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Abstract Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] = \mathbb{Q}[x_1, \dots, x_n]$ by variable permutation. The coinvariant algebra is the graded $\mathfrak{S}_n$-module $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$, where I_n is the ideal in $\mathbb{Q}[\mathbf{x}_n]$ generated by invariant polynomials with vanishing constant term. Haglund, Rhoades, and Shimozono introduced a new quotient $R_{n,k}$ of the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ depending on two positive integers $k \leqslant n$ which reduces to the classical coinvariant algebra of the symmetric group $\mathfrak{S}_n$ when $k = n$. The quotient $R_{n,k}$ carries the structure of a graded $\mathfrak{S}_n$-module; Haglund et. al. determine its graded isomorphism type and relate it to the Delta Conjecture in the theory of Macdonald polynomials. We introduce and study a related quotient $S_{n,k}$ of $\mathbb{F}[\mathbf{x}_n]$ which carries a graded action of the 0-Hecke algebra $H_n(0)$, where $\mathbb{F}$ is an arbitrary field. We prove 0-Hecke analogs of the results of Haglund, Rhoades, and Shimozono. In the classical case $k = n$, we recover earlier results of Huang concerning the 0-Hecke action on the coinvariant algebra.

1. Introduction

The purpose of this paper is to define and study a 0-Hecke analog of a recently defined graded module for the symmetric group [16]. Our construction has connections with the combinatorics of ordered set partitions and the Delta Conjecture [15] in the theory of Macdonald polynomials.

The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \dots, x_n]$ by variable permutation. The corresponding *invariant subring* $\mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n}$ consists of all $f \in \mathbb{Q}[\mathbf{x}_n]$ with $w(f) = f$ for all $w \in \mathfrak{S}_n$, and is generated by the elementary symmetric functions $e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n)$, where

$$(1) \qquad e_d(\mathbf{x}_n) = e_d(x_1, \dots, x_n) := \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} x_{i_1} \cdots x_{i_d}.$$

The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal generated by those invariants $\mathbb{Q}[\mathbf{x}_n]_+^{\mathfrak{S}_n}$ with vanishing constant term:

$$(2) \qquad I_n := \langle \mathbb{Q}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle = \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle.$$

The *coinvariant algebra* $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$ is the corresponding quotient ring.

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The coinvariant algebra R_n inherits a graded action of $\mathfrak{S}_n$ from $\mathbb{Q}[\mathbf{x}_n]$. This module is among the most important representations in algebraic and geometric combinatorics. Its algebraic properties are closely tied to the combinatorics of permutations in $\mathfrak{S}_n$; let us recall some of these properties.

- The quotient R_n has dimension $n!$ as a $\mathbb{Q}$-vector space. In fact, E. Artin [2] used Galois theory to prove that the set of 'sub-staircase' monomials $\mathcal{A}_n := \{x_1^{i_1} \cdots x_n^{i_n} : 0 \leqslant i_j < j\}$ descends to a basis for R_n.
- A different monomial basis $\mathcal{GS}_n$ of R_n was discovered by Garsia and Stanton [12]. Given a permutation $w = w(1) \dots w(n) \in \mathfrak{S}_n$, the corresponding GS monomial basis element is

$$gs_w := \prod_{w(i) > w(i+1)} x_{w(1)} \cdots x_{w(i)}.$$

- Chevalley [8] proved that R_n is isomorphic as an ungraded $\mathfrak{S}_n$-module to the regular representation $\mathbb{Q}[\mathfrak{S}_n]$.
- Lusztig (unpublished) and Stanley described the *graded* $\mathfrak{S}_n$-module structure of R_n using the major index statistic on standard Young tableaux [25].

Let $k \leqslant n$ be two positive integers. Haglund, Rhoades, and Shimozono [16, Defn. 1.1] introduced the ideal $I_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n]$ with generators

$$I_{n,k} := \langle x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle \tag{3}$$

and studied the corresponding quotient ring $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$. Since $I_{n,k}$ is homogeneous and stable under the action of $\mathfrak{S}_n$, the ring $R_{n,k}$ is a graded $\mathfrak{S}_n$-module. When $k = n$, we have $I_{n,n} = I_n$, so that $R_{n,n} = R_n$ and we recover the usual invariant ideal and coinvariant algebra.

To study $R_{n,k}$ one needs the notion of an *ordered set partition* of $[n] := \{1, 2, \dots, n\}$, which is a set partition of $[n]$ with a total order on its blocks. For example, we have an ordered set partition

$$\sigma = (25 \mid 6 \mid 134)$$

written in the 'bar notation'. The three blocks $\{2, 5\}$, $\{6\}$, and $\{1, 3, 4\}$ are ordered from left to right, and elements of each block are increasing.

Let $\mathcal{OP}_{n,k}$ denote the collection of ordered set partitions of $[n]$ with k blocks. We have

$$|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n, k), \tag{4}$$

where $\mathrm{Stir}(n, k)$ is the (signless) Stirling number of the second kind counting k-block set partitions of $[n]$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{OP}_{n,k}$ by permuting the letters $1, \dots, n$. For example, the permutation $w = 241365$, written in one-line notation, sends $(25 \mid 6 \mid 134)$ to $(46 \mid 5 \mid 123)$.

Just as the structure of the classical coinvariant module R_n is controlled by permutations in $\mathfrak{S}_n$, the structure of $R_{n,k}$ is governed by the collection $\mathcal{OP}_{n,k}$ of ordered set partitions of $[n]$ with k blocks [16].

- The dimension of $R_{n,k}$ is $|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n, k)$ [16, Thm. 4.11]. We have a generalization $\mathcal{A}_{n,k}$ of the Artin monomial basis to $R_{n,k}$ [16, Thm. 4.13].
- There is a generalization $\mathcal{GS}_{n,k}$ of the Garsia-Stanton monomial basis to $R_{n,k}$ [16, Thm. 5.3].
- The module $R_{n,k}$ is isomorphic as an *ungraded* $\mathfrak{S}_n$-representation to $\mathcal{OP}_{n,k}$ [16, Thm. 4.11].
- There are explicit descriptions of the *graded* $\mathfrak{S}_n$-module structure of $R_{n,k}$ which generalize the work of Lusztig–Stanley [16, Thm 6.11, Cor. 6.12, Cor. 6.13, Thm. 6.14].

Now let $\mathbb{F}$ be an arbitrary field and let n be a positive integer. The (*type A*) 0-*Hecke algebra* $H_n(0)$ is the unital associative $\mathbb{F}$-algebra with generators $\pi_1, \pi_2, \ldots, \pi_{n-1}$ and relations

$$\begin{cases} \pi_i^2 = \pi_i & 1 \leqslant i \leqslant n-1, \\ \pi_i \pi_j = \pi_j \pi_i & |i-j| > 1, \\ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} & 1 \leqslant i \leqslant n-2. \end{cases} \tag{5}$$

Recall that the symmetric group $\mathfrak{S}_n$ has Coxeter generators $\{s_1, s_2, \ldots, s_{n-1}\}$, where s_i is the adjacent transposition $s_i = (i, i+1)$. These generators satisfy similar relations as (5) except that $s_i^2 = 1$ for all i. If $w \in \mathfrak{S}_n$ is a permutation and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced (i.e., as short as possible) expression for w in the Coxeter generators $\{s_1, \ldots, s_{n-1}\}$, we define the 0-Hecke algebra element $\pi_w := \pi_{i_1} \cdots \pi_{i_\ell} \in H_n(0)$. It can be shown that the set $\{\pi_w : w \in \mathfrak{S}_n\}$ forms a basis for $H_n(0)$ as an $\mathbb{F}$-vector space, and in particular $H_n(0)$ has dimension $n!$. In contrast to the situation with the symmetric group, the representation theory of the 0-Hecke algebra is insensitive to the choice of ground field, which motivates our generalization from $\mathbb{Q}$ to $\mathbb{F}$.

The algebra $H_n(0)$ is a deformation of the symmetric group algebra $\mathbb{F}[\mathfrak{S}_n]$. Roughly speaking, whereas in a typical $\mathbb{F}[\mathfrak{S}_n]$-module the generator s_i acts by 'swapping' the letters i and $i+1$, in a typical $H_n(0)$-module the generator π_i acts by 'sorting' the letters i and $i+1$. Indeed, the relations satisfied by the π_i are precisely the relations satisfied by bubble sorting operators acting on a length n list of entries $x_1 \ldots x_n$ from a totally ordered alphabet:

$$\pi_i.(x_1 \ldots x_i x_{i+1} \ldots x_n) := \begin{cases} x_1 \ldots x_{i+1} x_i \ldots x_n & x_i > x_{i+1} \\ x_1 \ldots x_i x_{i+1} \ldots x_n & x_i \leqslant x_{i+1}. \end{cases} \tag{6}$$

Proving 0-Hecke analogs of module theoretic results concerning the symmetric group has received a great deal of recent study in algebraic combinatorics [4, 17, 18, 26]; let us recall the 0-Hecke analog of the variable permutation action of $\mathfrak{S}_n$ on a polynomial ring.

Let $\mathbb{F}[\mathbf{x}_n] := \mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{F}$. The algebra $H_n(0)$ acts on $\mathbb{F}[\mathbf{x}_n]$ by the *isobaric Demazure operators*:

$$\pi_i(f) := \frac{x_i f - x_{i+1}(s_i(f))}{x_i - x_{i+1}}, \quad 1 \leqslant i \leqslant n-1. \tag{7}$$

If $f \in \mathbb{F}[\mathbf{x}_n]$ is symmetric in the variables x_i and x_{i+1}, then $s_i(f) = f$ and thus $\pi_i(f) = f$. The isobaric Demazure operators give a 0-Hecke analog of variable permutation.

We also have a 0-Hecke analog of the permutation action of $\mathfrak{S}_n$ on $\mathcal{OP}_{n,k}$. It is well-known that the 0-Hecke algebra $H_n(0)$ has another generating set $\{\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}\}$ subject to the relations

$$\begin{cases} \overline{\pi}_i^2 = -\overline{\pi}_i & 1 \leqslant i \leqslant n-1, \\ \overline{\pi}_i \overline{\pi}_j = \overline{\pi}_j \overline{\pi}_i & |i-j| > 1, \\ \overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1} & 1 \leqslant i \leqslant n-2. \end{cases} \tag{8}$$

Here $\overline{\pi}_i := \pi_i - 1$ for all i. We will often use the relation $\overline{\pi}_i \pi_i = \pi_i \overline{\pi}_i = 0$. One can define $\overline{\pi}_w := \overline{\pi}_{i_1} \cdots \overline{\pi}_{i_\ell}$ for any $w \in \mathfrak{S}_n$ with a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ and show that the set $\{\overline{\pi}_w : w \in \mathfrak{S}_n\}$ is a basis for $H_n(0)$. Let $\mathbb{F}[\mathcal{OP}_{n,k}]$ be the $\mathbb{F}$-vector space with basis given by $\mathcal{OP}_{n,k}$. Then $H_n(0)$ acts on $\mathbb{F}[\mathcal{OP}_{n,k}]$ by the rule

$$\overline{\pi}_i.\sigma := \begin{cases} -\sigma, & \text{if } i+1 \text{ appears in a block to the left of } i \text{ in } \sigma, \\ s_i(\sigma), & \text{if } i+1 \text{ appears in a block to the right of } i \text{ in } \sigma, \\ 0, & \text{if } i+1 \text{ appears in the same block as } i \text{ in } \sigma, \end{cases} \tag{9}$$

For example, we have

$$\begin{aligned}\overline{\pi}_1(25 \mid 6 \mid 134) &= -(25 \mid 6 \mid 134),\\ \overline{\pi}_2(25 \mid 6 \mid 134) &= (35 \mid 6 \mid 124),\\ \overline{\pi}_3(25 \mid 6 \mid 134) &= 0.\end{aligned}$$

It is straightforward to check that these operators satisfy the relations (8) and so define an $H_n(0)$-action on $\mathbb{F}[\mathcal{OP}_{n,k}]$. In fact, this is a special case of an $H_n(0)$-action on generalized ribbon tableaux introduced in [18]. See also the proof of Lemma 5.2.

The coinvariant algebra R_n can be viewed as a 0-Hecke module. Indeed, the "Leibniz rule"

$$\overline{\pi}_i(fg) = \overline{\pi}_i(f)g + s_i(f)\overline{\pi}_i(g) \tag{10}$$

implies that the ideal $I_n \subseteq \mathbb{F}[\mathbf{x}_n]$ generated by $e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]$ is stable under the action of $H_n(0)$ on $\mathbb{F}[\mathbf{x}_n]$. Therefore, the quotient $R_n = \mathbb{F}[\mathbf{x}_n]/I_n$ inherits a 0-Hecke action. Huang gave explicit formulas for its degree-graded and length-degree-bigraded quasisymmetric 0-Hecke characteristic [17, Cor. 4.9]. The bivariant characteristic $\mathrm{Ch}_{q,t}(R_n)$ turns out to be a generating function for the pair of Mahonian statistics (inv, maj) on permutations in $\mathfrak{S}_n$, weighted by the Gessel fundamental quasisymmetric function $F_{\mathrm{iDes}(w)}$ corresponding to the inverse descent set $\mathrm{iDes}(w)$ of $w \in \mathfrak{S}_n$ [17, Cor. 4.9 (i)].

We will study a 0-Hecke analog of the rings $R_{n,k}$ of Haglund, Rhoades, and Shimozono [16]. For $k < n$ the ideal $I_{n,k}$ is not usually stable under the action of $H_n(0)$ on $\mathbb{F}[\mathbf{x}_n]$, so that the quotient ring $R_{n,k} = \mathbb{F}[\mathbf{x}_n]/I_{n,k}$ does not have the structure of an $H_n(0)$-module. To remedy this situation, we introduce the following modified family of ideals. Let

$$h_d(x_1, \ldots, x_i) := \sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_d \leqslant i} x_{j_1} \cdots x_{j_d} \tag{11}$$

be the complete homogeneous symmetric function of degree d in the variables $x_1, x_2, \ldots, x_i$.

Definition 1.1. *For two positive integers $k \leqslant n$, we define a quotient ring*

$$S_{n,k} := \mathbb{F}[\mathbf{x}_n]/J_{n,k}$$

where $J_{n,k} \subseteq \mathbb{F}[\mathbf{x}_n]$ is the ideal with generators

$$J_{n,k} := \langle h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)\rangle.$$

The ideal $J_{n,k}$ is homogeneous. We claim that $J_{n,k}$ is stable under the action of $H_n(0)$. Since $e_d(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]^{\mathfrak{S}_n}$ and $h_k(x_1, \ldots, x_i)$ is symmetric in x_j and x_{j+1} for $j \neq i$, thanks to Equation (10) this reduces to the observation that

$$\pi_i(h_k(x_1, \ldots, x_i)) = h_k(x_1, \ldots, x_i, x_{i+1}). \tag{12}$$

Equation (12) is clear when $i = 1$ and can be obtained from the following identity when $i \geqslant 2$:

$$h_k(x_1, \ldots, x_i) = \sum_{0 \leqslant j \leqslant k} x_i^j h_{k-j}(x_1, \ldots, x_{i-1}). \tag{13}$$

Thus the quotient $S_{n,k}$ has the structure of a graded $H_n(0)$-module.

It can be shown that $J_{n,n} = I_n$, so that $S_{n,n} = R_n$ is the classical coinvariant module. At the other extreme, we have $J_{n,1} = \langle x_1, x_2, \ldots, x_n\rangle$, so that $S_{n,1} \cong \mathbb{F}$ is the trivial $H_n(0)$-module in degree 0.

Let us remark on an analogy between the generating sets of $I_{n,k}$ and $J_{n,k}$ which may rationalize the more complicated generating set of $J_{n,k}$. The *defining representation* of $\mathfrak{S}_n$ on $[n]$ is (of course) given by $s_i(i) = i+1, s_i(i+1) = i$, and $s_i(j) = j$ otherwise. The generators of $I_{n,k}$ come in two flavors:

(1) high degree elementary invariants $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$, and
(2) a homogeneous system of parameters $\{x_1^k, x_2^k, \dots, x_n^k\}$ of degree k whose linear span is stable under the action of $\mathfrak{S}_n$ and isomorphic to the defining representation.

$$1 \xleftrightarrow{\ s_1\ } 2 \xleftrightarrow{\ s_2\ } \cdots \xleftrightarrow{\ s_{n-1}\ } n$$

$$x_1^k \xleftrightarrow{\ s_1\ } x_2^k \xleftrightarrow{\ s_2\ } \cdots \xleftrightarrow{\ s_{n-1}\ } x_n^k$$

The *defining representation* of $H_n(0)$ on $[n]$ is given by $\pi_i(i) = i+1$ and $\pi_i(j) = j$ otherwise (whereas s_i acts by *swapping* at i, π_i acts by *shifting* at i). The generators of $J_{n,k}$ come in two analogous flavors:

(1) high degree elementary invariants $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$, and
(2) a homogeneous system of parameters $\{h_k(x_1), \dots, h_k(x_1, x_2, \dots, x_n)\}$ of degree k whose linear span is stable under the action of $H_n(0)$ and isomorphic to the defining representation (see (12)).

$$1 \xrightarrow{\ \pi_1\ } 2 \xrightarrow{\ \pi_2\ } \cdots \xrightarrow{\ \pi_{n-1}\ } n$$

$$h_k(x_1) \xrightarrow{\ \pi_1\ } h_k(x_1, x_2) \xrightarrow{\ \pi_2\ } \cdots \xrightarrow{\ \pi_{n-1}\ } h_k(x_1, \dots, x_n)$$

Deferring various definitions to Section 2, let us state our main results on $S_{n,k}$.

- The module $S_{n,k}$ has dimension $|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$ as an $\mathbb{F}$-vector space (Theorem 3.8). There is a basis $\mathcal{C}_{n,k}$ for $S_{n,k}$, generalizing the Artin monomial basis of R_n. (Theorem 3.5, Corollary 3.6).
- There is a generalization $\mathcal{GS}_{n,k}$ of the the Garsia-Stanton monomial basis to $S_{n,k}$ (Corollary 4.3).
- As an *ungraded* $H_n(0)$-module, the quotient $S_{n,k}$ is isomorphic to $\mathbb{F}[\mathcal{OP}_{n,k}]$ (Theorem 5.9).
- As a *graded* $H_n(0)$-module, we have explicit formulas for the degree-graded characteristics $\mathrm{Ch}_t(S_{n,k})$ and $\mathbf{ch}_t(S_{n,k})$ and the length-degree-bigraded characteristic $\mathrm{Ch}_{q,t}(S_{n,k})$ of $S_{n,k}$ (Theorem 6.2, Corollary 6.4). The degree-graded quasisymmetric characteristic $\mathrm{Ch}_t(S_{n,k})$ is symmetric and coincides with the graded Frobenius character of the $\mathfrak{S}_n$-module $R_{n,k}$ (over $\mathbb{Q}$).

