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AN EXTENSION OF DELEEUW'S THEOREM TO THE N-DIMENSIONAL ROTATION GROUP

by A. H. DOOLEY (*) and G. I. GAUDRY

Introduction.

In his paper on extension, restriction and periodification of Fourier multipliers, deLeeuw [7] proved the following result:

THEOREM. — *Let* Φ *be a bounded continuous function on* $(-\infty, \infty)$. *Suppose that, for every* $\lambda > 0$, *the function* $\varphi^{(\lambda)}$:

$$\varphi^{(\lambda)}(n) = \Phi\left(\frac{n}{\lambda}\right)$$

on $\mathbf{Z}$ *is a multiplier of* $\mathscr{F}L^p(\mathbf{T})$, p *being fixed in the range* $[1,\infty]$. *Suppose also that*

$$\limsup_{\lambda \to 0^+} \|\varphi^{(\lambda)}\|_{M_p} = K < +\infty .$$

Then Φ *is a multiplier of* $\mathscr{F}L^p(\mathbf{R})$, *and* $\|\Phi\|_{M_p} \leqslant K$.

The aim of this paper is to establish an analogue of this theorem for the pair of groups $SO(n+1)$ and $M(n)$ — the group of rigid motions of $\mathbf{R}^n$.

The group $M(n)$ will play the role of $\mathbf{R}$, while the role of the circle group $\mathbf{T}$ will be taken by the rotation group.

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An earlier result of this kind was established for the groups SO(3) and M(2) by R. L. Rubin [8]. While our methods have some general features in common with his, there are substantial novelties. The principal of these is that we identify the essential structural link between the two groups via a family of so-called *contraction maps* from $M(n)$ to $SO(n+1)$. There results a natural passage from harmonic analysis on the one group to that on the other. Crucial to this is a formula which shows that irreducible representations of $M(n)$ are approximable by those of $SO(n+1)$. Such a formula is naturally obtained by realizing irreducible representations of $SO(n+1)$ by the Borel-Weil theorem (using holomorphic induction). The approximation result, along with the basic representation theory, are to be found in Dooley and Rice [2].

The theorem of deLeeuw has been extended to the noncommutative setting in ways other than those initiated by Rubin. Cf. Clerc [1].

Definitions and notation. Except where otherwise explained in this paper, we adopt the standard definitions and notations of abstract harmonic analysis given in Hewitt and Ross [3], [4].

1. Representation theory and harmonic analysis for the group M(n).

The group $M(n)$ is the semi-direct product $\mathbf{R}^n \rtimes SO(n)$, in which $SO(n)$ acts on $\mathbf{R}^n$ by rotation. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbf{R}^n$, and denote by $SO(n-1)$ the subgroup of $SO(n)$ that fixes e_n. Let $\mathbf{T}^{m-1}$ denote a maximal torus of $SO(n-1)$, and fix a choice P_- of positive roots of $SO(n-1)$ on $\mathbf{T}^{m-1}$ as in [2]. To each character α of $\mathbf{T}^{m-1}$ and each positive number R, we associate a unitary representation $\omega_{\alpha,R}$ of $M(n)$. The representation space of $\omega_{\alpha,R}$ is denoted $\mathscr{H}^\alpha$; it is the same for all $R > 0$. We denote by $\hat{\mathbf{T}}^{m-1}_+$ the set of $\alpha \in \hat{\mathbf{T}}^{m-1}$ such that $(d\alpha,\rho) \geqslant 0$ for all $\rho \in P_-$. The space $\mathscr{H}^\alpha$ is nontrivial iff $\alpha \in \hat{\mathbf{T}}^{m-1}_+$.

1.1. Definition. – *The space* $\mathscr{H}^\alpha$ $(=\mathscr{H}^\alpha(SO(n)))$ *consists of those functions* f *in* $L^2(SO(n))$ *such that*

$$(1) \qquad f(st) = \overline{\alpha(t)}f(s) \quad (s \in SO(n),\ t \in \mathbf{T}^{m-1});$$

(2) *for every* $s \in SO(n)$, *the function* $u \to f(su)$ *on* $SO(n-1)$ *is « holomorphic » with respect to* P_-.

Let σ_α be the representation of $SO(n-1)$ induced from the character α as in § 2. The representation $\omega_{\alpha,R}$ is that induced from the representation $\sigma_\alpha \otimes e^{iR\langle e_n,\cdot\rangle}$ of $SO(n-1) \times \mathbf{R}^n$. Its action on the space $\mathcal{H}^\alpha$ is given by the formula

$$(3) \qquad [\omega_{\alpha,R}(x,t)f](s) = e^{-iR\langle e_n, s^{-1}x\rangle} f(t^{-1}s)$$

$((x,t) \in M(n),\ s \in SO(n))$. Cf. [2].

Plancherel measure on the p.o.t. dual of $M(n)$. The set $X = \{\omega_{\alpha,R} : \alpha \in \hat{T}^{m-1},\ R>0,\ \mathcal{H}^\alpha \neq \{0\}\}$ does not comprise all of the irreducible representations of $M(n)$. Its elements are said to be representations of *principal orbit type*. The set X carries the Plancherel measure μ on the dual of $M(n)$. The measure μ is given by the formula

$$(4) \qquad d\mu(\alpha,R) = c_n R^{n-1}\, dR d_\alpha,$$

in which c_n denotes the surface measure of the unit sphere in $\mathbf{R}^n$, and d_α denotes the dimension of the representation space of the irreducible representation of $SO(n-1)$ obtained by holomorphic induction of the character α from T^{m-1} to $SO(n-1)$. Cf. § 2. A convenient reference for formula (4) is Kleppner and Lipsman [6], p. 473.

Haar measure on $M(n)$. This is just the product of Lebesque measure on $\mathbf{R}^n$ and normalized Haar measure on $SO(n)$; the group $M(n)$ is unimodular.

Fourier multipliers on $M(n)$. Let X and μ be as above.

1.2 Definition. — *If* $1 \leqslant p < \infty$, *and* Φ *is an operator-valued function on* X, *we say that* Φ *is a* Fourier multiplier *of* $L^p(M(n))$ *if*

(i) *for each* $\omega_{\alpha,R}$, $\Phi(\omega_{\alpha,R})$ *is a bounded operator on* $\mathcal{H}^\alpha$;

(ii) *the norms* $\|\Phi(\omega_{\alpha,R})\|$ *are uniformly bounded;*

(iii) *for each* α, *the function* $R \to \Phi(\omega_{\alpha,R})$ *is (Bochner) measurable;*

(iv) *there exists a number* $B \geqslant 0$ *such that*

$$(5) \qquad \left| \int_X \mathrm{Tr}\, \{\Phi(\omega_{\alpha,R})\hat{f}(\omega_{\alpha,R})\hat{g}(\omega_{\alpha,R})\}\, d\mu \right| \leqslant B\|f\|_p\|g\|_{p'}$$

for (say) all $f,\ g \in C_c(M(n))$.

The smallest number B for which (5) holds is called the *norm* of Φ, and is denoted $|||\Phi|||_p$.

Remark. — It can be shown that, if Φ is a Fourier multiplier of $L^p(M(n))$, then there is a bounded linear operator T_Φ on $L^p(M(n))$ such that

$$(T_\Phi f)\hat{}\,(\omega_{\alpha,R}) = \Phi(\omega_{\alpha,R})\hat{f}(\omega_{\alpha,R})$$

for (say) all $f \in C_c$ and all α, R.

The operator T_Φ commutes with right translations. A converse assertion can be established equally easily.

Note that $|||\Phi|||_p = \|T_\Phi\|_{op}$ and that $|||\Phi|||_p$ is, strictly speaking, only a semi-norm.

