an operation F is fully stratified, it is enough to do the following.

(1) Show that there is a set A such that the flags of $F(\mathcal{M})$ can be identified with all pairs (a,Φ) with $a \in A$ and Φ a flag of $\mathcal{M}$.

(2) Define the action of each $s_i \in W_m$ on (a, Φ) in a way that looks like

$$s_i(a,\Phi) = \begin{cases} (a', u\Phi) & \text{if (some condition on } i \text{ and } a), \\ \cdots & \\ (a'', v\Phi) & \text{if (some condition on } i \text{ and } a). \end{cases}$$

(We can extract the necessary function $\varphi : A \times S \to W_n$ from this definition.)

Example 3.10. We already remarked (in Example 3.5) that the flags of $\mathrm{Me}(\mathcal{M})$ consist of all pairs $(0, \Phi)$ and $(2, \Phi)$, where Φ ranges over all flags of $\mathcal{M}$. The definition of the action of s_i meets the requirements above, and so the operation $F(\mathcal{M}) = \mathrm{Me}(\mathcal{M})$ is fully stratified.

Example 3.11. Truncation of a map (as defined in [31, Section 4.2]) is also a fully stratified operation.

Example 3.12. Suppose $\mathcal{L}$ is an n-polytope, and let $F(\mathcal{M}) = \mathcal{M} \bigodot \mathcal{L}$, where $\bigodot$ is one of the products in [15]. Then F is fully stratified; see Section 6.1.

Example 3.13. The operations Cl_k and $\widetilde{Cl}_k$ from [32] are fully stratified; see Section 6.2.

Example 3.14. The *k-bubble of $\mathcal{P}$*, defined in [20], is a fully stratified operation; see Section 6.3.

3.4. Operations defined using parallel products. We would like to identify other operations F that satisfy the conclusion of Theorem 3.9. First, let us note that if F is fully stratified and if $\mathcal{M}$ and $\mathcal{L}$ are maniplexes such that $\mathrm{Conn}(\mathcal{M}) \cong \mathrm{Conn}(\mathcal{L})$, then Theorem 3.9 implies that $\mathrm{Conn}(F(\mathcal{M})) \cong \mathrm{Conn}(F(\mathcal{L}))$. In fact, this condition is equivalent to the conclusion of Theorem 3.9. Thus we make the following definition.

Definition 3.15. *A maniplex operation F is* connection-preserving *if, whenever $\mathrm{Conn}(\mathcal{M}) \cong \mathrm{Conn}(\mathcal{L})$, then $\mathrm{Conn}(F(\mathcal{M})) \cong \mathrm{Conn}(F(\mathcal{L}))$.*

Now, let us suppose that F is a stratified operation with set of strata A and with function $\varphi : A \times W_m \to W_n$. Let us further suppose that there is a group homomorphism $\psi : W_m \to W_n$ such that, for every $a \in A$, we have $\varphi(a, x) = \psi(x)$. That means that the action of W_m on a flag (a, Φ) of $F(\mathcal{M})$ may be written as

$$x(a,\Phi) = (\sigma_x a, \psi(x)\Phi),$$

so the action of W_m on the second component does not really depend on a. Thus x just acts component-wise; we have an action of W_m on the set A and an action (mediated by ψ) of W_m on the flags of $\mathcal{M}$. This is known as the *parallel product* of the two actions [38].

Maniplex operations involving parallel products often depend on the choice of a base flag for the input maniplex $\mathcal{M}$. In other words, these operations are more properly thought of as operations that take rooted maniplexes as input. The definition of a stratified operation works without modification for rooted maniplexes.

Proposition 3.16. *Suppose that $\psi : W_m \to W_n$ is a surjective group homomorphism, and that W_m acts transitively on $A \neq \varnothing$ via a homomorphism that sends each s_i to a permutation σ_i of A. Let a_0 be a fixed base point of A. For rooted n-maniplexes $(\mathcal{M}, \Phi_0)$, define $F(\mathcal{M}, \Phi_0)$ to be the rooted m-maniplex whose flags are the orbit of (a_0, Φ_0) under the action of W_m, where*

$$x(a,\Phi) = (\sigma_x a, \psi(x)\Phi).$$

Then F is a cover-preserving, stratified operation.

Proof. To show that F is stratified, the only part that is not immediate is that the set of flags of $F(\mathcal{M})$ needs to project surjectively onto A and Ω. This is true because W_m acts transitively on A, and because ψ is surjective and W_n acts transitively on Ω.

Now, we want to show that F is cover-preserving. Suppose that $(\mathcal{M}, \Phi_0)$ and $(\mathcal{L}, \Psi_0)$ are rooted n-maniplexes such that $\pi : \mathcal{M} \to \mathcal{L}$ is a covering that sends Φ_0 to Ψ_0. As in Proposition 3.8, we define $\tilde{\pi} : F(\mathcal{M}) \to F(\mathcal{L})$ by $(a, \Phi)\tilde{\pi} = (a, \Phi\pi)$. In order for this to be well-defined, we need to show that $(a, \Phi\pi)$ is a flag of $F(\mathcal{L})$ whenever (a, Φ) is a flag of $F(\mathcal{M})$. By definition, the flags of $F(\mathcal{M})$ consist of the orbit of (a_0, Φ_0) under the action of W_m. So for each (a, Φ), there is some $x \in W_m$ such that $(a, \Phi) = x(a_0, \Phi_0) = (\sigma_x a_0, \psi(x)\Phi_0)$. Then:

$$\begin{aligned} x(a_0, \Psi_0) &= (\sigma_x a_0, \psi(x)\Psi_0) \\ &= (a, \psi(x)(\Phi_0\pi)) \\ &= (a, (\psi(x)\Phi_0)\pi) \\ &= (a, \Phi\pi), \end{aligned}$$

and so $(a, \Phi\pi)$ is in the orbit of (a_0, Ψ_0) and thus a flag of $F(\mathcal{L})$. The rest of the proof (showing that $\tilde{\pi}$ commutes with connections) is the same as the proof of Proposition 3.8. □

