ANNALES DE L'INSTITUT FOURIER

LARS-INGE HEDBERG
THOMAS H. WOLFF

Thin sets in nonlinear potential theory

Annales de l'institut Fourier, tome 33, nº 4 (1983), p. 161-187
<http://www.numdam.org/item?id=AIF_1983__33_4_161_0>

Ann. Inst. Fourier, Grenoble
33, 4 (1983), 161-187.

THIN SETS IN NONLINEAR POTENTIAL THEORY

by L. I. HEDBERG [1] and Th. H. WOLFF [2]

1. Introduction and main results.

Let $L^q_\alpha(\mathrm{R}^d)$, $\alpha > 0$, $1 < q < \infty$ denote the space of Bessel potentials $f = G_\alpha * g$, $g \in L^q(\mathrm{R}^d)$, with norm $\|f\|_{\alpha,q} = \|g\|_{\bar{q}}$, and let $L^p_{-\alpha}(\mathrm{R}^d)$, $p + q = pq$ be its dual. Here G_α is the Bessel kernel, best defined as the inverse Fourier transform of $\hat{G}_\alpha(\xi) = (1+|\xi|^2)^{-\alpha/2}$. As is well known (A. P. Calderón [7]), when α is an integer there is a constant A so that

$$A^{-1}\|f\|_{\alpha,q} \leqslant \left(\sum_{0\leqslant|\beta|\leqslant\alpha}\int_{\mathrm{R}^d}|D^\beta f|^q\,dx\right)^{1/q} \leqslant A\,\|f\|_{\alpha,q},$$

and thus L^q_α coincides with the Sobolev space variously denoted $H^{\alpha,q}$, W^q_α, etc.

We shall be interested in the case $\alpha q \leqslant d$, when the functions in L^q_α are not in general continuous. Their lack of continuity can be measured by a set function called (α,q)-capacity which is most conveniently defined by

$$C_{\alpha,q}(E) = \inf\left\{\int_{\mathrm{R}^d} g^q\,dx; g\geqslant 0, G_\alpha * g \geqslant 1 \text{ for all } x\in E\right\}.$$

This definition is meaningful for arbitrary $E \subset \mathrm{R}^d$, because

$$G_\alpha * g(x) = \int_{\mathrm{R}^d} G_\alpha(x-y)g(y)\,dy$$

is always defined $(\leqslant +\infty)$ when $g \geqslant 0$.

(1) Supported by the Swedish Natural Science Research Council.
(2) This work was done while an NSF post doctoral fellow at the University of Chicago.

Classical potential theory is closely associated with the space L_1^2 or more generally L_α^2, $0 < 2\alpha \leqslant d$, and $C_{1,2}$ is nothing but a slight modification of Newtonian capacity (or logarithmic capacity for $d=2$). In fact, for $0 < 2\alpha \leqslant d$, $C_{\alpha,2}$ is classical capacity with respect to the kernel $G_{2\alpha}$, and for $0 < 2\alpha < d$ this kernel has the same singularity as the M. Riesz kernel $R_{2\alpha}(x) = |x|^{2\alpha-d}$.

The concept of a thin (effilé) set is fundamental in potential theory. A set $E \subset R^d$ is called $(\alpha,2)$-*thin* at a point x_0 if the following equivalent statements hold.

(A) *There exists a positive Radon measure* μ (briefly, $\mu \in \mathcal{M}^+$) *such that*

$$G_{2\alpha} * \mu(x_0) < \liminf_{\substack{x\to x_0\\ x\in E\setminus\{x_0\}}} G_{2\alpha} * \mu(x).$$

(B) $\displaystyle\int_0^1 \frac{C_{\alpha,2}(E\cap B_\delta(x_0))}{\delta^{d-2\alpha}}\frac{d\delta}{\delta} < \infty$. *Here* $B_\delta(x_0) = \{x; |x-x_0|<\delta\}$.

The equivalence of (A) and (B) is the content of the Wiener criterion. See e.g. Landkof [23] for $0 < \alpha \leqslant 1$, Fuglede [14] for the extension to $2\alpha \leqslant d$.

Let $e_{\alpha,2}(E) = \{x;\ E \text{ is } (\alpha,2)\text{-thin at } x\}$. Thus $e_{\alpha,2}(E)$ contains the exterior of E, and in general, part of the boundary ∂E. The set $e_{\alpha,2}(E)$ has the following important properties.

THE KELLOGG PROPERTY. — $C_{\alpha,2}(E\cap e_{\alpha,2}(E)) = 0$ *for any* $E \subset R^d$.

THE CHOQUET PROPERTY — *For any* $E \subset R^d$ *and any* $\varepsilon > 0$ *there is an open* $G \subset R^d$ *such that* $e_{\alpha,2}(E) \subset G$ *and* $C_{\alpha,2}(E\cap G) < \varepsilon$.

See Brelot [5,6], Choquet [9], Fuglede [14]. Clearly the Kellogg property follows from the Choquet property.

Our purpose is to generalize all these results to (α,q)-capacities, $1 < q \leqslant d/\alpha$. Before we can formulate these results and discuss earlier work in this direction we have to recall a few more facts about (α,q)-capacities. It is in the nature of things that for $q \neq 2$ the role of the potential $G_{2\alpha} * \mu$ will be played by a nonlinear function of μ, $G_\alpha * (G_\alpha * \mu)^{p-1}$, called a nonlinear potential and denoted $v_{\alpha,q}^\mu$. The following results are known.

Let $E \subset R^d$ be an arbitrary set with finite (α,q)-capacity. Then there is $\mu \in \mathcal{M}^+(\bar{E})$, the exterior capacity measure, such that

(*a*) $\mu(\bar{E}) = C_{\alpha,q}(E)$.

b) $(G_\alpha * \mu)^{p-1} = g \in L^q(R^d)$, and

$$\int g^q\,dx = \int (G_\alpha * \mu)^p\,dx = C_{\alpha,q}(E).$$

(*c*) $V^\mu_{\alpha,q}(x) = G_\alpha * (G_\alpha * \mu)^{p-1}(x) \geqslant 1$ on E, except, possibly on a set of zero (α,q)-capacity, i.e. (α,q)-quasieverywhere ((α,q)-q.e.) on E.

(*d*) $V^\mu_{\alpha,q} \leqslant 1$ everywhere on $\operatorname{supp}\mu$.

Moreover, there is a constant M such that for any $\mu \in \mathcal{M}^+$

(*e*) $\sup_x V^\mu_{\alpha,q}(x) \leqslant M \sup_{x \in \operatorname{supp}\mu} V^\mu_{\alpha,q}(x)$.

Choquet's capacitability theorem applies, so for Borel or Suslin sets E

(*f*) $C_{\alpha,q}(E) = \sup\{C_{\alpha,q}(K);\ K \subset E,\ K \text{ compact}\}$.

Furthermore, if E is Borel (Suslin)

(*g*) $C_{\alpha,q}(E)^{1/q} = \sup\{\mu(E);\ \mu \in \mathcal{M}^+, \operatorname{supp}\mu \subset E, \|G_\alpha * \mu\|_p \leqslant 1\}$.

(*h*) $C_{\alpha,q}(E) = \sup\{\mu(E);\ \mu \in \mathcal{M}^+, \operatorname{supp}\mu \subset E, \sup_{x \in \operatorname{supp}\mu} V^\mu_{\alpha,q}(x) \leqslant 1\}$.

In view of this the natural generalization of (A) is the following statement about a set $E \subset R^d$ and a point $x_0 \in R^d$.

(C) *There exists a* $\mu \in \mathcal{M}^+$ *such that*

$$V^\mu_{\alpha,q}(x_0) < \liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} V^\mu_{\alpha,q}(x).$$

Another possibility is the following.

(D) *There is a* $\mu \in \mathcal{M}^+$ *such that* $V^\mu_{\alpha,q}$ *is bounded, and*

$$V^\mu_{\alpha,q}(x_0) < \liminf_{x \to x_0,\, x \in \setminus\{x_0\}} V^\mu_{\alpha,q}(x).$$

It turns out that the natural generalization of (B) is

$$\text{(E)} \qquad \int_0^1 \left(\frac{C_{\alpha,q}(E \cap B_\delta(x_0))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} < \infty.$$

Unfortunately (C), (D) and (E) are not equivalent in general. In addition to the obvious implication (D) $\Rightarrow$ (C) the following are known :

$$\begin{array}{lll} \text{(C)} \Leftrightarrow \text{(D)} \Leftrightarrow \text{(E)} & \textit{for} & 2 - \frac{\alpha}{d} < q (\leqslant d/\alpha) \\[2ex] \text{(D)} \Rightarrow \text{(E)} & \textit{for} & 1 < q \leqslant 2 - \frac{\alpha}{d} \\[2ex] \text{(E)} \not\Rightarrow \text{(D)} & \textit{for} & 1 < q < 2 - \frac{\alpha}{d}. \end{array}$$

The reason for the difficulty is that for $q > 2 - \frac{\alpha}{d}$ there are estimates

$$A_1 \int_0^\infty \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} e^{-b_1\delta} \frac{d\delta}{\delta} \leqslant V^\mu_{\alpha,q}(x) \leqslant A_2 \int_0^\infty \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} e^{-b_2\delta} \frac{d\delta}{\delta},$$

but for $1 < q \leqslant 2 - \frac{\alpha}{d}$ only the lower estimate is true. To see that the upper estimate breaks down it is enough to take μ to be a point mass, since this gives $V^\mu_{\alpha,q} \equiv \infty$.

As a consequence extensions of the Kellogg and Choquet properties to the case $1 < q \leqslant 2 - \frac{\alpha}{d}$ have been lacking.

