ALGEBRAIC COMBINATORICS

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Four-Valent Oriented Graphs of Biquasiprimitive Type

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ABSTRACT Let $\mathcal{OG}(4)$ denote the family of all graph-group pairs (Γ, G) where Γ is 4-valent, connected and G-oriented (G-half-arc-transitive). Using a novel application of the structure theorem for biquasiprimitive permutation groups of the second author, we produce a description of all pairs $(\Gamma, G) \in \mathcal{OG}(4)$ for which every nontrivial normal subgroup of G has at most two orbits on the vertices of Γ, and at least one normal subgroup has two orbits. In particular we show that G has a unique minimal normal subgroup N and that $N \cong T^k$ for a simple group T and $k \in \{1, 2, 4, 8\}$. This provides a crucial step towards a general description of the long-studied family $\mathcal{OG}(4)$ in terms of a normal quotient reduction. We also give several methods for constructing pairs (Γ, G) of this type and provide many new infinite families of examples, covering each of the possible structures of the normal subgroup N.

1. Introduction

All graphs considered in this paper are simple, undirected and finite. A graph Γ is said to be *G-oriented* with respect to some group $G \leqslant \operatorname{Aut}(\Gamma)$, and some edge-orientation Δ, if G acts transitively on the vertices and edges of Γ and G preserves the edge-orientation Δ. Thus, although the group G will act transitively on the vertices and edges, it will not be transitive on the arcs of Γ, where an *arc* is an ordered pair of adjacent vertices. Conversely, any graph Γ admitting a vertex- and edge- but not arc-transitive group G of automorphisms will admit some G-invariant edge-orientation: simply take one of the two G-arc-orbits Δ and orient an edge $\{\alpha, \beta\}$ from α to β if and only if $(\alpha, \beta) \in \Delta$.

Every G-oriented graph necessarily has even valency and all connected components of a G-oriented graph are pairwise isomorphic. It is thus natural to restrict attention to G-oriented graphs which are connected. For each even integer $m \geqslant 2$, we let $\mathcal{OG}(m)$ denote the family of graph-group pairs (Γ, G) where Γ is connected, m-valent and G-oriented. Throughout this paper we will represent G-oriented graphs as pairs (Γ, G) contained in some family $\mathcal{OG}(m)$. Our notation suppresses the orientation Δ. Technically therefore the pairs $(\Gamma, G) \in \mathcal{OG}(m)$ are G-orientable rather than G-oriented, as the orientation Δ has not been chosen yet. However this orientation is determined by the pair (Γ, G) up to possibly reversing the orientation of all edges. The term G-oriented graph was suggested by B. D. Mckay and has been used, for example in [1, 2, 3, 20] for pairs in $\mathcal{OG}(m)$ with an understanding that this choice

between two orientations is yet to be made. A detailed discussion of G-oriented graphs is given in [3, Section 1].

It is easy to see that the family $\mathcal{OG}(2)$ consists only of oriented cycle graphs. On the other hand, study of the family $\mathcal{OG}(4)$ has been an active area of research for several decades and has taken a number of different directions (especially because of its connection with the embedding of graphs into Riemann surfaces). For a good summary of this research up to 1998 see [12], for a more recent overview see [3, Section 2].

A particularly useful tool for studying $\mathcal{OG}(4)$ was given in [11], where several important combinatorial parameters were defined for graphs in this family based on certain cyclic subgraphs called *G-alternating cycles*. This led to the formulation of an approach to studying $\mathcal{OG}(4)$ by considering various quotients defined in terms of the G-alternating cycles, see [14].

The combined results of [11, 14] provide a complete classification of some subfamilies of $\mathcal{OG}(4)$ and prove that pairs $(\Gamma, G) \in \mathcal{OG}(4)$ not contained in these subfamilies are covers of other members of $\mathcal{OG}(4)$ satisfying certain combinatorial conditions. In particular, this approach naturally identifies two subfamilies of $\mathcal{OG}(4)$ as "alternating-cycle-basic" in the sense that all 4-valent G-oriented graphs (other than those already classified) are covers of these basic members. However the analysis in [11, 14] provided no tools for studying the "basic" graphs relative to this reduction.

More recently, a new framework for studying the family $\mathcal{OG}(4)$ was proposed in [3] and developed further in [1, 2]. This new approach aims to analyse $\mathcal{OG}(4)$ using a normal quotient reduction, a method which has been successfully used to study other families of graphs with prescribed symmetry conditions, see for instance [15, 22, 23], but has never been applied to oriented graphs. The aim of this approach (explained in detail below) is to describe the family $\mathcal{OG}(4)$ in terms of graph quotients arising from normal subgroups of the groups contained in this family. In particular, it is again possible to identify three subfamilies of $\mathcal{OG}(4)$ which are "normal-quotient-basic" in the sense that all pairs $(\Gamma, G) \in \mathcal{OG}(4)$ are *normal* covers of at least one of these basic pairs (see Section 2.2).

It is likely that these two approaches may converge in a significant proportion of cases. The quotient graph $\Gamma_{\mathcal{B}}$ constructed in [14, Section 3] from a given pair $(\Gamma, G) \in \mathcal{OG}(4)$, related to the G-alternating cycles, has been studied again recently by Ramos Rivera and Šparl [26, Construction 5.4]. Provided a mild condition on parameters is satisfied (the attachment number should be less than the radius), they prove that $\Gamma_{\mathcal{B}}$ is a normal quotient [26, Theorem 5.6] and hence may be studied using the powerful theory developed in [1, 2, 3], supplemented by the results of this paper.

In this paper we answer [3, Problem 1.2] and provide a description of the pairs $(\Gamma, G) \in \mathcal{OG}(4)$ of biquasiprimitive type (one of the three families of pairs defined to be "basic" with respect to normal quotients). Our solution provides an important step towards a description of $\mathcal{OG}(4)$ in terms of normal quotients. For a detailed description of this programme and definitions of all basic pairs see Section 2.2.

BIQUASIPRIMITIVE BASIC PAIRS. A pair $(\Gamma, G) \in \mathcal{OG}(4)$ is said to be basic of *biquasiprimitive type* if every nontrivial normal subgroup of G has at most two orbits on the vertices of Γ, and G contains at least one normal subgroup, say N, with exactly two orbits on the vertices of Γ. In such a case, it is easy to see that Γ is bipartite: since Γ is connected there is an edge joining vertices in different N-orbits, and since G normalises N and Γ is G-edge-transitive, each edge joins vertices in different N-orbits. Thus the two orbits of N form a bipartition of Γ.

It follows that there is an index two subgroup G^+ of G which fixes the two parts of the bipartition of Γ setwise. The main result of this paper is the following theorem

which describes the biquasiprimitive basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ in a manner analogous to [3, Theorem 1.3] for the quasiprimitive case.

THEOREM 1.1. *Suppose that* $(\Gamma, G) \in \mathcal{OG}(4)$ *is basic of biquasiprimitive type. Then* G *has a unique minimal normal subgroup* $N = \mathrm{soc}(G)$*, and* N *is contained in a unique intransitive index 2 subgroup* $G^+ \leqslant G$*. Furthermore,* $N \cong T^k$ *where* T *is a finite simple group and exactly one of the following holds:*

(a) *T is abelian and $k \leqslant 2$, or*

(b) *T is nonabelian, $k \in \{1, 2, 4\}$, and N is the unique minimal normal subgroup of G^+, or*

(c) *T is nonabelian, $k = 2\ell$ with $\ell \in \{1, 2, 4\}$, G^+ has exactly two minimal normal subgroups each isomorphic to T^{ℓ}, and N is the direct product of these two subgroups.*

Moreover, there are infinitely many biquasiprimitive basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ described by each of the cases (a)–(c) *and each value of k in each case.*

The first part of this paper establishes that cases (a)–(c) of Theorem 1.1 must hold in two steps. In Section 3 we show that if $(\Gamma, G) \in \mathcal{OG}(4)$ is basic of biquasiprimitive type then G has a unique minimal normal subgroup N and one of the three cases (a), (b) or (c) holds for some $k \geqslant 1$. For this we use the structure theorem for biquasiprimitive groups given in [24]. Then in Section 4 we use combinatorial arguments to obtain the various possibilities for the value of k (the number of simple direct factors of $\mathrm{soc}(G)$) in each case.

In the second part of this paper (Section 5), we provide two detailed methods for constructing basic biquasiprimitive pairs (Section 5.1), and we also discuss the possibility of a third method for doing this which uses the separated box product of two digraphs as studied in [19], and which produces some, but not all, examples (Section 5.2). We conclude the paper by providing an infinite family of basic pairs for each of the cases described in Theorem 1.1, and for each possible value of k. This proves the final assertion of Theorem 1.1.

2. PRELIMINARIES

Unless otherwise stated we will let $V\Gamma$, $E\Gamma$ and $A\Gamma$ denote the vertex-, edge-, and arc-set of a given graph Γ (an *arc* is an ordered pair of adjacent vertices). Given a vertex $\alpha \in V\Gamma$ we let $\Gamma(\alpha)$ denote the neighbourhood of α in Γ. For fundamental graph-theoretic concepts we refer the reader to [7], and for group-theoretic concepts not defined here, please refer to [25].

Given a group G acting on a set X, we will always let G^X denote the subgroup of $\mathrm{Sym}(X)$ induced by the group G. Given elements $g \in G$ and $x \in X$, we let x^g denote the image of x under g. A permutation group G^X is said to be *semiregular* if only the identity element of G fixes a point in X, and is said to be *regular* if it is semiregular and transitive.

2.1. *G*-ORIENTED GRAPHS. If Γ is a G-oriented graph then the group G is transitive on the vertices and edges but not on the arcs of Γ. It follows that the group G has two orbits on the arc set of Γ and these two orbits are paired. (Every arc (u, v) in one orbit will have its reverse arc (v, u) in the other orbit.) Either of these two G-orbits on the arc set of Γ naturally gives rise to a G-invariant orientation of the edges of Γ: simply take any arc (u, v) of Γ and then orient each edge $\{x, y\}$ from x to y if and only if $(u, v)^g = (x, y)$ for some $g \in G$.

Given a pair $(\Gamma, G) \in \mathcal{OG}(4)$, any vertex $v_0 \in V\Gamma$ and any G-invariant orientation of $E\Gamma$, we will say that an *in-neighbour* of v_0 is any neighbour u for which the edge

$\{v_0, u\}$ is oriented from u to v_0, and an *out-neighbour* of v_0 is any neighbour w for which the edge $\{v_0, w\}$ is oriented from v_0 to w. The vertex v_0 will always have exactly two in-neighbours and two out-neighbours, and the stabiliser G_{v_0} of the vertex v_0 will always have two orbits of length two on the neighbourhood of v_0 corresponding to the in-neighbours and out-neighbours of v_0 with respect to the given orientation.

Given a connected, 4-valent, G-vertex-transitive graph Γ, we may show that $(\Gamma, G) \in \mathcal{OG}(4)$ by showing that G_{v_0} has two orbits of size 2 on $\Gamma(\alpha)$, and that no element of G can reverse an edge of Γ.

An *oriented s-arc* of a G-oriented graph with a fixed G-invariant orientation is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ of Γ, such that for each $i \in \{0, \ldots, s-1\}$, v_i and v_{i+1} are adjacent, and each edge $\{v_i, v_{i+1}\}$ is oriented from v_i to v_{i+1}. We will make use of the following important fact concerning oriented s-arcs of G-oriented graphs, the proof of which can be found in the first part of the proof of [3, Lemma 6.2].

LEMMA 2.1. *Let $(\Gamma, G) \in \mathcal{OG}(4)$ and let $s \geqslant 1$ be the largest integer such that G acts transitively on the oriented s-arcs of Γ. Then G acts regularly on the oriented s-arcs of Γ.*

Now let $(\Gamma, G) \in \mathcal{OG}(4)$ and take a vertex $\alpha \in V\Gamma$. Let s be as in the statement of Lemma 2.1 and consider an oriented s-arc $(\alpha, v_1, \ldots, v_s)$ of Γ. Since G is regular on the oriented s-arcs of Γ, it follows that the vertex-stabiliser G_α is regular on the oriented s-arcs starting at α. From this it follows that G_α has order 2^s, and for each i with $0 \leqslant i < s$, the subgroup $G_{\alpha, v_1, \ldots, v_{s-i}}$ has order 2^i. In particular, $|G_{\alpha, v_1, \ldots, v_{s-1}}| = 2$, and the stabiliser of a vertex G_α is a 2-group.

Note that the vertex stabilisers of pairs $(\Gamma, G) \in \mathcal{OG}(4)$ have been studied in several papers. See for instance [13, 18].

2.2. NORMAL QUOTIENTS. Given a pair $(\Gamma, G) \in \mathcal{OG}(4)$ and a normal subgroup N of G, we define a new graph Γ_N called a *G-normal-quotient* of Γ. The vertices of Γ_N are the N-orbits on the vertices of Γ, with an edge between two distinct N-orbits $\{B, C\}$ in Γ_N if and only if there is an edge of the form $\{\alpha, \beta\}$ in Γ, with $\alpha \in B$ and $\beta \in C$. The group G induces a group $G_N = G/K$ of automorphisms of Γ_N, where K is the kernel of the G-action on Γ_N. By definition $N \leqslant K$, and hence the K-orbits are the same as the N-orbits so $\Gamma_K = \Gamma_N$. However K may be strictly larger than N.

If (Γ_N, G_N) is itself a member of $\mathcal{OG}(4)$, that is, Γ_N is a 4-valent G_N-oriented graph, then Γ is said to be a *G-normal cover of Γ_N*. In general however, the pair (Γ_N, G_N) need not lie in $\mathcal{OG}(4)$, and the various possibilities for such normal quotient pairs (Γ_N, G_N) were identified in [3, Theorem 1.1]. In particular, it was proved that for any $(\Gamma, G) \in \mathcal{OG}(4)$, and any nontrivial normal subgroup N of G, either (Γ_N, G_N) is also in $\mathcal{OG}(4)$ and Γ is a G-normal cover of Γ_N, or Γ_N is isomorphic to K_1, K_2 or a cycle C_r, for some $r \geqslant 3$. A pair (Γ_N, G_N) where Γ_N is isomorphic to one of K_1, K_2 or C_r is defined to be *degenerate*, while a pair $(\Gamma, G) \in \mathcal{OG}(4)$ for which (Γ_N, G_N) is degenerate relative to every non-trivial normal subgroup N of G is defined to be *basic*.