Definition 3.17. *A maniplex operation F is* a parallel product *if it satisfies the conditions of Proposition* 3.16*.*

Let us look at several examples of parallel products.

Example 3.18. The I-doubles of a map (defined in [26]) can be seen as a parallel product, where A is a two element set with some prescribed action of W_m, and where W_m acts in the natural way on Ω.

Example 3.19. Let $\mathcal{L}$ be an n-maniplex with base flag Ψ_0. Consider the operation on rooted n-maniplexes defined by $F(\mathcal{M}, \Phi_0) = (\mathcal{M}, \Phi_0) \diamond (\mathcal{L}, \Psi_0)$. This is the *mix* of rooted maniplexes, as defined in [8, Section 3.2] and seen in other forms in many previous papers. The flags of $F(\mathcal{M}, \Phi_0)$ consist of (some) pairs (Φ, Ψ), with Φ a flag of $\mathcal{M}$ and Ψ a flag of $\mathcal{L}$, and for each i, $s_i(\Phi, \Psi) = (r_i\Phi, r_i\Psi)$. (In fact, in this case we have $m = n$, so maybe we ought to write $r_i(\Phi, \Psi) = (r_i\Phi, r_i\Psi)$.)

Example 3.20. Let $\mathcal{Q}$ be a vertex-bipartite polytope, and consider the operation on facet-bipartite polytopes $\mathcal{P}$ given by $F(\mathcal{P}) = \mathcal{P}|\mathcal{Q}$, the *flat amalgamation of $\mathcal{P}$ and $\mathcal{Q}$*. (See [6, Definition 4.2].) This is also a parallel product.

Now let us show that parallel products are connection-preserving.

Theorem 3.21. *Suppose that F is a parallel product. Let $\mathcal{M}$ be a maniplex with smallest reflexible cover $\mathcal{R}$. Then* $\mathrm{Conn}(F(\mathcal{R})) = \mathrm{Conn}(F(\mathcal{M}))$*.*

Proof. By Proposition 3.16, F is cover-preserving. So $F(\mathcal{R})$ covers $F(\mathcal{M})$, which implies that there is a surjective group homomorphism π from $\mathrm{Conn}(F(\mathcal{R}))$ to $\mathrm{Conn}(F(\mathcal{M}))$ that sends each generator of the former to the corresponding generator of the latter. It remains to show that this is an isomorphism.

Suppose that $x \in W_m$ such that the image of x in $\mathrm{Conn}(F(\mathcal{R}))$ is in the kernel of π. In other words, x fixes every flag of $F(\mathcal{M})$. Since F is stratified, every $a \in A$ appears as part of a flag (a, Φ) of $F(\mathcal{M})$. Then x must fix every $a \in A$. Now, every flag Φ appears as part of a flag (a, Φ), and so for every Φ there is some a such that $\varphi(a, x)$ fixes Φ. Since F is a parallel product, there is a function ψ such that $\varphi(a, x) = \psi(x)$. So, for every Φ we have that $\psi(x)$ fixes Φ. By Proposition 2.4, it follows that $\psi(x)$ fixes every flag of $\mathcal{R}$ as well. Thus we have shown that for every flag (Φ, a) of $F(\mathcal{R})$, we have

$x(a,\Phi) = (a,\Phi)$, which implies that x corresponds to the identity of $\mathrm{Conn}(F(\mathcal{R}))$. Therefore, the map from $\mathrm{Conn}(F(\mathcal{R}))$ to $\mathrm{Conn}(F(\mathcal{M}))$ is an isomorphism. □

3.5. Open problems on stratified and non-stratified operations. Let us look at an example of a stratified operation that is neither fully stratified nor a parallel product.

Example 3.22. Consider the operation $F(\mathcal{M}) = T_w(\mathcal{M})$, the general twist of [10]. We can take $A = \{0,1\}$ (or to be the colors "red" and "white"), and the flags of $F(\mathcal{M})$ consist of pairs (c,Φ) where c is the color of Φ. Then the definition of the action of each s_i is given in Section 4.3 of [10], and each s_i swaps colors. Also, F is stratified, but neither fully stratified nor a parallel product.

We naturally wonder whether T_w is connection-preserving. A simple example shows that it need not be.