The remainder of the paper is structured as follows. In Section 2 we give background and definitions related to compositions, ordered set partitions, Gröbner theory, and the representation theory of 0-Hecke algebras. In Section 3 we will prove that the quotient $S_{n,k}$ has dimension $|\mathcal{OP}_{n,k}|$ as an $\mathbb{F}$-vector space. We will derive a formula for the Hilbert series of $S_{n,k}$ and give a generalization of the Artin monomial basis to $S_{n,k}$. In Section 4 we will introduce a family of bases of $S_{n,k}$ which are related to the classical Garsia-Stanton basis in a unitriangular way when $k = n$. In Section 5 we will use one particular basis from this family to prove that the ungraded 0-Hecke structure of $S_{n,k}$ coincides with $\mathbb{F}[\mathcal{OP}_{n,k}]$. In Section 6 we derive formulas for the degree-graded quasisymmetric and noncommutative symmetric characteristics $\mathrm{Ch}_t(S_{n,k})$ and $\mathbf{ch}_t(S_{n,k})$, and the length-degree-bigraded quasisymmetric characteristics $\mathrm{Ch}_{q,t}(S_{n,k})$ of $S_{n,k}$. In Section 7 we make closing remarks.

2. Background

2.1. Compositions. Let n be a nonnegative integer. A *(strong) composition* α of n is a sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of positive integers with $\alpha_1 + \cdots + \alpha_\ell = n$. We call $\alpha_1, \dots, \alpha_\ell$ the *parts* of α. We write $\alpha \models n$ to mean that α is a composition of n. We also write $|\alpha| = n$ for the *size* of α and $\ell(\alpha) = \ell$ for the number of parts of α.

The *descent set* $\mathrm{Des}(\alpha)$ of a composition $\alpha = (\alpha_1, \dots, \alpha_\ell) \models n$ is the subset of $[n-1]$ given by

$$\mathrm{Des}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}. \tag{14}$$

The map $\alpha \mapsto \mathrm{Des}(\alpha)$ gives a bijection from the set of compositions of n to the collection of subsets of $[n-1]$. The *major index* of $\alpha = (\alpha_1, \dots, \alpha_\ell)$ is

$$\mathrm{maj}(\alpha) := \sum_{i \in \mathrm{Des}(\alpha)} i = (\ell - 1) \cdot \alpha_1 + \cdots + 1 \cdot \alpha_{\ell-1} + 0 \cdot \alpha_\ell. \tag{15}$$

Given two compositions $\alpha, \beta \models n$, we write $\alpha \preceq \beta$ if $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(\beta)$. Equivalently, we have $\alpha \preceq \beta$ if the composition α can be formed by merging adjacent parts of the composition β. If $\alpha \models n$, the *complement* $\alpha^c \models n$ of α is the unique composition of n which satisfies $\mathrm{Des}(\alpha^c) = [n-1] \smallsetminus \mathrm{Des}(\alpha)$.

As an example of these concepts, let $\alpha = (2,3,1,2) \models 8$. We have $\ell(\alpha) = 4$. The descent set of α is $\mathrm{Des}(\alpha) = \{2,5,6\}$. The major index is $\mathrm{maj}(\alpha) = 2+5+6 = 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 1 + 0 \cdot 2 = 13$. The complement of α is $\alpha^c = (1,2,1,3,1) \models 8$ with descent set $\mathrm{Des}(\alpha^c) = \{1,3,4,7\} = [7] \smallsetminus \{2,5,6\}$.

If $\mathbf{i} = (i_1, \dots, i_n)$ is any sequence of integers, the *descent set* $\mathrm{Des}(\mathbf{i})$ is given by

$$\mathrm{Des}(\mathbf{i}) := \{1 \leqslant j \leqslant n-1 \,:\, i_j > i_{j+1}\}. \tag{16}$$

The *descent number* of $\mathbf{i}$ is $\mathrm{des}(\mathbf{i}) := |\mathrm{Des}(\mathbf{i})|$ and the *major index* of $\mathbf{i}$ is $\mathrm{maj}(\mathbf{i}) := \sum_{j \in \mathrm{Des}(\mathbf{i})} j$. Finally, the *inversion number* $\mathrm{inv}(\mathbf{i})$

$$\mathrm{inv}(\mathbf{i}) := |\{(j,j') : 1 \leqslant j < j' \leqslant n,\ i_j > i_{j'}\}| \tag{17}$$

counts the number of inversion pairs in the sequence $\mathbf{i}$.

If a permutation $w \in \mathfrak{S}_n$ has one-line notation $w = w(1) \cdots w(n)$, we define $\mathrm{Des}(w)$, $\mathrm{maj}(w)$, $\mathrm{des}(w)$, and $\mathrm{inv}(w)$ as in the previous paragraph for the sequence $(w(1), \dots, w(n))$. It turns out that $\mathrm{inv}(w)$ is equal to the *Coxeter length* $\ell(w)$ of w, i.e., the length of a reduced expression for w in the generating set $\{s_1, \dots, s_{n-1}\}$ of $\mathfrak{S}_n$. Moreover, we have $i \in \mathrm{Des}(w)$ if and only if some reduced expression of w ends with s_i. We also let $\mathrm{iDes}(w) := \mathrm{Des}(w^{-1})$ be the descent set of the inverse of the permutation w.

The statistics maj and inv are equidistributed on $\mathfrak{S}_n$ and their common distribution has a nice form. Let us recall the standard q-analogs of numbers, factorials, and multinomial coefficients:

$$[n]_q := 1 + q + \cdots + q^{n-1} \qquad [n]!_q := [n]_q [n-1]_q \cdots [1]_q$$
$$\begin{bmatrix} n \\ a_1, \dots, a_r \end{bmatrix}_q := \frac{[n]!_q}{[a_1]!_q \cdots [a_r]!_q} \qquad \begin{bmatrix} n \\ a \end{bmatrix}_q := \frac{[n]!_q}{[a]!_q [n-a]!_q}.$$

MacMahon [20] proved

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} = [n]!_q, \tag{18}$$

and any statistic on $\mathfrak{S}_n$ which shares this distribution is called *Mahonian*. The joint distribution $\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)}$ of the pair of statistics $(\mathrm{inv}, \mathrm{maj})$ is called the *biMahonian distribution*.

If $\alpha \models n$ and $\mathbf{i} = (i_1, \ldots, i_n)$ is a sequence of integers of length n, we define $\alpha \cup \mathbf{i} \models n$ to be the unique composition of n which satisfies

$$\mathrm{Des}(\alpha \cup \mathbf{i}) = \mathrm{Des}(\alpha) \cup \mathrm{Des}(\mathbf{i}). \tag{19}$$

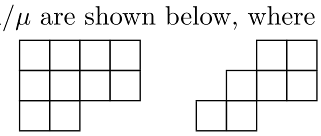
For example, let $\alpha = (3, 2, 3) \models 8$ and let $\mathbf{i} = (4, 5, 0, 0, 1, 0, 2, 2)$. We have

$$\mathrm{Des}(\alpha \cup \mathbf{i}) = \mathrm{Des}(\alpha) \cup \mathrm{Des}(\mathbf{i}) = \{3, 5\} \cup \{2, 5\} = \{2, 3, 5\},$$

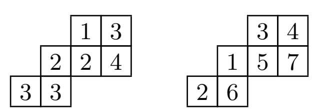
so that $\alpha \cup \mathbf{i} = (2, 1, 2, 3)$. Whenever $\alpha \models n$ and $\mathbf{i}$ is a length n sequence, we have the relation $\alpha \preceq \alpha \cup \mathbf{i}$.

A *partition* λ of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_\ell)$ of positive integers which satisfies $\lambda_1 + \cdots + \lambda_\ell = n$. We write $\lambda \vdash n$ to mean that λ is a partition of n. We also write $|\lambda| = n$ for the *size* of λ and $\ell(\lambda) = \ell$ for the number of *parts* of λ. The (*English*) *Ferrers diagram* of λ consists of λ_i left justified boxes in row i.

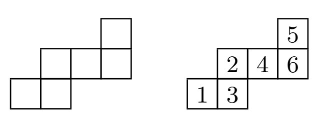
Identifying partitions with Ferrers diagrams, if $\mu \subseteq \lambda$ are a pair of partitions related by containment, the *skew partition* λ/μ is obtained by removing μ from λ. We write $|\lambda/\mu| := |\lambda| - |\mu|$ for the number of boxes in this skew diagram. For example, the Ferrers diagrams of λ and λ/μ are shown below, where $\lambda = (4, 4, 2)$ and $\mu = (2, 1)$.



A *semistandard tableau* of a skew shape λ/μ is a filling of the Ferrers diagram of λ/μ with positive integers which are weakly increasing across rows and strictly increasing down columns. A *standard tableau* of shape λ/μ is a bijective filling of the Ferrers diagram of λ/μ with the numbers $1, 2, \ldots, |\lambda/\mu|$ which is semistandard. An example of a semistandard tableau and a standard tableau of shape $(4, 4, 2)/(2, 1)$ are shown below.



A *ribbon* is an edgewise connected skew diagram which contains no 2×2 square. The set of compositions of n is in bijective correspondence with the set of size n ribbons: a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ corresponds to the ribbon whose i^{th} row from the bottom contains α_i boxes. We will identify compositions with ribbons in this way. For example, the ribbon corresponding to $\alpha = (2, 3, 1)$ is shown on the left below.



Let $\alpha \models n$ be a composition. We define a permutation $w_0(\alpha) \in \mathfrak{S}_n$ as follows. Starting at the leftmost column and working towards the right, and moving from top to bottom within each column, fill the ribbon diagram of α with the numbers $1, 2, \ldots, n$ (giving a standard tableau). The permutation $w_0(\alpha)$ has one-line notation obtained by reading along the ribbon from the bottom row to the top row, proceeding from left to right within each row. It can be shown that $w_0(\alpha)$ is the unique left weak Bruhat minimal permutation $w \in \mathfrak{S}_n$ which satisfies $\mathrm{Des}(w) = \mathrm{Des}(\alpha)$ (cf. Björner and Wachs [6]). For example, if $\alpha = (2, 3, 1)$, the figure on the above right shows $w_0(\alpha) = 132465 \in \mathfrak{S}_6$.

2.2. Ordered set partitions. As explained in Section 1, an *ordered set partition* σ of size n is a set partition of $[n]$ with a total order on its blocks. Let $\mathcal{OP}_{n,k}$ denote the collection of ordered set partitions of size n with k blocks. In particular, we may identify $\mathcal{OP}_{n,n}$ with $\mathfrak{S}_n$.

Also as in Section 1, we write an ordered set partition of $[n]$ as a permutation of $[n]$ with bars to separate blocks, such that letters within each block are increasing and blocks are ordered from left to right. For example, we have

$$\sigma = (245 \mid 6 \mid 13) \in \mathcal{OP}_{6,3}.$$

The *shape* of an ordered set partition $\sigma = (B_1 \mid \cdots \mid B_k)$ is the composition $\alpha = (|B_1|, \dots, |B_k|)$. For example, the above ordered set partition has shape $(3,1,2) \models 6$.

If $\alpha \models n$ is a composition, let $\mathcal{OP}_\alpha$ denote the collection of ordered set partitions of n with shape α. Given an ordered set partition $\sigma \in \mathcal{OP}_\alpha$, we can also represent σ as the pair (w, α), where $w = w(1) \cdots w(n)$ is the permutation in $\mathfrak{S}_n$ (in one-line notation) obtained by erasing the bars in σ. For example, the above ordered set partition becomes

$$\sigma = (245613, (3,1,2)).$$

This notation establishes a bijection between $\mathcal{OP}_{n,k}$ and pairs (w, α) where $\alpha \models n$ is a composition with $\ell(\alpha) = k$ and $w \in \mathfrak{S}_n$ is a permutation with $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$.

We extend the statistic maj from permutations to ordered set partitions as follows. Let $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}$ be an ordered set partition represented as a pair (w, α) as above. We define the *major index* $\mathrm{maj}(\sigma)$ to be the statistic

$$\mathrm{maj}(\sigma) = \mathrm{maj}(w, \alpha) := \mathrm{maj}(w) + \sum_{i \,:\, \max(B_i) < \min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i - i). \tag{20}$$

For example, if $\sigma = (24 \mid 57 \mid 136 \mid 8)$, then

$$\mathrm{maj}(\sigma) = \mathrm{maj}(24571368) + (2-1) + (2+2+3-3) = 4 + 1 + 4 = 9.$$

We caution the reader that our definition of maj is *not* equivalent to, or even equidistributed with, the corresponding statistics for ordered set partitions in [22, 16] and elsewhere. However, the distribution of our maj on $\mathcal{OP}_{n,k}$ is the reversal of the distribution of their maj.

The generating function for maj on $\mathcal{OP}_{n,k}$ may be described as follows. Let rev_q be the operator on polynomials in the variable q which reverses coefficient sequences. For example, we have

$$\mathrm{rev}_q(3q^3 + 2q^2 + 1) = q^3 + 2q + 3.$$

The *q-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined by the recursion

$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k) \tag{21}$$

and the initial condition $\mathrm{Stir}_q(0,k) = \delta_{0,k}$, where δ is the Kronecker delta.

Proposition 2.1. *Let $k \leqslant n$ be positive integers. We have*

$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)). \tag{22}$$

Proof. To see why this equation holds, consider the statistic maj' on an ordered set partition $\sigma = (B_1 \mid \cdots \mid B_k) = (w, \alpha) \in \mathcal{OP}_{n,k}$ defined by

$$\mathrm{maj}'(\sigma) = \mathrm{maj}'(w, \alpha) := \sum_{i=1}^{k} (i-1)(\alpha_i - 1) + \sum_{i \,:\, \min(B_i) > \max(B_{i+1})} i. \tag{23}$$

This is precisely the version of major index on ordered set partitions studied by Remmel and Wilson [22]. They proved [22, Eqn. 15, Prop. 5.1.1] that

$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}'(\sigma)} = [k]!_q \cdot \mathrm{Stir}_q(n,k). \tag{24}$$

On the other hand, for any $\sigma = (w, \alpha) = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}$ we have

$$\mathrm{maj}(w) = \sum_{i \,:\, \max(B_i) > \min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i). \tag{25}$$

This implies

$$\mathrm{maj}(\sigma) = \mathrm{maj}(w) + \sum_{i \,:\, \max(B_i) < \min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i - i) \tag{26}$$

$$= \sum_{i=1}^{k-1} [(k-i) \cdot \alpha_i] - \sum_{i \,:\, \max(B_i) < \min(B_{i+1})} i. \tag{27}$$

The longest element $w_0 = n \ldots 21$ (in one-line notation) of $\mathfrak{S}_n$ gives an involution on $\mathcal{OP}_\alpha$ by

$$\sigma = (B_1 \mid \cdots \mid B_k) \mapsto w_0(\sigma) = (w_0(B_1) \mid \cdots \mid w_0(B_k)).$$

If $\alpha \models n$ and $\ell(\alpha) = k$, then for $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_\alpha$ and any index $1 \leqslant i \leqslant k-1$ we have $\max(B_i) < \min(B_{i+1})$ if and only if $\min(w_0(B_i)) > \max(w_0(B_{i+1}))$. Therefore,

$$\mathrm{maj}'(\sigma) + \mathrm{maj}(w_0(\sigma)) = \sum_{i=1}^{k} [(i-1)(\alpha_i - 1) + (k-i) \cdot \alpha_i] \tag{28}$$

$$= \sum_{i=1}^{k} [-\alpha_i - i + 1 + k\alpha_i] \tag{29}$$

$$= (k-1)(n-k) + \binom{k}{2}. \tag{30}$$

On the other hand, it is easy to see that

$$\max\{\mathrm{maj}(\sigma) \,:\, \sigma \in \mathcal{OP}_{n,k}\} = (k-1)(n-k) - \binom{k}{2} = \max\{\mathrm{maj}'(\sigma) \,:\, \sigma \in \mathcal{OP}_{n,k}\}.$$

Applying Equation (24) gives

$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q \left[\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}'(\sigma)} \right] = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)). \tag{31}$$

□

We have an action of the 0-Hecke algebra $H_n(0)$ on $\mathbb{F}[\mathcal{OP}_{n,k}]$ given by Equation (9). This $H_n(0)$-action preserves $\mathbb{F}[\mathcal{OP}_\alpha]$ for each composition α of n.

2.3. Gröbner theory. We review material related to Gröbner bases of ideals $I \subseteq \mathbb{F}[\mathbf{x}_n]$ and standard monomial bases of the corresponding quotients $\mathbb{F}[\mathbf{x}_n]/I$. For a more leisurely introduction to this material, see [9].

A total order $\leqslant$ on the monomials in $\mathbb{F}[\mathbf{x}_n]$ is called a *term order* if

- $1 \leqslant m$ for all monomials $m \in \mathbb{F}[\mathbf{x}_n]$, and
- if $m, m', m'' \in \mathbb{F}[\mathbf{x}_n]$ are monomials, then $m \leqslant m'$ implies $m \cdot m'' \leqslant m' \cdot m''$.

In this paper, we will consider the lexicographic term order *with respect to the variable ordering* $x_n > \cdots > x_2 > x_1$. That is, we have

$$x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$$

if and only if there exists an integer $1 \leqslant j \leqslant n$ such that $a_{j+1} = b_{j+1}, \ldots, a_n = b_n$, and $a_j < b_j$. Following the notation of SAGE, we call this term order `neglex`.

Let $\leqslant$ be any term order on monomials in $\mathbb{F}[\mathbf{x}_n]$. If $f \in \mathbb{F}[\mathbf{x}_n]$ is a nonzero polynomial, let $\mathrm{in}_<(f)$ be the leading (i.e., largest) term of f with respect to $<$. If $I \subseteq \mathbb{F}[\mathbf{x}_n]$ is an ideal, the associated *initial ideal* is the monomial ideal

$$\mathrm{in}_<(I) := \langle \mathrm{in}_<(f) \,:\, f \in I - \{0\} \rangle. \tag{32}$$

The set of monomials

$$\{\text{monomials } m \in \mathbb{F}[\mathbf{x}_n] \,:\, m \notin \mathrm{in}_<(I)\} \tag{33}$$

descends to a $\mathbb{F}$-basis for the quotient $\mathbb{F}[\mathbf{x}_n]/I$; this basis is called the *standard monomial basis* (with respect to the term order $\leqslant$) [9, Prop. 1, p. 230].

Let $I \subseteq \mathbb{F}[\mathbf{x}_n]$ be any ideal and let $\leqslant$ be a term order. A finite set $G = \{g_1, \dots, g_r\} \subseteq I$ of nonzero polynomials in I is called a *Gröbner basis* of I if

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_r) \rangle. \tag{34}$$

If G is a Gröbner basis of I, then we have $I = \langle G \rangle$ [9, Cor. 6, p. 77].

Let G be a Gröbner basis for I with respect to the term order $\leqslant$. The basis G is called *minimal* if

- for any $g \in G$, the leading coefficient of g with respect to $\leqslant$ is 1, and
- for any $g \neq g'$ in G, the leading monomial of g does not divide the leading monomial of g'.