We shall show that (E) is the good choice of definition for an (α,q)-thin set, and prove that the Kellogg and Choquet properties are true in this strong sense. Furthermore, we shall show that (C) (and (D)) can be replaced by a modified statement which is equivalent to (E). The main new tool is an inequality which gives a characterization of the positive measures in $L^p_{-\alpha}(R^d)$. To state the results precisely we set

$$W^\mu_{\alpha,q}(x) = \int_0^1 \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} \frac{d\delta}{\delta}.$$

Like $V^\mu_{\alpha,q}$, $W^\mu_{\alpha,q}$ is a lower semicontinuous function of x. In fact, for any $\delta_0 > 0$, $\displaystyle\int_{\delta_0}^1 \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} \frac{d\delta}{\delta}$ is continuous.

By the above estimate there is $A > 0$ so that $W^\mu_{\alpha,q}(x) \leqslant AV^\mu_{\alpha,q}(x)$, and thus $\displaystyle\int W^\mu_{\alpha,q}\, d\mu \leqslant A \int V^\mu_{\alpha,q}\, d\mu = A \int_{R^d} (G_\alpha * \mu)^p\, dx$. The new result is that the converse inequality is true, although the pointwise estimate is false.

THEOREM 1. — *Let* $\mu \in \mathcal{M}^+$, $\alpha > 0$ *and* $1 < q \leqslant d/\alpha$. *Then there is* $A > 0$ *so that*

$$\int (G_\alpha * \mu)^p \, dx \leqslant A \int W^{\mu}_{\alpha,q} \, d\mu .$$

COROLLARY. — $\mu \in \mathcal{M}^+ \cap L^p_{-\alpha}(R^d)$ *if and only if* $\int W^{\mu}_{\alpha,q} \, d\mu < \infty$.

DEFINITION. — *A set* $E \subset R^d$ *is* (α,q)*-thin at* $x \in R^d$ *if* $\alpha q \leqslant d$ *and*

$$\int_0^1 \left(\frac{C_{\alpha,q}(E \cap B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} \frac{d\delta}{\delta} < \infty ,$$

$$e_{\alpha,q}(E) = \{x \in R^d ; \text{E is } (\alpha,q)\textit{-thin at } x\}.$$

THEOREM 2 (The Kellogg property). — *Let* $\alpha > 0$, $1 < q \leqslant d/\alpha$. *Then* $C_{\alpha,q}(e_{\alpha,q}(E) \cap E) = 0$ *for any* $E \subset R^d$.

Theorem 1 makes it possible to develop nonlinear potential theory using $\int W^{\mu}_{\alpha,q} \, d\mu$ instead of $\int V^{\mu}_{\alpha,q} \, d\mu$ as a generalized energy, and $W^{\mu}_{\alpha,q}$ as a nonlinear potential. In this way one can extend the Choquet property.

THEOREM 3 (The Choquet property). — *Let* $\alpha > 0$, $1 < q \leqslant d/\alpha$. *Then for any* $E \subset R^d$ *and any* $\varepsilon > 0$ *there is an open* G *such that*

$$e_{\alpha,q}(E) \subset G \qquad \textit{and} \qquad C_{\alpha,q}(E \cap G) < \varepsilon .$$

We also obtain an equivalent formulation of thinness in terms of $W^{\mu}_{\alpha,q}$.

THEOREM 4. — *Let* $\alpha > 0$, $1 < q \leqslant d/\alpha$. *A set* $E \subset R^d$ *is* (α,q)*-thin at* $x_0 \in \bar{E}$ *if and only if there is* $\mu \in \mathcal{M}^+$ *such that*

$$W^{\mu}_{\alpha,q}(x_0) < \liminf_{x \to x_0,\, x \in E \setminus \{x_0\}} W^{\mu}_{\alpha,q}(x) .$$

These results have a number of consequences which we formulate in the next section. In section 3 we prove theorem 1, and since theorem 2 can be deduced quickly from theorem 1 in the case when E is capacitable or, say, Borel, we give the deduction although theorem 2 is a consequence of theorem 3. Finally in section 4 we develop the potential theory for $W^{\mu}_{\alpha,q}$ which is needed to prove theorems 3 and 4.

Numerous references to earlier work on potentials of L^p functions and nonlinear potential theory by D. R. Adams, B. Fuglede, V. P. Havin, V. G. Maz'ja, N. G. Meyers, Ju. G. Rešetnyak, and others are found in the earlier papers [17-21] of the first author. (α,q)-thin sets were defined in [17] and independently by Adams-Meyers [2]. The relations between (C), (D) and (E) we proved in [2] (and partly in [17]) except for a slight extension given in [1]. The definition of (α,q)-thin sets given here was proposed by Meyers [25], who also studied the associated (α,q)-fine topology and properties of the function $W^{\mu}_{\alpha,q}$. In the case $2-\frac{\alpha}{d}<q\leqslant\frac{d}{\alpha}$ the Kellogg and Choquet properties were proved in [17].

The results given in section 3 are due to Wolff; the results in section 4 were found subsequently by Hedberg.

Throughout the paper A denotes various constants, whose value can change from one line to the next.

The second author is grateful to Peter Jones for drawing his attention to these problems and for valuable conversations.

2. Applications.

The main result in [20] was that all closed sets $F\subset R^d$ admit so called (m,q)-synthesis for any positive integer m and any $q>2-\frac{1}{d}\cdot$ It was pointed out there (and in [18]) that the result would follow for $q>1$ if the Kellogg property could be proved for all (m,q)-capacities, $q>1$. Thus, we can now state that result. We refer to [20] for the precise definition of the traces $f|_F, D^k f|_F$.

THEOREM 5. — *Let $q>1$, let m be a positive integer, let $F\subset R^d$ be closed, and let $f\in L^q_m(R^d)$. Then $f\in \mathring{L}^q_m(F^c)$ if and only if $D^k f|_F=0$ for all multiindices k, $0\leqslant|k|\leqslant m-1$.*

More precisely, if $D^k f|_F=0$, $0\leqslant k\leqslant m-1$ (in particular if $f\in\mathring{L}^q_m(F^c)$) then for any $\varepsilon>0$ there exists a function w, $0\leqslant w\leqslant 1$, such that $\operatorname{supp} w$ is compact and does not intersect F, and such that $wf\in L^\infty$, and $\|f-wf\|_{m,q}<\varepsilon$.

Of course, all the consequences of this result which were given in [20] for $q > 2 - \frac{1}{d}$ can now be extended to $q > 1$. We do not repeat these corollaries here.

We can now also solve a problem left open in [17]. Thus the following theorem extends theorem 7 in [17]. See also the survey [19].

THEOREM 6. — *Let* $\alpha > 0$, $q > 1$, $E \subset R^d$, *and* $S \subset E$. *Then the following are equivalent.*

(*a*) $C_{\alpha,q}(S \cap G) = C_{\alpha,q}(E \cap G)$ *for all open* G;

(*b*) *For some* $\eta > 0$, $C_{\alpha,q}(S \cap G) \geqslant \eta C_{\alpha,q}(E \cap G)$ *for all open* G;

(*c*) *For* (α,q)-q.e. $x \in E$, $\liminf_{\delta \to 0} \frac{C_{\alpha,q}(S \cap B_\delta(x))}{C_{\alpha,q}(E \cap B_\delta(x))} > 0$;

(*d*) $C_{\alpha,q}(e_{\alpha,q}(S) \backslash e_{\alpha,q}(E)) = 0$;

(*e*) $e_{\alpha,q}(E) = e_{\alpha,q}(S)$;

(*f*) $C_{\alpha,q}(e_{\alpha,q}(S) \cap E) = 0$.

Remark. — In the terminology of Choquet [10] a subset S of a set E is called $C_{\alpha,q}$-representative for E if (*a*) holds.

Proof. — The implications (*a*) $\Rightarrow$ (*b*) $\Rightarrow$ (*c*) $\Rightarrow$ (*d*) and (*a*) $\Rightarrow$ (*e*) are trivial. (*d*) implies (*f*) by the Kellogg property. (*f*) implies (*a*) by the implication (D) $\Rightarrow$ (E) of section 1. In fact, if (*f*) is assumed, it follows that the capacitary potential for S (which is known to be bounded) is $\geqslant 1$ (α,q) q.e. on E, which gives the result.

As in [17] we can apply Theorem 6 to solve a problem on rational approximation. See also [19]. Let $K \subset G$ be compact, let $L^p_a(K)$ denote the subspace of $L^p(K)$ consisting of functions analytic on K^0, the interior of K, and let $R^p(K)$ be the closure in $L^p(K)$ of the rational functions with poles off K. Then $R^p(K)$ is a subspace of $L^p_a(K)$, and it is well known (Havin [16]) that $R^p(K) = L^p_a(K)$ for $1 < p < 2$.

THEOREM 7. — *Let* $2 \leqslant p < \infty$. *The following are equivalent :*

(*a*) $R^p(K) = L^p_a(K)$;

(*b*) K *is* $(1,q)$-*stable, i.e. if* $\varphi \in W^q_1(R^2)$ *and* $\varphi = 0$ *on* K^c, *then* $\varphi \in \mathring{W}^q_1(K^c)$;

(*c*) $C_{1,q}(G \backslash K) = C_{1,q}(G \backslash K^0)$ *for all open* G;

(*d*) $C_{1,q}(\partial K \cap e_{1,q}(K^c)) = 0$.