Since [3, Theorem 1.1] ensures that every member of $\mathcal{OG}(4)$ is a normal cover of a basic pair, this result suggests a framework for studying the family $\mathcal{OG}(4)$ using normal quotient reduction. The goal of this framework is to improve understanding of this family by developing a theory to describe the basic pairs in $\mathcal{OG}(4)$, and subsequently developing a theory to describe the G-normal covers of these basic pairs.

Work in this direction was initiated in [3] where the basic pairs were further divided into three types and the basic pairs of quasiprimitive type were analysed. A pair $(\Gamma, G) \in \mathcal{OG}(4)$ is said to be basic of *quasiprimitive type* if all G-normal quotients Γ_N of Γ are isomorphic to K_1. This occurs precisely when all non-trivial normal subgroups of G are transitive on the vertices of Γ. A permutation group with

this property is said to be quasiprimitive, and there is a general structure theorem available for quasiprimitive groups analogous to the O'Nan–Scott Theorem for primitive permutation groups in [22]. Using this tool, as well as combinatorial properties of the family $\mathcal{OG}(4)$, it was shown [3, Theorem 1.3] that if $(\Gamma, G) \in \mathcal{OG}(4)$ is basic of quasiprimitive type, then G has a unique minimal normal subgroup $N \cong T^k$ where T is a nonabelian finite simple group and $k \leqslant 2$.

Of course, every pair $(\Gamma, G) \in \mathcal{OG}(4)$ will have at least one normal quotient Γ_N isomorphic to K_1 since we may take the quotient with respect to the full group G. If the only normal quotients of a pair $(\Gamma, G) \in \mathcal{OG}(4)$ are the graphs K_1 or K_2, and Γ has at least one G-normal quotient isomorphic to K_2, then (Γ, G) is basic of biquasiprimitive type. The group G here is biquasiprimitive: it is not quasiprimitive but each nontrivial normal subgroup has at most two orbits. Again, there is a structure theorem for biquasiprimitive groups available in [24].

The basic pairs in $\mathcal{OG}(4)$ which are neither quasiprimitive nor biquasiprimitive must have at least one normal quotient isomorphic to a cycle graph C_r, and hence are said to be of *cycle type.* Work towards describing the basic pairs of cycle type was initiated in [2] where several important families of these graphs, which have already been discussed in the literature, were analysed from a normal quotient point of view. A more general analysis of these pairs was done in [1], however further work is required to understand this type.

The above discussion outlining the three types of basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ is summarised in Table 1. This table also includes references to the papers where the corresponding basic pairs were previously studied. The objective of this paper is to describe the basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ of biquasiprimitive type, several families of which were constructed in [20].

Before proceeding, we note that G-oriented graphs may also be divided into two types depending on the action of their full automorphism group. For a pair $(\Gamma, G) \in \mathcal{OG}(m)$, the graph Γ is said to be *half-arc-transitive* if $(\Gamma, \mathrm{Aut}(\Gamma)) \in \mathcal{OG}(m)$. If this is not the case then $\mathrm{Aut}(\Gamma)$ preserves neither of the G-orientations of Γ, and therefore $\mathrm{Aut}(\Gamma)$ is transitive on the arc-set of Γ, that is, Γ is *arc-transitive.* The 4-valent half-arc-transitive graphs form a heavily studied yet elusive subfamily of $\mathcal{OG}(4)$, see [12, 13, 16, 26] for instance.

For a pair $(\Gamma, G) \in \mathcal{OG}(4)$ and a nondegenerate normal quotient (Γ_N, G_N), the graph Γ_N may be either half-arc-transitive or arc-transitive, [2, Proposition 3.1]. If Γ is arc-transitive and $N_{\mathrm{Aut}(\Gamma)}(N)$ is arc-transitive, then Γ_N is definitely arc-transitive, see [21, Lemma 1.1]. On the other hand, by considering the census of all 4-valent G-oriented graphs of order at most 1000 (see [17]), one can check that the half-arc-transitive graph HAT[168,9] has two normal quotients, one of which is the half-arc-transitive graph HAT[84,1], the other being an arc-transitive 4-valent graph of order 12. These normal quotients both arise from subgroups of the cyclic normal subgroup related to G-alternating cycles described in [26, Theorem 5.2, Theorem 5.6].

Half-arc-transitivity is thus difficult to discern when studying $\mathcal{OG}(4)$ via normal quotients. Still, once we have a good description of all basic pairs in $\mathcal{OG}(4)$ it would be interesting to study which of these pairs are arc-transitive and yet are normal quotients of half-arc transitive graphs.

2.3. Bi-Cayley Graphs. A *bi-Cayley graph* Γ is a graph which admits a semiregular group of automorphisms H with two orbits on the vertex set of Γ. These graphs are important for our purposes as for many of the pairs $(\Gamma, G) \in \mathcal{OG}(4)$ studied in this paper, the group G will have a normal subgroup N contained in G^+ which acts

Basic Type	Possible Γ_N for $1 \neq N \lhd G$	Conditions on G-action on vertices	Reference
Quasiprimitive	K_1 only	quasiprimitive	[3]
Biquasiprimitive	K_1 and K_2 only	biquasiprimitive	–
Cycle	At least one C_r $(r \geqslant 3)$	at least one quotient action D_{2r} or $\mathbb{Z}_r$	[1, 2]

TABLE 1. Types of Basic Pairs $(\Gamma, G) \in \mathcal{OG}(4)$.

semiregularly with two orbits on $V\Gamma$. In such cases, Γ is a bi-Cayley graph and the two N-orbits coincide with the two parts of the bipartition of Γ.

Every bi-Cayley graph of a group H may be constructed in the following way. Let R and L be inverse-closed subsets of H which do not contain the identity, and let S be a subset of H. Define the graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$ to be the graph whose vertex set is the union of the sets $H_0 = \{h_0 : h \in H\}$ and $H_1 = \{h_1 : h \in H\}$ (two copies of the group H), and whose edge set is the union of the *right edges* $\{\{h_0, g_0\} : gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} : gh^{-1} \in L\}$, and the *spokes* $\{\{h_0, g_1\} : gh^{-1} \in S\}$. Note that if Γ is connected then H is generated by $R \cup L \cup S$ (however the converse does not necessarily hold). The group H then acts by right multiplication on the vertices of Γ, and this action is semiregular with two orbits H_0 and H_1. See for instance [5, 27].

3. Biquasiprimitive Basic Pairs: two types.

Suppose now that $(\Gamma, G) \in \mathcal{OG}(4)$ is a basic pair of biquasiprimitive type and recall that this implies that Γ is bipartite. Let X denote the vertex set of Γ with $\{\Delta, \Delta'\}$ the bipartition of X, and let G^+ be the index 2 subgroup of G fixing the two biparts Δ and Δ' setwise. Since Γ is G-vertex-transitive it follows that G^+ is transitive on both Δ and Δ'.

In this section we will begin working towards the proof of Theorem 1.1. We start with a lemma about the intransitive normal subgroups of G.

LEMMA 3.1. *Let $(\Gamma, G) \in \mathcal{OG}(4)$ be basic of biquasiprimitive type, and let X denote the vertex set of Γ. Let G^+ be the subgroup of G of index two with orbits Δ, Δ' (the biparts of X). Then*

   (a) *G^+ is faithful on Δ (and Δ'), and*
   (b) *any non-trivial intransitive normal subgroup N of G must have the sets Δ and Δ' as its two orbits on X. In particular, N is contained in G^+.*

Proof. (a). Let K be the subgroup of G^+ fixing Δ pointwise and suppose that $K \neq 1$, and hence that K acts non-trivially on Δ'. If $g \in G \backslash G^+$ then K^g is the pointwise stabiliser of Δ' in G^+, and hence $K \cap K^g = 1$, so $\langle K, K^g \rangle \cong K \times K^g$.

Now since both K and K^g are normal in G^+, and since $g^2 \in G^+$ (because $|G : G^+| = 2$), it follows that $(K \times K^g)^g = K \times K^g$, and so $K \times K^g$ is a normal subgroup of G contained in G^+. Thus $K \times K^g$ has two orbits Δ and Δ' as (Γ, G) is basic of biquasiprimitive type. But this implies that K is transitive on Δ', which is impossible since for any $\alpha \in \Delta$ we have $K \leqslant G_\alpha$, and G_α is not transitive on $\Gamma(\alpha) \subset \Delta'$. Thus part (a) holds.

(b). Since $|V\Gamma| \geqslant |\{\alpha\} \cup \Gamma(\alpha)| = 5$, it follows that $|N| \geqslant \frac{1}{2}|V\Gamma| > 2$, hence $N \cap G^+ \neq 1$ since $|N : N \cap G^+| \leqslant 2$. Thus $N \cap G^+$ is a nontrivial intransitive normal

subgroup of G contained in G^+, so its orbits are Δ and Δ', and these must also be the orbits of the intransitive normal subgroup N. □

Next we introduce a convenient framework for investigating these graphs, based on the Imprimitive Wreath Embedding Theorem [25, Theorem 5.5] which identifies the vertex set X with $\{v_i \mid v \in V, i \in \{0,1\}\}$, and G with a transitive subgroup of $\operatorname{Sym}(V) \wr \operatorname{Sym}(2)$ in its natural imprimitive action, so that $\Delta = \{v_0 \mid v \in V\}$ and $\Delta' = \{v_1 \mid v \in V\}$. Since G is transitive, its subgroup G^+ induces transitive subgroups $(G^+)^\Delta$ and $(G^+)^{\Delta'}$ on Δ and Δ', each of which we identify with a transitive subgroup of $\operatorname{Sym}(V)$.

Let $\tau \in \operatorname{Sym}(V) \wr \operatorname{Sym}(2)$ generate the top group, that is, $\tau : v_\varepsilon \mapsto v_{1-\varepsilon}$ for each $v \in V, \varepsilon \in \{0,1\}$, and note that τ conjugates each element $(h_1, h_2) \in \operatorname{Sym}(V) \times \operatorname{Sym}(V)$ to its reverse (h_2, h_1). For a group H, $y \in H$, and $\varphi \in \operatorname{Aut}(H)$, we denote by ι_y the inner automorphism of H induced by y, that is $h \mapsto y^{-1}hy$, and by $\operatorname{Diag}_\varphi(H \times H) = \{(h, h^\varphi) \mid h \in H\}$ the diagonal subgroup of $H \times H$ corresponding to φ.

PROPOSITION 3.2. *Let $(\Gamma, G) \in \mathcal{OG}(4)$ be basic of biquasiprimitive type, and let X denote the vertex set of Γ. Let G^+ be the subgroup of G of index two with orbits Δ, Δ' in X, and let H be the permutation group induced by G^+ on Δ. Let $\alpha \in \Delta$ and $\beta \in \Gamma(\alpha) \subseteq \Delta'$. Then, if necessary, by replacing (Γ, G) with the pair (Γ^g, G^g), where $\Gamma^g \cong \Gamma$ has edge-set $(E\Gamma)^g$, for some $g \in \operatorname{Sym}(X)$, we may take $X = \{v_i \mid v \in V, i \in \{0,1\}\}, \Delta, \Delta'$ and $\alpha = u_0$ as above, for some $u \in V$, and we may identify H with a transitive subgroup of $\operatorname{Sym}(V)$, such that*

   (a) *$G \leqslant H \wr \operatorname{Sym}(2)$, so $H = (G^+)^\Delta = (G^+)^{\Delta'}$; and*
   (b) *for some $y \in H$ and $\varphi \in \operatorname{Aut}(H)$ with $\varphi^2 = \iota_y$, we have $G^+ = \operatorname{Diag}_\varphi(H \times H)$, and $G = \langle G^+, g \rangle$, where $g := (y, 1)\tau$, and $\beta = \alpha^g = (u^y)_1$. Also $G_\alpha = G_\alpha^+ \cong H_u$ is a 2-group.*

Proof. The assertion in part (a), that we may choose a conjugate of G and corresponding identifications of X, Δ, Δ', so that the transitive subgroups $(G^+)^\Delta$ and $(G^+)^{\Delta'}$ determine the same subgroup H of $\operatorname{Sym}(V)$, follows from the embedding theorem [25, Theorem 5.5]. Thus $G \leqslant H \wr \operatorname{Sym}(2) = (H \times H) \rtimes \langle \tau \rangle$, with τ as above, and G^+ is a subdirect subgroup of $H \times H$. By Lemma 3.1, G^+ is faithful on each of Δ and Δ', and hence G^+ is a diagonal subgroup of $H \times H$, so $G^+ = \operatorname{Diag}_\varphi(H \times H)$, for some $\varphi \in \operatorname{Aut}(H)$. Since $\alpha \in \Delta$, we have $\alpha = u_0$ for some $u \in V$. Also, since $G_\alpha < G^+ < G$, we have $G_\alpha = G_\alpha^+$, and since G^+ is a diagonal subgroup of $H \times H$, projection to the first coordinate induces a monomorphism $G_\alpha^+ \to H$ with image H_u. Thus $G_\alpha^+ \cong H_u$, and we know already that G_α is a 2-group.