A minimal Gröbner basis G is called *reduced* if in addition

- for any $g \neq g'$ in G, the leading monomial of g does not divide any term in the polynomial g'.

Up to a choice of term order, every ideal I has a unique reduced Gröbner basis [9, Prop. 6, p. 92].

2.4. Sym, QSym, and **NSym**. Let $\mathbf{x} = (x_1, x_2, \dots)$ be a totally ordered infinite set of variables and let Sym be the ($\mathbb{Z}$-)algebra of symmetric functions in $\mathbf{x}$ with coefficients in $\mathbb{Z}$. The algebra Sym is graded; its degree n component has basis given by the collection $\{s_\lambda \,:\, \lambda \vdash n\}$ of *Schur functions*. The Schur function s_λ may be defined as

$$s_\lambda = \sum_T \mathbf{x}^T, \tag{35}$$

where the sum is over all semistandard tableaux T of shape λ and $\mathbf{x}^T$ is the monomial

$$\mathbf{x}^T := x_1^{\#\text{of 1s in } T} x_2^{\#\text{of 2s in } T} \cdots . \tag{36}$$

Given partitions $\mu \subseteq \lambda$, we also let $s_{\lambda/\mu} \in \mathrm{Sym}$ denote the associated *skew Schur function*. The expansion of $s_{\lambda/\mu}$ in the $\mathbf{x}$ variables is also given by Equation (35). In particular, if α is a composition (thought of as a ribbon), we have the *ribbon Schur function* $s_\alpha \in \mathrm{Sym}$.

There is a coproduct of Sym given by replacing the variables $x_1, x_2, \dots$ with $x_1, x_2, \dots, y_1, y_2, \dots$ such that Sym becomes a graded Hopf algebra which is self-dual under the basis $\{s_\lambda\}$ [14, §2].

Let $\alpha = (\alpha_1, \dots, \alpha_k) \models n$ be a composition. The *monomial quasisymmetric function* is the formal power series

$$M_\alpha := \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}. \tag{37}$$

The graded algebra of *quasisymmetric functions* QSym is the $\mathbb{Z}$-linear span of the M_α, where α ranges over all compositions.

We will focus on a basis for QSym other than the monomial quasisymmetric functions M_α. If n is a positive integer and if $S \subseteq [n-1]$, the *Gessel fundamental quasisymmetric function* F_S attached to S is

$$F_S := \sum_{\substack{i_1 \leqslant \cdots \leqslant i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}. \tag{38}$$

In particular, if $w \in \mathfrak{S}_n$ is a permutation with inverse descent set $\mathrm{iDes}(w) \subseteq [n-1]$, we have the quasisymmetric function $F_{\mathrm{iDes}(w)}$. If $\alpha \models n$ is a composition, we extend this notation by setting $F_\alpha := F_{\mathrm{Des}(\alpha)}$.

Next, let **NSym** be the graded algebra of *noncommutative symmetric functions*. This is the free unital associative (noncommutative) algebra

$$\mathbf{NSym} := \mathbb{Z}\langle \mathbf{h}_1, \mathbf{h}_2, \dots \rangle \tag{39}$$

generated over $\mathbb{Z}$ by the symbols $\mathbf{h}_1, \mathbf{h}_2, \dots$, where $\mathbf{h}_d$ has degree d. The degree n component of **NSym** has $\mathbb{Z}$-basis given by $\{\mathbf{h}_\alpha : \alpha \models n\}$, where for $\alpha = (\alpha_1, \dots, \alpha_\ell) \models n$ we set

$$\mathbf{h}_\alpha := \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_\ell}. \tag{40}$$

Another basis of the degree n piece of **NSym** consists of the *ribbon Schur functions* $\{\mathbf{s}_\alpha : \alpha \models n\}$. The ribbon Schur function $\mathbf{s}_\alpha$ is defined by

$$\mathbf{s}_\alpha := \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathbf{h}_\beta. \tag{41}$$

Finally, there are coproducts for QSym and **NSym** such that they become dual graded Hopf algebras [14, §5].

2.5. Characteristic maps. Let A be a finite-dimensional algebra over a field $\mathbb{F}$. The *Grothendieck group* $G_0(A)$ *of the category of finitely-generated* A*-modules* is the quotient of the free abelian group generated by isomorphism classes $[M]$ of finitely-generated A-modules M by the subgroup generated by elements $[M]-[L]-[N]$ corresponding to short exact sequences $0 \to L \to M \to N \to 0$ of finitely-generated A-modules. The abelian group $G_0(A)$ has free basis given by the collection of (isomorphism classes of) irreducible A-modules. The *Grothendieck group* $K_0(A)$ *of the category of finitely-generated projective* A*-modules* is defined similarly, and has free basis given by the set of (isomorphism classes of) projective indecomposable A-modules. If A is semisimple then $G_0(A) = K_0(A)$. See [3] for more details on representation theory of finite dimensional algebras.

The symmetric group algebra $\mathbb{Q}[\mathfrak{S}_n]$ is semisimple and has irreducible representations S^λ indexed by partitions $\lambda \vdash n$. The *Grothendieck group* $G_0(\mathbb{Q}[\mathfrak{S}_\bullet])$ of the tower $\mathbb{Q}[\mathfrak{S}_\bullet] : \mathbb{Q}[\mathfrak{S}_0] \hookrightarrow \mathbb{Q}[\mathfrak{S}_1] \hookrightarrow \mathbb{Q}[\mathfrak{S}_2] \hookrightarrow \cdots$ of symmetric group algebras is the direct sum of $G_0(\mathbb{Q}[\mathfrak{S}_n])$ for all $n \geqslant 0$. It is a graded Hopf algebra with product and coproduct given by induction and restriction along the embeddings $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$. The structure constants of $G_0(\mathbb{Q}[\mathfrak{S}_\bullet])$ under the self-dual basis $\{S^\lambda\}$, where λ runs through all partitions, are the well-known *Littlewood-Richardson coefficients*.

The *Frobenius character*[(1)] $\mathrm{Frob}(V)$ of a finite-dimensional $\mathbb{Q}[\mathfrak{S}_n]$-module V is

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} [\, V : S^\lambda \,] \cdot s_\lambda \in \mathrm{Sym} \tag{42}$$

[(1)]The Frobenius character $\mathrm{Frob}(V)$ is indeed a "character" since the Schur functions are characters of irreducible polynomial representations of the general linear groups.

where $[\,V : S^\lambda\,]$ is the multiplicity of the simple module S^λ among the composition factors of V. The correspondence Frob gives a graded Hopf algebra isomorphism $G_0(\mathbb{Q}[\mathfrak{S}_\bullet]) \cong \mathrm{Sym}$ [14, §4.4].

One can refine Frob for graded representations of $\mathbb{Q}[\mathfrak{S}_n]$. Recall that the *Hilbert series* of a graded vector space $V = \bigoplus_{d\geqslant 0} V_d$ with each component V_d finite-dimensional is

$$\mathrm{Hilb}(V;q) := \sum_{d\geqslant 0} \dim(V_d)\cdot q^d. \tag{43}$$

If V carries a graded action of $\mathbb{Q}[\mathfrak{S}_n]$, we also define the *graded Frobenius series* by

$$\mathrm{grFrob}(V;q) := \sum_{d\geqslant 0} \mathrm{Frob}(V_d)\cdot q^d. \tag{44}$$

Now let us recall the 0-Hecke analog of the above story. Consider an arbitrary ground field $\mathbb{F}$. The representation theory of the $\mathbb{F}$-algebra $H_n(0)$ was studied by Norton [21] and has a different flavor from that of $\mathbb{Q}[\mathfrak{S}_n]$ since $H_n(0)$ is not semisimple. Norton [21] proved that the $H_n(0)$-modules

$$P_\alpha := H_n(0)\overline{\pi}_{w_0(\alpha)}\pi_{w_0(\alpha^c)}, \tag{45}$$

where α ranges over all compositions of n, form a complete list of nonisomorphic indecomposable projective $H_n(0)$-modules. For each $\alpha \models n$, the $H_n(0)$-module P_α has a basis

$$\{\overline{\pi}_w \pi_{w_0(\alpha^c)} : w \in \mathfrak{S}_n,\ \mathrm{Des}(w) = \mathrm{Des}(\alpha)\}.$$

Moreover, P_α has a unique maximal submodule spanned by all elements in the above basis except its cyclic generator $\overline{\pi}_{w_0(\alpha)}\pi_{w_0(\alpha^c)}$, and the quotient of P_α by this maximal submodule, denoted by C_α, is one-dimensional and admits an $H_n(0)$-action by $\overline{\pi}_i = -1$ for all $i \in \mathrm{Des}(\alpha)$ and $\overline{\pi}_i = 0$ for all $i \in \mathrm{Des}(\alpha^c)$. The collections $\{P_\alpha : \alpha \models n\}$ and $\{C_\alpha : \alpha \models n\}$ are complete lists of nonisomorphic projective indecomposable and irreducible $H_n(0)$-modules, respectively.

Just as the Frobenius character map gives a deep connection between the representation theory of symmetric groups and the ring Sym of symmetric functions, there are two characteristic maps Ch and **ch**, defined by Krob and Thibon [19], which facilitate the study of representations of $H_n(0)$ through the rings QSym and **NSym**. Let us recall their construction.

The two Grothendieck groups $G_0(H_n(0))$ and $K_0(H_n(0))$ have free $\mathbb{Z}$-bases $\{C_\alpha : \alpha \models n\}$ and $\{P_\alpha : \alpha \models n\}$, respectively. Associated to the tower of algebras $H_\bullet(0) : H_0(0) \hookrightarrow H_1(0) \hookrightarrow H_2(0) \hookrightarrow \cdots$ are the two Grothendieck groups

$$G_0(H_\bullet(0)) := \textstyle\bigoplus_{n\geqslant 0} G_0(H_n(0)) \text{ and } K_0(H_\bullet(0)) := \bigoplus_{n\geqslant 0} K_0(H_n(0)).$$

These groups are graded Hopf algebras with product and coproduct given by induction and restriction along the embeddings $H_n(0)\otimes H_m(0) \hookrightarrow H_{n+m}(0)$, and they are dual to each other via the pairing $\langle P_\alpha, C_\beta\rangle = \delta_{\alpha,\beta}$.

Analogously to the Frobenius correspondence, Krob and Thibon [19] defined two linear maps

$$\mathrm{Ch} : G_0(H_\bullet(0)) \to \mathrm{QSym} \text{ and } \mathbf{ch} : K_0(H_\bullet(0)) \to \mathbf{NSym}$$

by $\mathrm{Ch}(C_\alpha) := F_\alpha$ and $\mathbf{ch}(P_\alpha) := \mathbf{s}_\alpha$ for all compositions α. These maps are isomorphisms of graded Hopf algebras. Krob and Thibon also showed [19] that for any composition α, the characteristic $\mathrm{Ch}(P_\alpha)$ equals the corresponding ribbon Schur function $s_\alpha \in \mathrm{Sym}$:

$$\mathrm{Ch}(P_\alpha) = \sum_{w\in\mathfrak{S}_n\,:\,\mathrm{Des}(w)=\mathrm{Des}(\alpha)} F_{\mathrm{iDes}(w)} = s_\alpha. \tag{46}$$

We give graded extensions of the maps Ch and **ch** as follows. Suppose that $V = \bigoplus_{d\geqslant 0} V_d$ is a graded $H_n(0)$-module with finite-dimensional homogeneous components V_d. The *degree-graded noncommutative characteristic* and *degree-graded quasisymmetric characteristic* of V are defined by

$$\mathbf{ch}_t(V) := \sum_{d\geqslant 0} \mathbf{ch}(V_d)\cdot t^d \quad \text{and} \quad \mathrm{Ch}_t(V) := \sum_{d\geqslant 0} \mathrm{Ch}(V_d)\cdot t^d. \tag{47}$$

On the other hand, the 0-Hecke algebra $H_n(0)$ has a *length filtration* $H_n(0)^{(0)} \supseteq H_n(0)^{(1)} \supseteq H_n(0)^{(2)} \supseteq \cdots$ where $H_n(0)^{(\ell)}$ is the span of $\{\pi_w : w \in \mathfrak{S}_n,\ \ell(w) \geqslant \ell\}$. Let $V = H_n(0)v$ be a cyclic $H_n(0)$-module whose distinguished generator $v \in V$ is equipped with a *length* $a \geqslant 0$. The *length filtration* $V^{(a)} \supseteq V^{(a+1)} \supseteq V^{(a+2)} \supseteq \cdots$ of V is given by

$$V^{(\ell)} := H_n(0)^{(\ell-a)}v, \quad \ell \geqslant a. \tag{48}$$

Following Krob and Thibon [19], we define the *length-graded quasymmetric characteristic* of V as

$$\mathrm{Ch}_q(V) := \sum_{\ell\geqslant a} \mathrm{Ch}\left(V^{(\ell)}/V^{(\ell+1)}\right)\cdot q^\ell. \tag{49}$$

The freedom to choose a length $a \geqslant 0$ for the distinguished generator v will make certain formulas look nicer.

Now suppose $V = \bigoplus_{d\geqslant 0} V_d$ is a graded $H_n(0)$-module which is also cyclic with a length filtration $V^{(a)} \supseteq V^{(a+1)} \supseteq \cdots$ as in the above paragraph. Let $V_d^{(\ell)} := V^{(\ell)} \cap V_d$ for $\ell \geqslant a$ and $d \geqslant 0$. We define the *length-degree-bigraded quasisymmetric characteristic* of V to be

$$\mathrm{Ch}_{q,t}(V) := \sum_{\substack{\ell\geqslant a\\ d\geqslant 0}} \mathrm{Ch}\left(V_d^{(\ell)}\Big/V_d^{(\ell+1)}\right)\cdot q^\ell t^d. \tag{50}$$

Finally, if an $H_n(0)$-module $V = \bigoplus_{\alpha\in I} V_\alpha$ is a direct sum of cyclic graded $H_n(0)$-submodules V_α for α in some index set I, then we define $\mathrm{Ch}_{q,t}(V) := \sum_{\alpha\in I}\mathrm{Ch}_{q,t}(V_\alpha)$. Note that $\mathrm{Ch}_{q,t}(V)$ may depend on the choice of the direct sum decomposition of V into cyclic submodules. For example, Huang [17] showed that the coinvariant algebra R_n is isomorphic to the regular representation of $H_n(0)$ and obtained the length-degree-bigraded quasisymmetric characteristic

$$\mathrm{Ch}_{q,t}(R_n) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} F_{\mathrm{iDes}(w)} \tag{51}$$

using the cyclic generator of R_n corresponding to the element $1 \in H_n(0)$. However, if R_n is viewed as a direct sum of projective indecomposable submodules indexed by compositions of n then the length grading received by each $w \in \mathfrak{S}_n$ needs to be changed to $\mathrm{inv}(w) - \mathrm{inv}(w_0(\alpha))$ where $\alpha \models n$ is determined by $\mathrm{Des}(\alpha) = \mathrm{Des}(w)$. For our convenience, we will choose an approriate decomposition of V into cyclic submodules, and further adjust the length grading by a suitable constant for each cyclic submodule in the distinguished direct sum decomposition of V. This will give a length-degree-bigraded characteristic $\mathrm{Ch}_{q,t}(V)$, which specializes to $\mathrm{Ch}_{1,t}(V) = \mathrm{Ch}_t(V)$ and $\mathrm{Ch}_{q,1}(V) = \mathrm{Ch}_q(V)$, respectively.

3. Hilbert Series and Artin basis

3.1. The point sets $Z_{n,k}$. In this section we will prove that $\dim(S_{n,k}) = |\mathcal{OP}_{n,k}|$. To do this, we will use tools from elementary algebraic geometry. This basic method dates back to the work of Garsia and Procesi on Springer fibers and Tanisaki quotients [11].

Given a finite point set $Z \subseteq \mathbb{F}^n$, let $\mathbf{I}(Z) \subseteq \mathbb{F}[\mathbf{x}_n]$ be the ideal of polynomials which vanish on Z:

$$\mathbf{I}(Z) := \{f \in \mathbb{F}[\mathbf{x}_n] \,:\, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in Z\}. \tag{52}$$

There is a natural identification of the quotient $\mathbb{F}[\mathbf{x}_n]/\mathbf{I}(Z)$ with the collection of polynomial functions $Z \to \mathbb{F}$.

We claim that any function $Z \to \mathbb{F}$ may be realized as the restriction of a polynomial function. This essentially follows from Lagrange Interpolation. Indeed, since $Z \subseteq \mathbb{F}^n$ is finite, there exist field elements $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ such that $Z \subseteq \{\alpha_1, \dots, \alpha_m\}^n$. For any n-tuple of integers $(i_1, \dots, i_n)$ between 1 and m, the polynomial

$$\prod_{j_1 \neq i_1} (x_1 - \alpha_{j_1}) \cdots \prod_{j_n \neq i_n} (x_n - \alpha_{j_n}) \in \mathbb{F}[\mathbf{x}_n]$$

vanishes on every point of $\{\alpha_1, \dots, \alpha_m\}^n$ except for $(\alpha_{i_1}, \dots, \alpha_{i_n})$. Hence, an arbitrary $\mathbb{F}$-valued function on $\{\alpha_1, \dots, \alpha_m\}^n$ may be realized using a linear combination of polynomials of the above form. Since $Z \subseteq \{\alpha_1, \dots, \alpha_m\}^n$, the same is true for an arbitrary $\mathbb{F}$-valued function on Z.

By the last paragraph, we may identify the quotient $\mathbb{F}[\mathbf{x}_n]/\mathbf{I}(Z)$ with the collection of all functions $Z \to \mathbb{F}$. In particular, the dimension of this quotient as an $\mathbb{F}$-vector space is

$$\dim\left(\mathbb{F}[\mathbf{x}_n]/\mathbf{I}(Z)\right) = |Z|. \tag{53}$$

The ideal $\mathbf{I}(Z)$ is almost never homogeneous. To get a homogeneous ideal, we do the following. For any nonzero polynomial $f \in \mathbb{F}[\mathbf{x}_n]$, let $\tau(f)$ be the top degree component of f. That is, if $f = f_d + f_{d-1} + \cdots + f_0$ where f_i has homogeneous degree i for all i and $f_d \neq 0$, then $\tau(f) = f_d$. The ideal $\mathbf{T}(Z) \subseteq \mathbb{F}[\mathbf{x}_n]$ is generated by the top degree components of all nonzero polynomials in $\mathbf{I}(Z)$. In symbols:

$$\mathbf{T}(Z) := \langle \tau(f) \,:\, f \in \mathbf{I}(Z) - \{0\} \rangle. \tag{54}$$

The ideal $\mathbf{T}(Z)$ is homogeneous by definition, so that $\mathbb{F}[\mathbf{x}_n]/\mathbf{T}(Z)$ is a graded $\mathbb{F}$-vector space. Moreover, it is well known that

$$\dim\left(\mathbb{F}[\mathbf{x}_n]/\mathbf{T}(Z)\right) = \dim\left(\mathbb{F}[\mathbf{x}_n]/\mathbf{I}(Z)\right) = |Z|. \tag{55}$$

Our three-step strategy for proving $\dim(S_{n,k}) = |\mathcal{OP}_{n,k}|$ is as follows.