For $p = 2$ the equivalence of (a), (b) and (d) is due to Havin [16]. The equivalence of (b) and (c) is due to T. Bagby [3]. Havin's result was extended to $p < 3$ $\left(\text{i.e. } q > 2 - \frac{1}{d} = \frac{3}{2}\right)$ in [17; Th. 11 and Cor. 2, p. 316]; this was in fact the motivation for the introduction of (α,q)-thin sets. What is new is the extension of Havin's result to all $p < \infty$. The equivalence of (c) and (d) and other conditions which we omit is an immediate consequence of theorem 6.

Let f be a function which is defined (α,q)-q.e. Then f is called (α,q)-*quasicontinuous* if for every $\varepsilon > 0$ there is an open set G such that the restriction $f|_{G^c}$ is continuous as a function on G^c. f is called (α,q)-*finely continuous* at a point x_0 if it is defined there, and if for all $\varepsilon > 0$ the set $E_\varepsilon = \{x; |f(x) - f(x_0)| \geqslant \varepsilon\}$ is (α,q)-thin there.

It is known that every (α,q)-quasicontinuous function is (α,q)-finely continuous (α,q)-q.e. This result is due to Fuglede [13]. See also [17]. It is a general result of Brelot [6; Theorem IV : 7] that the Choquet property implies the converse. (We are grateful to T. Kolsrud for pointing out this fact.) Thus we have the following theorem.

THEOREM 8. — *A function is (α,q)-quasicontinuous if and only if it is (α,q)-finely continuous (α,q)-q.e.*

An application of this result is given by Kolsrud in [22].

3. The main inequality.

For the proof of Theorem 1 we shall use subdivisions of R^d into dyadic cubes. For each $n \geqslant 0$ we subdivide R^d into non-intersecting cubes of side 2^{-n}, so that each cube in one generation is split into 2^d cubes of the next generation. Q and Q' will always denote such dyadic cubes. The volume of Q is $|Q|$, its sidelength is $\ell(Q)$, and $\chi_Q(x)$ denotes the characteristic function of Q. The cube concentric to Q with sidelength $3\ell(Q)$ is denoted $\tilde{Q}$. Finally, the unit cube is denoted I_0.

As is well known there are constants a and A so that

$$G_\alpha(x) \leqslant A|x|^{\alpha-d}, \quad 0 < |x| \leqslant 1,$$

and

$$G_\alpha(x) \leqslant Ae^{-a|x|}, \qquad |x| > 1.$$

See e.g. Stein [26]. Note that $0 < \alpha < d$ since we are assuming $\alpha q \leqslant d$, $q > 1$.

Set

$$\tilde{R}_\alpha(x) = |x|^{\alpha - d}, \qquad 0 < |x| < 1,$$
$$\tilde{R}_\alpha(x) = 0, \qquad |x| \geqslant 1.$$

Our first observation is that it is enough to estimate

$$\int_{\mathbf{R}^d} (\tilde{R}_\alpha * \mu)^p \, dx.$$

In fact, for any fixed n,

$$\begin{aligned} G_\alpha * \mu(x) &\leqslant A\Big(\tilde{R}_\alpha * \mu(x) + \int_{|x-y|>1} e^{-a|x-y|} \, d\mu(y)\Big) \\ &\leqslant A\tilde{R}_\alpha * \mu(x) + A \sum_{\ell(Q)=2^{-n}} e^{-a \operatorname{dist}(x,Q)} \mu(Q). \end{aligned}$$

Thus

$$\begin{aligned} (G_\alpha * \mu)^p(x) &\leqslant A(\tilde{R}_\alpha * \mu)^p(x) + A\Big(\sum_{\ell(Q)=2^{-n}} e^{-a \operatorname{dist}(x,Q)} \mu(Q)\Big)^p \\ &\leqslant A(\tilde{R}_\alpha * \mu)^p(x) + A\Big(\sum_{\ell(Q)=2^{-n}} e^{-a \operatorname{dist}(x,Q)}\Big)^{p-1} \\ &\qquad\qquad \Big(\sum_{\ell(Q)=2^{-n}} e^{-a \operatorname{dist}(x,Q)} \mu(Q)^p\Big) \\ &\leqslant A(\tilde{R}_\alpha * \mu)^p(x) + A \sum_{(Q)=2^{-n}} e^{-a \operatorname{dist}(x,Q)} \mu(Q)^p, \end{aligned}$$

so that

$$\int_{\mathbf{R}^d} (G_\alpha * \mu)^p \, dx \leqslant A \int_{\mathbf{R}^d} (\tilde{R}_\alpha * \mu)^p \, dx + A \sum_{\ell(Q)=2^{-n}} \mu(Q)^p.$$

But if n is chosen so that $2^{-n}\sqrt{d} \leqslant 1/2$, we have for

$$x \in Q, \quad \ell(Q) = 2^{-n}: \qquad W^\mu(x) = \int_0^1 \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} \geqslant A\mu(Q)^{p-1},$$

so that

$$\int W^\mu \, d\mu = \sum_{\ell(Q)=2^{-n}} \int_Q W^\mu \, d\mu \geqslant A \sum_{\ell(Q)=2^{-n}} \mu(Q)^p,$$

which proves the assertion.

Secondly, we observe that it is no loss of generality to assume $\operatorname{supp}\mu \subset I_0$. In fact,

$$\int_{\mathbf{R}^d} (\tilde{\mathrm{R}}_\alpha * \mu)^p\, dx = \int_{\mathbf{R}^d} \Big(\sum_{\substack{\ell(Q)=1\\ x\in Q}} \int_Q \tilde{\mathrm{R}}_\alpha(x-y)\, d\mu(y) \Big)^p dx$$

$$\leqslant A \int_{\mathbf{R}^d} \sum_{\ell(Q)=1} \Big(\int_Q \tilde{\mathrm{R}}_\alpha(x-y)\, d\mu(y) \Big)^p dx.$$

It follows that if $\displaystyle\int_{\mathbf{R}^d} \Big(\int_Q \tilde{\mathrm{R}}_\alpha(x-y)\, d\mu(y)\Big)^p dx \leqslant A \int_Q W^{\mu|Q}\, d\mu$, then

$$\int_{\mathbf{R}^d} (\tilde{\mathrm{R}}_\alpha * \mu)^p\, dx \leqslant \sum_{\ell(Q)=1} \int_Q W^{\mu|Q}\, d\mu \leqslant A \int W^\mu\, d\mu.$$

It is easily seen that

$$\tilde{\mathrm{R}}_\alpha * \mu(x) \leqslant \int_{|x-y|\leqslant 1} \frac{d\mu(y)}{|x-y|^{d-\alpha}} \leqslant \sum_{n=0}^{\infty} 2^{(n+1)(d-\alpha)} \mu(B_{2^{-n}}(x))$$

$$\leqslant A \sum_{\ell(Q)\leqslant 1} \frac{\mu(\tilde{Q})}{\ell(Q)^{d-\alpha}} \chi_Q(x),$$

and that

$$W^\mu(x) = \int_0^1 \left(\frac{\mu(B_\delta(x))}{\delta^{d-\alpha q}} \right)^{p-1} \frac{d\delta}{\delta} \geqslant A \sum_{n=1}^{\infty} 2^{n(d-\alpha q)} \mu(B_{2^{-n}}(x))^{p-1}$$

$$\geqslant A \sum_{\ell(Q)\leqslant 1} \left(\frac{\mu(Q)}{\ell(Q)^{d-\alpha q}} \right)^{p-1} \chi_Q(x).$$

So

$$\int W^\mu\, d\mu \geqslant A \sum_{\ell(Q)\leqslant 1} \frac{\mu(Q)^p}{\ell(Q)^{d(p-1)-\alpha p}} = A \sum_{\ell(Q)\leqslant 1} \left(\frac{\mu(Q)}{\ell(Q)^{d-\alpha}} \right)^p |Q|$$

$$= A \int_{\mathbf{R}^d} \sum_{\ell(Q)\leqslant 1} \left(\frac{\mu(Q)}{\ell(Q)^{d-\alpha}} \right)^p \chi_Q(x)\, dx.$$

It is thus enough to prove

$$(*) \quad \int_{I_0} \Big(\sum_{\ell(Q)\leqslant 1} \ell(Q)^{\alpha-d} \mu(\tilde{Q}) \chi_Q(x) \Big)^p dx \leqslant A \sum_{\ell(Q)\leqslant 1} (\ell(Q)^{\alpha-d} \mu(Q))^p |Q|.$$

Our argument is rather similar to Hansson's proof of the strong type capacitary inequality [15]. We first prove a lemma. We set

$$\ell(Q)^{\alpha-d}\mu(Q) = b(Q).$$

LEMMA 1. — *Let* $s > 0$, $n \geqslant 1$. *Then for any dyadic cube* I,

$$\sum_{Q' \subset I} b(\tilde{Q}')^s \sum_{Q \subset Q'} b(\tilde{Q})^n |Q| \leqslant A \sum_{Q \subset I} b(\tilde{Q})^{n+s} |Q| .$$

Proof. — If $n = 1$ the left hand side is

$$\sum_{Q' \subset I} b(\tilde{Q}')^s \sum_{Q \subset Q'} b(\tilde{Q})|Q| = A \sum_{Q' \subset I} b(\tilde{Q}')^s \sum_{Q \subset Q'} \mu(\tilde{Q})\ell(Q)^\alpha$$

$$\leqslant A \sum_{Q' \subset I} b(Q')^s \mu(\tilde{Q}') \sum_{2^{-n} \leqslant \ell(Q')} 2^{-n\alpha} \leqslant A \sum_{Q' \subset I} b(\tilde{Q}')^{s+1} |Q'| .$$