Since G is transitive on X, there exists $g = (h_1, h_2)\tau \in G$ such that $\beta = \alpha^g$, and $G = \langle G^+, g \rangle$. Set $s := (1, h_2) \in H \times H$. Then s induces a graph isomorphism from Γ to the graph Γ^s with vertex set X and arc set consisting of all pairs $(v_\varepsilon^s, w_{1-\varepsilon}^s) = (v_\varepsilon, (w^{h_2})_{1-\varepsilon})$, where $(v_\varepsilon, w_{1-\varepsilon})$ is an arc of Γ. Moreover $(\Gamma^s, G^s) \in \mathcal{OG}(4)$, the group G^s is equal to $\langle (\operatorname{Diag}_\varphi(H \times H))^s, g^s \rangle$, and we have $(\operatorname{Diag}_\varphi(H \times H))^s = \operatorname{Diag}_{\varphi \iota_{h_2}}(H \times H)$ and

$$g^s = (1, h_2^{-1})(h_1, h_2)\tau(1, h_2) = (h_1 h_2, 1)\tau.$$

Set $y := h_1 h_2$. Then g^s maps α^s to its out-neighbour β^s in Γ^s, and we have $\alpha^s = \alpha$, and $\beta^s = (\alpha^g)^s = (\alpha^s)^{g^s} = \alpha^{g^s} = (u_0)^{(y,1)\tau} = (u^y)_1$.

Now replace $\Gamma, G, g, \varphi, \alpha, \beta$ by $\Gamma^s, G^s, g^s, \varphi \iota_{h_2}, \alpha, \beta^s$. Then all assertions are proved apart from the equality $\varphi^2 = \iota_y$, which we now prove (for the new φ). Since $g = (y, 1)\tau$ normalises $G^+ = \operatorname{Diag}_\varphi(H \times H)$, it follows that, for all $h \in H$, G^+

contains $(h, h^\varphi)^g = (h, h^\varphi)^{(y,1)\tau} = (h^\varphi, h^y)$ and hence we must have $h^y = (h^\varphi)^\varphi$ for all $h \in H$, that is to say, $\varphi^2 = \iota_y$. $\qquad\square$

Now we apply the structure theorem from [24] for biquasiprimitive groups. It turns out that only two of the various possible structures given in Theorem 1.1 of [24] can arise as groups of automorphisms of 4-valent oriented graphs of basic biquasiprimitive type. Note that the stabiliser $G_\alpha = \{(h, h^\varphi) \mid h \in H_u\} \cong H_u$.

PROPOSITION 3.3. *Under the assumptions of Proposition* 3.2*, the automorphism* φ *is nontrivial, and* G *has a unique minimal normal subgroup* $N = \mathrm{soc}(G)$*. Moreover* $N = \mathrm{Diag}_\varphi(M \times M) \cong M$ *where* $M = \mathrm{soc}(H) \cong T^k$ *for some simple group* T *and* $k \geqslant 1$*, and either*

  (a) *H is quasiprimitive and M is its unique minimal normal subgroup, or*
  (b) *H is not quasiprimitive and $M = R \times R^\varphi$ where R, R^φ are intransitive minimal normal subgroups of H. In this case G^+ has two minimal normal subgroups, namely $K := \mathrm{Diag}_\varphi(R \times R)$ and $L = \mathrm{Diag}_\varphi(R^\varphi \times R^\varphi)$, and these are the only minimal normal subgroups if T is nonabelian. Moreover, $N = K \times L$ (so $k = 2\ell$ and $K \cong L \cong R \cong T^\ell$).*

Proof. We examine the possibilities for the structure of G given in [24, Theorem 1.1]. Since G_α is a 2-group, cases (a)(iii), (b) and (c)(ii) do not arise, and since G^+ is faithful on Δ, the possible cases are (a)(i) and (c)(i).

Consider first case (a)(i). Since $|X| > 4$, it follows from [24, Lemma 3.1] that the element $g = (y, 1)\tau$ does not centralise G^+. A straightforward computation shows that $C_{G^+}(g)$ consists of all pairs (h, h^φ) such that $h \in C_H(\varphi)$. Thus φ is nontrivial, since g does not centralise G^+. Moreover in case (a)(i), H is quasiprimitive on V and the stabiliser $H_u \cong G_\alpha$ is a 2-group. We now apply the O'Nan–Scott Theorem for quasiprimitive groups from [22]. This theorem tells us that if H has more than one minimal normal subgroup then the stabiliser H_u is not solvable. Thus H has a unique minimal normal subgroup $M = \mathrm{soc}(H) \cong T^k$ where T is a simple group and $k \geqslant 1$. Since $G^+ \cong H$, it follows that $N = \mathrm{Diag}_\varphi(M \times M) \cong M$ is the unique minimal normal subgroup of G^+. Moreover, by [24, Lemma 3.1], no element of $G \setminus G^+$ centralises G^+, and hence $G \not\cong C_2 \times G^+$, so N is the unique minimal normal subgroup of G.

It remains to consider case (c)(i). By [24, Theorem 1.1], G again has a unique minimal normal subgroup $N = \mathrm{soc}(G)$, and N has the form $N = \mathrm{Diag}_\varphi(M \times M)$, where in this case $M = \mathrm{soc}(H) = R \times R^\varphi$ with R, R^φ distinct intransitive minimal normal subgroups of H. Thus $R \cong R^\varphi \cong T^\ell$ for some simple group T and $\ell \geqslant 1$, so that $N \cong M \cong T^k$ with $k = 2\ell$. Since $R^\varphi \neq R$, the map φ is nontrivial. Since $G^+ \cong H$, the subgroups $K := \mathrm{Diag}_\varphi(R \times R)$ and $L = \mathrm{Diag}_\varphi(R^\varphi \times R^\varphi)$ are minimal normal subgroups of G^+, each isomorphic to T^ℓ, and $\mathrm{soc}(G) = \mathrm{soc}(G^+) = K \times L$ (since $N = \mathrm{soc}(G) \leqslant G^+$). If T is a nonabelian simple group, then each minimal normal subgroup of $K = T^\ell$ is one of the ℓ simple direct factors of this direct decomposition, and G^+ permutes these ℓ simple groups transitively by conjugation. The same holds for L, and as $\mathrm{soc}(G) = K \times L$, it follows that K and L are the only minimal normal subgroups of G^+ (and are interchanged by g, noting that, for $(h, h^\varphi) \in K$, the conjugate $(h, h^\varphi)^g = (h^\varphi, h^y) \in L$, since $\varphi^2 = \iota_y$, and vice versa). On the other hand, if $T = C_p$, then as an H-module, M has two composition factors isomorphic to R as H-modules. In particular, H may have other minimal normal subgroups. However, for any such subgroup S we have $S \cong R$ (as groups) since there are just two composition factors and both are isomorphic to R as H-modules. Also since N is the unique minimal normal subgroup of G it follows that we would have $M = S \times S^\varphi$ also. $\qquad\square$

In summary if $(\Gamma, G) \in \mathcal{OG}(4)$ is basic and biquasiprimitive, then $N := \operatorname{soc}(G)$ is the unique minimal normal subgroup of G, and is contained in G^+. In particular N is transitive on the two G^+-orbits Δ and Δ^+, and since $G_\alpha = G_\alpha^+$, it follows that $G^+ = NG_\alpha$.

Using the framework of Proposition 3.2, we can specify the neighbours of $\alpha = u_0$ and of $\alpha^{g^{-1}} = u_1$. We denote by $\Gamma_{out}(\gamma)$ and $\Gamma_{in}(\gamma)$ the 2-subsets of out-neighbours and in-neighbours of a vertex γ, respectively. Each of these two sets is an orbit of the stabiliser G_γ, and we can always choose an element of G_γ that acts fixed-point-freely on $\Gamma(\gamma)$ (whether the induced group has order 2 or 4). For the vertex α, such an element is of the form $(z^{\varphi^{-1}}, z)$ for some $z \in (H_u)^\varphi$. Since we did not specify above, let us now *choose the G-orientation such that the vertex $\beta = (u^y)_1$ in Proposition 3.2 is an in-neighbour of α, that is, $\beta \in \Gamma_{in}(\alpha)$.*

LEMMA 3.4. *Use the notation of Proposition 3.2 (in particular that $g = (y,1)\tau$ and $\alpha = u_0$), and let $(z^{\varphi^{-1}}, z) \in G_\alpha$ be fixed-point-free on $\Gamma(\alpha)$, for some $z \in (H_u)^\varphi$. Then*

(a) *$\Gamma_{in}(\alpha) = \{(u^y)_1, (u^{yz})_1\}$ and $\Gamma_{out}(\alpha) = \{u_1, (u^z)_1\}$; and*

(b) *for $\gamma := \alpha^{g^{-1}} = u_1$, $\Gamma_{in}(\gamma) = \{u_0, (u^{yzy^{-1}})_0\}$ and $\Gamma_{out}(\gamma) = \{(u^{y^{-1}})_0, (u^{zy^{-1}})_0\}$.*

Proof. As mentioned above, we assume that the vertex $\beta = \alpha^g = (u^y)_1$ in Proposition 3.2 lies in $\Gamma_{in}(\alpha)$. As $(z^{\varphi^{-1}}, z) \in G_\alpha$ is fixed-point-free on $\Gamma(\alpha)$, the second vertex in $\Gamma_{in}(\alpha)$ is $\beta^{(z^{\varphi^{-1}}, z)} = (u^{yz})_1$. Note that $g^{-1} = (1, y^{-1})\tau$. Applying g^{-1} to $\{\alpha\} \cup \Gamma_{in}(\alpha)$ we find first that $\alpha^{g^{-1}} = u_1$ and then that $\Gamma_{in}(u_1)$ consists of the vertices $(u^y)_1^{g^{-1}} = u_0$ and $(u^{yz})_1^{g^{-1}} = (u^{yzy^{-1}})_1$. In particular $u_1 \in \Gamma_{out}(u_0)$ and the second vertex in this set is therefore $u_1^{(z^{\varphi^{-1}}, z)} = (u^z)_1$. This completes the proof of part (a). Finally applying g^{-1} to $\{\alpha\} \cup \Gamma_{out}(\alpha)$ we find that $\Gamma_{out}(u_1)$ consists of the vertices $(u)_1^{g^{-1}} = (u^{y^{-1}})_0$ and $(u^z)_1^{g^{-1}} = (u^{zy^{-1}})_0$. $\qquad\square$

4. BIQUASIPRIMITIVE BASIC PAIRS: RESTRICTING THE SOCLE.

We will now show that for any biquasiprimitive basic pair $(\Gamma, G) \in \mathcal{OG}(4)$, the unique minimal normal subgroup N of G is a direct product of k finite simple groups where k takes one of only several possible values depending on the structure of G. We deduce these values of k by separately considering the cases when N is abelian and nonabelian.

We first consider the case where the minimal normal subgroup $N = \operatorname{soc}(G)$ is abelian. Since N is contained in G^+, this implies that N acts transitively and hence regularly on Δ (and Δ'). In particular, Γ is a bi-Cayley graph over N, that is, $\Gamma = \operatorname{BiCay}(N, \varnothing, \varnothing, S)$ (as defined in Subsection 2.3), and $N = C_p^k$ for some $k \geqslant 1$. Since Γ has valency 4, the cardinality $|S| = 4$. Moreover, by [5, Proposition 2.1], we may assume that S contains the identity of N. We will write N additively so edges of Γ are of the form (h_0, g_1) where $g, h \in C_p^k$ and $g - h \in S$, that is, $g = h + s$ for some $s \in S$. Note that this means that a vertex adjacent to g_1 is of the form $(h')_0$ with $h' = g - s' = h + (s - s')$ for some $s' \in S$. It follows that Γ is connected if and only if $S - S := \{s - s' \mid s, s' \in S\}$ generates N, and since we are assuming that the identity $0 \in S$, this is equivalent to requiring that $S \setminus \{0\}$ generates N.

LEMMA 4.1. *Let $(\Gamma, G) \in \mathcal{OG}(4)$ be basic of biquasiprimitive type and suppose that $N = \operatorname{soc}(G)$ is abelian. Then $N = C_p^k$ with $k \leqslant 2$ and p an odd prime.*

Proof. As discussed above, $N = C_p^k$ for some $k \geqslant 1$, and $\Gamma \cong \operatorname{BiCay}(N, \varnothing, \varnothing, S)$, for some subset $S \subseteq N$ such that $0 \in S$. Since Γ is connected, N is generated by the

3-element subset $S \smallsetminus \{0\}$. Hence $k \leqslant 3$. Suppose next that $k = 3$. Then since k is odd, it follows from Proposition 3.3 that $G^+ = NG_\alpha$ is quasiprimitive on $\Delta = N$. In particular since N is regular on Δ, no proper non-trivial subgroup of N is normal in G^+. Since N acts trivially on itself by conjugation, this implies that conjugation by G_α fixes no proper non-trivial subgroup of N. However, G_α is a 2-group, and N has exactly $p^2 + p + 1$ subgroups of order p, which is odd. Thus some subgroup of order p must be left fixed under conjugation by G_α and hence must be normal in G^+, a contradiction. Therefore $k \leqslant 2$.

If $p = 2$ then the number of vertices is $|X| = 2p^k = 2^{k+1}$ and since $|X| > 4$ we must have $k = 2$. However $|G^+| = |\Delta|.|G_\alpha| = 2^2.|G_\alpha| \geqslant 8$, and G^+ is a 2-group, while by Proposition 3.2, $|G^+| = |H|$ for some transitive subgroup $H \leqslant \mathrm{Sym}(4)$. Hence $G^+ \cong D_8$, but then G^+ has a unique minimal normal subgroup of order 2 with four orbits in X, which is a contradiction. $\qquad\square$

The next lemma concerns the case when $N = \mathrm{soc}(G)$ is nonabelian. The proof develops ideas used to prove a similar result for quasiprimitive basic pairs in [3, Lemma 6.2].

Lemma 4.2. *Let $(\Gamma, G) \in \mathcal{OG}(4)$ be basic of biquasiprimitive type and suppose that $N = \mathrm{soc}(G)$ is nonabelian. Then either*

(a) *N is a minimal normal subgroup of G^+ and $N = T^k$, for some nonabelian simple group T and $k \in \{1, 2, 4\}$; or*

(b) *$N = K \times K^g$ where $g \in G \backslash G^+$, and $K = T^\ell$ is a minimal normal subgroup of G^+ with T a nonabelian simple group and $\ell \in \{1, 2, 4\}$. In particular, $N \cong T^k$ with $k = 2\ell$.*

Proof. Let $(\Gamma, G) \in \mathcal{OG}(4)$ and $N = \mathrm{soc}(G)$ be as in the statement of the theorem and fix a G-invariant orientation of the edges of Γ. The possible cases (a) and (b) here correspond directly to the two cases of Proposition 3.3. The group K in case (b) is the subgroup $K := \{(r, r^\varphi) : r \in R\}$ of Proposition 3.3, and so $K^g = \{(r^\varphi, r^y) : r \in R\}$, where R is an intransitive minimal normal subgroup of H.