1. Find a point set $Z_{n,k} \subseteq \mathbb{F}^n$ which is in bijective correspondence with $\mathcal{OP}_{n,k}$.
2. Prove that the generators of $J_{n,k}$ arise as top degree components of polynomials in $\mathbf{I}(Z_{n,k})$, so that $J_{n,k} \subseteq \mathbf{T}(Z_{n,k})$.
3. Use Gröbner theory to prove $\dim(S_{n,k}) \leqslant |\mathcal{OP}_{n,k}|$, forcing $\dim(S_{n,k}) = |\mathcal{OP}_{n,k}|$ by Steps 1 and 2.

A similar three-step strategy was used by Haglund, Rhoades, and Shimozono [16] in their analysis of the $\mathfrak{S}_n$-module structure of $R_{n,k}$. In our setting, since we do not have a group action, we can only use this strategy to deduce the vector space structure of $S_{n,k}$, rather than the $H_n(0)$-module structure of $S_{n,k}$.

To achieve Step 1 of our strategy, we need to find a candidate set $Z_{n,k} \subseteq \mathbb{F}^n$ which is in bijective correspondence with $\mathcal{OP}_{n,k}$. Here we run into a problem: to define our candidate point sets, we need the field $\mathbb{F}$ to contain at least $n + k - 1$ elements. This problem did not arise in the work of Haglund et. al. [16]; they worked exclusively over the field $\mathbb{Q}$. To get around this problem, we use the following trick.

Lemma 3.1. *Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension and $J = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{F}[\mathbf{x}_n]$ an ideal of $\mathbb{F}[\mathbf{x}_n]$ generated by $f_1, \ldots, f_r \in \mathbb{F}[\mathbf{x}_n]$. Then $\dim_{\mathbb{F}}(\mathbb{F}[\mathbf{x}_n]/J) = \dim_{\mathbb{K}}(\mathbb{K}[\mathbf{x}_n]/J')$ where $J' := \mathbb{K} \otimes_{\mathbb{F}} J$.*

Since $J_{n,k}$ is generated by polynomials with all coefficients equal to 1, the generating set of $J_{n,k}$ satisfies the conditions of Lemma 3.1.

Proof. Let $\leqslant$ be any term order. It suffices to show that the quotient rings $\mathbb{F}[\mathbf{x}_n]/J$ and $\mathbb{K}[\mathbf{x}_n]/J'$ have the same standard monomial bases with respect to $\leqslant$. To calculate the reduced Gröbner basis for the ideal J, we apply Buchberger's Algorithm [9, Ch. 2, §7] to the generating set $\{f_1, \ldots, f_r\}$. To calculate the Gröbner basis for the ideal J', we also apply Buchberger's Algorithm to the generating set $\{f_1, \ldots, f_r\}$. In either case, all of the coefficients involved in the polynomial long division are contained in the field $\mathbb{F}$. In particular, the reduced Gröbner bases of J and J' coincide. Hence, the standard monomial bases of $\mathbb{F}[\mathbf{x}_n]/J$ and $\mathbb{K}[\mathbf{x}_n]/J'$ also coincide. $\square$

We are ready to define our point sets $Z_{n,k}$. Thanks to Lemma 3.1, we may harmlessly assume that the field $\mathbb{F}$ has at least $n + k - 1$ elements by replacing $\mathbb{F}$ with an extension if necessary. We will have to choose a somewhat non-obvious point set $Z_{n,k} \subseteq \mathbb{F}^n$ in order to get the desired equality of ideals $\mathbf{T}(Z_{n,k}) = J_{n,k}$.

Definition 3.2. *Assume $\mathbb{F}$ has at least $n+k-1$ elements and let $\alpha_1, \alpha_2, \ldots, \alpha_{n+k-1} \in \mathbb{F}$ be a list of $n + k - 1$ distinct field elements. Define $Z_{n,k} \subseteq \mathbb{F}^n$ to be the collection of points $(z_1, z_2, \ldots, z_n)$ such that*

- *for $1 \leqslant i \leqslant n$ we have $z_i \in \{\alpha_1, \alpha_2, \ldots, \alpha_{k+i-1}\}$,*
- *the coordinates $z_1, z_2, \ldots, z_n$ are distinct, and*
- *we have $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \{z_1, z_2, \ldots, z_n\}$.*

We claim that $Z_{n,k}$ is in bijective correspondence with $\mathcal{OP}_{n,k}$. A bijection $\varphi : \mathcal{OP}_{n,k} \to Z_{n,k}$ may be obtained as follows. Let $\sigma = (B_1 \mid \cdots \mid B_k) \in \mathcal{OP}_{n,k}$ be an ordered set partition; we define $\varphi(\sigma) = (z_1, \ldots, z_n) \in Z_{n,k}$. For $1 \leqslant i \leqslant k$, we first set $z_j = \alpha_i$, where $j = \min(B_i)$. Write the set of unassigned indices of $(z_1, \ldots, z_n)$ as $S = [n] - \{\min(B_1), \ldots, \min(B_k)\} = \{s_1 < \cdots < s_{n-k}\}$. For $s \in S$, let ℓ_s be the number of blocks B weakly to the left of s in σ which satisfy $\min(B) < s$. Let $z_{s_1} = \alpha_{k+\ell_{s_1}}$. Assuming $z_{s_1}, z_{s_2}, \ldots, z_{s_{j-1}}$ have already been defined, let z_{s_j} be the $\ell_{s_j}^{th}$ term in the sequence formed by deleting $z_{s_1}, z_{s_2}, \ldots, z_{s_{j-1}}$ from the sequence $(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{n+k-1})$.

As an example of the map φ, let $\sigma = (7 \mid 248 \mid 13 \mid 569) \in \mathcal{OP}_{9,4}$. The following table computes the image $\varphi(\sigma) = (z_1, \ldots, z_9)$. We start by assigning the coordinates $(z_7, z_2, z_1, z_5) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the letters in the minimal blocks of σ. At the top row of the table, the coordinates corresponding to the letters $S = \{3, 4, 6, 8, 9\}$ which are not minimal in their blocks of σ are unassigned and we have the sequence of possible values $(\alpha_{k+1}, \ldots, \alpha_{n+k-1}) = (\alpha_5, \ldots, \alpha_{12})$. We add the elements of S to the blocks of σ one at a time, from smallest to largest. At each stage, we record the letter s added together with the number ℓ_s of blocks B weakly to the left of s in σ which satisfy $\min(B) < s$. We assign the coordinate z_s the value of the ℓ_s^{th} term in the list of unassigned values, and then erase the value from the list. In summary, we have

$$\varphi : (7 \mid 248 \mid 13 \mid 569) \mapsto (\alpha_3, \alpha_2, \alpha_6, \alpha_5, \alpha_4, \alpha_9, \alpha_1, \alpha_8, \alpha_{12}).$$

σ	letter s added	ℓ_s	unassigned α's	$\varphi(\sigma) = (z_1, \ldots, z_n)$
$(\mathbf{7} \mid \mathbf{2} \mid \mathbf{1} \mid \mathbf{5})$			$(\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12})$	$(\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_2, z_3, z_4, \boldsymbol{\alpha}_4, z_6, \boldsymbol{\alpha}_1, z_8, z_9)$
$(7 \mid 2 \mid 1\mathbf{3} \mid 5)$	3	$\ell_3 = 2$	$(\alpha_5, \boldsymbol{\alpha}_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12})$	$(\alpha_3, \alpha_2, \boldsymbol{\alpha}_6, z_4, \alpha_4, z_6, \alpha_1, z_8, z_9)$
$(7 \mid 2\mathbf{4} \mid 13 \mid 5)$	4	$\ell_4 = 1$	$(\boldsymbol{\alpha}_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12})$	$(\alpha_3, \alpha_2, \alpha_6, \boldsymbol{\alpha}_5, \alpha_4, z_6, \alpha_1, z_8, z_9)$
$(7 \mid 24 \mid 13 \mid 5\mathbf{6})$	6	$\ell_6 = 3$	$(\alpha_7, \alpha_8, \boldsymbol{\alpha}_9, \alpha_{10}, \alpha_{11}, \alpha_{12})$	$(\alpha_3, \alpha_2, \alpha_6, \alpha_5, \alpha_4, \boldsymbol{\alpha}_9, \alpha_1, z_8, z_9)$
$(7 \mid 24\mathbf{8} \mid 13 \mid 56)$	8	$\ell_8 = 2$	$(\alpha_7, \boldsymbol{\alpha}_8, \alpha_{10}, \alpha_{11}, \alpha_{12})$	$(\alpha_3, \alpha_2, \alpha_6, \alpha_5, \alpha_4, \alpha_9, \alpha_1, \boldsymbol{\alpha}_8, z_9)$
$(7 \mid 248 \mid 13 \mid 56\mathbf{9})$	9	$\ell_9 = 4$	$(\alpha_7, \alpha_{10}, \alpha_{11}, \boldsymbol{\alpha}_{12})$	$(\alpha_3, \alpha_2, \alpha_6, \alpha_5, \alpha_4, \alpha_9, \alpha_1, \alpha_8, \boldsymbol{\alpha}_{12})$

We leave it for the reader to check that $\varphi : \mathcal{OP}_{n,k} \to Z_{n,k}$ is well-defined and invertible. The point set $Z_{n,k}$ therefore achieves Step 1 of our strategy.

Achieving Step 2 of our strategy involves showing that the generators of $J_{n,k}$ arise as top degree components of strategically chosen polynomials vanishing on $Z_{n,k}$.

Lemma 3.3. *Assume $\mathbb{F}$ has at least $n + k - 1$ elements. We have $J_{n,k} \subseteq \mathbf{T}(Z_{n,k})$.*

Proof. It suffices to show that every generator of $J_{n,k}$ arises as the top degree component of a polynomial in $\mathbf{I}(Z_{n,k})$. Let us first consider the generators $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$.

For $1 \leqslant i \leqslant n$, we claim that

$$\sum_{j \geqslant 0} (-1)^j h_{k-j}(x_1, x_2, \ldots, x_i) e_j(\alpha_1, \alpha_2, \ldots, \alpha_{k+i-1}) \in \mathbf{I}(Z_{n,k}). \tag{56}$$

Indeed, this alternating sum is the coefficient of t^k in the power series expansion of the rational function

$$\frac{(1 - \alpha_1 t)(1 - \alpha_2 t) \cdots (1 - \alpha_{k+i-1} t)}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_i t)}. \tag{57}$$

If $(x_1, \ldots, x_n) \in Z_{n,k}$, by the definition of $Z_{n,k}$ the terms in the denominator cancel with i terms in the numerator, yielding a polynomial in t of degree $k - 1$. The assertion (56) follows. Taking the highest degree component, we get $h_k(x_1, x_2, \ldots, x_i) \in \mathbf{T}(Z_{n,k})$.

Next, we show $e_r(x_1, \ldots, x_n) \in \mathbf{T}(Z_{n,k})$ for $n - k < r \leqslant n$. To prove this, we claim that

$$\sum_{j \geqslant 0} (-1)^j e_{r-j}(x_1, \ldots, x_n) h_j(\alpha_1, \ldots, \alpha_{n+k-1}) \in \mathbf{I}(Z_{n,k}). \tag{58}$$

Indeed, this alternating sum is the coefficient of t^r in the rational function

$$\frac{(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t)}{(1 + \alpha_1 t)(1 + \alpha_2 t) \cdots (1 + \alpha_k t)}. \tag{59}$$

If $(x_1, \ldots, x_n) \in Z_{n,k}$, the terms in the denominator cancel with k terms in the numerator, yielding a polynomial in t of degree $n - k$. Since $r > n - k$, the assertion (58) follows. Taking the highest degree component, we get $e_r(x_1, \ldots, x_n) \in \mathbf{T}(Z_{n,k})$. □

3.2. The Hilbert series of $S_{n,k}$. Let $<$ be the `neglex` term order on $\mathbb{F}[\mathbf{x}_n]$. We are ready to execute Step 3 of our strategy and describe the standard monomial basis of the quotient $S_{n,k}$. To do so, we recall the definition of 'skip monomials' in $\mathbb{F}[\mathbf{x}_n]$ of [16].

Let $S = \{s_1 < \cdots < s_m\} \subseteq [n]$ be a set. Following [16, Defn. 3.2], the *skip monomial* $\mathbf{x}(S)$ is the monomial in $\mathbb{F}[\mathbf{x}_n]$ given by

$$\mathbf{x}(S) := x_{s_1}^{s_1} x_{s_2}^{s_2-1} \cdots x_{s_m}^{s_m-m+1}. \tag{60}$$

For example, we have $\mathbf{x}(2578) = x_2^2 x_5^4 x_7^5 x_8^5$. The adjective 'skip' refers to the fact that the exponent sequence $\mathbf{x}(S)$ increases whenever the set S skips a letter. Our variable order convention will require us to consider the *reverse skip monomial*

$$\mathbf{x}(S)^* := x_{n-s_1+1}^{s_1} x_{n-s_2+1}^{s_2-1} \cdots x_{n-s_m+1}^{s_m-m+1}. \tag{61}$$

For example, if $n = 9$ we have $\mathbf{x}(2578)^* = x_8^2 x_5^4 x_3^5 x_2^5$. The following definition is the reverse of [16, Defn. 4.4].

Definition 3.4. *Let $k \leqslant n$ be positive integers. A monomial $m \in \mathbb{F}[\mathbf{x}_n]$ is (n,k)-*reverse nonskip *if*

- *$x_i^k \nmid m$ for $1 \leqslant i \leqslant n$, and*
- *$\mathbf{x}(S)^* \nmid m$ for all $S \subseteq [n]$ with $|S| = n + k - 1$.*

Let $\mathcal{C}_{n,k}$ denote the collection of all (n,k)-reverse nonskip monomials in $\mathbb{F}[\mathbf{x}_n]$.

There is some redundancy in Definition 3.4. In particular, if $n \in S$, the power of x_1 in $\mathbf{x}(S)^*$ where $|S| = n - k + 1$ is x_1^k, so that we need only consider those sets S with $n \notin S$.

Theorem 3.5. *Let $\mathbb{F}$ be any field and $\leqslant$ be the* `neglex` *term order on $\mathbb{F}[\mathbf{x}_n]$. The standard monomial basis of $S_{n,k} = \mathbb{F}[\mathbf{x}_n]/J_{n,k}$ with respect to $\leqslant$ is $\mathcal{C}_{n,k}$.*

Proof. By the definition of `neglex`, we have

$$\mathrm{in}_<(h_k(x_1, x_2, \ldots, x_i)) = x_i^k \in \mathrm{in}_<(J_{n,k}). \tag{62}$$

By [16, Lem. 3.4, Lem. 3.5] we also have $\mathbf{x}(S)^* \in \mathrm{in}_<(J_{n,k})$ whenever $S \subseteq [n]$ satisfies $|S| = n - k + 1$. It follows that $\mathcal{C}_{n,k}$ contains the standard monomial basis of $S_{n,k}$.

To prove that $\mathcal{C}_{n,k}$ is the standard monomial basis of $S_{n,k}$, it suffices to show $|\mathcal{C}_{n,k}| \leqslant \dim(S_{n,k})$. Thanks to Lemma 3.1, we may replace $\mathbb{F}$ by an extension if necessary to assume that $\mathbb{F}$ contains at least $n + k - 1$ elements. By Lemma 3.3, we have

$$\dim(S_{n,k}) = \dim(\mathbb{F}[\mathbf{x}_n]/J_{n,k}) \geqslant \dim(\mathbb{F}[\mathbf{x}_n]/\mathbf{T}(Z_{n,k})) = |Z_{n,k}| = |\mathcal{OP}_{n,k}|. \tag{63}$$

On the other hand, [16, Thm. 4.9] implies (after reversing variables) that $|\mathcal{OP}_{n,k}| = |\mathcal{C}_{n,k}|$. □

When $k = n$, the collection $\mathcal{C}_{n,n}$ consists of sub-staircase monomials $x_1^{a_1} \cdots x_n^{a_n}$ whose exponent sequences satisfy $0 \leqslant a_i \leqslant n - i$; this is the basis for the coinvariant algebra obtained by E. Artin [2] using Galois theory. Let us mention an analogous characterization of $\mathcal{C}_{n,k}$ which was derived in [16].

Recall that a *shuffle* of two sequences $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ is an interleaving $(c_1, \ldots, c_{r+s})$ of these sequences which preserves the relative order of the a's and the b's. A (n,k)-*staircase* is a shuffle of the sequences $(k-1, k-2, \ldots, 1, 0)$ and $(k-1, k-1, \ldots, k-1)$, where the second sequence has $n - k$ copies of $k - 1$. For example, the $(5,3)$-staircases are the shuffles of $(2,1,0)$ and $(2,2)$:

$$(2,1,0,2,2), (2,1,2,0,2), (2,2,1,0,2), (2,1,2,2,0), (2,2,1,2,0), \text{ and } (2,2,2,1,0).$$

The following theorem is just the reversal of [16, Thm. 4.13].

Corollary 3.6 ([16, Thm. 4.13]). *The monomial basis $\mathcal{C}_{n,k}$ of $S_{n,k}$ is the set of monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in $\mathbb{F}[\mathbf{x}_n]$ whose exponent sequences $(a_1, a_2, \ldots, a_n)$ are componentwise $\leqslant$* some *(n,k)-staircase.*

For example, consider the case $(n,k) = (4,2)$. The $(4,2)$-staircases are the shuffles of $(1,0)$ and $(1,1)$:

$$(1,0,1,1), (1,1,0,1), \text{ and } (1,1,1,0).$$

It follows that

$$\mathcal{C}_{4,2} = \{1, x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3\}$$

is the standard monomial basis of $S_{4,2}$ with respect to `neglex`. Consequently, we have the Hilbert series

$$\mathrm{Hilb}(S_{4,2}; q) = 1 + 4q + 6q^2 + 3q^3.$$

We can also describe a Gröbner basis of the ideal $J_{n,k}$. For $\gamma = (\gamma_1, \ldots, \gamma_n)$ a *weak composition* (i.e., possibly containing 0's) of length n, let $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]$ be the associated *Demazure character* (see e.g. [16, Sec. 2.4]).

If $S \subseteq [n]$, let $\gamma(S) = (\gamma_1, \ldots, \gamma_n)$ be the exponent sequence of the corresponding skip monomial $\mathbf{x}(S)$. That is, if $S = \{s_1 < \cdots < s_m\}$ we have

$$\gamma_i = \begin{cases} s_j - j + 1 & \text{if } i = s_j \in S \\ 0 & \text{if } i \notin S. \end{cases} \tag{64}$$

Let $\gamma(S)^* = (\gamma_n, \ldots, \gamma_1)$ be the reverse of the weak composition $\gamma(S)$. In particular, we can consider the Demazure character $\kappa_{\gamma(S)^*}(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]$.