If $n > 1$, first assume $Q' = I_0$. Then, by Hölder's inequality there are ε, $\varepsilon' > 0$ so that

$$\sum_{Q \subset I_0} b(\tilde{Q})^n |Q| = A \sum_{Q \subset I_0} b(\tilde{Q})^{n-1} \ell(Q)^\alpha \mu(\tilde{Q})$$

$$\leqslant A \left(\sum_{Q \subset I_0} b(\tilde{Q})^{n-1+s} \ell(Q)^{\alpha+\varepsilon} \mu(\tilde{Q}) \right)^{\frac{n-1}{n-1+s}} \left(\sum_{Q \subset I_0} \ell(Q)^{\alpha-\varepsilon'} \mu(\tilde{Q}) \right)^{\frac{s}{n-1+s}}$$

$$\leqslant A \left(\sum_{Q \subset I_0} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon} \right)^{\frac{n-1}{n-1+s}} \mu(\tilde{I}_0)^{\frac{s}{n-1+s}} .$$

Multiplying by $\mu(\tilde{I}_0)^s$, and observing that

$$\mu(\tilde{I}_0)^{n+s} = b(\tilde{I}_0)^{n+s} \ell(I_0)^{d+\varepsilon} \leqslant \sum_{Q \subset I_0} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon},$$

we find

$$\mu(\tilde{I}_0)^s \sum_{Q \subset I_0} b(\tilde{Q})^n |Q| \leqslant A \left(\sum_{Q \subset I_0} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon} \right)^{\frac{n-1}{n-1+s}} \mu(\tilde{I}_0)^{\frac{s(n+s)}{n-1+s}}$$

$$\leqslant A \sum_{Q \subset I_0} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon} .$$

Replacing I_0 by Q' we obtain by homogeneity

$$b(\tilde{Q}')^s \ell(Q')^\varepsilon \sum_{Q \subset Q'} b(\tilde{Q})^n |Q| \leqslant A \sum_{Q \subset Q'} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon} .$$

Now divide by $\ell(Q')^{\varepsilon}$ and sum over $Q' \subset I$, obtaining

$$\sum_{Q' \subset I} b(\tilde{Q}')^s \sum_{Q \subset Q'} b(\tilde{Q})^n |Q| \leqslant A \sum_{Q \subset I} b(\tilde{Q})^{n+s} \ell(Q)^{d+\varepsilon} \sum_{Q' : Q \subset Q' \subset I} \ell(Q')^{-\varepsilon}$$
$$\leqslant A \sum_{Q \subset I} b(\tilde{Q})^{n+s} |Q| ,$$

which proves the lemma.

We can now prove (*). It is clearly enough to prove

$$\int_{I_0} \left(\sum_{x \in Q} b(\tilde{Q}) \right)^p dx \leqslant A \sum_{\ell(Q) \leqslant 1} b(\tilde{Q})^p |Q| .$$

Let $p = n + s$, $n =$ integer, $0 < s \leqslant 1$. Then

$$\int_{I_0} \left(\sum_{x \in Q} b(\tilde{Q}) \right)^p dx \leqslant A \int_{I_0} \sum_{Q_n \supset \ldots \supset Q_1 \ni x} b(\tilde{Q}_n) \ldots b(\tilde{Q}_1) \left(\sum_{x \in Q} b(\tilde{Q}) \right)^s dx .$$

For fixed $Q_1, \ldots, Q_n$ we split the sum over $\tilde{Q}$ according to which Q_i's are inside or outside Q. The case $Q \subset Q_1$ has to be considered separately. For the others, look at

$$\int \sum_{Q_n \supset \cdots \supset Q_1 \ni x} b(\tilde{Q}_n) \ldots b(\tilde{Q}_1) \left(\sum_{Q_k \subset Q \subset Q_{k+1}} b(\tilde{Q}) \right)^s dx$$
$$\leqslant A \sum_{Q_n \supset \cdots \supset Q_{k+1}} b(\tilde{Q}_n) \ldots b(\tilde{Q}_{k+1}) \sum_{Q \subset Q_{k+1}} b(\tilde{Q})^s$$
$$\sum_{Q \supset Q_k \supset \cdots \supset Q_1} b(\tilde{Q}_k) \ldots b(\tilde{Q}_1) |Q| ,$$

which by repeated application of the lemma is $\leqslant A \sum_Q b(\tilde{Q})^p |Q|$.

For $Q \subset Q_1$ we have

$$\int \sum_{Q_n \supset \cdots \supset Q_1} b(\tilde{Q}_n) \ldots b(\tilde{Q}_1) \left(\sum_{x \in Q \subset Q_1} b(\tilde{Q}) \right)^s dx$$
$$= \sum_{Q_n \supset \cdots \supset Q_1} b(\tilde{Q}_n) \ldots b(\tilde{Q}_1) \int_{Q_1} \left(\sum_{Q \subset Q_1} b(\tilde{Q}) \chi_Q(x) \right)^s dx .$$

By Hölder's inequality the integral over Q_1 is

$$\leqslant |Q_1|^{1-s}\left(\int_{Q_1}\sum_{Q\subset Q_1} b(\tilde{Q})\chi_Q(x)\,dx\right)^s = |Q_1|^{1-s}\left(\sum_{Q\subset Q_1} b(\tilde{Q})|Q|\right)^s$$
$$\leqslant A|Q_1|^{1-s}\left(\sum_{Q\subset Q_1}\mu(\tilde{Q})\ell(Q)^{\alpha}\right)^s \leqslant A|Q_1|^{1-s}(\mu(\tilde{Q}_1)\ell(Q_1)^{\alpha})^s$$
$$= Ab(\tilde{Q}_1)^s|Q_1|.$$

Thus the left hand side is

$$\leqslant A\sum_{Q_n\supset\cdots\supset Q_1} b(\tilde{Q}_n)\cdots b(\tilde{Q}_2)b(\tilde{Q}_1)^{1+s}|Q_1|.$$

Again repeated applications of the lemma show that this is

$$\leqslant A\sum_{Q} b(\tilde{Q})^p|Q|.$$

This proves Theorem 1.

In the case of Borel sets Theorem 2, the Kellogg property, now follows from Theorem 1 and the following lemma.

LEMMA 2. — *If there is a Borel set* E *without the Kellogg property then for any* $\varepsilon > 0$ *there is a compact* $E_\varepsilon \subset E$ *such that* $C_{\alpha,q}(E_\varepsilon) > 0$ *and*

$$\int_0^1\left(\frac{C_{\alpha,q}(E_\varepsilon\cap B_\delta(x))}{\delta^{d-\alpha q}}\right)^{p-1}\frac{d\delta}{\delta} < \varepsilon \qquad \textit{for all} \qquad x\in E_\varepsilon.$$

Proof. — Set $(\delta^{\alpha q-d}C_{\alpha,q}(E\cap B_\delta(x)))^{p-1}\,\delta^{-1} = C(E,\delta,x)$. Suppose that $\int_0^1 C(E,\delta,x)\,d\delta < \infty$ for all $x\in E\cap e_{\alpha,q}(E) = E'$, and $C_{\alpha,q}(E') > 0$. Then for some (large) $\varepsilon_0 > 0$ there is an $E_{\varepsilon_0}\subset E'$ such that $\int_0^1 C(E_{\varepsilon_0},\delta,x)\,d\delta < \varepsilon_0$, and $C_{\alpha,q}(E_{\varepsilon_0}) > 0$. By Choquet's theorem we can assume that E_{ε_0} is compact. We choose E_{ε_0} so that $C_{\alpha,q}(E_{\varepsilon_0}\cap B_\delta(x)) > 0$ for all $x\in E_{\varepsilon_0}$ and $\delta > 0$. Let $W(x) = \int_0^1 C(E_{\varepsilon_0},\delta,x)\,d\delta$, and let a be a point of E_{ε_0} with $W(a) > \sup_{x\in E_{\varepsilon_0}} W(x) - \varepsilon/4$. Then choose $\gamma > 0$ with

$$\int_\gamma^1 C(E_{\varepsilon_0},\delta,x)\,d\delta > W(a) - \varepsilon/4.$$

If $\rho < \gamma$ is small enough, then for all $x \in \overline{B_\rho(a)}$,

$$\int_\gamma^1 C(E_{\varepsilon_0},\delta,x)\, d\delta > W(a) - \varepsilon/4 > \sup_{x \in E_{\varepsilon_0}} W(x) - \frac{3\varepsilon}{4},$$

so
$$\int_0^\gamma C(E_{\varepsilon_0},\delta,x)\, d\delta = W(x) - \int_\gamma^1 C(E_{\varepsilon_0},\delta,x)\, d\delta < \frac{3\varepsilon}{4}.$$

Let $E_\varepsilon = E_{\varepsilon_0} \cap \overline{B_\rho(a)}$. Then E_ε is compact, $C_{\alpha,q}(E_\varepsilon) > 0$, and

$$\int_0^1 C(E_\varepsilon,\delta,x)\, d\delta < \frac{3\varepsilon}{4} + C_{\alpha,q}(\overline{B_\rho(a)})^{p-1} \int_\gamma^1 \delta^{(\alpha q - d)(p-1)-1}\, d\delta < \varepsilon$$

if ρ is small enough.

Proof of Theorem 2. — If the property fails, choose E_ε by the lemma. Let μ be its capacitary measure. Then

$$\mu(E_\varepsilon) = \int_{E_\varepsilon} V^\mu_{\alpha,q}\, d\mu = \int_{R^d} (G_\alpha * \mu)^p\, dx \leqslant A \int_{E_\varepsilon} W^\mu\, d\mu.$$

But by property (h) on p. 163 $\mu(E_\varepsilon \cap B_\delta(x)) \leqslant C_{\alpha,q}(E_\varepsilon \cap B_\delta(x))$, so

$$W^\mu(x) \leqslant \int_0^1 \left(\frac{C_{\alpha,q}(E_\varepsilon \cap B_\delta(x))}{\delta^{d-q\alpha}}\right)^{p-1} \frac{d\delta}{\delta} < \varepsilon \qquad \text{for} \qquad x \in E_\varepsilon.$$

Thus $\int V^\mu\, d\mu < A\varepsilon\mu(E_\varepsilon)$ which is a contradiction if $A\varepsilon < 1$. The Kellogg property follows.