2. Representation theory and harmonic analysis for the group SO(n + 1).

Identify the subgroup $SO(n-1)$ of $SO(n+1)$ with the set of rotations of $\mathbf{R}^{n+1}$ that fix both e_n and e_{n+1}. The elements of $(SO(n-1)$ can be thought of as block matrices in which the bottom-right 2×2 block is the identity matrix. If $\mathbf{T}^{m-1}$ is a maximal torus in $SO(n-1)$, and $\mathbf{T}$ denotes the set of block $(n+1) \times (n+1)$ matrices of the form

$$\left(\begin{array}{c|cc} I_{(n-1)\times(n-1)} & \multicolumn{2}{c}{0} \\ \hline 0 & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{array}\right)$$

then $T = \mathbf{T}^{m-1} \times \mathbf{T}$ is a maximal torus in $SO(n+1)$.

Let γ be the character of $\mathbf{T}^m$ which projects each element of $\mathbf{T}^m$ onto its bottom-right 2×2 rotation matrix. Then the set of characters of the maximal torus is the same as the set of functions of the form $\alpha\gamma^k$, in which α is a character of $\mathbf{T}^{m-1}$ and k is an integer.

The dual of $SO(n+1)$ can be obtained in the following way. For each character α of $\mathbf{T}^{m-1}$ and each integer k, consider the character $\chi = \alpha\gamma^k$

of the maximal torus T of $SO(n+1)$. Its differential $d\chi$ is $d\alpha + k\,d\gamma$, an element of $t_{\mathbb{C}}^*$, the dual of the complexification of the Lie algebra of T. Let a choice Φ^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, t_{\mathbb{C}})$ be made as in [2], and split Φ^+ as $P_- \cup P_+$, where P_- is the set of roots for $SO(n-1)$ chosen above, and P_+ is the complementary set, viz.

$$P_+ = \{\pm u^*_{n+1-2q,n-2q} + u^*_{n+1,n} : 0 < q \leqslant m-1\}$$

if $n+1 = 2m$, with the addition of $u^*_{n+1,n}$ if n is even.

Let (X,Y) denote $-\kappa(X,Y)$, κ the Killing form on $\mathfrak{g}_{\mathbb{C}}$. Then

$$(6) \qquad (dX,\rho) \geqslant 0 \text{ for all } \rho \in \Phi^+$$

iff

$$(7) \qquad \begin{cases} \text{(i)} \quad (d\alpha,\rho) \geqslant 0 \text{ for all } \rho \in P_-; \text{ and} \\ \text{(ii)} \quad k \geqslant k_0(\alpha), \text{ where} \end{cases}$$

$$k_0(\alpha) = \max_{0<q\leqslant m-1} |(d\alpha, u_{n+1-2q,n-2q})|.$$

To see the equivalence of (6) and (7) notice that $(d\gamma,\rho) = 0$ for all $\rho \in P_-$, so (1) is immediate from (6). To obtain (ii) from (6), notice that $(d\gamma, u_{n+1,n}) = 1$ and use the description of P_+ given above.

The condition (6) is necessary and sufficient for the representation of $SO(n+1)$ holomorphically induced from χ to be nontrivial.

For each character $\chi = \alpha\gamma^k$ satisfying (6), let $\mathscr{H}_{\alpha,k}(SO(n+1))$ denote the representation space of the representation $\sigma_{\alpha,k}$ induced holomorphically from χ. The main facts about $\sigma_{\alpha,k}$ and $\mathscr{H}_{\alpha,k}$ that we shall use are as follows. (See Dooley and Rice, loc. cit.).

(*a*) Each $\mathscr{H}_{\alpha,k}$ is a finite-dimensional invariant subspace of $L^2(SO(n+1))$; and $\sigma_{\alpha,k}$ is the restriction of the left regular representation to $\mathscr{H}_{\alpha,k}$.

(*b*) There is a function ψ on $SO(n+1)$ that takes the value 1 on $SO(n) = \{g \in SO(n+1) : ge_{n+1} = e_{n+1}\}$ and maps $\mathscr{H}_{\alpha,k}$ pointwise into $\mathscr{H}_{\alpha,k+1}$.

(*c*) It follows from (*b*) that the sequence of restriction spaces $(\mathscr{H}_{\alpha,k|SO(n)})_{k\geqslant k_0}$ forms an increasing sequence of finite-dimensional $SO(n)$-

invariant subspaces of $\mathscr{H}^\alpha(\mathrm{SO}(n))$. It is shown in [2] that $\bigcup_{k=k_0}^{\infty} \mathscr{H}_{\alpha,k|\mathrm{SO}(n)}$ is dense in $\mathscr{H}^\alpha(\mathrm{SO}(n))$.

(*d*) It follows from (*c*) and (7) that the representation space $\mathscr{H}^\alpha(\mathrm{SO}(n))$ of $\omega_{\alpha,\mathrm{R}}$ is nontrivial if and only if $(d\alpha,\rho) \geqslant 0$ for all positive roots ρ of $\mathrm{SO}(n-1)$; i.e. iff the representation σ_α of $\mathrm{SO}(n-1)$ induced from α is nontrivial.

Plancherel measure on the dual of $\mathrm{SO}(n+1)$. This is the discrete measure that assigns to each irreducible representation $\sigma_{\alpha,k}$ the mass $d_{\alpha,k} = d_{\sigma_{\alpha,k}}$, the dimension of $\mathscr{H}_{\alpha,k}$. By the Weyl dimension formula ([4], p. 139)

$$(8) \qquad d_{\sigma_\chi} = \prod_{\rho > 0} \frac{(d\chi+\delta,\rho)}{(\delta,\rho)},$$

δ denoting $\frac{1}{2} \sum_{\rho>0} \rho \in t_{\mathrm{C}}^*$.

At a later stage we shall need a relationship between the Plancherel measures on the duals of our two groups. Crucial to the establishment of this relationship is the following lemma.

2.1 LEMMA. – *Let* $\chi = \alpha\gamma^k$ *be as above, with* χ *satisfying* (6). *Denote by* $\sigma_\chi = \sigma_{\alpha,k}$ (resp. σ_α) *the representation of* $\mathrm{SO}(n+1)$ (resp. $\mathrm{SO}(n-1)$) *induced holomorphically from* χ *(resp.* α*) and by* $d_{\alpha,k}$ (resp. d_α) *their dimensions. Then*

$$(9) \qquad k^{n-1} d_\alpha = \mathrm{A} d_{\alpha,k}\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right)$$

as $k \to +\infty$, A *being a positive number independent of* α *and* k.

Proof. – Write $\Phi^+ = \mathrm{P}_- \cup \mathrm{P}_+$ as above. Notice that P_+ contains $(n-1)$ elements, that

$$\sum_{\rho \in \mathrm{P}_+} \rho = (n-1) u_{n+1,n}^*$$

and that

$$(u_{n+1,n}^*, \rho) = 0 \quad \text{if} \quad \rho \in \mathrm{P}_-.$$

Thus, setting $\delta_- = \frac{1}{2} \sum_{\rho \in P_-} \rho$ and using (8), we obtain

$$(10) \quad d_{\alpha,k} = \prod_{\rho \in P_-} \frac{(d\alpha + \delta_-, \rho)}{(\delta_-, \rho)} \cdot \prod_{\rho \in P_+}{}' \left\{ \frac{k(d\gamma,\rho) + (d\alpha,\rho)}{(\delta,\rho)} + 1 \right\}$$

since $(d\alpha, u^*_{n+1,n}) = 0$, and $(d\gamma, u^*_{p,q}) = 0$ if $1 \leqslant q < p < n+1$.

Now the first product in (10) is precisely d_α, since P_- is a set of positive roots for $SO(n-1)$. Furthermore $(d\gamma,\rho) = 1$ for all $\rho \in P_+$, and so

$$(11) \qquad d_{\alpha,k} = d_\alpha Q(\alpha,k)$$

where

$$Q(\alpha,k) = \prod_{i=1}^{n-1} (ka_i + b_i),$$

the numbers a_i being positive and independant of k, α. Dividing both sides of (11) by $\prod_{i=1}^{n-1} \left(a_i + \frac{b_i}{k}\right)$, we deduce (9), with $A = \prod_{i=1}^{n-1} a_i^{-1}$.