Theorem 3.7. *Let $k \leqslant n$ be positive integers and let $\leqslant$ be the* `neglex` *term order on $\mathbb{F}[\mathbf{x}_n]$. The polynomials*

$$h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$$

together with the Demazure characters

$$\kappa_{\gamma(S)^*}(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]$$

for all $S \subseteq [n-1]$ satisfying $|S| = n - k + 1$, form a Gröbner basis for the ideal $J_{n,k}$.

When $k < n$, this Gröbner basis is minimal.

For example, if $(n,k) = (6,4)$, a Gröbner basis of $J_{6,4} \subseteq \mathbb{F}[\mathbf{x}_6]$ is given by the polynomials

$$h_4(x_1), \quad h_4(x_1,x_2), \quad h_4(x_1,x_2,x_3), \quad h_4(x_1,x_2,x_3,x_4),$$
$$h_4(x_1,x_2,x_3,x_4,x_5) \quad \text{and} \quad h_4(x_1,x_2,x_3,x_4,x_5,x_6)$$

together with the Demazure characters

$$\kappa_{(0,0,0,1,1,1)}(\mathbf{x}_6), \kappa_{(0,0,2,0,1,1)}(\mathbf{x}_6), \kappa_{(0,3,0,0,1,1)}(\mathbf{x}_6), \kappa_{(0,0,2,2,0,1)}(\mathbf{x}_6), \kappa_{(0,3,0,2,0,1)}(\mathbf{x}_6),$$
$$\kappa_{(0,3,3,0,0,1)}(\mathbf{x}_6), \kappa_{(0,0,2,2,2,0)}(\mathbf{x}_6), \kappa_{(0,3,0,2,2,0)}(\mathbf{x}_6), \kappa_{(0,3,3,0,2,0)}(\mathbf{x}_6), \kappa_{(0,3,3,3,0,0)}(\mathbf{x}_6).$$

Proof. We need to show that the polynomials in question lie in the ideal $J_{n,k}$. This is clear for the polynomials $h_k(x_1, \ldots, x_i)$. For the Demazure characters, we apply [16, Lem. 3.4] (and in particular [16, Eqn. 3.4]) to see that $\kappa_{\gamma(S)^*}(\mathbf{x}_n) \in J_{n,k}$ whenever $S \subseteq [n-1]$ satisfies $|S| = n - k + 1$.

Next we examine the leading terms of the polynomials in question. It is evident that

$$\mathrm{in}_<(h_k(x_1, \ldots, x_i)) = x_i^k.$$

After applying variable reversal to [16, Lem. 3.5], we see that

$$\mathrm{in}_<(\kappa_{\gamma(S)^*}(\mathbf{x}_n)) = \mathbf{x}(S)^*.$$

By Theorem 3.5 and the remarks following Definition 3.4, it follows that these monomials generate the initial ideal $\mathrm{in}_<(J_{n,k})$ of $J_{n,k}$.

When $k < n$, observe that for $S \subseteq [n-1]$ with $|S| = n-k+1$, the monomial $\mathbf{x}(S)^*$ has support $\{i : n-i+1 \in S\}$. Moreover, the monomial $\mathbf{x}(S)^*$ does not contain any exponents $\geqslant k$ since $n \notin S$. The minimality of the Gröbner basis follows. □

Theorem 3.7 is the 0-Hecke analog of [16, Thm. 4.14]. Unlike the case of [16, Thm. 4.14], the Gröbner basis of Theorem 3.7 is not reduced. When $k = n$, the ideal $J_{n,n}$ is the classical invariant ideal I_n and has reduced Gröbner basis $\{h_1(x_1,\dots,x_n), h_2(x_1,\dots,x_{n-1}),\dots,h_n(x_1)\}$. The authors do not have a conjecture for the reduced Gröbner basis for the ideal $J_{n,k}$. The work of [16] gives us a formula for the Hilbert series of $S_{n,k}$.

Theorem 3.8. *Let $k \leqslant n$ be positive integers. We have* $\mathrm{Hilb}(S_{n,k};q) = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k))$.

Proof. By Theorem 3.5 and [16, Thm. 4.13], the Hilbert series of $S_{n,k}$ equals the Hilbert series of $R_{n,k}$. Applying [16, Thm. 4.10] finishes the proof. □

4. Garsia-Stanton type bases

Let $k \leqslant n$ be positive integers. Given a composition $\alpha \models n$ and a length n sequence $\mathbf{i} = (i_1,\dots,i_n)$ of nonnegative integers, define a monomial $\mathbf{x}_{\alpha,\mathbf{i}} \in \mathbb{F}[\mathbf{x}_n]$ by

$$\mathbf{x}_{\alpha,\mathbf{i}} := \left(\prod_{j \in \mathrm{Des}(\alpha)} x_1 x_2 \cdots x_j\right) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}. \tag{65}$$

If $w \in \mathfrak{S}_n$ is a permutation and $\mathbf{i} = (i_1,\dots,i_n)$ is a sequence of nonnegative integers, we define the *generalized Garsia-Stanton monomial* $gs_{w,\mathbf{i}} := w(\mathbf{x}_{\alpha,\mathbf{i}})$, where $\alpha \models n$ is characterized by $\mathrm{Des}(\alpha) = \mathrm{Des}(w)$. The degree of $gs_{w,\mathbf{i}}$ is given by $\deg(gs_{w,\mathbf{i}}) = \mathrm{maj}(w) + |\mathbf{i}|$, where $|\mathbf{i}| := i_1 + \cdots + i_n$.

For example, let $(n,k) = (9,5)$, $w = 254689137 \in \mathfrak{S}_9$ and $\mathbf{i} = (2,2,1,1,0,0,0,0,0)$. We have $\mathrm{Des}(w) = \{2,6\}$, so that the composition $\alpha \models 9$ with $\mathrm{Des}(\alpha) = \mathrm{Des}(w)$ is $\alpha = (2,4,3)$. It follows that

$$\mathbf{x}_{\alpha,\mathbf{i}} = (x_1x_2)(x_1x_2x_3x_4x_5x_6)(x_1^2x_2^2x_3^1x_4^1).$$

The corresponding generalized GS monomial is

$$gs_{w,\mathbf{i}} = (x_2x_5)(x_2x_5x_4x_6x_8x_9)(x_2^2x_5^2x_4^1x_6^1).$$

Haglund, Rhoades, and Shimozono introduced [16, Defn. 5.2] (using different notation) the following collection $\mathcal{GS}_{n,k}$ of monomials:

$$\mathcal{GS}_{n,k} := \{gs_{w,\mathbf{i}} : w \in \mathfrak{S}_n,\ k - \mathrm{des}(w) > i_1 \geqslant \cdots \geqslant i_{n-k} \geqslant 0 = i_{n-k+1} = \cdots = i_n\}.$$

When $k = n$, we have $gs_{w,\mathbf{i}} \in \mathcal{GS}_{n,n}$ if and only if $w \in \mathfrak{S}_n$ and $\mathbf{i} = 0^n$ is the sequence of n zeros. Garsia [10] proved that $\mathcal{GS}_{n,n}$ descends to a basis of the classical coinvariant algebra R_n. Garsia and Stanton [12] later studied $\mathcal{GS}_{n,n}$ in the context of Stanley-Reisner theory. Extending Garsia's result, Haglund et. al. proved that $\mathcal{GS}_{n,k}$ descends to a basis of $R_{n,k}$ [16, Thm. 5.3]. We will prove that $\mathcal{GS}_{n,k}$ also descends to a basis of $S_{n,k}$. In fact, we will prove that $\mathcal{GS}_{n,k}$ is just one of a family of bases of $S_{n,k}$.

Huang used isobaric Demazure operators to define a basis of the classical coinvariant algebra R_n which is related to the classical GS basis $\mathcal{GS}_{n,n}$ by a unitriangular transition matrix [17]. We will modify $\mathcal{GS}_{n,k}$ to get a new basis of $S_{n,k}$ in an analogous way. As in [17], our modified basis will be crucial in our analysis of the $H_n(0)$-module structure of $S_{n,k}$. This modified basis and $\mathcal{GS}_{n,k}$ itself will both belong to the following family of bases of $S_{n,k}$.

To describe these bases, we will need a partial order on monomials in $\mathbb{F}[\mathbf{x}_n]$. If $m = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in $\mathbb{F}[\mathbf{x}_n]$, let $\lambda(m) := \mathrm{sort}(a_1, \ldots, a_n)$ be the sequence obtained by sorting the exponent sequence of m into weakly decreasing order. Following Adin, Brenti, and Roichman [1], we associate a collection of objects to any monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ in $\mathbb{F}[\mathbf{x}_n]$ as follows. Let $\sigma(m) = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ be the permutation (in one-line notation) obtained by listing the indices of variables in weakly decreasing order of the exponents in m, breaking ties by listing smaller indexed variables first. Let $d(m) = (d_1, \ldots, d_n)$ be the integer sequence given by $d_j = |\mathrm{Des}(\sigma(m)) \cap \{j, j+1, \ldots, n\}|$. Adin, Brenti, and Roichman [1] showed that the componentwise difference $\lambda(m) - d(m)$ is an integer partition (i.e., has weakly decreasing components). Let $\mu(m)$ be the *conjugate* of this integer partition.

For example, if $m = x_1^3 x_2^4 x_3^0 x_4^2 x_5^2 x_6^0 x_7^0$, then $\lambda(m) = (4,3,2,2,0,0,0)$ and $\sigma(m) = 2145367$. It follows that $d(m) = (2,1,1,1,0,0,0), \lambda(m) - d(m) = (2,2,1,1,0,0,0)$, and $\mu(m) = (4,2)$.

Definition 4.1. *Let $\prec$ be the partial order on monomials in $\mathbb{F}[\mathbf{x}_n]$ defined by $m \prec m'$ if and only if $\lambda(m) < \lambda(m')$ in lexicographical order.*

Lemma 4.2. *Let $\mathcal{B}_{n,k} = \{b_{w,\mathbf{i}}\}$ be a set of polynomials indexed by pairs $(w, \mathbf{i})$ where $w \in \mathfrak{S}_n$ and $\mathbf{i} = (i_1, \ldots, i_n)$ satisfy*

$$k - \mathrm{des}(w) > i_1 \geqslant \cdots \geqslant i_{n-k} \geqslant 0 = i_{n-k+1} = \cdots = i_n.$$

Assume that any $b_{w,\mathbf{i}} \in \mathcal{B}_{n,k}$ has the form

$$b_{w,\mathbf{i}} = gs_{w,\mathbf{i}} + \sum_{m \prec gs_{w,\mathbf{i}}} c_m \cdot m, \tag{66}$$

where the $c_m \in \mathbb{F}$ are scalars which could depend on $(w, \mathbf{i})$ and $\prec$ is the partial order on monomials appearing in Definition 4.1. The set $\mathcal{B}_{n,k}$ descends to a basis of $S_{n,k}$.

Proof. By [16, Thm. 5.3], we know that $|\mathcal{B}_{n,k}| = |\mathcal{GS}_{n,k}| = |\mathcal{OP}_{n,k}|$. By Theorem 3.8, we have $\dim(S_{n,k}) = |\mathcal{OP}_{n,k}|$. Therefore, it is enough to show that $\mathcal{B}_{n,k}$ descends to a spanning set of $S_{n,k}$.

If $\mathcal{B}_{n,k}$ did not descend to a spanning set of $S_{n,k}$, then there would be a monomial $m \in \mathbb{F}[\mathbf{x}_n]$ whose image $m + J_{n,k}$ did not lie in the span of $\mathcal{B}_{n,k}$. Working towards a contradiction, suppose that such a monomial existed.

Let $m = x_1^{a_1} \cdots x_n^{a_n}$ be any monomial in $\mathbb{F}[\mathbf{x}_n]$. We argue that m is expressible modulo $J_{n,k}$ as a linear combination of monomials of the form $m' = x_1^{b_1} \cdots x_n^{b_n}$ with $b_i < k$ for all i. Indeed, if m does not already have this form, choose i maximal such that $a_i > k$. Since $h_k(x_1, \ldots, x_i) \in J_{n,k}$, modulo $J_{n,k}$ we have

$$m \equiv -(x_1^{a_1} \cdots x_i^{a_i - k} \cdots x_n^{a_n}) \sum_{\substack{1 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant i \\ j_1 \neq i}} x_{j_1} \cdots x_{j_k}. \tag{67}$$

If every monomial appearing on the right hand side is of the required form, we are done. Otherwise, we may iterate this procedure. Since $h_k(x_1) = x_1^k \in J_{n,k}$, iterating this procedure eventually yields 0 or a linear combination of monomials of the required form.

Let $\prec_{ABR}$ be the partial order on monomials in $\mathbb{F}[\mathbf{x}_n]$ defined by $m \prec_{ABR} m'$ if and only if $\lambda(m) < \lambda(m')$ in lexicographical order or ($\lambda(m) = \lambda(m')$ and $\mathrm{inv}(\sigma(m)) > \mathrm{inv}(\sigma(m'))$). In particular, the relation $m \prec m'$ implies $m \prec_{ABR} m'$.

Let $m = x_1^{a_1} \cdots x_n^{a_n}$ be any monomial in $\mathbb{F}[\mathbf{x}_n]$ such that $m + J_{n,k}$ does not lie in the span of $\mathcal{B}_{n,k}$. By the reasoning above, we may assume that $a_i < k$ for all $1 \leqslant i \leqslant n$. Choose such an m which is minimal with respect to the partial order $\prec_{ABR}$.

Adin, Brenti, and Roichman [1, Lem. 3.5] proved that we can 'straighten' the monomial m and write

$$m = gs_{\sigma(m)} e_{\mu(m)}(\mathbf{x}_n) - \Sigma, \tag{68}$$

where Σ is a linear combination of monomials which are $\prec_{ABR} m$. Here

$$gs_{\sigma(m)} := gs_{\sigma(m),0^n} = x_{\sigma_1}^{d_1} \cdots x_{\sigma_n}^{d_n} \tag{69}$$

is the 'classical' GS monomial indexed by $\sigma(m)$. Our assumption on m guarantees that Σ lies in the span of $\mathcal{B}_{n,k}$ modulo $J_{n,k}$.

If $\mu(m)_1 > n-k$, then $e_{\mu(m)}(\mathbf{x}_n) \equiv 0$ modulo $J_{n,k}$. It follows that m lies in the span of $\mathcal{B}_{n,k}$ modulo $J_{n,k}$, which is a contradiction.

If $\mu(m)_1 \leqslant n-k$, then by the definition of $\lambda(m)$, $d(m)$, and $\mu(m)$, we may write

$$m = gs_{\sigma(m)} \cdot x_{\sigma_1}^{\mu(m)'_1} \cdots x_{\sigma_{n-k}}^{\mu(m)'_{n-k}}, \tag{70}$$

where $\mu(m)'_1 \geqslant \cdots \geqslant \mu(m)'_{n-k} \geqslant 0$ is the conjugate of $\mu(m)$. Suppose $\mu(m)'_1 \geqslant k - \mathrm{des}(\sigma(m))$. Since the exponent of x_{σ_1} in $gs_{\sigma(m)}$ equals $\mathrm{des}(\sigma(m))$, we then have $x_{\sigma_1}^k \mid m$, which contradicts the assumption that m has no variables with power $\geqslant k$. Therefore, we have $\mu(m)'_1 < k - \mathrm{des}(\sigma(m))$. This means that $m \in \mathcal{GS}_{n,k}$ and $m = gs_{w,\mathbf{i}}$ for some pair $(w, \mathbf{i})$. (In fact, we can take $(w, \mathbf{i}) = (\sigma(m), \mu')$.) However, our assumption on $\mathcal{B}_{n,k}$ guarantees that

$$m = gs_{w,\mathbf{i}} = b_{w,\mathbf{i}} - \sum_{m' \prec m} c_{m'} \cdot m' \tag{71}$$

for some scalars $c_{m'} \in \mathbb{F}$. Then our assumption on m together with the fact ($m' \prec m \Rightarrow m' \prec_{ABR} m$) imply that m lies in the span of $\mathcal{B}_{n,k}$ modulo $J_{n,k}$, which is a contradiction. ☐

Corollary 4.3. *Let $k \leqslant n$ be positive integers. The set $\mathcal{GS}_{n,k}$ of generalized Garsia-Stanton monomials descends to a basis of $S_{n,k}$.*

For example, suppose $(n,k) = (7,5)$ and $w = 6123745$. Then $\mathrm{des}(w) = 2$ and the classical GS monomial is $gs_w = (x_6)(x_6x_1x_2x_3x_7)$. We have $n-k = 2$ and $k - \mathrm{des}(w) = 3$, so that this classical GS monomial gives rise to the following six elements of $\mathcal{GS}_{n,k}$:

$$\begin{array}{ccc} (x_6)(x_6x_1x_2x_3x_7) & (x_6)(x_6x_1x_2x_3x_7)(x_6) & (x_6)(x_6x_1x_2x_3x_7)(x_6^2) \\ (x_6)(x_6x_1x_2x_3x_7)(x_6x_2) & (x_6)(x_6x_1x_2x_3x_7)(x_6^2x_2) & (x_6)(x_6x_1x_2x_3x_7)(x_6^2x_2^2). \end{array}$$

5. Module structure over the 0-Hecke algebra

In this section we prove an isomorphism $S_{n,k} \cong \mathbb{F}[\mathcal{OP}_{n,k}]$ of (ungraded) $H_n(0)$-modules.

5.1. Ordered set partitions.

We first describe the $H_n(0)$-module structure of $\mathbb{F}[\mathcal{OP}_{n,k}]$. Recall that if $\alpha \models n$ is a composition, then P_α is the corresponding indecomposable projective $H_n(0)$-module. We need a family of projective $H_n(0)$-modules which are indexed by pairs of compositions related by refinement. Let $\alpha, \beta \models n$ be two compositions satisfying $\alpha \preceq \beta$. Let $P_{\alpha,\beta}$ be the $H_n(0)$-module given by

$$P_{\alpha,\beta} := H_n(0) \overline{\pi}_{w_0(\alpha)} \pi_{w_0(\beta^c)}. \tag{72}$$

In particular, we have $P_{\alpha,\alpha} = P_\alpha$. More generally, we have the following structural result on $P_{\alpha,\beta}$.

Lemma 5.1 (Huang [18, Thm. 3.2]). *Let $\alpha, \beta \models n$ and assume $\alpha \preceq \beta$. Then $P_{\alpha,\beta}$ has basis*

$$\{\overline{\pi}_w \pi_{w_0(\beta^c)} \,:\, w \in \mathfrak{S}_n,\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\beta)\} \tag{73}$$

and direct sum decomposition

$$P_{\alpha,\beta} \cong \bigoplus_{\alpha \preceq \gamma \preceq \beta} P_\gamma. \tag{74}$$

For example, the module $P_{(4,1),(1,2,1,1)}$ breaks up into projective indecomposable submodules as

$$P_{(4,1),(1,2,1,1)} \cong P_{(4,1)} \oplus P_{(1,3,1)} \oplus P_{(3,1,1)} \oplus P_{(1,2,1,1)}.$$

Recall that, for each composition $\alpha = (\alpha_1, \dots, \alpha_\ell) \models n$, we denote by $\mathcal{OP}_\alpha$ the collection of ordered set partitions of shape α, i.e., pairs (w, α) for all $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$.