Since $N = \mathrm{soc}(G)$ is nonabelian, it follows that N is a direct product of isomorphic nonabelian simple groups T. In particular, $N = T^k$ for $k \geqslant 1$, and in case (b), $k = 2\ell$ where $K = T^\ell$ and $\ell \geqslant 1$. We will now show that k divides 4 in case (a) and ℓ divides 4 in case (b). As $N = \mathrm{soc}(G)$, we will identify N with its group of inner automorphisms $\mathrm{Inn}(N)$, and regard G as a subgroup of $\mathrm{Aut}(N) \cong \mathrm{Aut}(T) \wr \mathrm{Sym}(k)$. The representations of elements will therefore be different from Proposition 3.3.

Let s be the largest integer such that G acts transitively on the oriented s-arcs of Γ, so $s \geqslant 1$. By Lemma 2.1, this implies that G is regular on the oriented s-arcs of Γ. Consider now an oriented s-arc $(v_0, v_1, \ldots, v_s)$ of Γ, and suppose that the pointwise stabiliser $G_{v_0,\ldots,v_{s-1}}$ of order 2 is generated by the element h_1, that is, $G_{v_0,\ldots,v_{s-1}} = \langle h_1 \rangle \cong C_2$.

Now let $g \in G \backslash G^+$ be an automorphism of Γ taking the oriented s-arc $(v_0, v_1, \ldots, v_s)$ to the oriented s-arc $(v_1, v_2, \ldots, v_s, v_{s+1})$ where v_{s+1} is some out-neighbour of v_s. For each $i = 2, \ldots, s$, define $h_i := h_{i-1}^{g^{-1}}$. It is clear that for each $i \leqslant s$ we have

$$G_{v_0,\ldots,v_{s-i}} = \langle h_1, \ldots, h_i \rangle.$$

We may write the automorphisms $h_1, g \in G$ as elements of $\mathrm{Aut}(N) \cong \mathrm{Aut}(T) \wr \mathrm{Sym}(k)$, so that $h_1 = f\sigma$ and $g = f'\tau$ where $f, f' \in \mathrm{Aut}(T)^k$ and $\sigma, \tau \in \mathrm{Sym}(k)$. In fact in case (b), $\sigma, \tau \in \mathrm{Sym}(\ell) \wr \mathrm{Sym}(2)$ (since in this case the ℓ-subsets of simple direct factors of the two minimal normal subgroups of G^+ form a G-invariant partition of

the set of k simple direct factors of N) with $\sigma \in \operatorname{Sym}(\ell) \times \operatorname{Sym}(\ell)$ (since $h_1 \in G^+$). In either case, $h_1^2 = 1$ implies that $\sigma^2 = 1$.

Now let π denote the projection map $\pi : \operatorname{Aut}(N) \to \operatorname{Sym}(k)$, so that $(h_1)\pi = \sigma$ and $(g)\pi = \tau$, and let $P := (G^+)\pi = (NG_{v_0})\pi = (G_{v_0})\pi$. Note that P is a 2-group since G_{v_0} is a 2-group, and moreover

$$P = (G_{v_0})\pi = \langle h_1, h_2, \ldots, h_s \rangle \pi = \langle \sigma, \sigma^{\tau^{-1}}, \ldots, \sigma^{\tau^{-(s-1)}} \rangle.$$

We claim that σ is not contained in any proper τ-invariant subgroup of P. Suppose to the contrary that $\bar{P}$ is a proper τ-invariant subgroup of P containing σ. Since $\bar{P}$ is τ-invariant it follows that $\sigma^{\tau^{-i}} \in \bar{P}$ for all $i \in \mathbb{Z}$, implying that $P \leqslant \bar{P}$ and hence that $P = \bar{P}$, a contradiction.

Notice that P is a subgroup of index 1 or 2 of $(G)\pi$, and the 2-group P is transitive in case (a) or has two orbits of length ℓ in case (b), so k divides $|P|$, or ℓ divides $|P|$ respectively. Hence, if $|P| \leqslant 4$ then the result follows. Thus we may assume that $|P| \geqslant 8$, and since P is generated by conjugates of σ this means that $\sigma \neq 1$, so σ has order 2. In particular, $P \neq \langle \sigma \rangle$, so there exists a maximal subgroup M of P containing $\langle \sigma \rangle$. Since P is a 2-group it follows that M is normal in P. If $P = (G)\pi$ then $\tau \in P$, and so M is τ-invariant, contradicting the fact that σ is not contained in any proper τ-invariant subgroup of P.

Therefore P is an index 2 subgroup of $(G)\pi$, so $\tau \in (G)\pi \backslash P$ and τ normalises P. Since $g^2 \in G^+$, we have $\tau^2 \in P$. Now τ does not normalise the maximal subgroup M of P containing σ, and so $M_2 := M^{\tau^{-1}}$ is a maximal subgroup of P distinct from M. Let $L := \Phi(P)$, the Frattini subgroup of P (the intersection of all maximal subgroups of P). In particular $L \leqslant M \cap M_2$, so P/L is elementary abelian of order at least 4. Also L is τ-invariant since τ normalises P, so $\sigma \notin L$. Setting $J := \langle L, \sigma \rangle$, it follows that $J \leqslant M$ and J/L has order 2, and conjugation by τ^{-1} maps J/L to $(J^{\tau^{-1}})/L$. However, J is normal in P since P/L is elementary abelian. In particular, since $\tau^2 \in P$, conjugation by τ^2 fixes J and J/L. Therefore repeated applications of conjugation by τ simply interchange the two (possibly equal) subgroups J/L and $(J^{\tau^{-1}})/L$ of P/L and each generator $\sigma^{\tau^{-i}}$ of P, lies in either J or $J^{\tau^{-1}}$. It follows that P/L is generated by J/L and $J^{\tau^{-1}}/L$, and it follows that $P/L \cong C_2^2$, that $M = J$, and $M_2 = J^{\tau^{-1}}$. Thus $M = \langle L, \sigma \rangle$, and it follows from [9, Satz III.3.2] that $M = \langle \sigma \rangle$. This implies that $|P| = 2|M| = 4$, which is a contradiction. This completes the proof. $\square$

The first assertions of Theorem 1.1 now follow directly from Proposition 3.3 together with Lemmas 4.1 and 4.2.

5. Constructing Biquasiprimitve Pairs

In this section we complete the proof of Theorem 1.1. We do this by explicitly constructing examples of biquasiprimitive pairs corresponding to the different cases of Theorem 1.1. In each of the three cases (a)–(c) of Theorem 1.1, the parameter k (the number of simple direct factors of the socle of G) can take several different values. In case (a) there are two possibilities for the value of k, while in each of the cases (b) and (c) there are three possibilities.

Thus Theorem 1.1 gives a total of eight different possibilities for the structure of $\operatorname{soc}(G)$ of a biquasiprimitive pair (Γ, G) where the number of simple direct factors is taken into account. To complete the proof, we therefore provide eight infinite families of biquasiprimitive basic pairs corresponding to these distinct cases.

In Subsection 5.1 we will describe two methods for constructing biquasiprimitive basic pairs. In short, Method 5.1 uses the standard bi-Cayley graph construction described in Subsection 2.3, while Method 5.7 is a more general coset graph construction

developed from Proposition 3.2. All of our constructions of biquasiprimitive pairs will use one of these two methods.

The examples constructed to complete the proof of Theorem 1.1 are given in Constructions 5.11–5.30 of this section. Table 2 shows all of these constructions along with the explicit simple group T used in each case. The "Method Used" column refers to one of the two methods developed in Subsection 5.1 for producing biquasiprimitive pairs. The construction numbers are included for easy reference.

Case described in Theorem 1.1	Value of k	Simple Group T	Construction #	Method Used
Case (a)	$k = 1$	$\mathbb{Z}_p$, $p \equiv 1 \bmod 4$	Construction 5.11	Method 5.1
	$k = 2$	$\mathbb{Z}_p$, $p \equiv 3 \bmod 4$	Construction 5.14	Method 5.1
Case (b)	$k = 1$	$\mathrm{Alt}(n)$, $n \geqslant 5$, odd	Construction 5.16	Method 5.1
	$k = 2$	$\mathrm{Alt}(n)$, $n \geqslant 5$, odd	Construction 5.19	Method 5.1
	$k = 4$	$\mathrm{PSL}(2, p)$, $p \geqslant 7$	Construction 5.23	Method 5.7
Case (c)	$k = 2$	$\mathrm{Alt}(n)$, $n \geqslant 5$, odd	Construction 5.26	Method 5.1
	$k = 4$	$\mathrm{PSL}(2, p)$, $p \geqslant 7$	Construction 5.28	Method 5.7
	$k = 8$	$\mathrm{PSL}(2, p)$, $p \geqslant 7$	Construction 5.30	Method 5.7

TABLE 2. Constructions of basic biquasiprimitive pairs (Γ, G) with $\mathrm{soc}(G) \cong T^k$ as described in the various cases of Theorem 1.1.

5.1. Two Methods for Constructing Biquasiprimitive Pairs. One way to construct biquasiprimitive pairs is using the "standard" bi-Cayley construction described in Subsection 2.3. Specifically, if $(\Gamma, G) \in \mathcal{OG}(4)$ is basic of biquasiprimitive type, and the unique minimal normal subgroup N of G contained in G^+ is semiregular (with two orbits) on $V\Gamma$, then we can take Γ to be a bi-Cayley graph $\Gamma := \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ (for some subset S of N of cardinality 4).

In our constructions involving bi-Cayley graphs presented in the form $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ we will always use the natural labelling of the vertex set $V\Gamma$. That is, we let $V\Gamma = N_0 \cup N_1$ consisting of two copies of the group N with each vertex labelled $(n)_\epsilon$ for $n \in N$ and $\epsilon \in \{0, 1\}$.

Suppose now that $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ where $S = S^{-1}$. Of course, such a graph is bipartite with N_0 and N_1 forming the bipartition. In order to show that Γ is connected, it suffices to show that the vertex set N_0 lies in a single connected component of Γ, or in other words that there is a path from $(1_N)_0$ to $(n)_0$ for any $n \in N$ (vertex-transitivity then ensures that this holds for N_1 also). Any such path must have even length and consist of repeated left multiplication in N by an element of S followed by an element of $S^{-1} = S$. In particular, the graph Γ is connected if $\langle S^2 \rangle = N$.

Hence we have the following simple method for constructing biquasiprimitive basic pairs (Γ, G).

METHOD 5.1. *Take a group $N = T^k$ where T is a simple group and $k \geqslant 1$, and construct a pair (Γ, G) with $N = \mathrm{soc}(G)$ as follows:*
  (1) *Let $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$, where $S \subset N$ such that $S = S^{-1}$, $|S| = 4$, and $\langle S \rangle = \langle S^2 \rangle = N$.*
  (2) *Take a group G with $N \leqslant G \leqslant N_{\mathrm{Aut}(\Gamma)}(N)$ for which Γ is G-oriented. This gives $(\Gamma, G) \in \mathcal{OG}(4)$.*

(3) *Show that N is the unique minimal normal subgroup of G to get that (Γ, G) is basic of biquasiprimitive type.*

Note that $N_{\mathrm{Aut}(\Gamma)}(N)$ (the normaliser of N in $\mathrm{Aut}(\Gamma)$) was determined in [27, Theorem 1.1]. In fact, in our constructions we will only use the following fact which follows from [27, Lemmas 3.2 and 3.3].

PROPOSITION 5.2. *Let $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ as defined in Subsection 2.3 with $S = S^{-1}$. Suppose $\alpha \in \mathrm{Aut}(N)$ with $S^\alpha = S$. Then the permutations δ_α and σ_α of $V\Gamma$ where $\delta_\alpha : x_\varepsilon \mapsto (x^\alpha)_{1-\varepsilon}$, and $\sigma_\alpha : x_\varepsilon \mapsto (x^\alpha)_\varepsilon$ for $x \in N$ and $\varepsilon \in \{0, 1\}$ are both automorphisms of Γ. Moreover both δ_α and σ_α normalise the semi-regular subgroup $N \leqslant \mathrm{Aut}(\Gamma)$.*

More generally, we may construct biquasiprimitive pairs (Γ, G) by using the coset graph construction. For a group G, a proper subgroup S, and an element $g \in G$, the *coset graph* $\Gamma = \mathrm{Cos}(G, S, g)$ is the undirected graph with vertex set $\{Sx : x \in G\}$ and edges $\{Sx, Sy\}$ if and only if xy^{-1} or $yx^{-1} \in SgS$. The group G acting by right multiplication on $V\Gamma$ induces a vertex-transitive and edge-transitive group of automorphisms of Γ, and this action is faithful if and only if S is core-free in G. Furthermore, the graph Γ is connected if and only if $\langle S, g \rangle = G$, and is G-oriented and 4-valent if and only if $g^{-1} \notin SgS$ and $|S : S \cap S^g| = 2$ (see discussion at the beginning of [3, Section 5]). In summary, if $\Gamma = \mathrm{Cos}(G, S, g)$, then $(\Gamma, G) \in \mathcal{OG}(4)$ if and only if

(1) $\quad S$ is core-free in $G, \qquad g^{-1} \notin SgS, \qquad |S : S \cap S^g| = 2, \quad \text{and} \quad \langle S, g \rangle = G.$

Moreover, for each pair $(\Gamma, G) \in \mathcal{OG}(4)$ there exist $S \leqslant G$ and $g \in G$ such that $\Gamma = \mathrm{Cos}(G, S, g)$ and (1) holds. Specifically, for a vertex $\alpha \in V\Gamma$, take $S := G_\alpha$ and take g to be an element of G mapping α to one of its neighbours with $\alpha^{g^2} \neq \alpha$.