Haar measure on $SO(n+1)$. With each element $g \in SO(n+1)$, associate the point $\xi = ge_{n+1}$ of S^n having spherical coordinates say $\theta_1, \ldots, \theta_n, \theta_n, 0 \leqslant \theta_j < \pi \; (j=2, \ldots, n)$, $0 \leqslant \theta_1 < 2\pi$. For each j, denote by s_{θ_j} the rotation of $\mathbf{R}^{n+1}$ which acts only in the (e_{j+1}, e_j)-plane, and rotates e_{j+1} towards e_j through angle θ_j. Let r_ξ be the rotation $s_{\theta_1} \ldots s_{\theta_n}$, and $h = r_\xi^{-1} g \in SO(n)$, so that $g = r_\xi h$. The procedure just described gives a well-defined way of writing g as a product $r_\xi h$, where $h \in SO(n)$ and r_ξ is determined by the point ξ of S^n. It is known, and simple to verify, that

$$(12) \qquad \int_{SO(n+1)} f(g)\, dg = \int_{S^n} \int_{SO(n)} f(r_\xi h)\, dh\, d\xi,$$

the integrations on the right side being with respect to normalized Haar and surface measures.

Fourier multipliers on $SO(n+1)$. The definition of Fourier multiplier of $L^p(SO(n+1))$ is that given in (35.1) of Hewitt and Ross [4]. Their definition can be easily reformulated in the shape of an appropriate « Parseval-type » inequality like that given for $M(n)$ above.

3. Elementary Lie theory: contractions.

The Lie algebra of the group $M(n)$ is isomorphic (as a vector space) with $\mathbf{R}^n \oplus \mathfrak{so}(n)$, and the Lie product is determined from the following formulas:

$$(1) \qquad [\mathbf{R}^n, \mathbf{R}^n] = \{0\};$$
$$(2) \qquad [X,Y] = XY - YX = [X,Y]_{\mathfrak{so}(n)}$$

for all $X, Y \in \mathfrak{so}(n)$;

$$(3) \qquad [u,X] = X(u) \quad (u \in \mathbf{R}^n, X \in \mathfrak{so}(n)).$$

In the formula (3), $X(u)$ is the result of applying the matrix (mapping) X to the vector u.

On the other hand, the Lie algebra of $SO(n+1)$ is the set $\mathfrak{so}(n+1)$ of real antisymmetric matrices with the standard Jacobi product. Note that $\mathfrak{so}(n+1)$ is isomorphic as a vector space to $\mathbf{R}^n \oplus \mathfrak{so}(n)$: to be specific think of $\mathfrak{so}(n)$ as the subspace of $\mathfrak{so}(n+1)$ supported in the upper-left $n \times n$ block, and of $\mathbf{R}^n$ as the subspace of elements carried by the bottom row and $(n+1)st$ column.

There is a natural sense in which the Lie product in $\mathfrak{m}(n)$ is the limit of the Lie products in a family of vector-space-isomorphic copies of $\mathfrak{so}\ (n+1)$. Viz. we let $\{\varphi_\lambda\}$ denote the family of mappings of $\mathfrak{so}(n+1) = \mathbf{R}^n \oplus \mathfrak{so}(n)$ onto itself given by the formulas

$$\varphi_\lambda(u \oplus X) = \frac{u}{\lambda} \oplus X.$$

It is simple to check that

$$(1) \qquad \lim_{\lambda \to +\infty} \varphi_\lambda^{-1}\{[\varphi_\lambda(S), \varphi_\lambda(T)]_{\mathfrak{so}(n+1)}\} = [S,T]_{\mathfrak{m}(n)}$$

for every pair (S,T) of elements of $\mathbf{R}^n \oplus \mathfrak{so}(n)$. The relation (1) captures the essence of the statement that the Lie algebra of $M(n)$ is a contraction of that of $SO(n+1)$.

Lie anti-derivatives: the mappings π_λ. For each $\lambda > 0$, there is a canonical mapping π_λ from $SO(n+1)$ onto $M(n)$ whose Lie derivative is the mapping φ_λ. It is given by the formula

$$\text{(2)} \qquad \pi_\lambda(x,t) = \exp_{SO(n+1)}\left(\frac{x}{\lambda}\right).t\,.$$

We examine in the next section the way the mappings π_λ intervene in the transfer (in the limit) of the harmonic analysis on (copies of) the group $SO(n+1)$ to that on the group $M(n)$. Full details of the representation-theoretic side of these matters are to be found in [2].

4. Transferring analysis from $SO(n+1)$ to $M(n)$.

In the case $n = 2$, the space $\mathscr{H}_{\alpha,k} = \mathscr{H}_{0,k}$ is easily identified with the space of trigonometric polynomials of degree at most k, a subspace of $\mathscr{H}^\alpha = \mathscr{H}^0 = L^2(\mathbf{T})$. This renders the subsequent analysis particularly simple. Cf. [8]. In the general case, it is necessary to show that the natural mapping R_k of $\mathscr{H}_{\alpha,k}(SO(n+1))$ into $\mathscr{H}^\alpha$ (which arises by restriction to $SO(n)$) is one-one. Moreover, the restriction mappings and their inverses figure in all of our results dealing with the transfer of analysis from $SO(n+1)$ to $M(n)$, in particular, in the analogue of deLeeuw's theorem itself.

Injection of the spaces $\mathscr{H}_{\alpha,k}$ *into* $\mathscr{H}_\alpha$. Let α be a character of T^{m-1} and k be a fixed integer. The functions in $\mathscr{H}_{\alpha,k}(SO(n+1))$ are continuous, so we may consider the *restriction mapping*

$$\text{(1)} \qquad R_k : f \to f_{|SO(n)}$$

of $\mathscr{H}_{\alpha,k}$ into $\mathscr{H}^\alpha$. We proceed to show that the mappings R_k are one-one. Actually, what we do is quite general, applying to a general compact, connected Lie group G with maximal torus T.

To prove that the mappings R_k are one-one, we introduce the complexification G^C of G and the canonical complex structure in $\mathfrak{g}_C$ defined by the mapping $J : J(X+iY) = -Y + iX (X,Y \in \mathfrak{g})$.

4.1 DEFINITION. — *Let* u *be a* C^∞ *function on* G^C. *We say that* f *is* holomorphic *if*

$$\tilde{X}u = -i(JX)^{\sim}u$$

for all $X \in \mathfrak{g}_C$. *(*$\tilde{X}u$ *denotes the derivative of* u *in the direction* X.).

Consider now the connected Lie subgroups H, U and B of G^C corresponding to the Lie subalgebras $\mathfrak{t}_C$, η^+ and $\mathfrak{b} = \mathfrak{t}_C + \eta^+$ of $\mathfrak{g}_C$. It is known [8, Ch. VIII, § 4] that $B = H \rtimes U$. Therefore, if χ is a character of T, we can consider its natural extension χ' to a homomorphism of H into C^*, and then the one-dimensional representation of B, also denoted χ', given by the formula

$$\chi'(h,u) = \chi'(h).$$

4.2 DEFINITION. — *Let* $\mathscr{E}_{\chi'}$ *denote the space of holomorphic functions* u *on* G^C *such that*

$$u(zb) = \chi'(b)^{-1}u(z) \tag{2}$$

for all $z \in G^C$, $b \in B$.

Recall that the space $\mathscr{H}_\chi(G)$ comprises those C^∞-functions f on G such that

$$f(xt) = \chi(t)^{-1}f(x) \qquad (x \in G, t \in T) \tag{3}$$

and

$$\frac{d}{ds}f(x \exp sX_1) + i\frac{d}{ds}f(x \exp sX_2)_{|s=0} = 0 \tag{4}$$

for all $X_1 + iX_2 \in \eta^+$. Notice also that every C^∞-function f on G has a unique holomorphic extension u to G^C.

4.3 LEMMA. — *Let* χ *be a character of the maximal torus* T *of the compact connected Lie group* G, *and suppose that* f *is a* C^∞*-function on* G. *Then* $f \in \mathscr{H}_\chi$ *iff its holomorphic extension to* G^C *belongs to* $\mathscr{E}_{\chi'}$.

Proof. — If $u \in \mathscr{E}_{\chi'}$, then $U_{|G}$ satisfies (4) in view of (2) and the fact that χ' is constant on U. The converse is equally easily established.