Lemma 5.2. *Let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be a composition of n. Then $\mathbb{F}[\mathcal{OP}_\alpha]$ is a cyclic $H_n(0)$-module generated by the ordered set partition $(12 \cdots n, \alpha)$ and is isomorphic to $P_{(n),\alpha}$ via the map defined by sending (w, α) to $\overline{\pi}_w \pi_{w_0(\alpha^c)}$ for all $w \in \mathfrak{S}_n$ with* $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$.

Proof. Huang [18, (3.3)] defined an action of $H_n(0)$ on the $\mathbb{F}$-span $P_{\alpha_1 \oplus \cdots \oplus \alpha_\ell}$ of standard tableaux of skew shape $\alpha_1 \oplus \cdots \oplus \alpha_\ell$, where $\alpha_1 \oplus \cdots \oplus \alpha_\ell$ is a disconnected union of rows of lengths $\alpha_1, \dots, \alpha_\ell$, ordered from southwest to northeast. There is an obvious isomorphism $\mathbb{F}[\mathcal{OP}_\alpha] \cong P_{\alpha_1 \oplus \cdots \oplus \alpha_\ell}$ by sending an ordered set partition $(B_1 | \cdots | B_k)$ to the tableau whose rows are $B_1, \dots, B_k$ from southwest to northeast. Combining this with the isomorphism $P_{\alpha_1 \oplus \cdots \oplus \alpha_\ell} \cong P_{(n),\alpha}$ provided by [18, Thm. 3.3] gives the desired result. □

Proposition 5.3. *Let $k \leqslant n$ be positive integers. Then we have isomorphisms of $H_n(0)$-modules:*

$$\mathbb{F}[\mathcal{OP}_{n,k}] \cong \bigoplus_{\substack{\alpha \models n \\ \ell(\alpha) = k}} \mathbb{F}[\mathcal{OP}_\alpha] \cong \bigoplus_{\beta \models n} P_\beta^{\oplus \binom{n-\ell(\beta)}{k-\ell(\beta)}}. \tag{75}$$

Proof. Since $\mathcal{OP}_{n,k}$ is the disjoint union of $\mathcal{OP}_\alpha$ for all compositions $\alpha \models n$ of length $\ell(\alpha) = k$, the first desired isomorphism follows. Applying Lemma 5.1 and Lemma 5.2 to each $\mathcal{OP}_\alpha$ gives a direct sum decomposition of $\mathbb{F}[\mathcal{OP}_{n,k}]$ into projective indecomposable modules. The multiplicity of P_β in this direct sum equals

$$|\{\beta \preceq \alpha : \ell(\alpha) = k\}| = \binom{n - \ell(\beta)}{k - \ell(\beta)}$$

for each $\beta \models n$. The second desired isomorphism follows. □

For example, when $n = 4$ and $k = 2$ we have $\mathbb{F}[\mathcal{OP}_{(1,3)}] \cong P_{(1,3)} \oplus P_{(4)}$, $\mathbb{F}[\mathcal{OP}_{(2,2)}] \cong P_{(2,2)} \oplus P_{(4)}$, $\mathbb{F}[\mathcal{OP}_{(3,1)}] \cong P_{(3,1)} \oplus P_{(4)}$, and summing these gives

$$\mathbb{F}[\mathcal{OP}_{4,2}] \cong P_{(1,3)} \oplus P_{(2,2)} \oplus P_{(3,1)} \oplus P_{(4)}^{\oplus 3}. \tag{76}$$

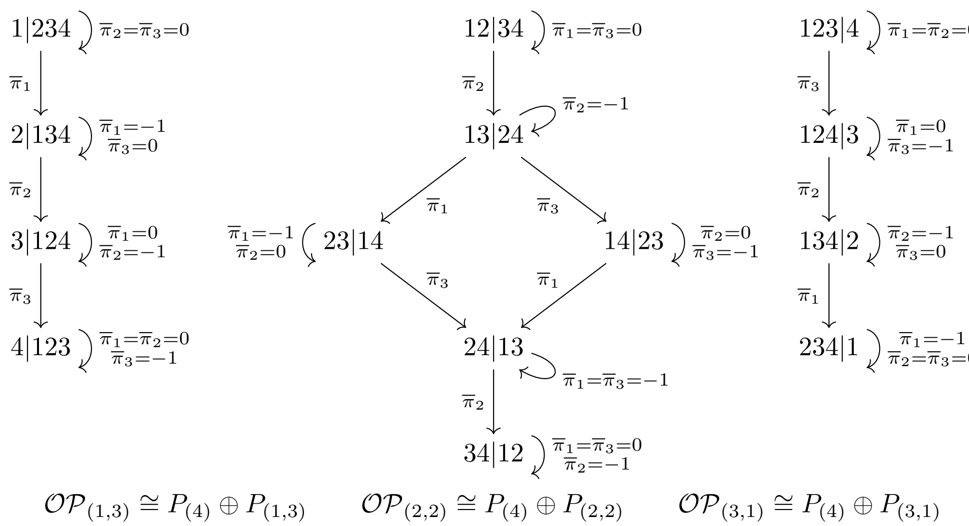


FIGURE 1. A decomposition of $\mathbb{F}[\mathcal{OP}_{4,2}]$

5.2. 0-HECKE ACTION ON POLYNOMIALS. Our next task is to show that $S_{n,k}$ has the same isomorphism type as the $H_n(0)$-module of Proposition 5.3. To do this, we will need to study the action of $H_n(0)$ on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ via the isobaric Demazure operators π_i defined in (7). Using the relation $\overline{\pi}_i = \pi_i - 1$, we have

$$\overline{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}(s_i(f))}{x_i - x_{i+1}}, \quad \forall i \in [n-1],\ \forall f \in \mathbb{F}[\mathbf{x}_n].$$

Thus for an arbitrary monomial $x_1^{a_1} \cdots x_n^{a_n}$, we have

(77)

$$\overline{\pi}_i(x_1^{a_1}\cdots x_n^{a_n}) = \begin{cases} (x_1^{a_1}\cdots x_{i-1}^{a_{i-1}} x_{i+2}^{a_{i+2}} \cdots x_n^{a_n}) \displaystyle\sum_{j=1}^{a_i - a_{i+1}} x_i^{a_i - j} x_{i+1}^{a_{i+1}+j} & a_i > a_{i+1} \\ 0 & a_i = a_{i+1} \\ -(x_1^{a_1}\cdots x_{i-1}^{a_{i-1}} x_{i+2}^{a_{i+2}} \cdots x_n^{a_n}) \displaystyle\sum_{j=0}^{a_{i+1} - a_i - 1} x_i^{a_{i+1}+j} x_{i+1}^{a_i - j} & a_i < a_{i+1}. \end{cases}$$

Using this we have the following triangularity result.

LEMMA 5.4. *Let $\mathbf{d} = (d_1 \geqslant \cdots \geqslant d_n)$ be a weakly decreasing vector of nonnegative integers and let $\mathbf{x}^{\mathbf{d}} = x_1^{d_1} \cdots x_n^{d_n}$ be the corresponding monomial in $\mathbb{F}[\mathbf{x}_n]$. Suppose $w \in \mathfrak{S}_n$ satisfies $\mathrm{Des}(w) \subseteq \mathrm{Des}(\mathbf{d})$. The polynomial $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}})$ has the form*

$$\overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) = w(\mathbf{x}^{\mathbf{d}}) + \sum_{m \prec w(\mathbf{x}^{\mathbf{d}})} c_m \cdot m \tag{78}$$

for some $c_m \in \mathbb{F}$.

Proof. The proof is similar to [17, Lem. 4.1]. Observe that a monomial m satisfies $m \prec \mathbf{x}^{\mathbf{d}}$ if and only if $m \prec w(\mathbf{x}^{\mathbf{d}})$ for any permutation $w \in \mathfrak{S}_n$.

We induct on the length $\ell(w)$ of the permutation w. If $\ell(w) = 0$, then w is the identity permutation and the lemma is trivial. Otherwise, we may write $w = s_j v$, where $j \in [n-1]$ and $v \in \mathfrak{S}_n$ satisfies $\ell(w) = \ell(v) + 1$. We have $j \in \mathrm{Des}(w^{-1})$,

$j \notin \mathrm{Des}(v^{-1})$, and $\mathrm{Des}(v) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\mathbf{d})$. By induction we have

$$\overline{\pi}_v(\mathbf{x}^{\mathbf{d}}) = v(\mathbf{x}^{\mathbf{d}}) + \sum_{m \prec \mathbf{x}^{\mathbf{d}}} a_m \cdot m \tag{79}$$

for some scalars $a_m \in \mathbb{F}$.

Since $j \notin \mathrm{Des}(v^{-1})$, we have $v^{-1}(j) < v^{-1}(j+1)$ and thus $d_{v^{-1}(j)} \geqslant d_{v^{-1}(j+1)}$. Since $wv^{-1}(j) = s_j(j) > s_j(j+1) = wv^{-1}(j+1)$, there exists an element of $[v^{-1}(j), v^{-1}(j+1)-1]$ which belongs to $\mathrm{Des}(w) \subseteq \mathrm{Des}(\mathbf{d})$. This implies $d_{v^{-1}(j)} > d_{v^{-1}(j+1)}$. Then by (77), applying $\overline{\pi}_j$ to $v(\mathbf{x}^{\mathbf{d}}) = x_{v(1)}^{d_1} \cdots x_{v(n)}^{d_n}$ we have

$$\overline{\pi}_j(v(\mathbf{x}^{\mathbf{d}})) = s_j v(\mathbf{x}^{\mathbf{d}}) + \sum_{m' \prec v(\mathbf{x}^{\mathbf{d}})} b_{m'} \cdot m' = w(\mathbf{x}^{\mathbf{d}}) + \sum_{m' \prec \mathbf{x}^{\mathbf{d}}} b_{m'} \cdot m' \tag{80}$$

for some scalars $b_{m'} \in \mathbb{F}$. On the other hand, (77) also implies that applying $\overline{\pi}_j$ to any monomial which is $\prec \mathbf{x}^{\mathbf{d}}$ will only yield terms which are also $\prec \mathbf{x}^{\mathbf{d}}$. Hence $\overline{\pi}_w(\mathbf{x}^d)$ has the desired form. □

We will decompose the quotient $S_{n,k}$ into a direct sum of projective modules of the form $P_{\alpha,\beta}$ defined in (72). This decomposition will ultimately rest on the following lemma.

Lemma 5.5. *Let* $\mathbf{d} = (d_1 \geqslant \cdots \geqslant d_n)$ *be a weakly decreasing sequence of nonnegative integers. Suppose* $\alpha, \beta \models n$ *such that* $\alpha \preceq \beta$ *and* $\mathrm{Des}(\mathbf{d}) = \mathrm{Des}(\beta)$*. Then* $H_n(0)\overline{\pi}_{w_0(\alpha)}\mathbf{x}^{\mathbf{d}}$ *has basis*

$$\left\{\overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) \,:\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\beta)\right\}. \tag{81}$$

Furthermore, sending each element $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}})$ *in the basis* (81) *to* $\overline{\pi}_w \pi_{w_0(\beta^c)}$ *gives an isomorphism* $H_n(0)\overline{\pi}_{w_0(\alpha)}\mathbf{x}^{\mathbf{d}} \cong P_{\alpha,\beta}$ *of* $H_n(0)$*-modules.*

Proof. Let $1 \leqslant i \leqslant n-1$. If $i \notin \mathrm{Des}(\beta)$, then the monomial $\mathbf{x}^{\mathbf{d}}$ is symmetric in x_i and x_{i+1}, so that $\overline{\pi}_i(\mathbf{x}^{\mathbf{d}}) = 0$ by (77). More generally, if $w \in \mathfrak{S}_n$ is such that $\mathrm{Des}(w) \not\subseteq \mathrm{Des}(\beta)$ then $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) = 0$ because there exists a reduced expression for w ending in s_i for some $i \in \mathrm{Des}(w) \setminus \mathrm{Des}(\beta)$.

By the last paragraph and the fact that $w_0(\alpha)$ is the left weak Bruhat minimal permutation with descent set α, the module $H_n(0)\overline{\pi}_{w_0(\alpha)}\mathbf{x}^{\mathbf{d}}$ is spanned by the set (81). This set is linearly independent and hence a basis for $H_n(0)\overline{\pi}_{w_0(\alpha)}\mathbf{x}^{\mathbf{d}}$, since by Lemma 5.4 and the equality $\mathrm{Des}(\mathbf{d}) = \mathrm{Des}(\beta)$, any two distinct elements $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}})$ and $\overline{\pi}_{w'}(\mathbf{x}^{\mathbf{d}})$ of this set have `neglex` leading monomials $w(\mathbf{x}^{\mathbf{d}})$ and $w'(\mathbf{x}^{\mathbf{d}})$, which are distinct by $\mathrm{Des}(w) \subseteq \mathrm{Des}(\mathbf{d})$ and $\mathrm{Des}(w') \subseteq \mathrm{Des}(\mathbf{d})$.

By Lemma 5.1, the module $P_{\alpha,\beta}$ has basis given by (73). Thus the assignment $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) \mapsto \overline{\pi}_w \pi_{w_0(\beta^c)}$ induces a linear isomorphism from $H_n(0)\overline{\pi}_{w_0(\alpha)}\mathbf{x}^{\mathbf{d}}$ to $P_{\alpha,\beta}$. To check that this is an isomorphism of $H_n(0)$-modules, let $1 \leqslant i \leqslant n-1$. We compare the action of $\overline{\pi}_i$ on the bases (81) and (73) as follows. Let $w \in \mathfrak{S}_n$ satisfy $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\beta)$.

If $i \in \mathrm{Des}(w^{-1})$, then there is a reduced expression for w starting with s_i and $\overline{\pi}_i$ acts by the scalar -1 on both $\overline{\pi}_w(\mathbf{x}^{\mathbf{d}})$ and $\overline{\pi}_w \pi_{w_0(\beta^c)}$ since $\overline{\pi}_i^2 = -\overline{\pi}_i$.

If $i \notin \mathrm{Des}(w^{-1})$ and $\mathrm{Des}(s_i w) \subseteq \mathrm{Des}(\beta)$, then the polynomial $\overline{\pi}_i \overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) = \overline{\pi}_{s_i w}(\mathbf{x}^{\mathbf{d}})$ lies in the basis (81) and the algebra element $\overline{\pi}_i \overline{\pi}_w \pi_{w_0(\beta^c)} = \overline{\pi}_{s_i w} \pi_{w_0(\beta^c)}$ lies in the basis (73).

If $i \notin \mathrm{Des}(w^{-1})$ and $\mathrm{Des}(s_i w) \not\subseteq \mathrm{Des}(\beta)$, we have $\overline{\pi}_i \overline{\pi}_w(\mathbf{x}^{\mathbf{d}}) = \overline{\pi}_{s_i w}(\mathbf{x}^{\mathbf{d}}) = 0$ by the observation in the first paragraph. On the other hand, we also have $\overline{\pi}_i \overline{\pi}_w \pi_{w_0(\beta^c)} = \overline{\pi}_{s_i w} \pi_{w_0(\beta^c)} = 0$, since $s_i w$ has a reduced expression ending with s_j for some $j \in \mathrm{Des}(\beta^c)$ and $\overline{\pi}_j \pi_{w_0(\beta^c)} = 0$ by the relation $\overline{\pi}_j \pi_j = 0$. □

5.3. Decomposition of $S_{n,k}$. We begin by introducing a family of $H_n(0)$-submodules of $S_{n,k}$.

Definition 5.6. *Let $A_{n,k}$ be the set of all pairs $(\alpha, \mathbf{i})$, where $\alpha \models n$ is a composition whose first part satisfies $\alpha_1 > n-k$ and $\mathbf{i} = (i_1, \dots, i_n)$ is a sequence of nonnegative integers satisfying*

$$k - \ell(\alpha) \geqslant i_1 \geqslant \cdots \geqslant i_{n-k} \geqslant 0 = i_{n-k+1} = \cdots = i_n.$$

Given a pair $(\alpha, \mathbf{i}) \in A_{n,k}$, let $N_{\alpha,\mathbf{i}}$ be the $H_n(0)$-module generated by the image of the polynomial $\overline{\pi}_{w_0(\alpha)}(\mathbf{x}_{\alpha,\mathbf{i}})$ in the quotient ring $S_{n,k}$.

For example, let $(n,k) = (6,3)$. Eliminating the $k = 3$ trailing zeros from the $\mathbf{i}$ sequences, and omitting parentheses and commas from compositions α and sequences $\mathbf{i}$, we have

$$A_{6,3} = \left\{ \begin{array}{c} (411,000),(42,111),(42,110),(42,100),(42,000),(51,111), \\ (51,110),(51,100),(51,000),(6,222),(6,221),(6,220), \\ (6,211),(6,210),(6,200),(6,111),(6,110),(6,100),(6,000) \end{array} \right\}.$$

Recall that, if $\alpha \models n$ and if $\mathbf{i}$ is a length n integer sequence, the composition $\alpha \cup \mathbf{i} \models n$ is characterized by $\mathrm{Des}(\alpha \cup \mathbf{i}) = \mathrm{Des}(\alpha) \cup \mathrm{Des}(\mathbf{i})$. When $(\alpha, \mathbf{i}) \in A_{n,k}$ we have the *disjoint* union decomposition $\mathrm{Des}(\alpha \cup \mathbf{i}) = \mathrm{Des}(\alpha) \sqcup \mathrm{Des}(\mathbf{i})$. In fact, each element of $\mathrm{Des}(\mathbf{i})$ lies in the interval $1 \leqslant j \leqslant n-k$ whereas each element of $\mathrm{Des}(\alpha)$ lies in the interval $n-k+1 \leqslant j \leqslant n-1$.