Proposition 3.23. *There is a twist operator T_w that is not connection-preserving.*

Proof. Let $\mathcal{M}$ be *Krughoff's cube* (see [10, Section 4.1]). There is a twist operator T_w such that $T_w(\mathcal{M})$ is reflexible (the trivial extension of a cube). Both $\mathcal{M}$ and $T_w(\mathcal{M})$ have 96 flags. Since $T_w(\mathcal{M})$ is reflexible, $|\mathrm{Conn}(T_w(\mathcal{M}))| = 96$. On the other hand, $\mathcal{M}$ is not reflexible, so $\mathrm{src}(\mathcal{M})$ has more flags than $\mathcal{M}$. It follows that $|\mathrm{Conn}(T_w(\mathrm{src}(\mathcal{M})))| > |\mathrm{Conn}(T_w(\mathcal{M}))|$. □

Problem 3.24. *Describe the necessary conditions for a stratified operation to be connection-preserving.*

With such a wide-ranging list of stratified operations, we start to wonder which operations are *not* stratified. The following result helps us demonstrate that some operations are not stratified. Let $|\mathcal{M}|$ denote the number of flags of a maniplex $\mathcal{M}$.

Proposition 3.25. *Suppose that F is a maniplex operation such that, if $\mathcal{M}$ is finite, then $F(\mathcal{M})$ is finite. Furthermore, suppose that $|F(\mathcal{M})|/|\mathcal{M}|$ is an unbounded function of $\mathcal{M}$ for finite maniplexes. Then F is not stratified.*

Proof. Suppose F is stratified and that $\mathcal{M}$ is finite. Then since the projection from the flag set of $F(\mathcal{M})$ to A is surjective, and $F(\mathcal{M})$ is finite when $\mathcal{M}$ is finite, this implies that A is finite. On the other hand, $|F(\mathcal{M})| \leqslant |\mathcal{M}| \cdot |A|$, and so $|F(\mathcal{M})|/|\mathcal{M}| \leqslant |A|$, and so $|F(\mathcal{M})|/|\mathcal{M}|$ is bounded. □

Example 3.26. Consider the operation $F(\mathcal{M}) = 2^{\mathcal{M}}$, first introduced in [9] for incidence complexes, and adapted in its dual form for maniplexes in [10]. Then F is not stratified – it yields $|F(\mathcal{M})|/|\mathcal{M}| = 2^v$, where v is the number of vertices of $\mathcal{M}$, and this is unbounded.

Problem 3.27. *Other than Proposition* 3.25*, what other conditions guarantee that an operation F is not stratified?*

Problem 3.28. *Even though $2^{\mathcal{M}}$ is not stratified, is it connection-preserving?*

4. Pyramids over equivelar toroidal maps

In this section we use the results of Section 3 to classify the smallest reflexible covers of the pyramids over the equivelar toroidal maps. To do this we will make use of the classifications of the symmetry types of equivelar toroidal maps in Theorems 7 and 8 of [23] (see also [3]), the classification of the minimal regular covers of equivelar toroidal maps obtained by Drach and Mixer in [11], and the classification of the minimal regular covers of the pyramids over the regular toroidal maps in [34] obtained by Pellicer and Williams.

4.1. Connection Groups of Pyramids Over Regular Toroidal Maps. First, we briefly summarize the main results from Pellicer and Williams' [34].

The main technique of that paper was to represent the connection group of a polytope as (isomorphic to) a subgroup of the wreath product of $\mathrm{Aut}(\mathcal{P})$ with S_k, where k is the number of flag orbits of $\mathcal{P}$, i.e. as a subgroup of $\Delta := S_k \wr_{[k]} \mathrm{Aut}(\mathcal{P}) = S_k \ltimes \mathrm{Aut}(\mathcal{P})^k$. To do this, we define an injective map $\iota : \mathrm{Conn}(\mathcal{P}) \hookrightarrow \Delta$ as follows. Let G be the symmetry type graph of $\mathcal{P}$ (see [7]) with flag graph $FG(\mathcal{P})$, and T a spanning tree of G. The lifts of T to $FG(\mathcal{P})$ are *chunks*; each flag lies in precisely one chunk, and each chunk contains exactly one flag from each orbit O_i of Ω under the action of $\mathrm{Aut}(\mathcal{P})$, and consequently the chunks form a partition on the flags of $\mathcal{P}$. Let Φ_1 be a representative of orbit O_1, and let $\Phi_i \in O_i$ be the representatives of the remaining orbits in the chunk containing Φ_1. Let $\omega \in \mathrm{Conn}(\mathcal{P})$. Then for each i there is a unique element $\alpha_i \in \mathrm{Aut}(\mathcal{P})$ taking Φ_1 to the representative of O_1 in the chunk containing Φ_i^ω, and an element $\sigma \in S_k$ corresponding to the permutation on the indices of the orbits induced by ω. We thus define $\iota : \mathrm{Conn}(\mathcal{P}) \to \Delta$ by

$$\iota(\omega) := (\sigma, [\alpha_1, \alpha_2, \ldots, \alpha_k]).$$

This map is an injective homomorphism ([34, Prop. 4.3]) and thus $|\mathrm{Conn}(\mathcal{P})| \leqslant |\mathrm{Aut}(\mathcal{P})|^k \cdot k!$ ([34, Lemma 4.4]). We summarize these results with the following lemma.