4.4 THEOREM. — *Let* $f \in \mathcal{H}_{\alpha,k}(\mathrm{SO}(n+1))$, *and suppose that* $f = 0$ *on* $\mathrm{SO}(n)$. *Then* $f \equiv 0$.

Proof. — Denote by χ the character $\alpha\gamma^k$, by G the group $\mathrm{SO}(n+1)$, and by K the subgroup $\mathrm{SO}(n)$. Let u be the holomorphic extension of f to G^{C}. By Lemma 4.4, $u \in \mathcal{E}_{\chi'}$. Since $f = 0$ on K, it follows that $u = 0$ on K^{C}. On the other hand, u satisfies (2). So $u = 0$ on $\mathrm{K}^{\mathrm{C}}\mathrm{B}$.

It remains to show that $\mathrm{K}^{\mathrm{C}}\mathrm{B}$ contains a neighbourhood of the identity in G^{C}. This follows from the fact that the set $\mathrm{K}^{\mathrm{C}}\mathrm{B}$, being the image of the direct product $\mathrm{K}^{\mathrm{C}} \times \mathrm{B}$, under a mapping of constant rank, is a submanifold of G^{C}; its tangent space is $\mathfrak{k}_{\mathrm{C}} + \mathfrak{b}$; and $\mathfrak{k}_{\mathrm{C}} + \mathfrak{b}$ is equal to $\mathfrak{g}_{\mathrm{C}}$, because it contains the elements $u_{n,n-2q} - iu_{n+1,n-2q}$ and $u_{n,n+1-2q} - iu_{n+1,n+1-2q}$ $(0<q<m-1)$.

4.5. COROLLARY. — *The restriction mapping* R_k *of* $\mathcal{H}_{\alpha,k}$ *into* $\mathcal{H}^{\alpha}$ *is one-one.*

Approximation of representations. A careful examination of the proof of Lemma 6 of [2] shows that Theorem 1 of that paper can be recast in a form more adapted to our present needs, as follows.

4.6 THEOREM. — *Let* $\alpha \in \hat{\mathrm{T}}_+^{m-1}$, *and suppose* $j \geqslant j_0(\alpha)$. *If* C *is a fixed compact subset of* $\mathbf{R}^n$, $\mathrm{U} \in \mathcal{H}_{\alpha,j}(\mathrm{SO}(n+1))$, $\mathrm{S} > 0$, *and*

$$\mathrm{E}(k,\lambda,x) = \|\mathrm{R}_k\sigma_{\alpha,k}(\pi_\lambda(x,t))\psi^{k-j}\mathrm{U} - \omega_{\alpha,\frac{k}{\lambda}}(x,t)\mathrm{R}_k\mathrm{U}\|_{\mathrm{L}^2(\mathrm{SO}(n))}$$

then there exists $\lambda_0 > 0$ *such that*

$$\lambda\mathrm{E}(k,\lambda,x,t,) \leqslant \mathrm{D}, \tag{5}$$

a fixed number, for all $\lambda \geqslant \lambda_0$, *all* $k \geqslant j$ *such that* $\frac{k}{\lambda} \leqslant \mathrm{S}$, *and all* $(x,t) \in \mathrm{C} \times \mathrm{SO}(n)$.

Remark. — In view of Theorem 4.4, the quantity $\mathrm{E}(k,\lambda,x,t)$ can be written as

$$\|\mathrm{R}_k\sigma_{\alpha,k}(\pi_\lambda(x,t))\mathrm{R}_k^{-1}u - \omega_{\alpha,\frac{k}{\lambda}}(x,t)u\| \tag{6}$$

for $u \in \mathcal{H}_{\alpha,j|\mathrm{SO}(n)}$. The statement (5) can be thought of as implying an « approximate intertwining » between the representations $\omega_{\alpha,\frac{k}{\lambda}}$ and the mappings $\sigma_{\alpha,k} \circ \pi_\lambda$.

The mappings π_λ *in spherical coordinates: approximation of Haar measures.* Adopting the notation explained for (12) of § 3, but with $s_{(\theta_1,\dots,\theta_n)}$ in place of $s_{\theta_1}, \dots s_{\theta_n}$, we express the action of $\exp_{SO(n+1)}$ on the subspace $\mathbf{R}^n$ of $\mathfrak{so}(n+1)$ as follows.

4.7 LEMMA. — *Let* $x = \mathrm{R}u$ *be a (nonzero) point of* $\mathbf{R}^n$, *where* u *is a point on* S^{n-1} *with spherical coordinates* $(\theta_1, \dots, \theta_{n-1})$. *Then*

$$(7)\quad \exp_{SO(n+1)} \left[\begin{array}{c|c} 0 & \mathrm{R}u \\ \hline -\mathrm{R}u^t & 0 \end{array}\right] = s_{(\theta_1,\dots,\theta_{n-1},0)}' s_{(\theta,\dots,\mathrm{R})} s_{(\theta_1,\dots,\theta_{n-1},0)} .$$

Proof. — It is merely a matter of checking that the one parameter mappings defined by the two sides of (7) have the same derivative at 0. □

4.8 COROLLARY. — *Let* λ *be positive and suppose that* x *is a point of* $\mathbf{R}^n$ *such that* $\|x\| < \lambda\pi$.
Then for every $t \in SO(n)$

$$(8)\qquad \pi_\lambda(x,t) = s_{(\theta_1,\dots,\theta_{n-1},\|x\|/\lambda)}\, s^{-1}_{(\theta_1,\dots,\theta_{n-1},0)}\, t .$$

Approximation of Haar measures.

4.9 DEFINITION. — *Let* f *be a function in* $\mathrm{C}_c(\mathrm{M}(n))$, *and suppose that* $\lambda > 0$ *is so large that*

$$\operatorname{supp} f \subseteq \mathrm{B}_\lambda = \{(x,t) : \|x\| < \lambda\pi\} .$$

Observing that the mapping π_λ is a one-one mapping of B_λ onto $SO(n+1)$, denote by f_λ the function on $SO(n+1)$ such that

$$(9)\qquad f_\lambda \circ \pi_\lambda = f .$$

4.10 THEOREM. — *Let* f *be a given function in* $\mathrm{C}_c(\mathrm{M}(n))$. *Then*

$$(10)\qquad c_n\lambda^n \int_{SO(n+1)} f_\lambda - \int_{\mathrm{M}(n)} f = \mathrm{O}(\lambda^{-2})$$

as $\lambda \to +\infty$, c_n *denoting the surface measure of the unit sphere* S^n, *the bound in the* $\mathrm{O}(\lambda^{-2})$ *term depending only on* $\int |f|$.

Proof. — The relation (10) can be verified by using (8) and (9) along with (12) of § 2, writing the integral over S^n in (12) for f_λ in terms of spherical coordinates. Cf. Lemma 5.7.

5. The multiplier theorem.

Denote by P_k the orthogonal projection of the space $\mathscr{H}^\alpha$ onto $\mathscr{H}_{\alpha, k|SO(n)}$.

5.1 THEOREM. — *Suppose that* $1 \leqslant p < +\infty$. *Let* Φ *be a function on the* p.o.t. *dual* X *of* $M(n)$ *such that*

(i) *for each* $a \in \hat{T}_+^{m-1}$, *the function* $R \to \Phi(\omega_{\alpha,R})$ *is a bounded continuous function with values in* $\mathscr{B}(\mathscr{H}^\alpha)$;

(*ii*) *for each* $\lambda > 0$, *the function* $\varphi^{(\lambda)} : \varphi^{(\lambda)}(\sigma_{\alpha,k}) = R_k^{-1} P_k \Phi(\omega_{\alpha,\frac{k}{\lambda}}) R_k$

$(\alpha \in \hat{T}_+^{m-1}, k \geqslant k_0(\alpha))$ *is a Fourier multiplier of* $L^p(SO(n+1))$, *and*

$$|||\varphi^{(\lambda)}|||_p \leqslant K$$

for all (*sufficiently small*) λ. *Then* Φ *is a Fourier multiplier of* $L^p(M(n))$.