Remark. — John L. Lewis has given a different, simpler proof of Theorem 1, obtaining the good-λ inequality

$$|\{x;R_\alpha * \mu(x) > C_1\lambda\}| \leqslant C_2\varepsilon|\{x;R_\alpha * \mu(x) > \lambda\}| + \left|\left\{x;\left(\int_0^\infty (\delta^{\alpha-d}\mu(B_\delta(x)))^p \frac{d\delta}{\delta}\right)^{1/p} > \varepsilon\lambda\right\}\right|$$

where C_1 and C_2 are positive absolute constants and $\lambda > 0$ and $0 < \varepsilon < 1/2$ are arbitrary.

Remark added in April 1983. — Per Nilsson recently observed the inequality

$$A^{-1}(R_\alpha * \mu)^{\#}(x) \leqslant \left(\int_0^\infty (\delta^{\alpha-d}\mu(B_\delta(x)))^p \frac{d\delta}{\delta}\right)^{1/p} \leqslant A(R_\alpha * \mu)(x),$$

where $\#$ denotes the Fefferman-Stein « sharp » operation. Theorem 1 then follows from the Fefferman-Stein inequality for the sharp function. See *Acta Math.* 129 (1972), 137-193, Theorem 5.

Nilsson also noticed that Theorem 1 means that the positive cones in $L^p_{-\alpha}$ and in the Besov space $B_p^{-\alpha,p}$ coincide. See J. Peetre, *New thoughts on Besov spaces*, Duke University, Durham, N. C., 1976, Theorem 4 in Ch. 8.

4. Nonlinear potential theory revisited.

We shall use the dyadic cubes Q and expanded cubes $\tilde{Q}$ of section 3. For any such Q we let φ_Q be a C^∞ function such that $\chi_Q \leqslant \varphi_Q \leqslant \chi_{\tilde{Q}}$. We write $\int \varphi_Q \, d\mu = \mu(\varphi_Q)$. As in section 3 one easily finds that there are constants A, ℓ_1 and ℓ_2 such that for $\mu \in \mathcal{M}^+$

$$A^{-1} \sum_{\ell(Q) \leqslant \ell_1} \left(\frac{\mu(\varphi_Q)}{\ell(Q)^{d-\alpha q}} \right)^{p-1} \varphi_Q(x) \leqslant W^\mu_{\alpha,q}(x) \leqslant A \sum_{\ell(Q) \leqslant \ell_2} \left(\frac{\mu(\varphi_{\tilde{Q}})}{\ell(Q)^{d-\alpha q}} \right)^{p-1} \varphi_Q(x).$$

We set $\mathscr{W}^\mu_{\alpha,q}(x) = \sum_{\ell(Q) \leqslant 1} \left(\frac{\mu(\varphi_Q)}{\ell(Q)^{d-\alpha q}} \right)^{p-1} \varphi_Q(x)$, and

$$\int \mathscr{W}^\mu_{\alpha,q} \, d\mu = \mathscr{I}(\mu).$$

The reason for introducing the functions φ_Q is that in this way we get a lower semicontinuous $\mathscr{W}^\mu$. By Theorem 1 we have

$$A^{-1} \int \mathscr{W}^\mu_{\alpha,q} \, d\mu \leqslant \int V^\mu_{\alpha,q} \, d\mu = \int (G_\alpha * \mu)^p \, dx \leqslant A \int \mathscr{W}^\mu_{\alpha,q} \, d\mu .$$

In the classical potential theory of Gauss, Frostman, H. Cartan, etc., the energy of a measure μ plays a fundamental role. In our situation, where we use Bessel potentials, the energy of a measure μ in $\mathcal{M}^+ \cap L^2_{-\alpha}$ is

$$I(\mu) = \int G_{2\alpha} * \mu \, d\mu = \int (G_\alpha * \mu)^2 \, dx .$$

In the nonlinear situation, the generalized energy,

$$I(\mu) = \int V^\mu_{\alpha,q} \, d\mu = \int (G_\alpha * \mu)^p \, dx$$

plays a similar part, although the difficulties mentioned in the introduction arise. Here we shall show that using the equivalent functional $\mathscr{I}(\mu) = \int \mathscr{W}^{\mu}_{\alpha,q}\, d\mu$ as the energy one can develop nonlinear potential theory in a way which completely parallels the classical theory as exposed in e.g. Carleson [8, ch. III] or Landkof [23, ch. II]. In particular the problems associated with the number $q = 2 - \dfrac{\alpha}{d}$ disappear.

This programm is carried out in a series of propositions. These may be of some interest in themselves, but our motivation is that they lead to proofs of Theorems 2, 3 and 4, and we carry the theory as far as we need to achieve that goal.

In what follows α and q will be fixed, and we usually drop those indices.

We have $\mathscr{I}(\mu) = \int \mathscr{W}^{\mu}\, d\mu = \sum_{\ell(Q)\leqslant 1} \ell(Q)^{(d-\alpha q)(1-p)}\mu(\varphi_Q)^p$, which we abbreviate by writing

$$\mathscr{I}(\mu) = \sum_{\ell(Q)\leqslant 1} a_Q \mu(\varphi_Q)^p .$$

For a compact K we now define the capacity $\mathscr{C}_{\alpha,q}(K) = \mathscr{C}(K)$ by

$$\mathscr{C}(K)^{1/q} = \sup\{\mu(K);\, \mu \in \mathscr{M}^+(K),\, \mathscr{I}(\mu) \leqslant 1\},$$

or equivalently

$$\mathscr{C}(K)^{-1} = \inf\{\mathscr{I}(\mu)^{q-1};\, \mu \in \mathscr{M}_e^+(K)\},$$

where we denote $\{\mu \in \mathscr{M}^+(K);\, \mu(K)=1\} = \mathscr{M}_e^+(K)$.

$\mathscr{C}(\cdot)$ is extended to arbitrary sets as an outer capacity in the usual way :

$$\mathscr{C}(G) = \sup\{\mathscr{C}(K);\, K \text{ compact},\, K \subset G\} \quad \text{if } G \text{ open.}$$
$$\mathscr{C}(E) = \inf\{\mathscr{C}(G);\; G \text{ open},\, G \supset E\}, \quad E \text{ arbitrary.}$$

By Theorem 1, $C_{\alpha,q}$ and $\mathscr{C}_{\alpha,q}$ are equivalent capacities, i.e.

$$A^{-1}\, C_{\alpha,q}(E) \leqslant \mathscr{C}_{\alpha,q}(E) \leqslant A C_{\alpha,q}(E).$$

PROPOSITION 1. — *Let* $K \subset R^d$ *be compact. Then there is a* $\gamma \in \mathscr{M}_e^+(K)$ *which minimizes* $\mathscr{I}(\cdot)$, *so that* $\mathscr{I}(\gamma) = \mathscr{C}_{\alpha,q}(K)^{1-p}$. *Moreover* $\mathscr{W}^{\gamma}_{\alpha,q}(x) \geqslant \mathscr{I}(\gamma)$ $(\alpha,q)-$q.e. *on* K.

Proof. – Only the last assertion needs proof. Suppose $\mathscr{W}^{\gamma}(x) < \mathscr{I}(\gamma)$ on a subset of K with positive capacity. Then there is an $\varepsilon > 0$ and a compact

$$F \subset K \cap \{x; \mathscr{W}^{\gamma}(x) < \mathscr{I}(\gamma) - \varepsilon\}$$

such that $\mathscr{C}(F) > 0$. Let $\tau \in \mathscr{M}_e^+(F)$ and $\mathscr{I}(\tau) < \infty$, and set

$$\mu_\delta = (1-\delta)\gamma + \delta\tau, \qquad 0 \leqslant \delta \leqslant 1,$$

so that $\mu_\delta \in \mathscr{M}_e^+(K)$, and thus $\mathscr{I}(\mu_\delta) \geqslant \mathscr{I}(\gamma)$. This leads to a contradiction. In fact, consider

$$\mathscr{I}(\mu_\delta) = \mathscr{I}(\gamma + \delta(\tau-\gamma)) = \sum_{\ell(Q)\leqslant 1} a_Q(\gamma(\varphi_Q) + \delta(\tau(\varphi_Q) - \gamma(\varphi_Q)))^p.$$

By the mean value theorem

$$(\gamma(\varphi_Q) + \delta(\tau(\varphi_Q) - \gamma(\varphi_Q)))^p$$
$$= \gamma(\varphi_Q)^p + \delta(\tau(\varphi_Q) - \gamma(\varphi_Q))p(\gamma(\varphi_Q) + \xi_Q(\tau(\varphi_Q) - \gamma(\varphi_Q)))^{p-1}, \text{ where } 0 < \xi_Q < \delta.$$

Moreover

$$0 \leqslant \gamma(\varphi_Q) + \xi_Q(\tau(\varphi_Q) - \gamma(\varphi_Q)) \leqslant \gamma(\varphi_Q) + \tau(\varphi_Q).$$

Thus

$$\mathscr{I}(\mu_\delta) = \mathscr{I}(\gamma) + \delta p \Sigma a_Q(\tau(\varphi_Q) - \gamma(\varphi_Q))(\gamma(\varphi_Q) + \xi_Q(\tau(\varphi_Q) - \gamma(\varphi_Q))^{p-1}.$$

Now the sum of the absolute values is majorized by

$$\Sigma a_Q(\gamma(\varphi_Q) + \tau(\varphi_Q))^p \leqslant A\mathscr{I}(\gamma) + A\mathscr{I}(\tau) < \infty.$$

It follows that we can let $\xi_Q \to 0$ in the sum, and obtain

$$\mathscr{I}(\mu_\delta) = \mathscr{I}(\gamma) + \delta p(\Sigma a_Q \tau(\varphi_Q)\gamma(\varphi_Q)^{p-1} - \Sigma a_Q \gamma(\varphi_Q)^p) + o(\delta)$$
$$= \mathscr{I}(\gamma) + \delta p\left(\int \mathscr{W}^{\gamma}\, d\tau - \mathscr{I}(\gamma)\right) + o(\delta) \leqslant \mathscr{I}(\gamma) - \delta p\varepsilon + o(\delta) < \mathscr{I}(\gamma)$$

if δ is small enough.