Lemma 4.1. *Let $\mathcal{P}$ be a polytope with automorphism group $\mathrm{Aut}(\mathcal{P})$, and let k be the number of orbits in $F(\mathcal{P})$ under the action of $\mathrm{Aut}(\mathcal{P})$. Then $\mathrm{Conn}(\mathcal{P})$ is isomorphic to a subgroup of $S_k \wr_{[k]} \mathrm{Aut}(\mathcal{P})$.*

The pyramid operation is a fully stratified operation on a regular polytope $\mathcal{P}$ of rank n (here viewed as a rooted maniplex), wherein the indexing set for the orbits in the automorphism group of the pyramid over $\mathcal{P}$ correspond to the strata (but are shifted by 1 from Section 3.1 due to the differences between the indexing sets; those used in [34] reflected the standard notation for S_k). This is actually quite nice, because in general the set of strata need not correspond to orbits. For example, the prism operation over regular n-polytopes is stratified, and there are $2n + 2$ strata formed by the construction, but at most $n + 1$ orbits. In the context of the pyramids over the equivelar toroidal maps, the pyramiding operation is a stratified operation on 3-maniplexes that produces a 4-maniplex. Following Definition 3.6, we thus have $k = 5$ strata. If S is the set of generators of W_4, then the action σ_i of s_i on A is the first component of $\iota(s_i)$. If Φ is in orbit a, then $\phi(a, s_i)$ is the (induced) left action of α_a on a flag $(1, \Phi) \sim \Phi_1$ described above. In other words, it is the a-th component of the second component of $\iota(s_i)$. Here we are taking strong advantage of the identification of the automorphism group of the regular toroidal base with its connection group.

4.2. The Regular Covers of the Equivelar Toroidal Maps. In [23] a classification of the equivelar toroidal maps is described. All such maps are quotients of the regular Euclidean tessellations $\{4, 4\}$, $\{3, 6\}$ and $\{6, 3\}$. Since the last two families are dual to each other, it suffices to confine our discussion to the classification of the equivelar toroidal maps of Schläfli type $\{4, 4\}$ and $\{3, 6\}$. Classes of toroidal maps are organized according to the number of orbits k, and by two translation vectors that generate the translation subgroup of the corresponding universal regular polytope needed to form the equivelar toroidal map as its quotient. (But note that there are many equivalent choices for generating a particular translation subgroup.) The two-orbit equivelar toroidal maps are further classified according to which adjacent flags belong to the same orbit, i.e. none of the adjacent flags belong to the same orbit for a two-orbit map labeled 2, while a two-orbit map in which the 0-adjacent and 2-adjacent flags belong to the same orbit is denoted $2_{0,2}$ (see also [24, 21, 31]). The classification of equivelar toroidal maps of type $\{3,6\}$ requires only the number of orbits and the two translation vectors.

For the $\{4,4\}$ equivelar toroidal maps, it is convenient to start with a regular tessellation of type $\{4,4\}$ whose vertices correspond to the Gaussian integer lattice in the complex plane given by $\mathbb{Z}[i]=\{a+bi : a,b\in\mathbb{Z}\}$. Thus vectors corresponding to translational symmetries of the tiling may be written as elements of $\mathbb{Z}[i]$, and any subgroup of translational symmetries will be determined by two linearly independent vectors (a,b) and (c,d), which in turn can be represented by the Gaussian integers $\alpha=a+bi$ and $\beta=c+di$, respectively. Every equivelar toroidal map of type $\{4,4\}$ is a quotient of $\{4,4\}$ by such a translational subgroup.

The $\{3,6\}$ equivelar toroidal maps may be described similarly, this time by associating the vertices of the tiling $\{3,6\}$ with the Eisenstein integers $\mathbb{Z}[\omega]$ where ω is the sixth root[(1)] of unity $\frac{1+i\sqrt{3}}{2}$. We may associate to each proper translational subgroup of the symmetries of $\{3,6\}$ a pair of linearly independent vectors with (a,b) and (c,d) relative to the basis $(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})$. We may then encode these vectors as Eisenstein integers $\alpha=a+b\omega$ and $\beta=c+d\omega$ for ease in computing minimal regular covers, noting that $\omega^2=\omega-1$. Every equivelar tiling of Schläfli type $\{3,6\}$ arises as a quotient by such a translational subgroup of the regular polytope $\{3,6\}$.

Throughout what follows, σ is either i or ω as appropriate. Following [11], we define $N(\alpha)=\alpha\overline{\alpha}$, where $\overline{a+b\sigma}$ is defined to be $a+b\overline{\sigma}$ with $\overline{\sigma}$ the usual complex conjugate of σ. In particular, $N(1+i)=2$ and $N(1+\omega)=3$.

Traditionally (cf. [27, 37]) the regular toroidal maps are written as $\{p,q\}_{(n,0)}$ or $\{p,q\}_{(n,n)}$, where $n\in\mathbb{Z}$, with the subscript $(n,0)$ indicating that the translational subgroup is generated by $(n,0)$ and $(0,n)$, and the subscript (n,n) indicated that the translational subgroup is generated by (n,n) and $(-n,n)$ in the $\{4,4\}$ case and (n,n) and $(-n,2n)$ in the $\{3,6\}$ case (written with respect to the basis of the lattice). If we let $\mathcal{T}$ be our regular tiling of the plane of type $\{3,6\}$ or $\{4,4\}$, then mimicking the classical notation for these toroidal maps we may write $\mathcal{T}_\eta$ to denote a regular toroidal map where $\eta=n$ or $\eta=n+n\sigma$ as appropriate.