The proof will be broken down into a succession of steps, presented in the form of Lemmas. We beging noting that, in proving that Φ is a Fourier multiplier of $L^p(M(n))$, it will be enough to establish the existence of a number B such that inequality (5) of Definition 1.2 holds for all $f, g \in C_c(M(n))$ which are linear combinations of functions of the form $F \otimes h$, in which $F \in C_c(\mathbf{R}^n)$, and h is a trigonometric polynomial on $SO(n)$. In the remainder of this paper, we shall simplify the notation by omitting the mappings P_k from the various formulas. It will be apparent from the context that for example, $R_k^{-1}\omega_{\alpha,R}R_k$ should be read as $R_k^{-1}P_k\omega_{\alpha,R}R_k$.

Trigonometric polynomials being linear combinations of entry functions, it will be convenient to choose a complete family of irreducible representations of $SO(n)$ that are related to the spaces $\mathscr{H}_{\alpha,k}$, as follows.

For each $\alpha \in \hat{T}^{m-1}$, and integer $k \geqslant 0$, the space $\mathscr{H}_{\alpha,k}(SO(n+1))$ is defined, though trivial unless $\alpha \in \hat{T}_+^{m-1}$ and $k \geqslant k_0(\alpha)$. In any event

$$\mathscr{H}_{\alpha,0|SO(n)} \subseteq \mathscr{H}_{\alpha,1|SO(n)} \subseteq \dots \subseteq \mathscr{H}^\alpha .$$

5.2 LEMMA. — *If* $\alpha \neq \beta$, *then* $\mathscr{H}^\alpha \perp \mathscr{H}^\beta$.

Proof. — If $f \in \mathscr{H}^\alpha$ and $g \in \mathscr{H}^\beta$, then

$$\begin{aligned}\int_{SO(n)} f(s)\overline{g(s)}\,ds &= \int_{SO(n)}\int_{T^{m-1}} f(st)\overline{g(st)}\,dt\,ds \\ &= \int_{SO(n)} f(s)\overline{g(s)}\left(\int_{T^{m-1}} \overline{\alpha(t)}\beta(t)\,dt\right) ds \\ &= 0,\end{aligned}$$

in view of the T^{m-1}-invariance property of the functions in $\mathscr{H}^\alpha$ (resp. $\mathscr{H}^\beta$).

□

Consider now the left regular representation of $SO(n)$. The space $L^2(SO(n))$ can be thought of as being decomposed

$$\left(\bigoplus_\alpha \mathscr{H}^\alpha\right) \oplus \left(L^2 \oslash \bigoplus_\alpha \mathscr{H}^\alpha\right),$$

the index α ranging over

$$\hat{T}_+^{m-1} = \{\alpha \in \hat{T}^{m-1} : (d\alpha,\rho) \geqslant 0, \rho \in P_-\}.$$

Each of the summands in the first bracket decomposes:

$$\mathscr{H}^\alpha = \mathscr{H}_{\alpha,0|SO(n)} \oplus (\mathscr{H}_{\alpha,1|SO(n)} \oslash \mathscr{H}_{\alpha,0|SO(n)}) \oplus \cdots$$

where the summands are chosen to be $SO(n)$-invariant. It follows therefore that we can choose a complete family of irreducible unitary representations of $SO(n)$ that falls into natural sub-families: those that act in (minimal) subspaces of $L^2 \oslash (\bigoplus_\alpha \mathscr{H}^\alpha)$; and, for each α, those that act in subspaces of $\mathscr{H}^\alpha$. Given α, the minimal invariant subspaces of $\mathscr{H}^\alpha$ can be chosen to lie in $\mathscr{H}_{\alpha,0|SO(n)}$ or in one of $\mathscr{H}_{\alpha,j+1|SO(n)} \oslash \mathscr{H}_{\alpha,j|SO(n)}$ ($j=0, 1, \ldots$).

5.3 LEMMA. — *Let* f *be a function in* $C_c(M(n))$ *of the form* $f = F \otimes h$, *with* $F \in C_c(\mathbf{R}^n)$, *and* h *a trigonometric polynomial on* $SO(n)$. *Then the family of operators* $R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}$ *vanishes off a fixed finite-dimensional subspace of the span of the spaces* $\mathscr{H}_{\beta,j|SO(n)}$, *and is zero except for finitely many* α.

Proof. – Using the notation of (12), § 2, we have that if $\varphi \in \mathscr{H}_{\alpha,k}(\mathrm{SO}(n+1))$,

$$(1)\quad R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}(R_k\varphi) = R_k \int_{S^n\times \mathrm{SO}(n)} F(s_{(\theta_1,\dots,\theta_{n-1},\lambda\theta_n)})h(s_{(\theta_1,\dots,\theta_{n-1})}t)\, \sigma_{\alpha,k}(s_{(\theta_1,\dots,\theta_{n-1})}t)\,d\xi\,dt\,\varphi$$

$$= R_k \int_{S^n} F(s_{(\theta_1,\dots,\theta_{n-1},\lambda\theta_n)})\sigma_{\alpha,k}(s_{(\theta_1,\dots,\theta_{n-1})})\,d\xi \int_{\mathrm{SO}(n)} h(s_{(\theta_1,\dots,\theta_{n-1})}t)\sigma_{\alpha,k}(t)\,\varphi\,dt.$$

The function

$$\int_{\mathrm{SO}(n)} h(s_{(\theta_1,\dots,\theta_{n-1})}t)\sigma_{\alpha,k}(t)\varphi\,dt$$

is in $\mathscr{H}_{\alpha,k}$; and its restriction to $\mathrm{SO}(n)$ is the same as

$$\int_{\mathrm{SO}(n)} h(s_{(\theta_1,\dots,\theta_{n-1})}t)\sigma_{\alpha,k}(t)(R_k\varphi)\,dt.$$

This is because $R_k(\sigma_{\alpha,k}(t)\varphi) = \sigma_{\alpha,k}(t)R_k\varphi$ for all $t\in \mathrm{SO}(n)$. In other words, we can write the last integral in (1) as

$$(2)\qquad R_k^{-1}\int_{\mathrm{SO}(n)} h(s_\theta t)\sigma_{\alpha,k}(t)R_k\varphi\,dt,$$

θ denoting $(\theta_1,\dots,\theta_{n-1})$. But for every θ, the function $P_\theta : t \to h(s_\theta t)$ is a trigonometric polynomial on $\mathrm{SO}(n)$ whose spectrum is in a fixed finite set of irreducibles of $\mathrm{SO}(n)$ (independent of θ). Furthermore, $\sigma_{\alpha,k}(t)$ is just $\rho(t)$, ρ denoting the left regular representation. In view of the discussion concerning a specific choice of a complete family of irreducibles of $\mathrm{SO}(n)$, given above, and of the orthogonality relations, it is clear that the integral in (2) is nonzero only if $R_k\varphi$ belongs to $\bigoplus_{\beta\in S}\mathscr{H}_{\beta,j|\mathrm{SO}(n)}$, say, S being finite and $j \geqslant 0$ fixed, both determined by the spectrum of h. □

In order to simplify the presentation of the remaining argument, we introduce further notation.

5.4 DEFINITION. — *For each* $\lambda > 0$, *character* $\alpha \in \hat{\mathrm{T}}^{m-1}$, *and integer* $k \geqslant 0$, *write*

$$\tau_{\alpha,k,\lambda} = \mathrm{R}_k(\sigma_{\alpha,k} \circ \pi_\lambda)\mathrm{R}_k^{-1}. \tag{3}$$

For each $(x,t) \in \mathrm{M}(n)$, $\tau_{\alpha,k,\lambda}(x,t)$ *is an operator on* $\mathscr{H}_{\alpha,k|\mathrm{SO}(n)}$ *(this space being possibly trivial).*

5.5 LEMMA. — *Let* f *be as in* 5.3. *Then*

(i) *for all* α, k, λ,

$$\hat{f}(\tau_{\alpha,k,\lambda}) = \int_{\mathrm{M}(n)} f(x,t)\tau_{\alpha,k,\lambda}(x,t)\,dx\,dt$$

vanishes off a fixed finite-dimensional subspace of the span of the spaces $\mathscr{H}_{\beta,j|\mathrm{SO}(n)}$, *and is zero but for finitely-many* α;

(ii) *the same is true of* $\hat{f}(\omega_{\alpha,\mathrm{R}})$ *for all* $\mathrm{R} > 0$.