We can use Proposition 3.2 on the structure of biquasiprimitive basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ together with the coset graph construction given above to find examples of coset graphs of biquasiprimitive type. We begin by providing a general construction which uses a permutation group H (with some prescribed properties) to produce a pair (Γ, G) where Γ is a coset graph for G, and G has an index 2 subgroup isomorphic to H. In the remainder of this section we will show that under certain conditions the pairs (Γ, G) constructed in this way are basic of biquasiprimitive type.

CONSTRUCTION 5.3. *Take a permutation group H, a proper subgroup $V < H$, a non-identity element $y \in H$, and an automorphism $\varphi \in \mathrm{Aut}(H)$ such that $\varphi^2 = \iota_y$, $y^\varphi = y$, and $\varphi \neq \iota_u$ for any $u \in H$ such that $u^2 = y$.*

Now consider the group $H \wr S_2$ and define two of its subgroups $G^+ := \mathrm{Diag}_\varphi(H \times H)$, and $S := \mathrm{Diag}_\varphi(V \times V)$. Also define an element $g := (y, 1)(12) \in H \wr S_2$. Finally construct the graph-group pair (Γ, G) where $G := \langle G^+, g \rangle \leqslant H \wr S_2$ and $\Gamma := \mathrm{Cos}(G, S, g)$.

It is clear that the construction of the group G in this way corresponds to the formulation of the biquasiprimitive permutation group G given in Proposition 3.2. Notice in particular that using this construction, the pair (Γ, G) is completely determined by the choices of appropriate H, V, y and φ. Hence we will say that a tuple (H, V, y, φ) is *appropriate* if H, V, y and φ satisfy the conditions of Construction 5.3. In many of the constructions that follow, we will simply apply Construction 5.3 on an appropriate (H, V, y, φ) to create pairs (Γ, G). The following lemma gives a sufficient condition for (Γ, G) constructed in this way to be a member of $\mathcal{OG}(4)$.

LEMMA 5.4. *Let (Γ, G) be a graph-group pair constructed using Construction 5.3 on an appropriate (H, V, y, φ). Then $(\Gamma, G) \in \mathcal{OG}(4)$ if*

(2) $\quad V$ *is core-free in* $H, \qquad y \notin VV^\varphi, \qquad |V : V \cap V^\varphi| = 2, \quad \text{and} \quad \langle V, y \rangle = H.$

Proof. Let G^+ and S be the subgroups of G defined in the construction, and let $\Gamma = \text{Cos}(G, S, g)$. Suppose that (2) holds. We will show that $(\Gamma, G) \in \mathcal{OG}(4)$ by showing that (1) holds also.

First, since $H \cong G^+$, $S \cong V$, and V is core-free in H, it follows S is core-free in G^+ and hence is core-free in G. Next, we will show that $y \notin VV^\varphi$ implies that $g^{-1} \notin SgS$. Notice that $g^{-1} = (1, y^{-1})(12)$, while for any element $z \in SgS$, $z = (s, s^\varphi)(y, 1)(12)(t, t^\varphi) = (syt^\varphi, s^\varphi t)(12)$ for some $s, t \in V$. Thus if $g^{-1} = z$ for some $z \in SgS$, then $1 = syt^\varphi$ and hence $y \in VV^\varphi$.

For the last two conditions notice that if we take $x \in G^+$ then $x^g = (h, h^\varphi)^g = (h^\varphi, h^y)$ for some $h \in H$. In particular, for $s \in S$ we have $s^g = (t^\varphi, t^y)$ where $t \in V$. So $s^g \in S$ if and only if $t^\varphi \in V$. Since $V \cong S$ we get that $|S : S \cap S^g| = |V : V \cap V^\varphi|$.

Finally, it is easy to check that $g^2 = (y, y)$ and since $y^\varphi = y$ it follows that $g^2 \in G^+$. Hence if $\langle V, y \rangle = H$, then $\langle S, g^2 \rangle = \text{Diag}_\varphi(\langle V, y \rangle \times \langle V, y \rangle) = \text{Diag}_\varphi(H \times H) = G^+$ and so $\langle S, g \rangle = G$. $\qquad\square$

Hence we have an easy condition for ensuring that pairs (Γ, G) formed using Construction 5.3 are contained in $\mathcal{OG}(4)$. Our next goal is to provide a simple condition under which such pairs are biquasiprimitive. For $u \in H$ we denote by ι_u the inner automorphism of H induced by conjugation by u.

LEMMA 5.5. *Let (Γ, G) be a graph-group pair constructed using Construction 5.3 on an appropriate (H, V, y, φ). Let G^+ and S be as defined in that construction. Then every minimal normal subgroup of G is contained in G^+. In particular, if $\text{soc}(G^+) \cong \text{soc}(H)$ is a minimal normal subgroup of G then it is the unique minimal normal subgroup of G.*

Proof. Notice that $|G : G^+| = 2$ since $G = \langle G^+, g \rangle$, g normalises G^+, and $g^2 = (y, y) \in G^+$. Suppose there exists a minimal normal subgroup N of G that is not contained in G^+. Then, by the minimality of N it follows that $G^+ \cap N = 1$ and hence $G = G^+ \times N$. It follows that $N = \langle x \rangle \cong C_2$, for some $x \in G \smallsetminus G^+$. The element x has the form

$$x = (u^{-1}, (u^{-1})^\varphi)g = (u^{-1}y, (u^{-1})^\varphi)(12), \quad \text{for some } u \in H.$$

Now x centralises G^+, and so, for all $h \in H$ we have $x(h, h^\varphi) = (h, h^\varphi)x$, or equivalently

$$\left(u^{-1}yh^\varphi, (u^{-1})^\varphi h\right)(12) = \left(hu^{-1}y, h^\varphi(u^{-1})^\varphi\right)(12).$$

This holds if and only if, for all $h \in H$ we have $(u^{-1})^\varphi h = h^\varphi(u^{-1})^\varphi$ (on equating the second entries, and noting that equality in the first entries follows from this on applying φ). This is equivalent to requiring $h = (uhu^{-1})^\varphi$ for all $h \in H$, that is to say, φ is equal to ι_u. In particular $(u^{-1})^\varphi = u^{-1}$ so $x = (u^{-1}y, u^{-1})(12)$, and since $x^2 = (u^{-1}yu^{-1}, u^{-2}y) = 1$, we have $y = u^2$. But by our choice of element y and automorphism φ in Construction 1, no such $u \in H$ exists. Therefore every minimal normal subgroup of G is contained in G^+ and hence if $\text{soc}(G^+)$ is a minimal normal subgroup of G, then it is the unique minimal normal subgroup of G. $\qquad\square$

The above result gives the following corollary.

COROLLARY 5.6. *Suppose that $(\Gamma, G) \in \mathcal{OG}(4)$ where (Γ, G) arises from applying Construction 5.3 to an appropriate (H, V, y, φ). Let G^+, and S be as defined in that construction. Suppose further that $H = MV$ where $M = \text{soc}(H) \cong T^k$ for some nonabelian simple group T and $k \geqslant 1$. If $\text{soc}(G^+) \cong \text{soc}(H)$ is a minimal normal subgroup of G, then (Γ, G) is biquasiprimitive.*

Proof. The vertex set of Γ is the set of right cosets of S in G. Hence there are two G^+-orbits, namely $\Delta = \{Sx : x \in G^+\}$ and $\Delta' = \{Sgx : x \in G^+\}$. If $N = \mathrm{soc}(G^+) \cong M$ is a minimal normal subgroup of G then N is the unique such subgroup by Lemma 5.5. Moreover, the condition $H = MV$ implies that $G^+ \cong NS$ so N is transitive on the two G^+-orbits Δ and Δ', and hence G is biquasiprimitive on $V\Gamma$. □

The above results now provide the following method for constructing biquasiprimitive pairs in $\mathcal{OG}(4)$.

METHOD 5.7. *Take a group $M = T^k$ for some nonabelian simple group T and $k \geqslant 1$, and define a group $H := MV$ where $M = \mathrm{soc}(H)$ and V is a proper subgroup $V \leqslant H$. Also take a non-identity $y \in H$ and an automorphism $\varphi \in \mathrm{Aut}(H)$, such that (H, V, y, φ) is appropriate.*

(1) *Apply Construction 5.3 on (H, V, y, φ) to create a pair (Γ, G).*
(2) *Show that H, V, y and φ satisfy condition (2) of Lemma 5.4 to get that $(\Gamma, G) \in \mathcal{OG}(4)$.*
(3) *Show that $\mathrm{soc}(G^+) \cong M$ is a minimal normal subgroup of G to get that (Γ, G) is biquasiprimitive (by Corollary 5.6).*

5.2. A THIRD METHOD: THE SEPARATED BOX PRODUCT. While all of our explicit constructions in Subsection 5.3 use one of the two methods in Subsection 5.1, there is a third way to construct basic biquasiprimitive pairs, namely by taking the "separated box product" of quasiprimitive basic pairs (see [19]). However, as we show below in Proposition 5.9, this third method fails to produce infinitely many of the basic biquasiprimitive pairs.

Suppose that $(\Sigma, H) \in \mathcal{OG}(4)$ is basic of quasiprimitive type and let Δ be the 2-valent H-arc-transitive orbital digraph obtained by taking one of the two paired orbits of H on the arc-set of Σ. As in [19, Section 3], the *separated box product* $\Gamma^\Delta := \mathrm{SBP}(\Delta, \Delta)$ is defined to be the digraph with vertex set $V\Gamma^\Delta = V\Delta \times V\Delta \times \mathbb{Z}_2$, and arcs being all ordered pairs

$$((\alpha, \gamma, 0), (\beta, \gamma, 1)) \text{ with } (\alpha, \beta) \text{ an arc of } \Delta \text{ and } \gamma \in V\Gamma,$$

together with all ordered pairs

$$((\beta, \gamma, 1), (\beta, \delta, 0)) \text{ with } (\gamma, \delta) \text{ an arc of } \Delta \text{ and } \beta \in V\Gamma.$$

Then Γ^Δ is a (possibly disconnected) bipartite 2-valent digraph, and by [19, Corollary 3.4], the group $G \cong H \wr S_2$ acts naturally on Γ^Δ as a subgroup of automorphisms, G is transitive on the arcs of Γ^Δ, and its index 2 subgroup $H \times H$ fixes setwise the two parts of the bipartition. A discussion of connectivity of Γ^Δ is given in [19, Section 3.3]. If Γ^Δ is connected and Γ is its underlying graph, then our discussion shows that $(\Gamma, G) \in \mathcal{OG}(4)$. Moreover, since $(\Sigma, H) \in \mathcal{OG}(4)$ is basic of quasiprimitive type, it follows from [3, Theorem 1.3] that H has a unique minimal normal subgroup, say N, and N is transitive on $V\Sigma = V\Delta$. Hence G also has a unique minimal normal subgroup $N \times N$, and $N \times N$ is transitive on each part of the bipartition of $V\Gamma$. This implies that G is biquasiprimitive on $V\Gamma$, and so (Γ, G) is basic of biquasiprimitive type. Thus we have the following method for constructing a basic pair $(\Gamma, G) \in \mathcal{OG}(4)$ of biquasiprimitive type.

METHOD 5.8. *Let $(\Sigma, H) \in \mathcal{OG}(4)$ be basic of quasiprimitive type, and let Δ be the 2-valent H-arc-transitive orbital digraph obtained by taking as arc-set one of the two paired orbits of H on the arc-set of Σ.*

(1) *Take $\Gamma^\Delta := \mathrm{SBP}(\Delta, \Delta)$, see [19, Section 3].*
(2) *Take $G \cong H \wr S_2$ as defined in [19, Corollary 3.4].*
(3) *Take Γ to be the underlying graph of Γ^Δ.*

(4) *Show that Γ is connected, and if this is the case then return (Γ, G).*

Our discussion shows that each pair (Γ, G) produced by Method 5.8 lies in $\mathcal{OG}(4)$ and is basic of biquasiprimitive type. It turns out, however, that many families of basic pairs of biquasiprimitive type cannot be constructed using Method 5.8. To explain why, we note that, by [3, Theorem 1.3], if $(\Sigma, H) \in \mathcal{OG}(4)$ is basic of quasiprimitive type, then H has a unique minimal normal subgroup $N = T^\ell$ such that T is a nonabelian simple group and $\ell \leqslant 2$. Hence for any basic biquasiprimitive pair (Γ, G) constructed by Method 5.8, the group G will have a unique minimal normal subgroup $\operatorname{soc}(G) = T^{2\ell}$ with $\ell \leqslant 2$. To start with, we will not obtain the examples (Γ, G) we give in Constructions 5.11 and 5.14 where the group G has an abelian socle. More than this is true. Our next result shows we will not obtain the examples arising from Constructions 5.16 and 5.30 where the socle of G is T^k with $k = 1$ and $k = 8$, respectively.

PROPOSITION 5.9. *Suppose that $(\Gamma, K) \in \mathcal{OG}(4)$ is basic of biquasiprimitive type and that $\operatorname{soc}(K) = T^k$ where T is a nonabelian simple group and $k \in \{1, 8\}$. Then the oriented graph Γ cannot be constructed using Method 5.8.*

Proof. Let (Γ, K) be as in the statement of the proposition, let $S := \operatorname{soc}(K)$, and note that S is the unique minimal normal subgroup of K, by Theorem 1.1. Now suppose for a contradiction that Method 5.8, applied to a basic quasiprimitive pair $(\Sigma, H) \in \mathcal{OG}(4)$ produces a basic biquasiprimitive pair $(\Gamma, G) \in \mathcal{OG}(4)$, where the groups $G = H \wr S_2$ and K preserve the same orientation of Γ. By the discussion preceding the proposition $U := \operatorname{soc}(G) = L^{2\ell}$, where L is a nonabelian simple group and $2\ell \in \{2, 4\}$. In particular $G \neq K$. We will show that this leads to a contradiction.