In [11], using factorization properties of the Gaussian and Eisenstein integers, it was shown that the minimal regular cover of an equivelar tiling is easily determined from the following result.

Theorem 4.2 ([11, Theorem 3.6]). *Let $\mathcal{T}_{\alpha,\beta}$ be an equivelar toroidal map represented as a quotient of a regular planar tessellation $\mathcal{T}$ by a translation subgroup $\langle\alpha,\beta\rangle$ of the translational symmetry subgroup of $\mathcal{T}$ generated by two non-collinear vectors, corresponding to the complex numbers $\alpha,\beta\in\mathbb{Z}[\sigma]$ with $\alpha=a+b\sigma, \beta=c+d\sigma$. Let $g=\mathrm{GCD}(a,b,c,d)$. Then for $\mathcal{T}_{\alpha,\beta}$ there exists a unique minimal regular covering map $\mathcal{T}_\eta$ with*

$$\eta=\begin{cases}\frac{|ad-bc|}{N(1+\sigma)g}(1+\sigma), & \text{if } \frac{a}{g}\equiv\frac{b}{g} \text{ and } \frac{c}{g}\equiv\frac{d}{g} \bmod N(1+\sigma);\\ \frac{|ad-bc|}{g}, & \text{otherwise.}\end{cases}$$

Moreover, the number K of fundamental regions of $\mathcal{T}_{\alpha,\beta}$ glued together in order to obtain $\mathcal{T}_\eta$ is equal to

$$K=\begin{cases}\frac{|ad-bc|}{N(1+\sigma)g^2}, & \text{if } \frac{a}{g}\equiv\frac{b}{g} \text{ and } \frac{c}{g}\equiv\frac{d}{g} \bmod N(1+\sigma);\\ \frac{|ad-bc|}{g^2}, & \text{otherwise.}\end{cases}$$

(1)Classically, the Eisenstein integers are defined using the third root of unity, but the rings are the same and here we are following the treatment in [11] and [23].

We have collected the classification of regular covers of the equivelar toroidal maps according to their symmetry classes from [23] in Table 1.

Table 1. Minimal regular covers of the equivelar toroidal maps. In the final column, $\mathcal{P}$ is the pyramid over the equivelar map. In this classification we assume that $a > b > 0, c, d > 0$. Note that the generators of the toroidal maps of type $\{4,4\}$ in Class 4 must additionally satisfy $c \geqslant a-b, c \neq 2a, a \neq 2c$ and if $b|a,c$, then $\frac{c}{b} \nmid 1 \pm \frac{a^2}{b^2}$. Generators of the toroidal maps of type $\{3,6\}$ in Class 6 must additionally satisfy $c \geqslant a-b, c \nmid 2a+b$ and if $b \mid a, c$, then $\frac{c}{b} \nmid 1 - \frac{a^2}{b^2}$ and $\frac{c}{b} \nmid 1 + \frac{a}{b} + \frac{a^2}{b^2}$.

Schäfli Type	Class	Generators	Additional Conditions	Regular Cover Type	src($\mathcal{P}$) is a polytope
$\{4,4\}$	1	$(a,0),(0,a)$	none	$(a,0)$	a even
		$(a,a),(-a,a)$	none	(a,a)	$a \geqslant 2$
	2	$(a,b),(-b,a)$	$\frac{a}{g}, \frac{b}{g} \equiv 1 \pmod 2$	$\left(\frac{a^2+b^2}{2g}, \frac{a^2+b^2}{2g}\right)$	always
			else	$\left(\frac{a^2+b^2}{g}, 0\right)$	a and b even
	$2_{0,2}$	$(a,0),(0,b)$	none	$\left(\frac{ab}{g}, 0\right)$	a or b even
		$(a,b),(a,-b)$	$\frac{a}{g}, \frac{b}{g} \equiv 1 \pmod 2$	$\left(\frac{ab}{g}, \frac{ab}{g}\right)$	always
			else	$\left(\frac{2ab}{g}, 0\right)$	always
	2_1	$(a,a)(-b,b)$	none	$\left(\frac{ab}{g}, \frac{ab}{g}\right)$	always
		$(a,b)(b,a)$	$\frac{a}{g}, \frac{b}{g} \equiv 1 \pmod 2$	$\left(\frac{a^2-b^2}{2g}, \frac{a^2-b^2}{2g}\right)$	always
			else	$\left(\frac{a^2-b^2}{g}, 0\right)$	a and b even
	4	$(a,b),(c,0)$	$\frac{a}{g} \equiv \frac{b}{g}, \frac{c}{g} \equiv 0 \pmod 2$	$\left(\frac{bc}{2g}, \frac{bc}{2g}\right)$	always
			else	$\left(\frac{bc}{g}, 0\right)$	b or c even
$\{3,6\}$	1	$(a,0),(0,a)$	none	$(a,0)$	$a > 1$
		$(a,a),(2a,-a)$	none	(a,a)	always
	2	$(a,b),(-b,a+b)$	$\frac{a}{g} \equiv \frac{b}{g} \pmod 3$	$\left(\frac{a^2+ab+b^2}{3g}, \frac{a^2+ab+b^2}{3g}\right)$	always
			else	$\left(\frac{a^2+ab+b^2}{g}, 0\right)$	always
	3	$(a,0),(-c,2c)$	$\frac{a}{g} \equiv 0 \pmod 3$	$\left(\frac{2ac}{3g}, \frac{2ac}{3g}\right)$	always
			else	$\left(\frac{2ac}{g}, 0\right)$	always
		$(a,d),(a+d,-d)$	$\frac{a}{g} \equiv \frac{d}{g} \pmod 3$	$\left(\frac{2ad+d^2}{3g}, \frac{2ad+d^2}{3g}\right)$	always
			else	$\left(\frac{2ad+d^2}{g}, 0\right)$	always
	6	$(a,b),(c,0)$	$\frac{a}{g} \equiv \frac{b}{g}, \frac{c}{g} \equiv 0 \pmod 3$	$\left(\frac{bc}{3g}, \frac{bc}{3g}\right)$	always
			else	$\left(\frac{bc}{g}, 0\right)$	always