It will turn out that the $N_{\alpha,\mathbf{i}}$ modules are special cases of the $P_{\alpha,\beta}$ modules. We will prove that if $(\alpha, \mathbf{i}) \in A_{n,k}$, then $N_{\alpha,\mathbf{i}} \cong P_{\alpha,\alpha\cup\mathbf{i}}$. To prove this fact, we will need a modification of the GS basis $\mathcal{GS}_{n,k}$ of $S_{n,k}$. This modified basis will come from the following lemma, which states that the collection of GS basis elements $\mathcal{GS}_{n,k}$ is related in a unitriangular way with the collection of polynomials

$$\{\overline{\pi}_w(x_{\alpha,\mathbf{i}}) \,:\, (\alpha,\mathbf{i}) \in A_{n,k},\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})\}.$$

Lemma 5.7. *Let $k \leqslant n$ be positive integers and endow monomials in $\mathbb{F}[\mathbf{x}_n]$ with the partial order $\prec$.*

(i) *Let $(\alpha, \mathbf{i}) \in A_{n,k}$ and $w \in \mathfrak{S}_n$ be such that $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})$. Then the unique $\prec$-leading term of $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ is $w(\mathbf{x}_{\alpha,\mathbf{i}}) = gs_{w,\mathbf{i}'} \in \mathcal{GS}_{n,k}$, where $\mathbf{i}' = (i'_1, \dots, i'_n)$ is related to $\mathbf{i} = (i_1, \dots, i_n)$ by*

$$i'_j = i_j - |\{r \in \mathrm{Des}(w) \cap [n-k] \,:\, r \geqslant j\}|. \tag{82}$$

(ii) *Let $gs_{w,\mathbf{i}'} \in \mathcal{GS}_{n,k}$ be a GS basis element. Then $gs_{w,\mathbf{i}'}$ is the unique $\prec$-leading term of $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ for some $w \in \mathfrak{S}_n$ and some $(\alpha, \mathbf{i}) \in A_{n,k}$ satisfying $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})$ if and only if*
   - *$\alpha \models n$ is characterized by $\mathrm{Des}(\alpha) = \mathrm{Des}(w) \setminus [n-k]$, and*
   - *the sequence $\mathbf{i} = (i_1, \dots, i_n)$ is related to the sequence $\mathbf{i}' = (i'_1, \dots, i'_n)$ by Equation (82).*

Proof. (i) Since $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})$, Lemma 5.4 applies to show that the unique $\prec$-leading term of $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ is $w(\mathbf{x}_{\alpha,\mathbf{i}})$. We need to show that

- the sequence $\mathbf{i}' = (i'_1, \dots, i'_n)$ is nonnegative, weakly decreasing, and satisfies $i'_1 < k - \mathrm{des}(w)$ and $i'_{n-k+1} = \cdots = i'_n = 0$ so that the GS monomial $gs_{w,\mathbf{i}'}$ makes sense and lies in $\mathcal{GS}_{n,k}$, and
- we have $w(\mathbf{x}_{\alpha,\mathbf{i}}) = gs_{w,\mathbf{i}'}$.

It is clear that $i'_j = i_j = 0$ for $j > n-k$. We check that the sequence $\mathbf{i}'$ is weakly decreasing. To see this, let $1 \leqslant j \leqslant n-k$ and note that

$$i'_j - i'_{j+1} = \begin{cases} i_j - i_{j+1} - 1 & j \in \mathrm{Des}(w) \cap [n-k], \\ i_j - i_{j+1} & j \notin \mathrm{Des}(w) \cap [n-k]. \end{cases} \tag{83}$$

Since $\mathbf{i}$ is a weakly decreasing sequence and $i_j = i_{j+1}$ implies $j \notin \mathrm{Des}(\alpha \cup \mathbf{i}) \supseteq \mathrm{Des}(w)$, we conclude that $i'_j \geqslant i'_{j+1}$. Finally, we have $\mathrm{Des}(w) \cap [n-k] = \mathrm{Des}(w) \smallsetminus \mathrm{Des}(\alpha)$ since the definition of $A_{n,k}$ implies $D(\alpha) \cap [n-k] = \varnothing$. Then

$$i'_1 = i_1 - |\mathrm{Des}(w) \cap [n-k]| = i_1 - \mathrm{des}(w) + \ell(\alpha) - 1 < k - \mathrm{des}(w), \tag{84}$$

so that $gs_{w,\mathbf{i}'} \in \mathcal{GS}_{n,k}$ is a genuine GS basis element.

Next, we show $w(\mathbf{x}_{\alpha,\mathbf{i}}) = gs_{w,\mathbf{i}'}$. Let $1 \leqslant j \leqslant n$. Since $\mathrm{Des}(w) \cap [n-k] = \mathrm{Des}(w) \smallsetminus \mathrm{Des}(\alpha)$, it follows from (82) that

$$|\{r \in \mathrm{Des}(\alpha) \,:\, r \geqslant j\}| + i_j = |\{r \in \mathrm{Des}(w) \,:\, r \geqslant j\}| + i'_j. \tag{85}$$

This means that the variable $x_{w(j)}$ has the same exponent in $w(\mathbf{x}_{\alpha,\mathbf{i}})$ as $gs_{w,\mathbf{i}'}$. We conclude that $w(\mathbf{x}_{\alpha,\mathbf{i}}) = gs_{w,\mathbf{i}'}$.

(ii) Let $gs_{w,\mathbf{i}'} \in \mathcal{GS}_{n,k}$. Suppose $gs_{w,\mathbf{i}'}$ is the unique $\prec$-leading term of $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ for some $w \in \mathfrak{S}_n$ and some $(\alpha, \mathbf{i}) \in A_{n,k}$ satisfying $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})$.

The definition of $A_{n,k}$ implies $\mathrm{Des}(\alpha) \cap [n-k] = \varnothing$ and $\mathrm{Des}(\mathbf{i}) \subseteq [n-k]$. Thus $\mathrm{Des}(\alpha) = \mathrm{Des}(w) \smallsetminus [n-k]$ and $\mathrm{Des}(w) \smallsetminus \mathrm{Des}(\alpha) = \mathrm{Des}(w) \cap [n-k]$. Lemma 5.4 guarantees that $gs_{w,\mathbf{i}'} = w(x_{\alpha,\mathbf{i}})$. Comparing the power of the variable $x_{w(j)}$ on both sides of this equality gives (82) for all $1 \leqslant j \leqslant n$.

Conversely, given $gs_{w,\mathbf{i}'} \in \mathcal{GS}_{n,k}$, define α and $\mathbf{i}$ as in the statement of the lemma. We have $(\alpha, \mathbf{i}) \in A_{n,k}$ and the unique $\prec$-leading term of $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ is $gs_{w,\mathbf{i}'}$ by similar arguments to those above. ☐

Lemma 5.7 can be used to derive a new basis for the quotient $S_{n,k}$. This basis will be helpful in decomposing $S_{n,k}$ into a direct sum of $H_n(0)$-modules of the form $N_{\alpha,\mathbf{i}}$.

Lemma 5.8. *Let $k \leqslant n$ be positive integers. The set of polynomials*

$$\{\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) \,:\, (\alpha, \mathbf{i}) \in A_{n,k},\, w \in \mathfrak{S}_n,\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})\} \tag{86}$$

in $\mathbb{F}[\mathbf{x}_n]$ descends to a vector space basis of the quotient ring $S_{n,k}$. Moreover, for any $(\alpha, \mathbf{i}) \in A_{n,k}$ and any $w \in \mathfrak{S}_n$ with $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})$ we have

$$\deg(\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})) = \deg(\mathbf{x}_{\alpha,\mathbf{i}}) = \mathrm{maj}(\alpha) + |\mathbf{i}|. \tag{87}$$

Proof. By Lemma 5.7, the polynomials in the statement satisfy the conditions of Lemma 4.2, and hence descend to a basis for $S_{n,k}$. The degree formula is clear. ☐

In the coinvariant algebra case $k = n$, the basis of Lemma 5.8 appeared in [17]. As in [17], this modified GS-basis will facilitate analysis of the $H_n(0)$-structure of $S_{n,k}$.

Theorem 5.9. *Let $k \leqslant n$ be positive integers. For each $(\alpha, \mathbf{i}) \in A_{n,k}$, the set of polynomials*

$$\{\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) \,:\, w \in \mathfrak{S}_n,\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})\} \tag{88}$$

descends to a basis for $N_{\alpha,\mathbf{i}}$, and we have an isomorphism $N_{\alpha,\mathbf{i}} \cong P_{\alpha,\alpha\cup\mathbf{i}}$ of $H_n(0)$-modules by $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) \mapsto \overline{\pi}_w \pi_{w_0((\alpha\cup\mathbf{i})^c)}$. Moreover, the $H_n(0)$-module $S_{n,k}$ satisfies

$$S_{n,k} = \bigoplus_{(\alpha,\mathbf{i}) \in A_{n,k}} N_{\alpha,\mathbf{i}} \cong \bigoplus_{\beta \models n} P_\beta^{\oplus \binom{n-\ell(\beta)}{k-\ell(\beta)}} \cong \mathbb{F}[\mathcal{OP}_{n,k}]. \tag{89}$$

Proof. By Lemma 5.8, $S_{n,k}$ has a basis given by (86), which is the disjoint union of (88) for all $(\alpha, \mathbf{i}) \in A_{n,k}$. Combining this with Lemma 5.5, we have the basis (88) for $N_{\alpha,\mathbf{i}}$ and the desired isomorphism $N_{\alpha,\mathbf{i}} \cong P_{\alpha,\alpha\cup\mathbf{i}}$ for all $(\alpha,\mathbf{i}) \in A_{n,k}$. The decomposition $S_{n,k} = \bigoplus_{(\alpha,\mathbf{i})\in A_{n,k}} N_{\alpha,\mathbf{i}}$ follows.

Next, let $\beta \models n$ and count the multiplicity of P_β as a direct summand in $S_{n,k}$. Suppose P_β is a direct summand of $N_{\alpha,\mathbf{i}}$ for some $(\alpha,\mathbf{i}) \in A_{n,k}$. Since $\mathrm{Des}(\alpha \cup \mathbf{i})$ is the disjoint union $\mathrm{Des}(\alpha) \sqcup \mathrm{Des}(\mathbf{i})$ and $\alpha \preceq \beta \preceq \alpha \cup \mathbf{i}$, we must have $\mathrm{Des}(\alpha) = \mathrm{Des}(\beta) \setminus [n-k]$. It follows that the multiplicity of P_β in $S_{n,k}$ equals the number of choices of $\mathbf{i}$ such that $(\alpha,\mathbf{i}) \in A_{n,k}$ and $\alpha \preceq \beta \preceq \alpha \cup \mathbf{i}$, where α is characterized by $\mathrm{Des}(\alpha) = \mathrm{Des}(\beta) \setminus [n-k]$.

We count the sequences $\mathbf{i} = (i_1, \ldots, i_n)$ of the above paragraph as follows. Since $\mathrm{Des}(\beta) \cap [n-k] \subseteq \mathrm{Des}(\mathbf{i})$, subtracting 1 from $i_1, \ldots, i_r$ for all $r \in \mathrm{Des}(\beta) \cap [n-k]$ gives a weakly decreasing sequence $\mathbf{i}' = (i'_1, \ldots, i'_n)$ satisfying $i'_{n-k+1} = \cdots = i'_n = 0$ and

$$i'_1 \leqslant k - \ell(\alpha) - |\mathrm{Des}(\beta) \cap [n-k]| = k - \ell(\beta).$$

This gives a bijection from the collection of sequences $\mathbf{i}$ of the last paragraph and sequences $\mathbf{i}'$ satisfying the conditions of the last sentence. The number of such sequences $\mathbf{i}'$ is $\binom{n-\ell(\beta)}{k-\ell(\beta)}$, which equals the multiplicity of P_β in $S_{n,k}$. Then Proposition 5.3 gives us $S_{n,k} \cong \mathbb{F}[\mathcal{OP}_{n,k}]$, as desired. □

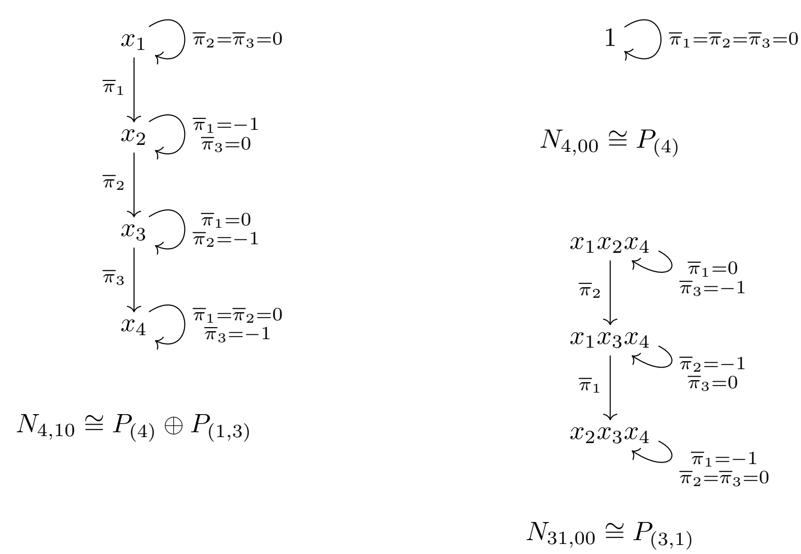
For example, let $(n,k) = (4,2)$. We have

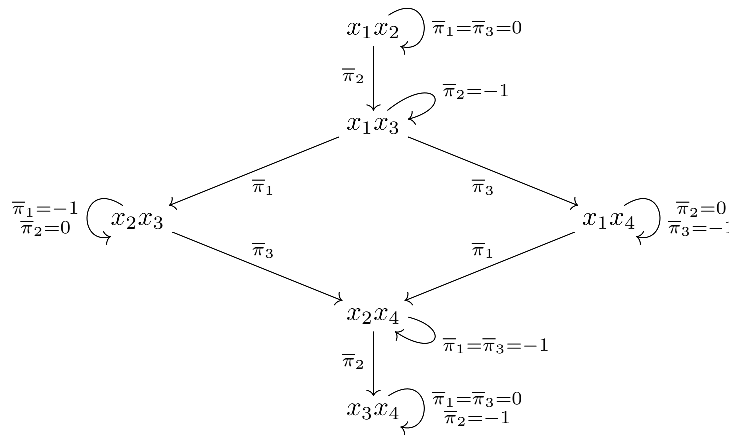
$$A_{4,2} = \{(31, 0000), (4, 1100), (4, 1000)(4, 0000)\}.$$

We get the corresponding $N_{\alpha,\mathbf{i}}$ modules

$$N_{31,0000} \cong P_{(3,1),(3,1)} \cong P_{(3,1)} \qquad N_{4,1100} \cong P_{(4),(2,2)} \cong P_{(4)} \oplus P_{(2,2)}$$
$$N_{4,1000} \cong P_{(4),(1,3)} \cong P_{(4)} \oplus P_{(1,3)} \qquad N_{4,0000} \cong P_{(4),(4)} \cong P_{(4)}.$$

Combining this with Theorem 5.9, we have $S_{4,2} \cong P_{(2,2)} \oplus P_{(1,3)} \oplus P_{(3,1)} \oplus P_{(4)}^{\oplus 3} \cong \mathbb{F}[\mathcal{OP}_{4,2}]$. The following picture illustrates this isomorphism via the action of $H_4(0)$ on the basis (86) of $S_{4,2}$ in Lemma 5.8. Note that the elements in this basis are polynomials in general, although they happen to be monomials in this example.





$$N_{4,1100} \cong P_{(4)} \oplus P_{(2,2)}$$

6. Characteristic formulas

In this section we derive formulas for the quasisymmetric and noncommutative symmetric characteristics of the modules $S_{n,k}$. To warm up, we calculate the degree-graded characteristics of the $N_{\alpha,\mathbf{i}}$ modules.

Recall that for $(\alpha,\mathbf{i}) \in A_{n,k}$ the module $N_{\alpha,\mathbf{i}}$ is the cyclic $H_n(0)$-module generated by the image of the polynomial $\overline{\pi}_{w_0(\alpha)}(\mathbf{x}_{\alpha,\mathbf{i}})$ in the quotient $S_{n,k}$.

> *We adopt the length grading convention that the distinguished generator $\overline{\pi}_{w_0(\alpha)}(\mathbf{x}_{\alpha,\mathbf{i}})$ of $N_{\alpha,\mathbf{i}}$ has length* $\mathrm{inv}(w_0(\alpha))$.

Lemma 6.1. *Let $k \leqslant n$ be positive integers and let $(\alpha,\mathbf{i}) \in A_{n,k}$. The module $N_{\alpha,\mathbf{i}}$ is projective and the characteristics $\mathbf{ch}_t(N_{\alpha,\mathbf{i}})$ and $\mathrm{Ch}_{q,t}(N_{\alpha,\mathbf{i}})$ have the following expressions:*

$$\mathbf{ch}_t(N_{\alpha,\mathbf{i}}) = t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} \sum_{\alpha \preceq \beta \preceq \alpha\cup\mathbf{i}} \mathbf{s}_\beta, \tag{90}$$

$$\mathrm{Ch}_{q,t}(N_{\alpha,\mathbf{i}}) = t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} \sum_{\substack{w\in\mathfrak{S}_n \\ \mathrm{Des}(\alpha)\subseteq\mathrm{Des}(w)\subseteq\mathrm{Des}(\alpha\cup\mathbf{i})}} q^{\mathrm{inv}(w)} F_{\mathrm{iDes}(w)}, \tag{91}$$

where in the second formula we view $N_{\alpha,\mathbf{i}}$ as a cyclic module generated by $\overline{\pi}_{w_0(\alpha)}(\mathbf{x}_{\alpha,\mathbf{i}})$.

Proof. Theorem 5.9 and Lemma 5.1 show that $N_{\alpha,\mathbf{i}} \cong P_{\alpha,\alpha\cup\mathbf{i}} \cong \bigoplus_{\alpha\preceq\gamma\preceq\alpha\cup\mathbf{i}} P_\gamma$ is a direct sum of projective modules, so that $N_{\alpha,\mathbf{i}}$ is projective. As observed in the proof of Theorem 5.9, the set

$$\{\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) \,:\, w\in\mathfrak{S}_n,\, \mathrm{Des}(\alpha)\subseteq\mathrm{Des}(w)\subseteq\mathrm{Des}(\alpha\cup\mathbf{i})\}$$

is a basis for $N_{\alpha,\mathbf{i}}$. Since the degree of the polynomial $\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}})$ is $\mathrm{maj}(\alpha)+|\mathbf{i}|$, the formula for $\mathbf{ch}_t(N_{\alpha,\mathbf{i}})$ follows from Theorem 5.9. For any $\ell \geqslant 0$, the term $N_{\alpha,\mathbf{i}}^{(\ell)}$ in the length filtration of $N_{\alpha,\mathbf{i}}$ has basis

$$\{\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) \,:\, w\in\mathfrak{S}_n,\, \mathrm{Des}(\alpha)\subseteq\mathrm{Des}(w)\subseteq\mathrm{Des}(\alpha\cup\mathbf{i}),\, \ell(w)\geqslant\ell\}.$$

The formula for $\mathrm{Ch}_{q,t}(N_{\alpha,\mathbf{i}})$ follows. ☐

THEOREM 6.2. *Let $k \leqslant n$ be positive integers. We have*

$$\mathbf{ch}_t(S_{n,k}) = \sum_{\alpha \models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n-\ell(\alpha) \\ k-\ell(\alpha) \end{bmatrix}_t \mathbf{s}_\alpha, \tag{92}$$

$$\mathrm{Ch}_{q,t}(S_{n,k}) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)} \tag{93}$$

$$= \sum_{(w,\alpha) \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w,\alpha)} F_{\mathrm{iDes}(w)}. \tag{94}$$

Proof. Theorem 5.9 gives a decomposition

$$S_{n,k} = \bigoplus_{(\alpha,\mathbf{i}) \in A_{n,k}} N_{\alpha,\mathbf{i}}. \tag{95}$$

Combining this with Lemma 6.1 we have

$$\begin{aligned} \mathbf{ch}_t(S_{n,k}) &= \sum_{(\alpha,\mathbf{i}) \in A_{n,k}} t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} \sum_{\alpha \preceq \beta \preceq \alpha \cup \mathbf{i}} \mathbf{s}_\beta \\ &= \sum_{\beta \models n} \sum_{\substack{(\alpha,\mathbf{i}) \in A_{n,k} \\ \alpha \preceq \beta \preceq \alpha \cup \mathbf{i}}} t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} \mathbf{s}_\beta. \end{aligned}$$

For each fixed composition $\beta \models n$, there exists $(\alpha, \mathbf{i}) \in A_{n,k}$ such that $\alpha \preceq \beta \preceq \beta \cup \mathbf{i}$ if and only if

- $\mathrm{Des}(\alpha) = \mathrm{Des}(\beta) \setminus [n-k]$ (so that α is uniquely determined by β),
- the sequence $\mathbf{i} = (i_1, \ldots, i_n)$ satisfies $\mathrm{Des}(\mathbf{i}) = \mathrm{Des}(\beta) \cap [n-k]$, and
- we have $k - \ell(\alpha) \geqslant i_1 \geqslant \cdots \geqslant i_{n-k} \geqslant 0 = i_{n-k+1} = \cdots = i_n$.