As usual, we denote by K^+ the index 2 subgroup of K which preserves the two parts of the bipartition of Γ. Taking a vertex $\alpha \in V\Gamma$ we then have, $K^+ = SK_\alpha$ and K_α is a 2-group, say $|K_\alpha| = 2^{a'} \geqslant 2$. Hence $|K| = 2|K^+| = 2|S|.|K_\alpha|/|S_\alpha| = 2^a|T|^k$ for some a such that $1 \leqslant a \leqslant a' + 1$. Also $|V\Gamma| = |K|/2^{a'} = 2^{a-a'}|T|^k$.

Now let $X = \langle G, K \rangle$, and note that $X \leqslant \operatorname{Aut}(\Gamma)$ and X preserves the same orientation of Γ preserved by K and G. Since $K \neq G$, the group K is a proper subgroup of X, and since X_α is a 2-group, it follows that $|X| = |V\Gamma||X_\alpha| = 2^b|T|^k$ for some $b \geqslant a + 1$.

Let Y be the last term in the derived series for X. Then X/Y is solvable, and hence its subgroup $SY/Y \cong S/(S \cap Y)$ is soluble, which implies that S is contained in Y. Now let N be a normal subgroup of X properly contained in Y, and suppose that N is maximal with respect to these properties. Then Y/N is a minimal normal subgroup of X/N. If Y/N is abelian then $[Y, Y] \leqslant N$, while from the definition of Y we have $[Y, Y] = Y$, a contradiction. Hence Y/N is nonabelian and thus $Y/N \cong R^c$ for some nonabelian simple group R and $c \geqslant 1$.

Since the index $|Y : S|$ is a 2-power, it follows that S is not contained in N (as otherwise Y/N would be a 2-group). Thus $S \cap N$ is a proper subgroup of S. Moreover, since both S and N are K-invariant, it follows that $S \cap N$ is normalised by K. However, since S is minimal normal in K we conclude that $S \cap N = 1$. Therefore $S \cong SN/N \leqslant Y/N \cong R^c$ and so $S = T^k$ is isomorphic to a subgroup of R^c. Note that since $S \cap N = 1$ it also follows that $N \cap K = 1$, as S is the unique minimal normal subgroup of K.

Now write $R^c = R_1 \times \cdots \times R_c$, and for each i let $\pi_i : R^c \to R_i$ denote the natural projection map. Notice that if $\pi_i(S) \neq 1$ for some i, then $\pi_i(S) \cong T^j$ for some $j \geqslant 1$, and the index $|R_i : \pi_i(S)|$ is a power of 2. Thus it follows from [8, Theorem 1] that $j = 1$ and either $R = T$; or $R = A_n, T = A_{n-1}$ and $|R : T| = n = 2^d$ for some d. In either case, since $|R^c|/|T|^k$ is a power of 2, it follows that $k = c$.

We can apply exactly these arguments with $G, U, L, 2\ell$ in place of K, S, T, k, and this yields in particular that $c = 2\ell$. Thus we conclude that $k = 2\ell$ which is a contradiction as either $k = 1 < 2\ell$ or $k = 8 > 2\ell$. $\qquad\square$

REMARK 5.10. The proof of Proposition 5.9 raises several questions about whether a nonabelian group K, where $(\Gamma, K) \in \mathcal{OG}(4)$ is basic of biquasiprimitive type, could be embedded into a larger group $X \leqslant \mathrm{Aut}(\Gamma)$ which preserves the same orientation of Γ as K does. In particular, taking T^k, N, and R^c to be as defined in the proof of Proposition 5.9, the arguments there imply that $k = c$, N is a 2-group, and either $R = T$ or $(R, T, |R : T|) = (A_n, A_{n-1}, 2^d)$. Presumably the latter case does not arise (we have no examples)? Also are there examples with N nontrivial, where $R = T$?

In summary, none of the basic biquasiprimitive pairs $(\Gamma, G) \in \mathcal{OG}(4)$ with $\mathrm{soc}(G)$ abelian, or with $\mathrm{soc}(G) = T^k$ nonabelian and $k \in \{1, 8\}$ can be constructed using Method 5.8. Further investigation is required to determine how useful Method 5.8 is in constructing biquasiprimitive pairs of the remaining types. Note that Method 5.8 returns a group $G \cong H \wr S_2$. If such a group appears in a basic pair (Γ, G) of biquasiprimitive type then $\mathrm{Aut}(H)$ must also contain a rather special automorphism φ as described in Proposition 3.2. It is thus not clear whether the basic biquasiprimitive pairs (Γ, G) obtained by Method 5.8 cover most, or all, or just some, of the examples with $\mathrm{soc}(G) = T^2$ or T^4 (and T nonabelian simple).

5.3. CONSTRUCTING EXAMPLES. We now provide constructions of basic biquasiprimitve pairs $(\Gamma, G) \in \mathcal{OG}(4)$ with the various possible structures for $\mathrm{soc}(G)$ as described in cases (a)–(c) of Theorem 1.1. We will use both the bi-Cayley graph construction described in Subsection 2.3 (Method 5.1) and the coset graph construction developed in the last part of the previous section (Method 5.7).

We begin with examples of biquasiprimitive basic pairs (Γ, G) with $\mathrm{soc}(G)$ abelian. Note that all 4-valent bi-Cayley graphs over an abelian group are arc-transitive [5, Proposition 1.3].

CONSTRUCTION 5.11. *Take a prime $p \equiv 1 \pmod 4$ and let $q \in \mathbb{Z}_p$ such that $q^2 \equiv -1$ mod p. Let $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ with vertex set $N_0 \cup N_1$, where $N = \mathbb{Z}_p$ and $S = \{\pm 1, \pm q\}$. Define a permutation δ of the vertices of Γ by $x_\varepsilon^\delta = (x \cdot q)_{1-\varepsilon}$ for $\varepsilon \in \{0, 1\}$, and set $G := N \rtimes \langle \delta \rangle$.*

REMARK 5.12. Note that for a pair (Γ, G) obtained using Construction 5.11, the graph Γ is a circulant and $\Gamma \cong \mathrm{Cay}(\mathbb{Z}_{2p}, \{\pm(p+2), \pm(p+2q)\})$. These pairs are somewhat exceptional in the family $\mathcal{OG}(4)$ in that the graph Γ contains precisely two G-alternating cycles, both spanning $V\Gamma$ (see [11, Proposition 2.4]).

LEMMA 5.13. *For Γ, G as in Construction 5.11, $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of biquasiprimitive type with $\mathrm{soc}(G)$ as described in Theorem 1.1 case (a) with $k = 1$.*

Proof. Since $|S| = 4$ and $\langle S \rangle = \langle S^2 \rangle = N$ it follows that Γ is 4-valent and connected. Also by Proposition 5.2, $\delta \in \mathrm{Aut}(\Gamma)$ since it is induced by an automorphism of N fixing S setwise. Notice that the automorphism δ has order 4 and that the stabiliser in G of the vertex $(0)_0$ is $\langle \delta^2 \rangle \cong C_2$. This group has two orbits of length two on the neighbourhood of $(0)_0$, namely $\{(1)_1, (-1)_1\}$ and $\{(q)_1, (-q)_1\}$.

Now, any automorphism $g \in G$ is of the form $g = n\delta^i$ with $n \in N$ and $i \in \{1..4\}$. In particular, any automorphism taking the vertex $(0)_0$ to its neighbour $(1)_1$ must be of the form $g = n\delta^i$ with $n \in N$ and $i \in \{1, 3\}$. This gives just two possibilities for such an automorphism namely $g_1 = q^3\delta$ and $g_2 = q\delta^3$ where $q \in N$. These two automorphisms map $(1)_1$ to $(1+q)_0$ and $(1-q)_0$ respectively. Thus no element of G can reverse edges and Γ is G-oriented.

Since the only proper non-trivial normal subgroups of G are N and $N\langle\delta^2\rangle$ it follows that (Γ, G) is basic of biquasiprimitve type. □

CONSTRUCTION 5.14. *Let* $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$ *where* $N = \mathbb{Z}_p^2$ *for a prime* $p \equiv 3$ *mod 4, and* $S = \{\pm(1,0), \pm(0,1)\}$. *Let* δ *be a permutation of* $V\Gamma$ *taking a vertex* $(x, y)_\epsilon$ *to* $(y, -x)_{1-\epsilon}$ *where* $x, y \in \mathbb{Z}_p$ *and* $\varepsilon \in \{0, 1\}$, *and let* $G := N \rtimes \langle\delta\rangle$.

LEMMA 5.15. *For* Γ, G *as in Construction 5.14,* $(\Gamma, G) \in \mathcal{OG}(4)$ *and is basic of biquasiprimitive type with* $\mathrm{soc}(G)$ *as described in Theorem 1.1 case* (a) *with* $k = 2$.

Proof. First note that $|S| = 4$ and $\langle S \rangle = \langle S^2 \rangle = N$ so Γ is 4-valent and connected. Also by Proposition 5.2, $\delta \in \mathrm{Aut}(\Gamma)$. Furthermore, the automorphism δ has order 4, and for the vertex $\alpha = (0,0)_0$, we have $G_\alpha = \langle\delta^2\rangle \cong C_2$ with two orbits of length two on the neighbourhood of α. Moreover any automorphism in G taking the vertex $(0,0)_0$ to its neighbour $(1,0)_1$ must be either $g_1 = (0,1)\delta$ or $g_2 = (0,-1)\delta^3$ where $(0,1)$ and $(0,-1)$ are elements of N. However, neither of these automorphisms maps $(1,0)_1$ to $(0,0)_0$ and so no $g \in G$ can reverse edges of Γ and $(\Gamma, G) \in \mathcal{OG}(4)$.

To show that (Γ, G) is basic of biquasiprimitive type, notice that the setwise stabiliser G^+ in G of the two parts N_0 and N_1 of $V\Gamma$ is $N \rtimes \langle\delta^2\rangle$, with δ^2 acting as inversion on N. Hence the nontrivial normal subgroups of G^+ are N, and the subgroups of N isomorphic to $\mathbb{Z}_p$ (all intransitive on N_0 since N is regular). Therefore we need to check that none of the subgroups of N of order p is normal in G.

To this end, notice that the subgroups corresponding to the direct factors of N are swapped by conjugation by δ in G, and hence aren't normal. All other nontrivial proper subgroups of N are of the form $\langle(1,x)\rangle$ with $x \in \mathbb{Z}_p^*$. Hence if $\langle(1,x)\rangle^\delta = \langle(x,-1)\rangle = \langle(1,x)\rangle$, then $c(1,x) = (x,-1)$ for some $c \in \mathbb{Z}_p^*$. It follows that $c = x$ and so $x^2 \equiv -1 \bmod p$, but this is impossible since $p \equiv 3 \bmod 4$. Thus the only proper non-trivial normal subgroups of G are N and G^+, both of which are transitive on the two biparts of $V\Gamma$. □

Next we give constructions of biquasiprimitive basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ with $\mathrm{soc}(G)$ nonabelian. Note that any nonabelian simple group T can be generated by an involution and an element of prime order [10]. In particular all nonabelian simple groups can be generated by two elements. In each of our constructions of biquasiprimitive pairs with nonabelian socle we will use a simple group T and a generating pair $\{a, b\}$ with prescribed properties.

We begin with constructions of biquasiprimitive basic pairs $(\Gamma, G) \in \mathcal{OG}(4)$ with $\mathrm{soc}(G)$ nonabelian and as described in Theorem 1.1 case (b).

CONSTRUCTION 5.16. *Let* T *be a nonabelian simple group, and let* $\{a, b\}$ *be a generating set for* T *where* a *is an involution and the elements* b *and* ab *have odd order. Let* $N = T$, $S_0 = \{ab, ba\}$, $S = S_0 \cup S_0^{-1}$, *and let* $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$. *Define two permutations* δ *and* σ *of* $V\Gamma$ *where* $x_\varepsilon^\delta = (x^a)_{1-\varepsilon}$, *and* $x_\varepsilon^\sigma = (x^a)_\varepsilon$ *for* $\varepsilon \in \{0, 1\}$, *and set* $G := N \rtimes \langle\sigma, \delta\rangle$.

REMARK 5.17. For an explicit example of a simple group T and generating set $\{a, b\}$ as in Construction 5.16 take T to be the alternating group $\mathrm{Alt}(n)$ for odd $n \geqslant 5$, and let $a = (12)(34)$ and $b = (12 \ldots n)$.

LEMMA 5.18. *For* Γ, G *as in Construction 5.19,* $(\Gamma, G) \in \mathcal{OG}(4)$ *and is basic of biquasiprimitive type with* $\mathrm{soc}(G)$ *as described in Theorem 1.1 case* (b) *with* $k = 1$.

Proof. Since N is nonabelian and the orders of b and ab are odd, it follows that $S_0 \cap S_0^{-1} = \varnothing$ (as $ab \neq (ab)^{-1}$) and hence that $|S| = 4$ and Γ is 4-valent. Again, using the fact that b has odd order it is easy to check that $a, b \in \langle S \rangle$ and hence that $\langle S \rangle = N$.

Now consider S^2. This set contains the elements $abab, b^2$ and $baba$. In particular, $\langle S^2 \rangle$ contains b and hence also contains aba. Since aba and $abab$ are contained in $\langle S^2 \rangle$ and the order of ab is odd, it follows that $a \in \langle S^2 \rangle$ and hence $\langle S^2 \rangle = N$. Therefore Γ is connected.

Next, notice that both σ and δ are induced by conjugation by a in N and this automorphism fixes S setwise. Hence σ and δ are automorphisms of Γ by Proposition 5.2. The stabiliser in G of the vertex $(1_N)_0$ is $\langle \sigma \rangle$ with two orbits on the neighbours of $(1_N)_0$, namely $\{(ab)_1, (ba)_1\}$ and $\{(b^{-1}a)_1, (ab^{-1})_1\}$. Furthermore a straightforward check shows that the only automorphisms in G mapping 1_0 to $(ab)_1$ are $g_1 = (ab)\sigma\delta$ and $g_2 = (ba)\delta$ (where (ab) and (ba) are automorphisms contained in N) and neither of these map $(ab)_1$ to 1_0. This implies that Γ is G-oriented and hence that $(\Gamma, G) \in \mathcal{OG}(4)$.