4.3. Connection groups of pyramids over equivelar toroidal maps.

In [34] the connection groups of pyramids over the regular toroidal maps were classified, and in particular, it was shown that the connection groups of all pyramids over regular toroidal maps of type $\{3,6\}$ and $\{6,3\}$ are string C-groups [34, Prop. 6.10], as are the connection groups of pyramids over toroidal maps of type $\{4,4\}$, except for the pyramids over toroidal maps of type $\{4,4\}_{(n,0)}$ with n odd [34, Prop. 6.2-6.4]. In summary,

Theorem 4.3. *The smallest reflexible cover of the pyramid over a regular toroidal map is a regular polytope except when the map is of type $\{4,4\}_{(n,0)}$ with n odd.*

Consequently, by Theorem 3.4, it is easy to see that the pyramid over an equivelar toroidal map has a polytopal minimal regular cover except when the underlying map is of type $\{4,4\}_{(n,0)}$ with n odd.

Example 4.4. Consider the 2-orbit equivelar toroidal maps $\{4,4\}_{(6,10),(10,6)}$ and $\{4,4\}_{(4,5),(5,4)}$ of type 2_1. In the former case the GCD of the coefficients is $g=2$, and $a=6, b=10$. Hence $\frac{a}{g}=3\equiv 1 \pmod 2$ and $\frac{b}{g}=5\equiv 1 \pmod 2$, and so the smallest reflexible cover is the polytopal minimal regular cover of $\mathrm{Pyr}(\{4,4\}_{(16,16)})$, which has $2^{56}\cdot 3\cdot 5$ flags. In the latter case, the GCD of the coefficients is 1, so the smallest reflexible cover of $\mathrm{Pyr}(\{4,4\}_{(4,5),(5,4)})$ is the maniplex corresponding to the smallest reflexible cover of $\mathrm{Pyr}(\{4,4\}_{(9,0)})$, which has $2^{12}\cdot 3^{21}\cdot 5$ flags, and is not a polytope.

5. Concluding remarks

Section 4 shows one way to use Theorem 3.9; namely, if we want to know something about $\mathrm{Conn}(F(\mathcal{M}))$, then we can work with $\mathrm{Conn}(F(\mathcal{R}))$ instead, where $\mathcal{R}=\mathrm{src}(\mathcal{M})$. Since the theory of reflexible maniplexes is much more well-developed than the theory of non-reflexible maniplexes, it is often easier to deal with $\mathrm{Conn}(F(\mathcal{R}))$, and working this way is especially appealing for classes of maniplexes where the smallest reflexible covers are well-understood.

There are also good reasons to use Theorem 3.9 in reverse. That is, suppose we would like to know something about $\mathrm{Conn}(F(\mathcal{R}))$ where $\mathcal{R}$ is reflexible, and suppose we know that $\mathcal{M}$ is a maniplex with smallest reflexible cover $\mathcal{R}$. For computational experimentation, working with the smaller maniplex $\mathcal{M}$ will usually be faster.

For example, let us consider the prisms over the regular toroidal maps $\{4,4\}_{(n,0)}$. Using some GAP [13] code written by the first author, we calculated the connection group of the prism over $\{4,4\}_{(n,0)}$ for $n\leqslant 30$ and determined that all such groups were string C-groups. This calculation took about 160 seconds on the first author's computer. Now, since $\{4,4\}_{(n,0)}$ is the smallest reflexible cover of $\{4,4\}_{(n,0),(0,1)}$ (that is, the quotient of the regular tessellation of type $\{4,4\}$ by the group generated by the translations by vectors $(n,0)$ and $(0,1)$), we can also determine the connection group of the prisms over the latter. For $n\leqslant 30$, we again determined that all such groups were string C-groups; this time the calculation took about 3 seconds. In fact, in about 32 seconds we were able to cover $n\leqslant 100$ using the second method. The source code for computing the connection groups of toroidal maps as well as pyramids and prisms over them can be found at `http://www.gabrielcunningham.com/connection.gap`.

6. Appendix

6.1. Products of polytopes. Four different products of polytopes were defined in [15]. Here we will consider the *join product* and show that we can use it to define a stratified operation. The other products also define stratified operations, and the proofs are analogous.