PROPOSITION 2. – *$\mathscr{W}^{\gamma}_{\alpha,q}(x) \leqslant \mathscr{I}(\gamma)$ everywhere on* $\operatorname{supp}\gamma$.

Proof. – Suppose $\mathscr{W}^{\gamma}(x_0) > \mathscr{I}(\gamma)$. Then $\mathscr{W}^{\gamma}(x) > \mathscr{I}(\gamma)$ on a neighborhood G of x_0. If $x_0 \in \operatorname{supp}\gamma$, then $\gamma(G) > 0$. But by proposition 1, $\mathscr{W}^{\gamma}(x) \geqslant \mathscr{I}(\gamma)$ q.e. on K, and thus a.e. (γ), whence the contradiction $\int \mathscr{W}^{\gamma}\, d\gamma > \mathscr{I}(\gamma)$.

If $\mu \in \mathscr{M}$ is a signed measure, $\mu = \mu_+ - \mu_-$, such that $\mathscr{I}(\mu_+ + \mu_-) < \infty$ we define

$$\mathscr{W}^{\mu}(x) = \sum_{\ell(Q) \leqslant 1} a_Q |\mu(\varphi_Q)|^{p-2} \mu(\varphi_Q)\varphi_Q(x);$$

$$\mathscr{I}(\mu) = \int \mathscr{W}^{\mu}\, d\mu = \sum_{\ell(Q) \leqslant 1} a_Q |\mu(\varphi_Q)|^p .$$

Let $E_\lambda = \{x; \mathscr{W}^{\mu}(x) > \lambda \text{ or } \mathscr{W}^{\mu_+ + \mu_-}(x) = +\infty\}$.

PROPOSITION 3. — *With the above notation* $\mathscr{C}_{\alpha,q}(K) \leqslant \dfrac{1}{\lambda^q}\mathscr{I}(\mu)$ *for any compact* $K \subset E_\lambda$.

Remark. — We have not proved that

$$\mathscr{C}_{\alpha,q}(E_\lambda) = \sup\{\mathscr{C}_{\alpha,q}(K); K \subset E_\lambda, K \text{ compact}\},$$

so we cannot immediately replace K by E_λ in the proposition. But since Choquet's capacitability theorem applies to $C_{\alpha,q}$ (see e.g. [24]), and since $C_{\alpha,q}$ is equivalent to $\mathscr{C}_{\alpha,q}$, we have $\mathscr{C}_{\alpha,q}(E_\lambda) \leqslant \dfrac{A}{\lambda^q}\mathscr{I}(\mu)$. It is of course not difficult to prove that Choquet's theorem applies to $\mathscr{C}_{\alpha,q}$ as well. Cf. [23, ch. II].

Proof. — First let $\mu \in \mathscr{M}^+$, and let σ be a probability measure with $\mathscr{I}(\sigma) < \infty$ and supp $\sigma \subset E_\lambda$. Then

$$\lambda \leqslant \int \mathscr{W}^{\mu}\, d\sigma = \Sigma a_Q \mu(\varphi_Q)^{p-1}\,\sigma(\varphi_Q) < \mathscr{I}(\mu)^{1/q}\mathscr{I}(\sigma)^{1/p},$$

thus
$$\mathscr{I}(\sigma)^{1-q} \leqslant \lambda^{-q}\mathscr{I}(\mu).$$

Choosing σ_K as extremal measures for compact subsets K of the open set E_λ we find

$$\mathscr{C}_{\alpha,q}(E_\lambda) = \sup_{K \subset E_\lambda} \mathscr{C}_{\alpha,q}(K) = \sup_{K \subset E_\lambda} \mathscr{I}(\sigma_K)^{1-q} \leqslant \lambda^{-q}\mathscr{I}(\mu).$$

It follows in particular that $\mathscr{C}_{\alpha,q}(\{x; \mathscr{W}^{\mu}(x) = \infty\}) = 0$.

Now we let μ be a signed measure, and we let $K \subset E_\lambda$. By the above we can assume $\mathscr{W}^{\mu_+ + \mu_-} < \infty$ on K. If σ_K again is the extremal probability measure for K we find in the same way

$$\lambda \leqslant \int \mathscr{W}^\mu \, d\sigma_K = \Sigma a_Q |\mu(\varphi_Q)|^{p-2} \mu(\varphi_Q) \sigma_K(\varphi_Q) \leqslant \mathscr{I}(\mu)^{1/q} \mathscr{I}(\sigma_K)^{1/p},$$

which proves the proposition.

Proposition 4. — *Let* K *be compact. Then*

$$\mathscr{C}_{\alpha,q}(K) = \inf \{\mathscr{I}(\mu); \mu \in \mathscr{M}^+, \mathscr{W}^\mu_{\alpha,q}(x) \geqslant 1 \ (\alpha,q)\text{-q.e. } on \ K\}.$$

Proof. — By the argument in the proof of Proposition 3

$$\mathscr{C}_{\alpha,q}(K) \leqslant \inf \{\mathscr{I}(\mu); \ \mu \in \mathscr{M}^+, \mathscr{W}^\mu(x) \geqslant 1 \text{ q.e. } \ on \ K\}.$$

But the capacitary measure γ_K, normalized so that

$$\gamma_K(K) = \mathscr{C}_{\alpha,q}(K),$$

gives $\mathscr{W}^{\gamma_K}(x) \geqslant 1$ q.e. on K by Proposition 1.

Proposition 5. — *Let* K *be compact. Then*

$$\mathscr{C}_{\alpha,q}(K) = \sup \{\mu(K); \mu \in \mathscr{M}^+(K), \mathscr{W}^\mu_{\alpha,q}(x) \leqslant 1 \quad on \ \operatorname{supp} \mu\}.$$

Proof. — Equality holds for $\mu = \gamma_K$. Suppose $\mu \in \mathscr{M}^+(K)$, $\mathscr{W}^\mu(x) \leqslant 1$ on $\operatorname{supp} \mu$. Then $\mathscr{I}(\mu) = \int \mathscr{W}^\mu \, d\mu \leqslant \mu(K)$, so that

$$\mu(K) \leqslant \int \mathscr{W}^{\gamma_K} \, d\mu = \Sigma a_Q \gamma_K(\varphi)^{p-1} \mu(\varphi_Q) \leqslant \mathscr{I}(\gamma_K)^{1/q} \mathscr{I}(\mu)^{1/p}$$
$$\leqslant \mathscr{C}_{\alpha,q}(K)^{1/q} \mu(K)^{1/p},$$

and
$$\mu(K) \leqslant \mathscr{C}_{\alpha,q}(K).$$

Proposition 6. — *If* $\mu \in \mathscr{M}^+$ *and* $\mathscr{I}(\mu) < \infty$, *then* $\mathscr{W}^\mu_{\alpha,q}$ *is* (α,q)-*quasicontinuous, i.e. for any* $\varepsilon > 0$ *there is an open* G *with* $\mathscr{C}_{\alpha,q}(G) < \varepsilon$ *such that* $\mathscr{W}^\mu|_{G^c}$ *is continuous on* G^c.

Proof. — $\mathscr{W}^\mu(x) = \sum_{\ell(Q) \leqslant 1} a_Q \mu(\varphi_Q)^{p-1} \varphi_Q(x).$

Set

$$\mathscr{W}_n^\mu(x) = \sum_{2^{-n} \leqslant \ell(Q) \leqslant 1} a_Q \mu(\varphi_Q)^{p-1} \varphi_Q(x),$$

so that $\mathscr{W}_n^\mu$ is continuous. Then

$$\int (\mathscr{W}^\mu - \mathscr{W}_n^\mu)\, d\mu = \sum_{\ell(Q) \leqslant 2^{-n}} a_Q \mu(\varphi_Q)^p = \varepsilon_n \to 0 \qquad \text{as} \quad n \to \infty .$$

Let

$$E_\lambda = \{x ; \mathscr{W}^\mu(x) - \mathscr{W}_n^\mu(x) > \lambda\} .$$

Then as in the proof of Proposition 3, $\mathscr{C}_{\alpha,q}(E_\lambda) \leqslant \lambda^{-q} \varepsilon_n$. Let $\varepsilon > 0$, choose $n_j \to \infty$ and $\lambda_j \to 0$ so that $\sum_{j=1}^{\infty} \lambda_j^{-q} \varepsilon_{n_j} < \varepsilon$. Set $G = \bigcup_{j=1}^{\infty} E_{\lambda_j}$, so that G is open and $\mathscr{C}_{\alpha,q}(G) < \varepsilon$. Then, outside G we have $0 \leqslant \mathscr{W}^\mu(x) - \mathscr{W}_{n_j}^\mu(x) \leqslant \lambda_j$ for all j, so $\mathscr{W}_{n_j}^\mu$ converges uniformly to $\mathscr{W}^\mu(x)$, which proves the proposition.