Now notice that neither $\langle \sigma \rangle$ nor $\langle \delta \rangle$ is normal in G. On the other hand, N is a normal (and hence minimal normal) subgroup of G, and is the unique such subgroup. Since N clearly has two orbits on $V\Gamma$, it follows that G is biquasiprimitive on the vertices of Γ. $\qquad\square$

CONSTRUCTION 5.19. *Let T be a nonabelian simple group, and let $\{a, b\}$ be a generating set for T such that no automorphism of T swaps a and b, and the elements a and b have odd order. Let $N = T \times T$, $S_0 = \{(a, b), (b, a)\}$, $S = S_0 \cup S_0^{-1}$, and let $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$. Define two permutations δ and σ of $V\Gamma$ where $(x, y)_\varepsilon^\delta = (y, x)_{1-\varepsilon}$, and $(x, y)_\varepsilon^\sigma = (y, x)_\varepsilon$ for $\varepsilon \in \{0, 1\}$. Set $G := N \rtimes \langle \sigma, \delta \rangle$.*

REMARK 5.20. For an explicit example of a simple group T and generating set $\{a, b\}$ as in Construction 5.19 take T to be the alternating group $\mathrm{Alt}(n)$ for odd n, and let $a = (123)$ and $b = (12 \ldots n)$.

LEMMA 5.21. *For Γ, G as in Construction 5.19, $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of biquasiprimitive type with $\mathrm{soc}(G)$ as described in Theorem 1.1 case (b) with $k = 2$.*

Proof. First notice that $S_0 \cap S_0^{-1} = \varnothing$ since if $(a, b)^{-1} = (a, b)$, then both a and b are involutions, while if $(a, b)^{-1} = (b, a)$ then $\langle a, b \rangle = \langle a \rangle$ is cyclic, and neither of these is possible. In particular, $|S| = 4$ and Γ is 4-valent.

To see that Γ is connected consider the following. The projections of $\langle S \rangle$ onto the simple direct factors of $N = T \times T$ are both equal to the group $\langle a, b \rangle = T$. Hence either $\langle S \rangle = N$ or $\langle S \rangle = \{(t, t^\varphi), t \in T\}$ for some $\varphi \in \mathrm{Aut}(T)$. In the latter case, $(a, b) = (a, a^\varphi)$ so $b = a^\varphi$, but also $(b, a) = (b, b^\varphi)$ so $a = b^\varphi$, but by our assumption no such automorphism φ exists. Hence $N = \langle S \rangle$. Finally, notice that since both a and b have odd order, we have $(a, b) \in \langle (a^2, b^2) \rangle$ (and similarly $(b, a) \in \langle (b^2, a^2) \rangle$). In particular both (a, b) and (b, a) are contained in $\langle S^2 \rangle$, so $N = \langle S \rangle = \langle S^2 \rangle$, and Γ is connected.

Once again Proposition 5.2 implies that $\sigma, \delta \in \mathrm{Aut}(\Gamma)$. Now it is clear that G acts transitively on the vertices of Γ and the stabiliser in G of the vertex $(1_N)_0$ is exactly $\langle \sigma \rangle \cong C_2$ with two orbits on the neighbourhood of $(1_N)_0$. Moreover, it is easy to check that no automorphism can reverse edges as follows. The only automorphisms taking $(1_N)_0$ to $(a, b)_1$ are $g_1 = n_1\sigma\delta$ and $g_2 = n_2\delta$ where $n_1 = (a, b)$ and $n_2 = (b, a)$ are elements of N. Since neither of these maps $(a, b)_1$ to $(1_N)_0$, it follows that Γ is G-oriented and $(\Gamma, G) \in \mathcal{OG}(4)$.

Finally, since conjugation by σ in G interchanges the two simple direct factors of N, it follows that N is a minimal normal subgroup of G and so is the unique minimal normal subgroup. Of course, N has two orbits on $V\Gamma$, thus G is biquasiprimitive on the vertices of Γ. $\qquad\square$

Next we give a construction of biquasiprimitive basic pairs as described in Theorem 1.1 case (b) with $k = 4$. This time we will use Method 5.7. We will use the same simple group T and generating pair $\{a, b\}$ in Constructions 5.23, 5.28 and 5.30. Hence we begin with the following important remark.

REMARK 5.22. For a prime $p \geqslant 7$ let T denote the simple group $\mathrm{PSL}(2, p)$. Then T is generated by two elements a and b where

$$a := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Moreover a and b have orders 2 and 3 respectively, while ab and ab^2 have order p, [6, Section 7.5].

CONSTRUCTION 5.23. *For a prime $p \geqslant 7$ let T denote the simple group $\mathrm{PSL}(2, p)$ generated by two elements a and b such that a and b have orders 2 and 3 respectively while ab and ab^2 have order p. Take the group $T \wr S_4$ with S_4 acting by permuting the four direct factors of T^4 and define the following elements of this group*

$$\tilde{\varphi} := (b, ba, ab, aba)(13),$$
$$y := \tilde{\varphi}^2 = (bab, baba, ab^2, ab^2 a),$$
$$h_1 := (a, a, a, a)(12)(34),$$
$$h_2 := h_1^{\tilde{\varphi}} = (b^{-1}aba, ab^{-1}ab, b^{-1}aba, ab^{-1}ab)(14)(23).$$

Now let $V := \langle h_1, h_2 \rangle$ and define the subgroup $H := T^4 \rtimes V \leqslant T \wr S_4$. Notice that conjugation by $\tilde{\varphi}$ in $T \wr S_4$ induces an automorphism $\varphi \in \mathrm{Aut}(H)$, in particular φ^2 is the inner automorphism of H corresponding to conjugation by $y \in H$.

Finally apply Construction 5.3 using H, V, y and φ to get the pair (Γ, G).

REMARK 5.24. Constructions 5.23, 5.28 and 5.30 all produce basic pairs of biquasiprimitive type via Method 5.7. In order to apply this method we must first show that the quadruple (H, V, y, φ) given in these constructions is appropriate. In each of these constructions, the fact that the automorphism $\varphi^2 = \iota_y$ and the fact that $y^\varphi = y$ follow immediately from the choices of $\varphi \in \mathrm{Aut}(H)$ and $y \in H$. Thus showing that (H, V, y, φ) is appropriate amounts to showing that $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$.

In each of these three constructions the group H is of the form $T^k \rtimes V \leqslant T^k \wr S_k$, where $k \geqslant 1$ and V is an elementary abelian 2-group. We may thus define two types of projection maps as follows. Let ρ be the projection map $\rho : T \wr S_k \to S_k$ and, for each $i \in [1, k]$, let π_i be the projection map $\pi_i : T_1 \times \cdots T_k \to T_i$.

Now if $\varphi = \iota_u$ for some $u \in H$ with $y = u^2$, then we have $u = ts$ where $t \in T^k$, $s \in \rho(H)$ and $s^2 = 1$. Furthermore, since $u^2 = y$ it follows that $u = ts$ and y commute, which implies that $y^{ts} = y$ and hence that $y^t = y^s$. We then have $\pi_{s(i)}(y) = \pi_i(y^s) = \pi_i(y^t) = \pi_i(y)^{\pi_i(t)}$ and it follows that, if s interchanges the ith and jth simple direct factors of T^k, then $\pi_j(y)$ and $\pi_i(y)$ lie in the same conjugacy class of T, a fact which we will use in proving that each of these constructions gives appropriate (H, V, y, φ).

LEMMA 5.25. *Let Γ, G be as in Construction 5.23. Then $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of biquasiprimitive type, with $\mathrm{soc}(G)$ as described in Theorem 1.1 case (b) with $k = 4$.*

Proof. We begin by showing that the quadruple (H, V, y, φ) as given in Construction 5.23 is appropriate. By Remark 5.24 we only need to show that $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$. Suppose then, for a contradiction, that $\varphi = \iota_u$ for some $u \in H$ such that $u^2 = y$. By Remark 5.24 we know that $u = ts$ for some $t \in T^4$ and $s \in \rho(H) = \rho(V)$ where, if s interchanges i and j, then $\pi_i(y)$ and $\pi_j(y)$ lie in

the same conjugacy class of T. Consider the order $|\pi_\ell(y)|$ for each ℓ. It is easy to check that $(|bab|, |baba|, |ab^2|, |ab^2a|) = (p, p, p, 3)$, and hence that ab^2a cannot lie in the same conjugacy class as any of the other entries of y. On the other hand, since each nontrivial element $c \in \rho(H)$ interchanges two pairs of elements of $\{1, 2, 3, 4\}$, it follows that $s = 1$. Thus $\varphi = \iota_u$ for some $u \in T^4$. However, it is clear that if we take $z := (1, 1, a, 1) \in H$, then $z^\varphi \neq z^u$ for any $u \in T^4$ since $\pi_1(z^u) = 1$ for each $u \in T^4$, while $\pi_1(z^\varphi) = a^{ab} \neq 1$. Thus $\varphi \neq \iota_u$ for any $u \in T^4$, a contradiction. Hence by Remark 5.24, the quadruple (H, V, y, φ) as given in Construction 5.23 is appropriate.

Thus Construction 5.23 is a special case of Construction 5.3. Hence, in order to show that $(\Gamma, G) \in \mathcal{OG}(4)$ it suffices to show that condition (2) of Lemma 5.4 is satisfied. First notice that $V \cong \mathbb{Z}_2^2$ since h_1 and h_2 are commuting involutions. Also V is core-free in H since for instance $V \cap V^y = 1$. It is also easy to check that $V \cap V^\varphi = \langle h_2 \rangle$, and so $|V : V \cap V^\varphi| = 2$.

Now suppose that $y \in VV^\varphi$ so that $y = vu$ for some $v \in V$, and $u \in V^\varphi$. This implies that $vu \in T^4$, and hence, if we again take ρ to be the projection map $T \wr S_4 \to S_4$, then $\rho(v) = \rho(u)$. Hence the only possibilities for (v, u) such that $y = vu$ that need to be considered are $(h_1, h_2^\varphi), (h_2, h_2)$, and $(h_1h_2, h_2h_2^\varphi)$. The second possibility gives $h_2^2 = 1 \neq y$, while the first and third possibilities both give $y = h_1h_2^\varphi$. It is easy to check however that $h_1h_2^\varphi = h_1h_1^y$ has bab^2 in its third coordinate while y has ab^2 in its third coordinate. Hence $y \notin VV^\varphi$.

It remains to show that $\langle V, y \rangle = H$, and to prove this it is sufficient to show that $T^4 \leqslant \langle V, y \rangle$. To this end, let $y_1 := y^{h_1}$ and $y_2 := y^{h_2}$, so that we have

$$y = (bab, baba, ab^2, ab^2a),$$
$$y_1 = (abab, ababa, b^2, b^2a), \text{ and}$$
$$y_2 = (b^2ab^2ab, ab^2ababa, b^2abab^2, ab^2).$$

We claim that $T^4 = \langle y, y_1, y_2 \rangle \leqslant \langle V, y \rangle$.

First, it is straightforward to check that the group $\langle y, y_1, y_2 \rangle$ projects onto each simple direct factor of T^4. Consider now the elements of T appearing as coordinates of y, y_1 and y_2. It is easy to see that the three elements ab^2a, b^2, and b^2ab^2ab have order 3. On the other hand, using the fact that ab and ab^2 have order p, we can check that $abab, bab$ and b^2abab^2 also have order p. The remaining elements appearing as coordinates of y, y_1 and y_2 are conjugates of these elements of order p and hence also have the same order. In particular, since the only elements of order 3 (ab^2a, b^2, and b^2ab^2ab), appear in the fourth, third and first coordinates of y, y_1 and y_2 respectively, and $\langle y, y_1, y_2 \rangle$ is a subdirect subgroup of T^4, it follows that $T^4 = \langle y, y_1, y_2 \rangle$ and so $\langle V, y \rangle = H$. Hence, by Lemma 5.4, $(\Gamma, G) \in \mathcal{OG}(4)$.

Finally we show that (Γ, G) is basic of biquasiprimitive type. Since H acts transitively on the simple direct factors of T^4, it follows that T^4 is a minimal normal subgroup of H, and is the unique such subgroup. Hence $N = \mathrm{Diag}_\varphi(T^4 \times T^4) \cong T^4$ is the unique minimal normal subgroup of G^+, and must be the unique minimal normal subgroup of G. Hence (Γ, G) is biquasiprimitive by Corollary 5.6. □

We conclude this section by giving constructions of basic biquasiprimitive $(\Gamma, G) \in \mathcal{OG}(4)$ as described in Theorem 1.1 case (c). The first construction is similar to Construction 5.19. As in that construction, the alternating group $\mathrm{Alt}(n)$ with n odd, and generators $a = (123)$ and $b = (12 \ldots n)$ will have the required properties.

CONSTRUCTION 5.26. *Let T be a nonabelian simple group, and let $\{a, b\}$ be a generating set for T such that no automorphism of T swaps a and b, and the elements a and b have odd order. Suppose further that there is an automorphism $\theta \in \mathrm{Aut}(T)$ which*

inverts both generators a and b. Let $N = T \times T$, $S_0 = \{(a, b), (b, a)\}$, $S = S_0 \cup S_0^{-1}$, and let $\Gamma = \mathrm{BiCay}(N, \varnothing, \varnothing, S)$. Define two permutations δ and σ of $V\Gamma$, where $(x, y)_\varepsilon^\delta = (y, x)_{1-\varepsilon}$, and $(x, y)_\varepsilon^\sigma = (x^\theta, y^\theta)_\varepsilon$ for $\varepsilon \in \{0, 1\}$. Set $G := N \rtimes \langle \sigma, \delta \rangle$.

LEMMA 5.27. *For Γ, G as in Construction 5.26, $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of bi-quasiprimitive type with $\mathrm{soc}(G)$ as described in Theorem 1.1 case (c) with $\ell = 1$.*

Proof. Since Γ is the same graph from Construction 5.19, it follows from Lemma 5.21 that Γ is 4-valent and connected. Again σ and δ are induced by automorphisms of N which fix S and hence are automorphisms of Γ by Proposition 5.2. Moreover it is a straightforward check that the stabiliser in G of the vertex $(1_N)_0$ is $\langle \sigma \rangle \cong C_2$ and also that there are only two automorphisms in G mapping the vertex $(1_N)_0$ to its neighbour $(a, b)_1$ but neither of these reverses the edge $\{(1_N)_0, (a, b)_1\}$. Hence Γ is G-oriented and $(\Gamma, G) \in \mathcal{OG}(4)$.