The join product of an n-polytope $\mathcal{P}$ with an m-polytope $\mathcal{Q}$ is an $(m+n+1)$-polytope, denoted $\mathcal{P}\bowtie\mathcal{Q}$. Section 6 of [15] describes the flags as triples (Φ,Ψ,a), where Φ is a flag of $\mathcal{P}$, Ψ is a flag of $\mathcal{Q}$, and $a=(a_0,a_1,\ldots,a_{m+n+1})$ is a tuple of elements from $\{1,2\}$ such that 2 appears $m+1$ times. This determines the faces of a flag as follows. For any flag Δ, let $\Delta(i)$ denote the i-face of Δ. Consider the flag $\Lambda=(\Phi,\Psi,a)$.

Then:

$$\Lambda(-1) = (\Phi(-1), \Psi(-1)).$$

$$\Lambda(0) = \begin{cases} (\Phi(0), \Psi(-1)) & \text{if } a_0 = 1, \\ (\Phi(-1), \Psi(0)) & \text{if } a_0 = 2. \end{cases}$$

In general, in order to determine $\Lambda(i+1)$, we take $\Lambda(i)$ (which will look like $(\Phi(j), \Psi(k))$ for some j and k), and if $a_{i+1} = 1$, we increment j; otherwise we increment k.

For the join product of two polytopes, rather than using tuples a, we may as well just record the position of every 2 in a into a subset I of $\{0, \dots, m+n+1\}$. So every flag of $\mathcal{P} \bowtie \mathcal{Q}$ can be represented as a triple $\Lambda = (\Phi, \Psi, I)$, where $I \subset \{0, \dots, n+m+1\}$ with $|I| = m+1$, and then

$$\Lambda(j) = (\Phi(j - |I \cap \{0, \dots, j\}|), \Psi(-1 + |I \cap \{0, \dots, j\}|)).$$

Conversely, every triple of this type determines a flag.

Now let us determine the flag that is j-adjacent to a given $\Lambda = (\Phi, \Psi, I)$. First, suppose that $j \in I$ and $j+1 \notin I$. Then if we compare $\Lambda(j-1)$ to $\Lambda(j+1)$, we see that $\Lambda(j+1)$ has both components different from $\Lambda(j-1)$. Then the section $\Lambda(j+1)/\Lambda(j-1)$ of the j-adjacent flag to Λ must just increase the two components in the opposite order compared to Λ, and so it has $j+1 \in I$ instead of $j \in I$. Thus, if exactly one of j and $j+1$ is in I, we have that $\Lambda^j = (\Phi, \Psi, I \triangle \{j, j+1\})$ (where $I \triangle \{j, j+1\}$ denotes the symmetric difference of those two sets).

If both j and $j+1$ are in I, then in moving from $\Lambda(j-1)$ to $\Lambda(j+1)$, we increase the second component twice, and so the first component remains the same. This implies that Λ^j has the same subset I and the same flag Φ. For some k, the second component of $\Lambda(j-1)$ is $\Psi(k-1)$, the second component of $\Lambda(j)$ is $\Psi(k)$, and the second component of $\Lambda(j+1)$ is $\Psi(k+1)$. Then the only other flag that differs from Λ only in its j-face must use Ψ^k in place of Ψ. In fact, as described above, $k = -1 + |I \cap \{0, \dots, j\}|$, and so

$$\Lambda^j = (\Phi, \Psi^{-1+|I \cap \{0, \dots, j\}|}, I).$$

A similar argument shows that if neither j nor $j+1$ is in I, then

$$\Lambda^j = (\Phi^{j - |I \cap \{0, \dots, j\}|}, \Psi, I).$$

Putting it all together, we get:

$$s_j(\Phi, \Psi, I) = \begin{cases} (r_{j-|I \cap \{0, \dots, j\}|}\Phi, \Psi, I), & \text{if } j, j+1 \notin I, \\ (\Phi, \Psi, I \triangle \{j, j+1\}), & \text{if } |I \cap \{j, j+1\}| = 1, \\ (\Phi, r_{-1+|I \cap \{0, \dots, j\}|}\Psi, I), & \text{if } j, j+1 \in I. \end{cases}$$

Now, suppose $\mathcal{Q}$ is an m-polytope and define an operation F on n-polytopes F by $F(\mathcal{P}) = \mathcal{P} \bowtie \mathcal{Q}$. We can see F as a fully stratified operation by taking A to be all pairs (Ψ, I).

We have shown:

Theorem 6.1. *If $\mathcal{Q}$ is an m-polytope and $\odot$ is one of the four products defined in* [15]*, then the operation F defined by $F(\mathcal{P}) = \mathcal{P} \odot \mathcal{Q}$ is fully stratified.*

6.2. Cleaved polytopes. Given an n-polytope $\mathcal{P}$, its k-th cleaved polytope and its partially k-th cleaved polytope were defined in [32]. They consist of

$$Cl_k(\mathcal{P}) := \{F_{-1}\} \cup \{(F, G) \,:\, F \leqslant G; \operatorname{rank}(F) \leqslant k-1; \operatorname{rank}(G) \geqslant k\}$$

and

$\widetilde{Cl}_k(\mathcal{P}) := \{F_{-1}, F_{n-1}\} \cup \{(F,G) : F \leqslant G; 0 \leqslant \mathrm{rank}(F) \leqslant k-1; k \leqslant \mathrm{rank}(G) \leqslant n-1\}$,

respectively, where F_{-1} is less than all other faces, F_{n-1} is greater than all other faces of $\widetilde{Cl}_k(\mathcal{P})$, and in both polytopes $(F,G) \leqslant (F',G')$ if and only if $F' \leqslant F \leqslant G \leqslant G'$ in $\mathcal{P}$.