In the next two propositions we extend Proposition 1 to arbitrary sets. Cf. Maz'ja-Havin [24, Theorem 5.5].

PROPOSITION 7. – *Let* G *be open with* $\mathscr{C}_{\alpha,q}(G) < \infty$. *Then there is a measure* $\gamma \in \mathscr{M}_e^+(\bar{G})$ *such that*

$$\mathscr{W}_{\alpha,q}^\gamma(x) \geqslant \mathscr{I}(\gamma) = \mathscr{C}_{\alpha,q}(G)^{1-p} \quad (\alpha,q)-\text{q.e. } on\ G .$$

Remark. – As we shall see later in Theorem 4 the inequality is true everywhere on G.

Proof. – Let $K_n \nearrow G$ be compact sets such that as $n \to \infty$, $\mathscr{C}(K_n)$ tends to $\sup_{K \subset G} \mathscr{C}(K) = \mathscr{C}(G)$, and let $\gamma_n \in \mathscr{M}_e^+(K_n)$ be the corresponding capacitary measures. By weak compactness there is a subsequence $\{\gamma_{n_i}\}$ that converges weakly to $\gamma \in \mathscr{M}^+(\bar{G})$ with $\mathscr{I}(\gamma) \leqslant \liminf \mathscr{I}(\gamma_n)$.

If $m < n$, then

$$1/2\,(\gamma_m + \gamma_n) \in \mathscr{M}_e^+(K_n), \qquad \text{and thus} \quad \mathscr{I}(1/2(\gamma_m + \gamma_n)) \geqslant \mathscr{I}(\gamma_n) .$$

As $m, n \to \infty$, $\mathscr{I}(\gamma_m)$ and $\mathscr{I}(\gamma_n)$ decrease to $\mathscr{C}(G)^{1-p}$. It follows from uniform convexity (Clarkson's inequalities, Clarkson [11]) that

$$\mathscr{I}(\gamma_m-\gamma_n) = \Sigma a_Q|\gamma_m(\varphi_Q) - \gamma_n(\varphi_Q)|^p \to 0.$$

Choosing $\varepsilon > 0$ and letting $n \to \infty$ we find

$$\mathscr{I}(\gamma_m-\gamma) \leqslant \liminf_{n\to\infty} \mathscr{I}(\gamma_m-\gamma_n) < \varepsilon$$

for m large enough, and thus $\lim_{m\to\infty} \mathscr{I}(\gamma_m-\gamma) = 0$. It follows easily from Proposition 3 and the remark following it that a subsequence $\{\mathscr{W}^{\gamma_{m_i}}\}$ converges to $\mathscr{W}^{\gamma}(x)$ q.e., thus that $\mathscr{W}^{\gamma}(x) \geqslant \mathscr{I}(\gamma)$. Then γ has to be a probability measure, and the proposition is proved.

PROPOSITION 8. – *Let* E *be an arbitrary set with* $0 < \mathscr{C}_{\alpha,q}(\mathrm{E}) < \infty$. *Then there is a* $\gamma \in \mathscr{M}_e^+(\bar{\mathrm{E}})$ *such that*

$$\mathscr{W}^{\gamma}_{\alpha,q}(x) \geqslant \mathscr{I}(\gamma) = \mathscr{C}_{\alpha,q}(\mathrm{E})^{1-p} \quad (\alpha,q)-\text{q.e. } on \text{ E}.$$

Moreover $\mathscr{W}^{\gamma}_{\alpha,q}(x) \leqslant \mathscr{I}(\gamma)$ *everywhere on* supp γ.

Proof. – Let G_n, $\mathrm{G}_n \supset \mathrm{E}$, be a decreasing sequence of open sets such that $\overline{\mathrm{G}_{n+1}} \subset \mathrm{G}_n$, $\bar{\mathrm{E}} = \bigcap_1^\infty \overline{\mathrm{G}_n}$, and $\mathscr{C}_{\alpha,q}(\mathrm{G}_n) \to \mathscr{C}_{\alpha,q}(\mathrm{E})$, and let $\gamma_n \in \mathscr{M}_e^+(\overline{\mathrm{G}_n})$ be the corresponding capacitary measures (Proposition 7). If $m < n$ then $\mathscr{I}(\gamma_m) \leqslant \mathscr{I}(\gamma_n)$, and $1/2(\gamma_m+\gamma_n) \in \mathscr{M}_e^+(\overline{\mathrm{G}_m})$, so $\mathscr{I}(1/2(\gamma_m+\gamma_n)) \geqslant \mathscr{I}(\gamma_m)$. Again $\mathscr{I}(\gamma_m-\gamma_n) \to 0$ by Clarkson's inequalities, and there is a $\gamma \in \mathscr{M}^+(\bar{\mathrm{E}})$ such that $\mathscr{I}(\gamma_m-\gamma) \to 0$ as $m \to \infty$. That $\mathscr{W}^{\gamma}(x) \geqslant \mathscr{I}(\gamma)$ q.e. on E and $\mathscr{W}^{\gamma}(x) \leqslant \mathscr{I}(\gamma)$ on supp γ follows easily as before.

PROPOSITION 9. – *Let* E *be an arbitrary set with* $0 < \mathscr{C}_{\alpha,q}(\mathrm{E}) < \infty$. *Let* μ *be the measure constructed in Proposition* 8, *but normalized so that* $\mu(\bar{\mathrm{E}}) = \mathscr{C}_{\alpha,q}(\mathrm{E})$. *Then* $\mu(\mathrm{S}) \leqslant \mathscr{C}_{\alpha,q}(\mathrm{E}\cap\mathrm{S})$ *for any Borel set* S.

Proof. – First suppose E is open. Let $\{\mathrm{F}_n\}_1^\infty$ be compact subsets of E such that the corresponding capacitary measures μ_n with $\mu_n(\mathrm{F}_n) = \mathscr{C}_{\alpha,q}(\mathrm{F}_n)$ converge weakly to μ. Let K be a compact subset of S. Then $\sigma_n = \mu_n|_{\mathrm{K}}$ satisfies $\mathscr{W}^{\sigma_n}(x) \leqslant 1$ on supp $\sigma_n \subset \mathrm{K} \cap \mathrm{F}_n$, so

$$\mu_n(\mathrm{K}) = \sigma_n(\mathrm{K}) \leqslant \mathscr{C}_{\alpha,q}(\mathrm{K}\cap\mathrm{F}_n) \leqslant \mathscr{C}_{\alpha,q}(\mathrm{S}\cap\mathrm{E}).$$

By weak convergence it follows that $\mu(K) \leqslant \mathscr{C}_{\alpha,q}(S\cap E)$. Since S is Borel it follows that $\mu(S) \leqslant \mathscr{C}_{\alpha,q}(S\cap E)$.

Now let E be arbitrary. Let V be an open neighborhood of $S \cap E$ such that $\mathscr{C}_{\alpha,q}(V) < \mathscr{C}_{\alpha,q}(S\cap E) + \varepsilon$, for some $\varepsilon > 0$. Let $\{G_n\}$ be open sets containing E such that $\mathscr{C}_{\alpha,q}(G_n) \to \mathscr{C}_{\alpha,q}(E)$ and such that $S \cap G_n \subset V$. Let μ_n be the capacitary measures for G_n, and suppose $\mu_n \to \mu$ weakly. By the above $\mu_n(S) \leqslant \mathscr{C}_{\alpha,q}(S\cap G_n) \leqslant \mathscr{C}_{\alpha,q}(V) < \mathscr{C}_{\alpha,q}(S\cap V) + \varepsilon$. The lemma follows.

PROPOSITION 10. — *Let* E *be an arbitrary set. Let* $x_0 \in e_{\alpha,q}(E) \cap \bar{E}$, *let* $\varepsilon > 0$, *and suppose that* $\mathscr{C}_{\alpha,q}(E\cap G) > 0$ *for all neighborhoods* G *of* x_0. *Then for every sufficiently small* G *the (normalized) capacitary measure* μ *for* $E \cap G$ *satisfies* $\mathscr{W}^{\mu}_{\alpha,q}(x_0) < \varepsilon$ *and* $\mathscr{W}^{\mu}_{\alpha,q}(x) \geqslant 1$ (α,q)-q.e. *on* $E \cap G$.

Proof. — By definition $\mathscr{W}^{\mu}(x_0) \leqslant A \sum_{n=0}^{\infty} (2^{n(d-\alpha q)}\mu(B_{2^{-n}}(x_0)))^{p-1}$. Choose $\varepsilon' > 0$ and choose N so large that

$$\sum_{n=N}^{\infty} (2^{n(d-\alpha q)}\mathscr{C}_{\alpha,q}(E\cap B_{2^{-n}}(x_0)))^{p-1} < \varepsilon' .$$

It follows from Proposition 9 that if $G \subset B_{2^{-N}}(x_0)$, then

$$\mathscr{W}^{\mu}(x_0) \leqslant A\left(\sum_{n=0}^{N-1} 2^{n(d-\alpha q)(p-1)}\right)\mathscr{C}_{\alpha,q}(E\cap B_{2^{-N}}(x_0))^{p-1} + A\varepsilon' \leqslant 2A\varepsilon' ,$$

if N is large enough. Here we have used the observation that convergence of the series

$$\sum_{n=0}^{\infty} (2^{n(d-\alpha q)}\mathscr{C}_{\alpha,q}(E\cap B_{2^{-n}}(x_0)))^{p-1}$$

implies that $\lim_{N\to\infty} (2^{N(d-\alpha q)}\mathscr{C}_{\alpha,q}(E\cap B_{2^{-N}}(x_0)))^{p-1} = 0$ in case $\alpha q < d$, and that $\lim_{N\to\infty} N\mathscr{C}_{\alpha,q}(E\cap B_{2^{-N}}(x_0))^{p-1} = 0$ in case $\alpha q = d$.