Now notice the setwise stabiliser in G of $\Delta := N_0$ is $G^+ = N\langle \sigma \rangle$ and that $T \times 1 \leqslant N$ is a normal subgroup of G^+ which is intransitive on Δ. In particular, G^+ is not quasiprimitve on Δ. Moreover δ interchanges the two simple direct factors of N, and hence N is the unique minimal normal subgroup of G. Since N is contained in G^+ it follows that Γ is basic of biquasiprimitive type as in Lemma 4.2 case (b). $\square$

The next two constructions both provide pairs (Γ, G) as described in Theorem 1.1 case (c) with $\ell = 2$ and $\ell = 4$ respectively. In both cases $\mathrm{soc}(G) = T^{2\ell}$ where T is the simple group $\mathrm{PSL}(2, p)$. In both cases we may use the same generating pairs $\{a, b\}$ as those used in Construction 5.23 (see Remark 5.22).

CONSTRUCTION 5.28. *For a prime $p \geqslant 7$ let T denote the simple group $\mathrm{PSL}(2, p)$ generated by two elements a and b such that a and b have orders 2 and 3 respectively while ab and ab^2 have order p. Take the group $T \wr S_4$ with S_4 acting by permuting the four direct factors of T^4 and define the following elements of this group*

$$\tilde{\varphi} := (b^2 ab, ab^2, b^2, a)(13)(24),$$
$$y := \tilde{\varphi}^2 = (b^2 a, ab^2 a, bab, b^2),$$
$$h_1 := (a, a, a, a)(12)(34).$$

Now let $V := \langle h_1 \rangle$ and define the subgroup $H := T^4 \rtimes V \leqslant T \wr S_4$. Notice that conjugation by $\tilde{\varphi}$ in $T \wr S_4$ induces an automorphism $\varphi \in \mathrm{Aut}(H)$, in particular φ^2 is the inner automorphism of H corresponding to conjugation by $y \in H$.

Finally apply Construction 5.3 using H, V, y and φ to get the pair (Γ, G).

LEMMA 5.29. *Let Γ, G be as in Construction 5.28. Then $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of biquasiprimitive type, with $\mathrm{soc}(G)$ as described in Theorem 1.1 case (c) with $\ell = 2$.*

Proof. We begin by showing that the quadruple (H, V, y, φ) as given in Construction 5.28 is appropriate. By Remark 5.24 we only need to show that $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$. Suppose then, for a contradiction, that $\varphi = \iota_u$ for some $u \in H$ such that $u^2 = y$. By Remark 5.24 we know that $u = ts$ for some $t \in T^4$ and $s \in \rho(H) = \rho(V)$, where if s interchanges i and j then $\pi_i(y)$ and $\pi_j(y)$ lie in the same conjugacy class of T. Now consider the order $|\pi_\ell(y)|$ for each ℓ. It is easy to check that $(|b^2 a|, |ab^2 a|, |bab|, |b^2|) = (p, 3, p, 3)$, implying that s cannot interchange 1 and 2. However, since $s \in \rho(H) = \langle(12)(34)\rangle$ it follows that $s = 1$, and so $\varphi = \iota_u$ for some $u \in T^4$. It is easy to see that, if we take $z := (1, 1, a, 1) \in H$ then $z^\varphi \neq z^u$ for any $u \in T^4$, since $\pi_1(z^u) = 1$ for all $u \in T^4$ while $\pi_1(z^\varphi) = a^{b^2} \neq 1$. This contradiction implies that $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$. Hence by Remark 5.24, the quadruple (H, V, y, φ) as given in Construction 5.28 is appropriate.

Thus Construction 5.28 is a special case of Construction 5.3. Hence, in order to show that $(\Gamma, G) \in \mathcal{OG}(4)$ it is sufficient to show that condition (2) of Lemma 5.4 is satisfied. Here $V \cong C_2$ and since $h_1^\varphi \notin V$ we have that V is core-free in H and $|V : V \cap V^\varphi| = 2$. It is also easy to check that $y \notin VV^\varphi$ in this case, by noticing that $y \neq h_1 h_1^\varphi$.

It remains to show that $\langle V, y \rangle = H$. In fact, we will show that $T^4 \leqslant \langle y_1, y \rangle$ where $y_1 := y^{h_1} = (b^2, ab^2, ab^2 a, ababa)$, from which it follows that $\langle V, y \rangle = H$. First, $\langle y, y_1 \rangle$ projects onto each simple direct factor of T^4, so we only need to make sure that $\langle y, y_1 \rangle$ is not a product of diagonal subgroups of T^4. Now y has elements of order p in its first and third coordinates and elements of order 3 in its second and fourth coordinates. Thus all we need to check is that no automorphism of T can map $b^2 a$ to bab and b^2 to $ab^2 a$ and that no automorphism can map $ab^2 a$ to b^2 and ab^2 to $ababa$. In the first case, such an automorphism must map a to $(ab)^3$ which is impossible since a is an involution. In the second case, such an automorphism must map a to $(ab^2)^3$, which again is not an involution. Hence $T^4 \leqslant \langle y_1, y \rangle$ and so $\langle V, y \rangle = H$. By Lemma 5.4 we conclude that $(\Gamma, G) \in \mathcal{OG}(4)$.

It is clear that the action of H on the simple direct factors of T^4 has two orbits of length 2. Thus H has two minimal normal subgroups isomorphic to T^2, and these are the only minimal normal subgroups of H. Furthermore, it is clear that the automorphism φ of H interchanges these normal subgroups. Let R and R^φ denote these two minimal normal subgroups of H.

Since $G^+ \cong H$, G^+ also has two minimal normal subgroups isomorphic to $R, R^\varphi \cong T^2$. Let K and L denote these minimal normal subgroups of G^+ so $K = \operatorname{Diag}_\varphi(R \times R)$ and $L = \operatorname{Diag}_\varphi(R^\varphi \times R^\varphi)$. Then conjugation by g in G, interchanges K and L and so G acts transitively on the direct factors of $\operatorname{soc}(G^+) = K \times L \cong T^4$. Hence $\operatorname{soc}(G^+)$ is a minimal normal subgroup of G and (Γ, G) is biquasiprimitive by Corollary 5.6. $\square$

CONSTRUCTION 5.30. *For a prime $p \geqslant 7$ let T denote the simple group $\operatorname{PSL}(2, p)$ generated by two elements a and b such that a and b have orders 2 and 3 respectively while ab and ab^2 have order p. Take the group $T \wr S_8$ with S_8 acting by permuting the eight direct factors of T^8 and define the following elements of this group*

$$\tilde{\varphi} := (b, ba, ab, aba, b^2, ab, ba, ab^2 a)(15)(28)(37)(46),$$
$$y := \tilde{\varphi}^2 = (1, a, ab^2 a, ab^2, 1, ababa, b^2, ab^2 aba),$$
$$h_1 := (a, a, a, a, a, a, a, a)(12)(34)(56)(78),$$
$$h_2 := h_1^{\tilde{\varphi}} = (b^2, ab^2 a, aba, b, b^2 aba, ab^2 ab, b^2 aba, ab^2 ab)(14)(23)(58)(67).$$

Now let $V := \langle h_1, h_2 \rangle$ and define the subgroup $H := T^8 \rtimes V \leqslant T \wr S_8$. Notice that conjugation by $\tilde{\varphi}$ in $T \wr S_8$ induces an automorphism $\varphi \in \operatorname{Aut}(H)$, in particular φ^2 is the inner automorphism of H corresponding to conjugation by $y \in H$.

Finally apply Construction 5.3 using H, V, y and φ to get the pair (Γ, G).

LEMMA 5.31. *Let Γ, G be as in Construction 5.30. Then $(\Gamma, G) \in \mathcal{OG}(4)$ and is basic of biquasiprimitive type with $\operatorname{soc}(G)$ as described in Theorem 1.1 case (c) with $\ell = 4$.*

Proof. We begin by showing that the quadruple (H, V, y, φ) as given in Construction 5.30 is appropriate. By Remark 5.24 we only need to show that $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$. Suppose then, for a contradiction, that $\varphi = \iota_u$ for some $u \in H$ such that $u^2 = y$. By Remark 5.24 we know that $u = ts$ for some $t \in T^8$ and $s \in \rho(H) = \rho(V)$, where if s interchanges i and j then $\pi_i(y)$ and $\pi_j(y)$ lie in the same conjugacy class of T. Now consider the order $|\pi_\ell(y)|$ for each ℓ. It is easy to check that

$$(|1|, |a|, |ab^2 a|, |ab^2|, |1|, |ababa|, |b^2|, |ab^2 aba|) = (1, 2, 3, p, 1, p, 3, 2).$$

This implies that s cannot interchange any two elements of $\{1,2,3,4\}$ or of $\{5,6,7,8\}$. However $s \in \rho(H)$ and the orbits of $\rho(H)$ are precisely $\{1,2,3,4\}$ and $\{5,6,7,8\}$. Thus s must fix all elements in each of these orbits, and so $s = 1$ and $\varphi = \iota_u$ for some $u \in T^8$. However, it is easy to see that if we take $z := (1,1,1,1,a,1,1,1) \in H$, then $z^\varphi \neq z^u$ for any $u \in T^8$ since $\pi_1(z^u) = 1$ for all $u \in T^8$ while $\pi_1(z^\varphi) = a^{b^2} \neq 1$. Thus $\varphi \neq \iota_u$ for any $u \in H$ with $u^2 = y$. Hence by Remark 5.24, the quadruple (H, V, y, φ) as given in Construction 5.30 is appropriate.

Hence, as in previous constructions, we only need to check that condition (2) of Lemma 5.4 is satisfied to show that $(\Gamma, G) \in \mathcal{OG}(4)$. Here we have $V = \langle h_1, h_2 \rangle \cong \mathbb{Z}_2^2$ with $V \cap V^\varphi = \langle h_2 \rangle$ and $V \cap V^y = 1$. Hence V is core-free in H and $|V : V \cap V^\varphi| = 2$. To check that $y \notin V V^\varphi$ it is sufficient to check that $y \neq h_1 h_1^y$ and this is clearly true since $y \neq 1$.

It remains to show that $\langle V, y \rangle = H$, and for this it is sufficient to show that $T^8 \leqslant \langle V, y \rangle$. In fact we will show that $T^8 = \langle y, y_1, y_2 \rangle \leqslant \langle V, y \rangle$ where $y_1 := y^{h_1}$, and $y_2 := y^{h_2}$. We have

$$
\begin{aligned}
y &= (1, a, ab^2a, ab^2, 1, ababa, b^2, ab^2aba), \\
y_1 &= (a, 1, b^2a, b^2, bab, 1, b^2ab, ab^2a), \text{ and} \\
y_2 &= (b^2a, ab^2a, abab^2a, 1, b^2ab, ab^2ab^2aba, b^2a, 1).
\end{aligned}
$$

It is easy to check that $\langle y, y_1, y_2 \rangle$ projects onto each simple direct factor of T^8. Furthermore, we note that the identity element occurs in the first and fifth coordinates of y, the second and sixth coordinates of y_1, and the fourth and eighth coordinates of y_2. So if $\langle y, y_1, y_2 \rangle$ is a product of diagonal subgroups of $T^8 = \Pi_{i=1}^8 T_i$ then each direct factor of $\langle y, y_1, y_2 \rangle$ must be either a full subgroup T_j for some $1 \leqslant j \leqslant 8$, or a diagonal subgroup of a subproduct $T_m \times T_n$ where $(m,n) \in \{(1,5),(2,6),(3,7),(4,8)\}$.

However, the elements in the first and fifth coordinates of y_2 have orders p and 2 respectively, the elements in the second and sixth coordinates of y have orders 2 and p respectively, the elements in the third and seventh coordinates of y_1 have orders p and 2 respectively, and the elements in the fourth and eighth coordinates of y have orders p and 2 respectively. Therefore $\langle y, y_1, y_2 \rangle = \Pi_{i=1}^8 T_i = T^8$ and hence $H = \langle V, y \rangle$. Lemma 5.4 now implies that $(\Gamma, G) \in \mathcal{OG}(4)$.

Note that the action of H on the simple direct factors of T^8 has two orbits of length 4. Thus H has two minimal normal subgroups isomorphic to T^4, and these subgroups are interchanged by the automorphism $\varphi \in \text{Aut}(H)$. As in previous constructions, we let R and R^φ denote these two minimal normal subgroups of H.

Since $G^+ \cong H$, G^+ also has two minimal normal subgroups, namely $K = \text{Diag}_\varphi(R \times R)$ and $L = \text{Diag}_\varphi(R^\varphi \times R^\varphi)$, and conjugation by $g \in G$ interchanges K and L, implying that G acts transitively on the direct factors of $K \times L \cong T^8$. In particular, $\text{soc}(G^+) = K \times L$ is a minimal normal subgroup of G, and hence (Γ, G) is biquasiprimitive by Corollary 5.6. This shows that (Γ, G) is basic of biquasiprimitive type, as described in Lemma 4.2 case (b) with $\ell = 4$. $\qquad \square$

Constructions 5.11–5.30 together with Lemmas 5.13–5.31, and the remarks in this section which give explicit simple groups and generating pairs for each construction, and therefore complete the proof of Theorem 1.1.

Note that in each of the explicit examples of biquasiprimitive pairs (Γ, G) provided here, the group G contains a subgroup N acting semi-regularly with two orbits on $V\Gamma$, hence all of these examples are bi-Cayley graphs. Of course, it should not be too difficult to construct non-bi-Cayley examples using Method 5.7.

An interesting further question would be to determine which nonabelian simple groups T can occur as the simple direct factors of the socle of G where $(\Gamma, G) \in \mathcal{OG}(4)$ is biquasiprimitive.

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