Proof. — (i) Suppose that $\varphi \in \mathscr{H}_{\alpha,k}$. Then by Corollary 4.8,

$$\begin{aligned}\hat{f}(\tau_{\alpha,k,\lambda})(\mathrm{R}_k\varphi) &= \int_{\mathbf{R}^n} \mathrm{F}(x) \int_{\mathrm{SO}(n)} h(t)\mathrm{R}_k\sigma_{\alpha,k}\left(\exp_{\mathrm{SO}(n+1)}\left(\frac{x}{\lambda}\right)t\right)\varphi\,dx\,dt \\ &= \int_0^{+\infty} \|x\|^{n-1}d\|x\| \int_{\mathrm{S}^{n-1}} d\theta \\ &\qquad \int_{\mathrm{SO}(n)} h(t)\mathrm{R}_k\sigma_{\alpha,k}(s_{(\theta_1,\ldots,\theta_{n-1},\|x\|/\lambda)}s_\theta^{-1}t)\varphi\,dt\end{aligned} \tag{4}$$

(in which $\theta = (\theta_1, \ldots, \theta_{n-1})$ is identified with a point of S^{n-1}). The formula (4) can be rewritten

$$\hat{f}(\tau_{\alpha,k,\lambda})(\mathrm{R}_k\varphi) = \int_0^{+\infty} y^{n-1}\,dy \int_{S^{n-1}} d\theta\, \mathrm{R}_k\rho(s_{(\theta_1,\ldots,\theta_{n-1},y/\lambda)}) \int_{\mathrm{SO}(n)} h(t)\rho(s_\theta^{-1}t)\varphi\,dt. \tag{5}$$

Now if $u \in \mathrm{SO}(n)$.

$$\rho(u)\varphi = \mathrm{R}_k^{-1}[\rho(u)\mathrm{R}_k\varphi].$$

So the last integral in (5) is equal to

$$(6) \qquad R_k^{-1} \int_{SO(n)} h(t)\rho(s_\theta^{-1}t)R_k\varphi \, dt .$$

The remainder of the proof is just like the closing stages of the proof of Lemma 5.3, and shows that (6) vanishes except when $R_k\varphi$ belongs to a finite-dimensional space determined by the trigonometric polynomial h.

(ii) This follows in a straightforward way from 1.1 (3). □

5.6 LEMMA. — *Let f be as in 5.3, and suppose that $\alpha \in \hat{T}_+^{m-1}$. Then if $S > 0$, there exist $\lambda_0 > 0$ and a positive integer j_0 such that*

$$(7) \qquad \lambda \| \hat{f}(\omega_{\alpha, \frac{k}{\lambda}}) - \hat{f}(\tau_{\alpha, k, \lambda}) \|_{\mathscr{B}(\mathscr{H}^\alpha)}$$

is uniformly bounded for all $\lambda \geqslant \lambda_0$, and all $k \geqslant j_0$ such that $\frac{k}{\lambda} \leqslant S$.

Proof. — Choose the compact set $C \subseteq \mathbf{R}^n$ so large that f is supported in $C \times SO(n)$. Then

$$(8) \quad \hat{f}(\omega_{\alpha, \frac{k}{\lambda}}) - \hat{f}(\tau_{\alpha, k, \lambda}) = \int_{C \times SO(n)} f(x,t)[\omega_{\alpha, \frac{k}{\lambda}}(x,t) - \tau_{\alpha, k, \lambda}(x,t)] \, dx \, dt.$$

In view of Lemma 5.5, the operators in (8) are zero on the complement of say $\mathscr{H}_{\alpha, j_0 | SO(n)}$, a finite-dimensional space. It is therefore sufficient to estimate

$$\lambda \| \hat{f}(\omega_{\alpha, \frac{k}{\lambda}})u - \hat{f}(\tau_{\alpha, k, \lambda})u \|_{\mathscr{H}^\alpha}$$

for $u \in \mathscr{H}_{a, j_0 | SO(n)}$. The existence of a uniform bound for these quantities follows from Theorem 4.5.

□

5.7 LEMMA. — *Let f be as in 5.3. Then*

$$\| c_n \lambda^n R_k \hat{f}_\lambda(\sigma_{\alpha, k}) R_k^{-1} - \hat{f}(\tau_{\alpha, k, \lambda}) \|_{\mathscr{B}(\mathscr{H}^\alpha)} = O(\lambda^{-2})$$

as $\lambda \to +\infty$, uniformly with respect to α, k.

Proof. — In view of Lemmas 5.3 and 5.5, it will suffice to prove the estimate

$$\|c_n\lambda^n R_k \hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}u - \hat{f}(\tau_{\alpha,k,\lambda})u\|_2 = O(\lambda^{-2})$$

for a fixed u in say $\mathcal{H}_{\alpha,j_0|SO(n)}$.

The first-term within the norm signs can be written, using spherical coordinates and invariance of Haar measure on $SO(n)$, as

$$I_\lambda = \frac{c_n\lambda^n}{c_n}\int_0^{2\pi} d\theta_1 \int_0^{\pi} \sin\theta_2\, d\theta_2 \ldots$$

$$\int_0^{\pi} \sin^{n-1}\theta_n\, d\theta_n \int_{SO(n)} f_\lambda(s_{(\theta_1,\ldots,\theta_n)}s^{-1}_{(\theta_1,\ldots,\theta_{n-1},0)}t)$$

$$R_k\sigma_{\alpha,k}(s_{(\theta_1,\ldots,\theta_n)}s^{-1}_{(\theta_1,\ldots,\theta_{n-1},0)}t)R_k^{-1}u\, dt.$$

If $\operatorname{supp} f \subseteq \{(x,t) : \|x\| \leqslant A\}$, and $\tau \in SO(n)$, then $f_\lambda(s_{(\theta_1,\ldots,\theta_n)}\tau) = 0$ unless $\lambda\theta_n \leqslant A$. Therefore if we change variables by setting $\theta_n = \|x\|/\lambda$, we get

$$I_\lambda = \lambda^{n-1}\int_0^{2\pi} d\theta_1 \ldots \int_0^{A} \sin^{n-1}\left(\frac{\|x\|}{\lambda}\right) d\|x\|$$

$$\int_{SO(n)} f_\lambda(\pi_\lambda(x,t))R_k\sigma_{\alpha,k}(\pi_\lambda(x,t))R_k^{-1}u\, dt$$

$$= \int_0^{2\pi} d\theta_1 \ldots \int_0^{A} \|x\|^{n-1} + [\lambda^{n-1}\sin^{n-1}\left(\frac{\|x\|}{\lambda}\right) - \|x\|^{n-1}] \int_{SO(n)} \ldots$$

$$= \hat{f}(\tau_{\alpha,k,\lambda})u + \int_0^{2\pi} d\theta_1 \ldots \int_0^{A} [\lambda^{n-1}\sin^{n-1}\left(\frac{\|x\|}{\lambda}\right)$$

$$- \|x\|^{n-1}]\, d\|x\| \int_{SO(n)} \ldots$$

The error term can be estimated by

$$(10) \quad \int_0^{2\pi} d\theta_1 \ldots \int_0^{A} C\, \frac{\|x\|^{n+1}}{\lambda^2}\, d\|x\| \int_{SO(n)} |f(x,t)|$$

$$\sup_{(x,t)}\|\tau_{\alpha,k,\lambda}(x,t)u\|_2\, dt.$$

Since $u \in \mathscr{H}_{a,j_0|SO(n)}$, we can write, for $k \geqslant j_0$,

$$(11)\quad R_k\sigma_{\alpha,k}(\pi_\lambda(x,t))R_k^{-1}u = R_k\sigma_{\alpha,k}(\pi_\lambda(x,t))\psi^{k-j_0}(R_{j_0}^{-1}u) = R\rho(\pi_\lambda(x,t))\psi^{k-j_0}(R_{j_0}^{-1}u),$$

R denoting restriction to $SO(n)$, and ρ the left regular representation. Since $\|\psi\|_\infty \leqslant 1$, it is clear from (11) that

$$(12)\qquad \sup_{(x,t)} \|\tau_{\alpha,k,\lambda}(x,t)u\|_2 \leqslant \sup_{(x,t)} \|R_{j_0}\sigma_{\alpha,j_0}(\pi_\lambda(x,t))R_{j_0}^{-1}u\|_2$$

and the right-hand side of (12) is independent of α and k since the σ_{α,j_0} are unitary. It follows that the expression (10) is $O(\lambda^{-2})$ uniformly with respect to α, k as $\lambda \to +\infty$.