We obtain a sequence $\mathbf{i}' = (i'_1, \ldots, i'_n)$ from $\mathbf{i}$ by subtracting 1 from $i_1, \ldots, i_j$ for all $j \in \mathrm{Des}(\mathbf{i})$. This gives a bijection between the sequences $\mathbf{i}$ satisfying the above requirements and the sequences $\mathbf{i}' = (i'_1, \ldots, i'_n)$ such that

$$k - \ell(\beta) \geqslant i'_1 \geqslant \cdots \geqslant i'_{n-k} \geqslant 0 = i_{n-k+1} = \cdots = i_n.$$

We also have

$$\mathrm{maj}(\alpha) + |\mathbf{i}| = \mathrm{maj}(\beta) + |\mathbf{i}| - \mathrm{maj}(\mathbf{i}) = \mathrm{maj}(\beta) + |\mathbf{i}'|.$$

It follows that

$$\mathbf{ch}_t(S_{n,k}) = \sum_{\beta \models n} t^{\mathrm{maj}(\beta)} \begin{bmatrix} n-\ell(\beta) \\ k-\ell(\beta) \end{bmatrix}_t \mathbf{s}_\beta.$$

Lemma 6.1 and the decomposition (95) yield

$$\begin{aligned} \mathrm{Ch}_{q,t}(S_{n,k}) &= \sum_{(\alpha,\mathbf{i}) \in A_{n,k}} t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} \sum_{\substack{w \in \mathfrak{S}_n: \\ \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})}} q^{\mathrm{inv}(w)} F_{\mathrm{iDes}(w)} \\ &= \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} \sum_{\substack{(\alpha,\mathbf{i}) \in A_{n,k}: \\ \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha \cup \mathbf{i})}} t^{\mathrm{maj}(\alpha)+|\mathbf{i}|} F_{\mathrm{iDes}(w)} \\ &= \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)} \end{aligned}$$

where the last equality follows from the previous argument for $\mathbf{ch}_t(S_{n,k})$ by setting $\mathrm{Des}(\beta) = \mathrm{Des}(w)$.

Now recall that for an ordered set partition $(w,\alpha) = (B_1|B_2|\cdots|B_k) \in \mathcal{OP}_{n,k}$ we have

$$\mathrm{maj}(w,\alpha) := \mathrm{maj}(w) + \sum_{i\,:\,\max(B_i)<\min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i - i).$$

For a fixed $w \in \mathfrak{S}_n$, there exists $\alpha \models n$ such that $(w,\alpha) \in \mathcal{OP}_{n,k}$ if and only if $|\mathrm{Des}(w)| < k$ and $\mathrm{Des}(\alpha)$ contains all descents of w together with $k-1-\mathrm{des}(w)$ many elements of $[n-1] \setminus \mathrm{Des}(w)$. Given a set of $k-1-\mathrm{des}(w)$ elements of $[n-1]\setminus\mathrm{Des}(w)$, we have $(w,\alpha) = (B_1|\cdots|B_k) \in \mathcal{OP}_{n,k}$ determined in the above way, and this set corresponds to a lattice path from the lower-left corner to the upper-right corner of a $(k-1-\mathrm{des}(w)) \times (n-k)$ rectangle. The areas of the rows above this path are given by $\alpha_1 + \cdots + \alpha_i - i$ for all $i \in [k-1]$ satisfying $\max(B_i) < \min(B_{i+1})$. Thus

$$\sum_{(w,\alpha)\in\mathcal{OP}_{n,k}} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w,\alpha)} F_{\mathrm{iDes}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)}.$$

This completes the proof. □

Remark 6.3. We can get the same characteristic $\mathrm{Ch}_{q,t}(S_{n,k})$ as in Theorem 6.2 using a different decomposition of $S_{n,k}$ into cyclic modules coming from the $H_n(0)$-module isomorphisms

$$S_{n,k} \cong \mathbb{F}[\mathcal{OP}_{n,k}] \cong \bigoplus_{\substack{\alpha\models n \\ \ell(\alpha)=k}} \mathbb{F}[\mathcal{OP}_\alpha]$$

provided by Theorem 5.9 and Proposition 5.3, without adjusting the length grading of each copy of the cyclic module $\mathbb{F}[\mathcal{OP}_\alpha]$ in $S_{n,k}$. The proof is somewhat messy and hence skipped.

The first expression for $\mathrm{Ch}_{q,t}(S_{n,k})$ presented in Theorem 6.2 is related to an extension of the biMahonian distribution to ordered set partitions. More precisely, let $\sigma \in \mathcal{OP}_{n,k}$ be an ordered set partition and represent σ as (w,α), where $w \in \mathfrak{S}_n$ is a permutation which satisfies $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$. We define the *length* statistic $\ell(\sigma)$ by

(96) $$\ell(\sigma) = \ell(w,\alpha) := \mathrm{inv}(w).$$

In the language of Coxeter groups, the permutation w is the Bruhat minimal representative of the parabolic coset $w\mathfrak{S}_\alpha = w(\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$, so that $\ell(\sigma)$ is the Coxeter length of this minimal element.

We have

(97) $$\sum_{\sigma\in\mathcal{OP}_\alpha} q^{\ell(\sigma)} = \begin{bmatrix} n \\ \alpha_1,\ldots,\alpha_k \end{bmatrix}_q.$$

Summing Equation (97) over all $\alpha \models n$ with $\ell(\alpha) = k$ gives a *different* distribution than the generating function of maj:

(98) $$\sum_{\sigma\in\mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)),$$

although these distributions both equal $[n]!_q$ in the case $k = n$.[(2)]

By Theorem 6.2 we have

(99) $$\mathrm{Ch}_{q,t}(S_{n,k}) = \sum_{\sigma\in\mathcal{OP}_{n,k}} q^{\ell(\sigma)} t^{\mathrm{maj}(\sigma)} F_{\mathrm{iDes}(\sigma)},$$

[(2)] There is a different extension of the inversion/length statistic on $\mathfrak{S}_n$ to $\mathcal{OP}_{n,k}$ [22, 27, 23, 15, 16] whose distribution is $[k]!_q \cdot \mathrm{Stir}_q(n,k)$.

where $F_{\mathrm{iDes}(\sigma)} := F_{\mathrm{iDes}(w)}$ for $\sigma = (w, \alpha)$. In other words, we have that $\mathrm{Ch}_{q,t}(S_{n,k})$ is the generating function for the 'biMahonian pair' (ℓ, maj) on $\mathcal{OP}_{n,k}$ with quasisymmetric function weight $F_{\mathrm{iDes}(\sigma)}$.

We may also derive expressions for the degree-graded quasisymmetric characteristic $\mathrm{Ch}_t(S_{n,k})$. It turns out that this quasisymmetric characteristic is actually a symmetric function since $S_{n,k}$ is projective and $\mathrm{Ch}(P_\alpha) = s_\alpha \in \mathrm{Sym}$ as given in (46). We give an explicit expansion of $\mathrm{Ch}_t(S_{n,k})$ in the Schur basis.

Corollary 6.4. *Let $k \leqslant n$ be positive integers. We have*

$$\mathrm{Ch}_t(S_{n,k}) = \sum_{(w,\alpha)\in\mathcal{OP}_{n,k}} t^{\mathrm{maj}(w,\alpha)} F_{\mathrm{iDes}(w)} \tag{100}$$

$$= \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)} \tag{101}$$

$$= \sum_{\alpha\models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n-\ell(\alpha) \\ k-\ell(\alpha) \end{bmatrix}_t s_\alpha. \tag{102}$$

Moreover, the above symmetric function has expansion in the Schur basis given by

$$\mathrm{Ch}_t(S_{n,k}) = \sum_{Q\in\mathrm{SYT}(n)} t^{\mathrm{maj}(Q)} \begin{bmatrix} n-\mathrm{des}(Q)-1 \\ k-\mathrm{des}(Q)-1 \end{bmatrix}_t s_{\mathrm{shape}(Q)}. \tag{103}$$

Proof. The first and second expressions for $\mathrm{Ch}_t(S_{n,k})$ follow from Theorem 6.2 by setting $q = 1$ in the expressions for $\mathrm{Ch}_{q,t}(S_{n,k})$ given there. The third expression for $\mathrm{Ch}_t(S_{n,k})$ follows from replacing $\mathbf{s}_\alpha$ by s_α in $\mathbf{ch}_t(S_{n,k})$.

To derive Equation (103), we start with

$$\mathrm{Ch}_t(S_{n,k}) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)}$$

and apply the Schensted correspondence. More precisely, the (row insertion) Schensted correspondence gives a bijection $w \mapsto (P(w), Q(w))$ from the symmetric group $\mathfrak{S}_n$ to ordered pairs of standard Young tableaux with n boxes having the same shape. An example is given below.

$$25714683 \;\mapsto\; \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 8 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}\;,\quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}$$

A *descent* of a standard tableau P is a letter i which appears in a row above the row containing $i+1$ in P. We let $\mathrm{Des}(P)$ denote the set of descents of P, and define the corresponding descent number $\mathrm{des}(P) := |\mathrm{Des}(P)|$ and major index $\mathrm{maj}(P) := \sum_{i\in\mathrm{Des}(P)} i$. Under the Schensted bijection we have $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$, so that $\mathrm{des}(w) = \mathrm{des}(Q(w))$ and $\mathrm{maj}(w) = \mathrm{maj}(Q(w))$. Moreover, we have $w^{-1} \mapsto (Q(w), P(w))$, so that $\mathrm{iDes}(w) = \mathrm{Des}(P(w))$.

Applying the Schensted correspondence, we see that

$$\mathrm{Ch}_t(S_{n,k}) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{iDes}(w)} \tag{104}$$

$$= \sum_{(P,Q)} t^{\mathrm{maj}(Q)} \begin{bmatrix} n-\mathrm{des}(Q)-1 \\ k-\mathrm{des}(Q)-1 \end{bmatrix}_t F_{\mathrm{Des}(P)}, \tag{105}$$

where the second sum is over all pairs (P,Q) of standard Young tableaux with n boxes satisfying $\mathrm{shape}(P)=\mathrm{shape}(Q)$. Gessel [13] proved that for any $\lambda\vdash n$,

$$\sum_{P\in\mathrm{SYT}(\lambda)} F_{\mathrm{Des}(P)} = s_\lambda, \tag{106}$$

where the sum is over all standard tableaux P of shape λ. Applying Equation (106) gives

$$\sum_{(P,Q)} t^{\mathrm{maj}(Q)} \begin{bmatrix} n-\mathrm{des}(Q)-1 \\ k-\mathrm{des}(Q)-1\end{bmatrix}_t F_{\mathrm{Des}(P)} =$$

$$= \sum_{Q} t^{\mathrm{maj}(Q)} \begin{bmatrix} n-\mathrm{des}(Q)-1 \\ k-\mathrm{des}(Q)-1\end{bmatrix}_t \sum_{P\in\mathrm{SYT}(\mathrm{shape}(Q))} F_{\mathrm{Des}(P)}$$

$$= \sum_{Q} t^{\mathrm{maj}(Q)} \begin{bmatrix} n-\mathrm{des}(Q)-1 \\ k-\mathrm{des}(Q)-1\end{bmatrix}_t s_{\mathrm{shape}(Q)},$$

as desired. □

The Schur expansion of $\mathrm{Ch}_t(S_{n,k})$ given in Corollary 6.4 coincides (after setting $q=t$) with the Schur expansion [16, Cor. 6.13] of the Frobenius image of the graded $\mathfrak{S}_n$-module $R_{n,k}$. That is, we have

$$\mathrm{Ch}_t(S_{n,k}) = \mathrm{grFrob}(R_{n,k};t). \tag{107}$$

7. Conclusion

7.1. Macdonald polynomials and Delta conjecture. Equation (107) gives a connection between our work and the theory of Macdonald polynomials. More precisely, the *Delta Conjecture* of Haglund, Remmel, and Wilson [15] predicts that

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k-1}(\mathbf{x};q,t) = \mathrm{Val}_{n,k-1}(\mathbf{x};q,t), \tag{108}$$

where $\Delta'_{e_{k-1}}$ is the Macdonald eigenoperator defined by

$$\Delta'_{e_{k-1}} : \tilde{H}_\mu \mapsto e_{k-1}[B_\mu(q,t)-1]\cdot \tilde{H}_\mu \tag{109}$$

and $\mathrm{Rise}_{n,k-1}(\mathbf{x};q,t)$ and $\mathrm{Val}_{n,k-1}(\mathbf{x};q,t)$ are certain combinatorially defined quasisymmetric functions; see [15] for definitions. By the work of Wilson [27] and Rhoades [23], we have the following consequence of the Delta Conjecture:

$$\mathrm{Rise}_{n,k-1}(\mathbf{x};q,0) = \mathrm{Rise}_{n,k-1}(\mathbf{x};0,q) = \mathrm{Val}_{n,k-1}(\mathbf{x};q,0) = \mathrm{Val}_{n,k-1}(\mathbf{x};0,q). \tag{110}$$

If we let $C_{n,k}(\mathbf{x};q)$ denote the common symmetric function in Equation (110), the work of Haglund, Rhoades, and Shimozono [16, Thm. 6.11] implies that

$$\mathrm{grFrob}(R_{n,k};q) = (\mathrm{rev}_q\circ\omega) C_{n,k}(\mathbf{x};q), \tag{111}$$

where ω is the standard involution on Sym sending h_d to e_d for all $d\geqslant 0$. Equation (107) implies that

$$\mathrm{Ch}_t(S_{n,k}) = (\mathrm{rev}_t\circ\omega) C_{n,k}(\mathbf{x};t). \tag{112}$$

The derivation of $\mathrm{grFrob}(R_{n,k};q)$ in [16] has a different flavor from our derivation of $\mathrm{Ch}_t(S_{n,k})$; the definition of the rings $R_{n,k}$ is extended to include a family $R_{n,k,s}$ involving a third parameter s. The $R_{n,k,s}$ rings are related to the image of the $R_{n,k}$ rings under a certain idempotent in the symmetric group algebra $\mathbb{Q}[\mathfrak{S}_n]$; this relationship forms the basis of an inductive derivation of $\mathrm{grFrob}(R_{n,k};q)$. The coincidence of $\mathrm{Ch}_t(S_{n,k})$ and $\mathrm{grFrob}(R_{n,k};t)$ is mysterious to the authors.

Problem 7.1. *Find a conceptual explanation of the identity*

$$\mathrm{Ch}_t(S_{n,k}) = \mathrm{grFrob}(R_{n,k}; t).$$

7.2. Tanisaki ideals. Given a partition $\lambda \vdash n$, let $I_\lambda \subseteq \mathbb{F}[\mathbf{x}_n]$ denote the corresponding *Tanisaki ideal* (see [11] for a generating set of I_λ). When $\mathbb{F} = \mathbb{Q}$, the quotient $R_\lambda := \mathbb{F}[\mathbf{x}_n]/I_\lambda$ is isomorphic to the cohomology ring of the Springer fiber attached to λ. The quotient R_λ is a graded $\mathfrak{S}_n$-module. It is well known [11] that $\mathrm{grFrob}(R_\lambda; q) = \mathrm{rev}_q(Q'_\lambda(\mathbf{x}; q))$, where $Q'_\lambda(\mathbf{x}; q)$ is the dual Hall-Littlewood polynomial indexed by λ.

Huang proved that I_λ is closed under the action of $H_n(0)$ on $\mathbb{F}[\mathbf{x}_n]$ if and only if λ is a hook, so that the quotient R_λ has the structure of a graded 0-Hecke module for hook shapes λ [17, Prop. 8.2]. Moreover, when $\lambda \vdash n$ is a hook, [17, Cor. 8.4] implies that $\mathrm{Ch}_t(R_\lambda) = \mathrm{grFrob}(R_\lambda; t) = \mathrm{rev}_t(Q'_\lambda(\mathbf{x}; t))$. When $\lambda \vdash n$ is not a hook, the quotient R_λ does not inherit a 0-Hecke action.

In this paper, we modified the ideal $I_{n,k}$ of [16] to obtain a new ideal $J_{n,k} \subseteq \mathbb{F}[\mathbf{x}_n]$ which is stable under the action of $H_n(0)$ on $\mathbb{F}[\mathbf{x}_n]$. Moreover, we have $\mathrm{Ch}_t\left(\mathbb{F}[\mathbf{x}_n]/J_{n,k}\right) = \mathrm{grFrob}\left(\mathbb{Q}[\mathbf{x}_n]/I_{n,k}; t\right)$. This suggests the following problem.

Problem 7.2. *Let $\lambda \vdash n$. Define a homogeneous ideal $J_\lambda \subseteq \mathbb{F}[\mathbf{x}_n]$ which is stable under the 0-Hecke action on $\mathbb{F}[\mathbf{x}_n]$ such that*

$$\mathrm{Ch}_t\left(\mathbb{F}[\mathbf{x}_n]/J_\lambda\right) = \mathrm{grFrob}(R_\lambda; t) = \mathrm{rev}_t(Q'_\lambda(\mathbf{x}; t)). \tag{113}$$

When λ is a hook, the Tanisaki ideal I_λ is a solution to Problem 7.2.

7.3. Generalization to reflection groups. Let W be a Weyl group. There is an action of the 0-Hecke algebra $H_W(0)$ attached to W on the Laurent ring of the weight lattice Q of W. If W has rank r, this Laurent ring is isomorphic to $\mathbb{F}[x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}]$. Huang described the 0-Hecke structure of the corresponding coinvariant algebra [17, Thm. 5.3]. On the other hand, Chan and Rhoades [7] described a generalization of the ideal $I_{n,k}$ of [16] for the complex reflection groups $G(r, 1, n) \cong \mathbb{Z}_r \wr \mathfrak{S}_n$. It would be interesting to give an analog of the work in this paper for a wider class of reflection groups.

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