Theorem 6.2. *Cl_k is fully stratified.*

Proof. According to [32, Remark 5], the set A required is the set of vectors with k entries "$-$" and $n-k$ entries "$+$".

Given a vector $\bar{a} := (a_1, \dots, a_n)$ with $a_i \in \{+,-\}$, let $\bar{a}^{i,j} := (a'_1, \dots, a'_n)$ be the vector in $\{+,-\}^n$ with $a_k = a'_k$ if and only if $k \notin \{i,j\}$. Furthermore, for $j \geqslant 2$ let $[-](\bar{a}, j)$ and $[+](\bar{a}, j)$ denote the number of entries "$-$" and the number of entries "$+$" in $\{a_1, \dots, a_{j-1}\}$, respectively. Clearly, $[-](\bar{a}, j) + [+](\bar{a}, j) = j-1$. Then

$$s_0(\bar{a}, \Phi) = \begin{cases} (\bar{a}, r_{k-1}\Phi) & \text{if } a_1 = -, \\ (\bar{a}, r_k \Phi) & \text{if } a_1 = +, \end{cases}$$

and if $j \geqslant 2$ then

$$s_j(\bar{a}, \Phi) = \begin{cases} (\bar{a}^{j,j+1}, \Phi) & \text{if } a_j \neq a_{j+1}, \\ (\bar{a}, r_{k-2-[-](\bar{a},j)}\Phi) & \text{if } a_j = a_{j+1} = -, \\ (\bar{a}, r_{k+1+[+](\bar{a},j)}\Phi) & \text{if } a_j = a_{j+1} = +. \end{cases} \qquad \square$$

Theorem 6.3. *$\widetilde{Cl}_k$ is fully stratified.*

Proof. Same as in the previous theorem, except that:

- $\bar{a}$ is a vector with only $n-2$ entries, $k-1$ of which are "$-$" and the remaining $n-k-1$ are "$+$",
- s_{n-2} is defined as

$$s_{n-2}(\bar{a}, \Phi) = \begin{cases} (\bar{a}, r_0 \Phi) & \text{if } a_{n-2} = -, \\ (\bar{a}, r_{n-1}\Phi) & \text{if } a_{n-2} = +. \end{cases} \qquad \square$$

Note that $\widetilde{Cl}_k$ is our first example of a fully stratified operation that reduces the rank of the input.

Problem 6.4. *Describe other fully stratified operations $F : \mathsf{M}_n \to \mathsf{M}_m$ with $m < n$.*

6.3. k-bubble of a polytope. The *k-bubble of* $\mathcal{P}$, denoted $[\mathcal{P}]_k$, can be viewed as a generalization of vertex truncation (in a different way from cleaved polytopes). As a poset, $[\mathcal{P}]_k$ is defined as follows. (See Def. 2.1 and Lemma 2.6 from [20].)

Definition 6.5. *We define a ranked poset $[\mathcal{P}]_k$ of the same rank as $\mathcal{P}$ layer by layer.*

1. *For $0 \leqslant i \leqslant k-1$, the set of i-faces of $[\mathcal{P}]_k$ is the same as the set of i-faces of $\mathcal{P}$.*
2. *Each k-face is a pair (F,G), where F is a k-face of $\mathcal{P}$ and G is an incident $(k+1)$-face of $\mathcal{P}$.*
3. *For $k+1 \leqslant i \leqslant n-1$, there are two types of i-faces; i-faces H of $\mathcal{P}$, and pairs (F,G) of a k-face and an incident $(i+1)$-face.*

We then adjoin a minimal and maximal element.

The partial order can be defined as follows:

1. *For faces of $[\mathcal{P}]_k$ that correspond to faces of $\mathcal{P}$, the partial order is the same.*
2. *$(F,G) \leqslant (F',G')$ if and only if $F = F'$ and $G \leqslant G'$.*

(3) $H \leqslant (F, G)$ *if and only if* $H \leqslant F$.
(4) $(F, G) \leqslant H$ *if and only if* $G \leqslant H$.

Theorem 6.6. *The operation that takes $\mathcal{P}$ to $[\mathcal{P}]_k$ is fully stratified.*

Proof. Lemma 2.8 of [20] gives a description of the flags of $[\mathcal{P}]_k$. Every flag has the form

$$(F_{-1}, F_0, \ldots, F_{k-1}, (F_k, F_{k+1}), \ldots, (F_k, F_i), F_i, \ldots, F_n),$$

where $(F_{-1}, \ldots, F_n)$ is a flag of $\mathcal{P}$ and $k+1 \leqslant i \leqslant n$. Thus, we may specify a flag of $[\mathcal{P}]_k$ as (i, Φ), and all such pairs define a flag of $[\mathcal{P}]_k$.

From this, it is straightforward to prove that:

$$s_j(i, \Phi) = \begin{cases} (i, r_j \Phi) & \text{if } 0 \leqslant j \leqslant k-1 \text{ or } j \geqslant i+1, \\ (i, r_{j+1} \Phi) & \text{if } k \leqslant j \leqslant i-2, \\ (i-1, \Phi) & \text{if } j = i-1, \\ (i+1, \Phi) & \text{if } j = i, \end{cases}$$

showing that the operation is fully stratified. □

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