□

5.8 LEMMA. — *Let f and g be functions on $M(n)$, each a finite linear combination of functions of the form $F \otimes h$, where $F \in C_c(\mathbf{R}^n)$ and h is a trigonometric polynomial on $SO(n)$. If $S > 0$, there exist $\lambda_0 > 0$, an integer $j_1 \geqslant 0$, and a constant $K = K(f,g)$ such that*

$$(13)\quad |\mathrm{Tr}\,\{A\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\} - c_n^2\lambda^{2n}\,\mathrm{Tr}\{R_k^{-1}AR_k\hat{f}_\lambda(\sigma_{\alpha,k})\hat{g}(\sigma_{\alpha,k})\}| \leqslant K\|A\|_{\mathscr{B}(\mathscr{H}^\alpha)}\left(\frac{1}{\lambda} + c_n\lambda^{n-1}\|\hat{f}_\lambda(\sigma_{\alpha,k})\|_{\varphi_2}\right)$$

for all $\alpha \in \hat{\mathbf{T}}_+^{m-1}$, $\lambda \geqslant \lambda_0$ and all $k \geqslant j_1$ for which $\dfrac{k}{\lambda} \leqslant S$.

Proof. — Write

$$(14)\quad \mathrm{Tr}\,\{A\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\} = \mathrm{Tr}\,\{A[\hat{f}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}]\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\}$$
$$+\mathrm{Tr}\,\{A[c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}][\hat{g}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n(R_k\hat{g}_\lambda(\sigma_{\alpha,k})R_k^{-1}]\}$$
$$+ c_n^2\lambda^{2n}\,\mathrm{Tr}\,\{AR_k\hat{f}_\lambda(\sigma_{\alpha,k})\hat{g}_\lambda(\sigma_{\alpha,k})R_k^{-1}\}.$$

It suffices to estimate appropriately the first two terms on the right of (14). In doing so we shall use a number of inequalities from [4], Appendix D, concerning the von Neumann norms $\|\cdot\|_{\varphi_p}$.

Now

$$(15)\quad |\mathrm{Tr}\,\{A[\hat{f}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n R_k \hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}]\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\}|$$

$$\leqslant \|A\|_{\mathscr{B}(\mathscr{H}^{\alpha})}\|\hat{f}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}\|_{\mathscr{B}(\mathscr{H}^{\alpha})}\|\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\|_{\varphi_1}.$$

By Lemmas 5.6 and 5.7, there exist $j_0 > 0$ and $\lambda_0 \geqslant 0$ such that the right side of (15) is bounded by

$$(16)\quad \frac{K_1}{\lambda}\,\|A\|\,\|\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\|_{\varphi_1} \leqslant \frac{K_2}{\lambda}\,\|A\|\,\|\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\|_{\varphi_\infty} \quad \text{(Lemma 5.5 (ii))}$$

$$\leqslant \frac{K_2}{\lambda}\,\|A\|\,\|g\|_1$$

for all $\alpha \in \hat{T}_+^{m-1}$, $\lambda \geqslant \lambda_0$ and $j \geqslant j_0$ such that $\frac{j}{\lambda} \leqslant S$.

For the second difference, we have the estimate

$$(17)\quad |\mathrm{Tr}\,\{A[c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}][\hat{g}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n R_k\hat{g}_\lambda(\sigma_{\alpha,k})R_k^{-1}]|$$

$$\leqslant \|A\|\,\|c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}\|_{\varphi_1}\|\hat{g}(\omega_{\alpha,\frac{k}{\lambda}}) - c_n\lambda^n R_k\hat{g}_\lambda(\sigma_{\alpha,k})R_k^{-1}\|_{\varphi_\infty}$$

$$\leqslant \frac{K_3}{\lambda}\,\|A\|\,\|c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}\|_{\varphi_1} \quad \text{(Lemmas 5.6 and 5.7).}$$

But for all α, k and $\lambda > 0$, we may choose a finite-dimensional space $M = M(\lambda,\alpha,k)$ such that

$$R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1} = P_M R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}P_M,$$

P_M denoting orthogonal projection onto M; and thanks to Lemma 5.3, we may assume that $\dim M \leqslant D$, a fixed number, for all α, k, λ. Hence

$$(18)\qquad \|c_n\lambda^n R_k\hat{f}_\lambda(\sigma_{\alpha,k})R_k^{-1}\|_{\varphi_1} \leqslant c_n\lambda^n D^{\frac{1}{2}}\|\hat{f}_\lambda(\sigma_{\alpha,k})\|_{\varphi_2}$$

under the same conditions as for (16). The inequality (13) comes from combining (15)-(18).

□

In order to be able to make effective use of Lemmas 5.2-5.8, we employ an appropriate summability process. This has the effect of reducing the discussion to one in which Φ has compact support. We shall use the results of J.-L. Clerc [1] concerning Riesz means.

Let $-\mu(\alpha\gamma^k)$ denote the eigenvalue, corresponding to the character $\alpha\gamma^k \in \hat{T}^m_+$, of a fixed bi-invariant Laplacian on $SO(n+1)$. Then according to [1], the family of functions

$$\left\{\left(1 - \frac{\mu(\alpha\gamma^k)}{R^2}\right)^N_+\right\}_{\alpha\gamma^k \in \hat{T}^m_+} \qquad (R > 0)$$

is a uniformly bounded family of multipliers of $\mathscr{F}L^q(SO(n+1))$ for every $q : 1 \leqslant q < \infty$, N denoting the dimension of $SO(n+1)$. We shall also need the fact [1, p. 152] that

$$\mu(\alpha\gamma^k) = \langle k\, d\gamma + d\alpha + \delta, k\, d\gamma + d\alpha + \delta\rangle - \langle \delta, \delta\rangle$$

δ denoting one-half of the sum of the positive roots, and $\langle .,. \rangle$ the Cartan-Killing form. Hence,

$$\mu(\alpha\gamma^k) = k^2 + Ak + B, \tag{19}$$

where A and B depend only on α.

5.9 LEMMA. — *(a) In order to prove Theorem* 5.1, *it will suffice to establish the existence of a constant* B *such that*

$$\left|\sum_{\alpha \in \hat{T}^{m-1}_+} d_\alpha \int_0^{+\infty} dR\, R^{n-1}\left(1 - \frac{R^2}{S^2}\right)^N_+ \operatorname{Tr}\{\Phi(\omega_{\alpha,R})\hat{f}(\omega_{\alpha,R})\hat{g}(\omega_{\alpha,R})\}\right| \leqslant B\|f\|_p\|g\|_{p'} \tag{20}$$

for all f, g *having the form specified in Lemma* 5.8, *and all sufficiently large* $S > 0$,

(b) The left side of (20) *is a positive scalar multiple of*

$$\lim_{\lambda \to +\infty} \left|\sum_{\alpha, k} d_{\alpha,k}\left(1 - \frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)^N_+ \frac{1}{\lambda^n} \operatorname{Tr}\{\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\}\right|.$$

Proof. — (*a*) is a simple consequence of the dominated convergence theorem.

(*b*) The idea here is to replace the integral by Riemann sums, but with $\left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+$ replacing the quantity $\left(1-\frac{k^2}{\lambda^2 S^2}\right)_+$ that would naturally appear; and also to replace $d_\alpha k^{n-1}$ by $\mathrm{A}d_{\alpha,k}$ (Lemma 2.1). We begin by studying the relationship between the former two quantities.

Thanks to (19), we have that

$$(21)\quad \left|\left(1-\frac{k^2}{\lambda^2 S^2}\right)_+ - \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+\right| \leqslant \left|\frac{k^2}{\lambda^2 S^2} - \frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right| = \frac{1}{\lambda^2 S^2}\,|A_1 k + B_1| \leqslant \frac{C_1 k}{\lambda^2 S^2},$$

say, C_1 denoting a fixed number, dependent only on α. Therefore, if $\frac{k^2}{\lambda^2 S^2} \leqslant 1$,

$$(22)\quad \left|\left(1-\frac{k^2}{\lambda^2 S^2}\right)_+^N - \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N\right| \leqslant \frac{C_2}{\lambda S} = O\left(\frac{1}{\lambda}\right)$$

uniformly with respect to $S \geqslant 1$, as $\lambda \to +\infty$. On the other hand, it follows from (19) that

$$\mu(\alpha\gamma^k) \geqslant \frac{1}{4}\,k^2$$

for all but finitely many k, say for $k \geqslant k_0$. If, therefore, $\mu(\alpha\gamma^k) < \lambda^2 S^2$ and $k \geqslant k_0$, then

$$(23)\quad \lambda^2 S^2 > \frac{1}{4}\,k^2;$$

hence (21)

$$(24)\quad \left|\left(1-\frac{k^2}{\lambda^2 S^2}\right)_+^N - \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N\right| \leqslant \frac{C_3}{\lambda S} = \underline{O}\left(\frac{1}{\lambda}\right)$$

uniformly with respect to $S \geqslant 1$, as $\lambda \to +\infty$.

We deduce from (22)-(24) that

$$(25)\quad \lim_{\lambda\to+\infty} \sum_k \left| \left(1-\frac{k^2}{\lambda^2 S^2}\right)_+^N - \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N \right| \left(\frac{k}{\lambda}\right)^{n-1} \frac{1}{\lambda} \left| \operatorname{Tr}\{\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\right|$$

$$\leqslant \lim_{\lambda\to+\infty} O\left(\frac{1}{\lambda}\right) \sum_{k\leqslant 2\lambda S} \left(\frac{k}{\lambda}\right)^{n-1} \frac{1}{\lambda} |\operatorname{Tr}\{\dots\}|$$

$$\leqslant \lim_{\lambda\to+\infty} O\left(\frac{1}{\lambda}\right) \int_0^{2S} dR\, R^{n-1} |\operatorname{Tr}\{\Phi(\omega_{\alpha,R})\hat{f}(\omega_{\alpha,R})\hat{g}(\omega_{\alpha,R})\}|$$

$$= 0.$$

One can prove in a similar, but simpler, fashion that

$$(26)\quad \lim_{\lambda\to+\infty} d_\alpha \sum_k \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N \left(\frac{k}{\lambda}\right)^{n-1} \frac{1}{\lambda} \operatorname{Tr}\{\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\}$$

$$= A \lim_{\lambda\to+\infty} \sum_k \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N d_{\alpha,k} \frac{1}{\lambda^n} \operatorname{Tr}\{\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\},$$

A denoting the constant appearing in Lemma 2.1.

The statement (*b*) follows from (25), (26), Lemma 5.5 and the definition of Riemann integral.

Proof of Theorem 5.1. – Let f and g be as in the statement of Lemma 5.8. Now there is a finite set E such that $\hat{f}(\omega_{\alpha,R}) = 0$ for all $\alpha \notin E$. By Lemma 5.8 and (23),

$$\overline{\lim_{\lambda\to+\infty}} \sum_{\alpha\in E} \sum_k \frac{d_{\alpha,k}}{\lambda^n} \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N |\operatorname{Tr}\{\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}(\omega_{\alpha,\frac{k}{\lambda}})\hat{g}(\omega_{\alpha,\frac{k}{\lambda}})\} - c_n^2\lambda^{2n} \operatorname{Tr}\{R_k^{-1}\Phi(\omega_{\alpha,\frac{k}{\lambda}})R_k\hat{f}_\lambda(\sigma_{\alpha,k})\hat{g}_\lambda(\sigma_{\alpha,k})\}|$$

$$(27)\quad \leqslant \overline{\lim_{\lambda\to+\infty}} \sum_{\alpha\in E} \sum_{j_0\leqslant k\leqslant 2\lambda S} \frac{d_{\alpha,k}}{\lambda^n} |\operatorname{Tr}\{\dots\} - c_n^2\lambda^{2n}\operatorname{Tr}\{\dots\}|$$

$$\leqslant C \overline{\lim_{\lambda\to+\infty}} \sum_{\alpha\in E} \sum_{j_0\leqslant k\leqslant 2\lambda S} \frac{d_{\alpha,k}}{\lambda^n} \left(\frac{1}{\lambda} + c_n\lambda^{n-1}\|\hat{f}_\lambda(\sigma_{\alpha,k})\|_{\varphi_2}\right).$$

Now, by Lemma 2.1,

$$(28)\qquad \sum_{\alpha\in E}\ \sum_{j_0\leqslant k\leqslant 2\lambda S} \frac{d_{\alpha,k}}{\lambda^{n+1}} \leqslant D \sup_{\alpha\in E} d_\alpha \sum_{1}^{2\lambda S} \frac{k^{n-1}}{\lambda^{n+1}} = O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$. Furthermore,

$$(29)\qquad \sum_{\alpha\in E}\ \sum_{j_0\leqslant k\leqslant 2\lambda S} \frac{d_{\alpha,k}}{\lambda} \|\hat{f}_\lambda(\sigma_{\alpha,k})\|_{\varphi_2}$$
$$\leqslant \frac{1}{\lambda}\left(\sum_{\alpha\in E}\ \sum_{j_0\leqslant k\leqslant 2\lambda S} d_{\alpha,k}\right)^{1/2}\left(\sum_{\alpha\in E}\ \sum_{j_0}^{+\infty} d_{\alpha,k}\|\hat{f}_\lambda(\sigma_{\alpha,k})\|_{\varphi_2}^2\right)^{1/2}$$
$$\leqslant \frac{D'}{\lambda}\lambda^{n/2}\|f_\lambda\|_2$$

by the Cauchy-Schwartz inequality and the Peter-Weyl theorem. Yet Theorem 4.10 implies that $\|f_\lambda\|_2 = O(\lambda^{-n/2})$ as $\lambda \to +\infty$; so it follows from (28) and (29) that the error estimate in (27) is in fact $\overline{\lim}\, O\left(\frac{1}{\lambda}\right) = 0$.

Finally,

$$(30)\qquad \lim_{\lambda\to+\infty}\left|\sum_{\alpha\in E}\sum_{k}\left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N c_n^2\lambda^{2n}\right.$$
$$\left.\mathrm{Tr}\,\{R_k^{-1}\Phi(\omega_{\alpha,\frac{k}{\lambda}})\hat{f}_\lambda(\sigma_{\alpha,k})\hat{g}_\lambda(\sigma_{\alpha,k})\}\right|$$
$$\leqslant \overline{\lim_{\lambda\to+\infty}}\ c_n^2\lambda^n K\|\mathscr{S}_{\lambda s}^N(f_\lambda)\|_p\|g_\lambda\|_{p'},$$

$\mathscr{S}_{\lambda S}^N(f_\lambda)$ denoting the function on $SO(n+1)$ such that

$$\mathscr{F}\mathscr{S}_{\lambda S}^N(f_\lambda)(\sigma_{\alpha,k}) = \left(1-\frac{\mu(\alpha\gamma^k)}{\lambda^2 S^2}\right)_+^N \hat{f}_\lambda(\sigma_{\alpha,k}).$$

Since the Riesz-mean operators $\mathscr{S}_{\lambda S}^N$ are uniformly bounded, the last expression in (30) is estimated by

$$\overline{\lim_{\lambda\to+\infty}}\ c_n^2\lambda^n K'\|f_\lambda\|_p\|g_\lambda\|_{p'} = c_n^2 K'\|f\|_p\|g\|_{p'},$$

by Theorem 4.8.

□

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