Proof of Theorem 3. — We can now copy Choquet's original proof [9]. See also [17]. Let $E \subset \mathbf{R}^d$, and let $\{O_n\}_1^{\infty}$ be an enumeration of all rational balls (i.e. $O_n = \{x; |x-x_n|<r_n\}$ with x_n and r_n rational) that intersect E. Let

$\mathcal{W}_n$ be the capacitary potential for $E \cap O_n$, if $\mathcal{C}_{\alpha,q}(E \cap O_n) > 0$, so that $\mathcal{W}_n(x) \geqslant 1$ q.e. on $E \cap O_n$, and set $A_n = \{x \in \bar{E} \cap O_n ; \mathcal{W}_n(x) < 1\}$. Set $A_n = \bar{E} \cap O_n$ if $\mathcal{C}_{\alpha,q}(E \cap O_n) = 0$. Then, by Proposition 10,

$$e_{\alpha,q}(E) \subset (\operatorname{ext} E) \cup \left(\bigcup_1^\infty A_n \right).$$

(We shall see below in Theorem 4 that equality holds.)

Let $\varepsilon > 0$. Since $\mathcal{W}_n(x) \geqslant 1$ q.e. on $E \cap O_n$ and since $\mathcal{W}_n$ is quasicontinuous, there is an open ω_n, $\mathcal{C}_{\alpha,q}(\omega_n) < \varepsilon 2^{-n}$, such that $\mathcal{W}_n|_{\omega_n^c}$ is continuous on ω_n^c, and $\mathcal{W}_n(x) \geqslant 1$ on $E \cap O_n \cap \omega_n^c$. Set $F = E \cap \left(\bigcap_{n=1}^\infty \omega_n^c \right)$, so that $\bar{F} = \bar{E} \cap \left(\bigcap_{n=1}^\infty \omega_n^c \right)$. Then $G = (\bar{F})^c$ has the required properties. In fact, $\mathcal{W}_n$ is continuous on $\bar{F}$ and $\mathcal{W}_n(x) \geqslant 1$ on $F \cap O_n$, so it follows that $\mathcal{W}_n(x) \geqslant 1$ on $\bar{F} \cap O_n$, and thus $\bar{F} \cap A_n = \varnothing$. Thus $e(E) \subset (\bar{F})^c$. Moreover

$$(\bar{F})^c \cap E \subset F^c \cap E \subset \bigcup_1^\infty \omega_n, \qquad \text{so} \qquad \mathcal{C}_{\alpha,q}((\bar{F})^c \cap E) < \varepsilon, \text{ q.e.d.}$$

In order to prove Theorem 4 we need one more proposition. Since we are interested not just in measures μ with bounded $\mathcal{W}^\mu$ or finite $\mathcal{I}(\mu)$, we have to replace Proposition 3 with the following.

PROPOSITION 11. — *Let* $\mu \in \mathcal{M}^+$. *Then*

$$\mathcal{C}_{\alpha,q}(\mathcal{W}_{\alpha,q}^\mu(x) > \lambda\}) \leqslant \frac{A}{\lambda^{q-1}} \mu(1).$$

Remark. — This estimate is false for $V_{\alpha,q}^\mu$ if $1 < q \leqslant 2 - \frac{\alpha}{d}$. In fact, in this case $V_{\alpha,q}^\mu \equiv +\infty$ if μ is a point mass. The estimate was proved for $V_{\alpha,q}^\mu$, $q \geqslant 2$, by Adams-Meyers [2; Prop. 4.4], and extended to $q > 2 - \frac{\alpha}{d}$ by Adams-Hedberg [1; Lemma 4]. For $q = 2$ it is of course classical. See e.g. Carleson [8; Theorem III.5]. The observation that $\mathcal{W}_{\alpha,q}^\mu(x) < \infty$ (α,q)-q.e. is due to Meyers [25; Theorem 2.1] and the proof given here (and in [1]) is just a modification of his proof.

Proof. — It is enough to prove the estimate for $W_{\alpha,q}^\mu$. Let γ be the capacitary measure for a compact subset K of $\{W_{\alpha,q}^\mu > \lambda\}$, normalized so that $\gamma(K) = \mathscr{C}_{\alpha,q}(K)$. To simplify the proof we assume that $K \subset B_{\frac{1}{2}}(0)$. Let $x \in \operatorname{supp} \gamma$, and set $M_\gamma\mu(x) = \sup\limits_{r>0} \dfrac{\mu(B_{r/5}(x))}{\gamma(B_r(x))}$. Then

$$W^\mu_{\alpha,q}(x) = \int_0^1 \left(\frac{\mu(B_r(x))}{r^{d-\alpha q}}\right)^{p-1} \frac{dr}{r} \leqslant AM_\gamma\mu(x)^{p-1}\, W^\gamma_{\alpha,q}(x) \leqslant AM_\gamma\mu(x)^{p-1},$$

since $W^\gamma_{\alpha,q}(x) \leqslant A\mathscr{W}^\gamma_{\alpha,q}(x)$, and $\mathscr{W}^\gamma_{\alpha,q}(x) \leqslant 1$ on $\operatorname{supp}\gamma$. Thus

$$\operatorname{supp}\gamma \subset \{AM_\gamma\mu(x)^{p-1} > \lambda\} = \left\{M_\gamma\mu(x) > \left(\frac{\lambda}{A}\right)^{q-1}\right\}.$$

By a well-known covering lemma (see e.g. Stein [26; Ch. I. 1.6]) one can cover $\operatorname{supp}\gamma$ with a union of balls $B_i = B_{r_i}(x_i)$ so that the balls $\frac{1}{5}B_i = B_{r_i/5}(x_i)$ are disjoint, and

$$\frac{\mu(1/5\ B_i)}{\gamma(B_i)} > \left(\frac{\lambda}{A}\right)^{q-1}.$$

It follows that

$$\mathscr{C}_{\alpha,q}(K) = \gamma(K) \leqslant \sum_i \gamma(B_i) < \left(\frac{A}{\lambda}\right)^{q-1} \sum_i \mu\left(\frac{1}{5}B_i\right) \leqslant \left(\frac{A}{\lambda}\right)^{q-1} \mu(1),$$

q.e.d.

Remark. — One can avoid having recourse to the balls $\frac{1}{5}B_i$ by appealing to the deeper covering theorem of Besicovitch [4]. What we have proved is enough for the proof of Theorem 4.

Proof of Theorem 4. — In one direction the theorem follows from Proposition 10, and (in case $C_{\alpha,q}(E \cap G) = 0$ for all neighborhoods G of x_0) from the observation that if E is thin at x_0, then there is an open set containing $E\backslash\{x_0\}$ which is also thin at x_0.

The proof of the converse follows Adams-Meyers [2; Theorem 5.1]. See also Frostman [12]. (The proof given in [17] requires W^μ to be bounded.)

Let $x_0 \in \bar{E} \subset R^d$, and suppose there is a $\mu \in \mathscr{M}^+$ such that

$$\liminf_{x \to x_0,\, x \in E\backslash\{x_0\}} W^\mu_{\alpha,q}(x) - W^\mu_{\alpha,q}(x_0) = \eta > 0.$$

We claim this implies that $x_0 \in e_{\alpha,q}(E)$, i.e.

$$\int_0^1 \delta^{(\alpha q-d)(p-1)} C_{\alpha,q}(E \cap B_\delta(x_0))^{p-1} \delta^{-1}\, d\delta < \infty .$$

We first observe that the assumptions imply that for any $\varepsilon > 0$ and $r > 0$ there is a $\gamma \in \mathscr{M}^+(B(x_0,r))$ such that $W^\gamma(x_0) < \varepsilon$ and

$$\liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} W^\gamma(x) \geqslant 1 .$$

To see this let $r > 0$ and set $\mu_r = \mu|_{B_r(x_0)}$. Let $|x-x_0| < \frac{r}{2}\cdot$ Then $\mu_r(B_\delta(x)) = \mu(B_\delta(x))$ if $\delta < \frac{r}{2}$, so

$$W^\mu(x) - W^{\mu_r}(x) = \int_{r/2}^1 \frac{\mu(B_\delta(x))^{p-1} - \mu_r(B_\delta(x))^{p-1}}{\delta^{(d-\alpha q)(p-1)}} \frac{d\delta}{\delta},$$

which is a continuous function of x. Thus

$$\liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} W^{\mu_r}(x) - W^{\mu_r}(x_0) = \liminf_{x \to x_0,\, x \in E\setminus\{x_0\}} W^\mu(x) - W^\mu(x_0) = \eta .$$

On the other hand $W^{\mu_r}(x_0) \to 0$ as $r \to 0$, so we can choose r so small that $W^{\mu_r}(x_0) < \eta\varepsilon$. Now choose $\gamma = \eta^{1-q}\mu_r$.

With this γ we let $\rho < r$ and set $\gamma_\rho = \gamma|_{B_\rho(x_0)}$, $\gamma = \gamma_\rho + \gamma'_\rho$. For $|x-x_0| < \frac{\rho}{2}$ we have

$$\begin{aligned} W^{\gamma'_\rho}(x) &\leqslant \int_{\rho/2}^1 \left(\frac{\gamma'_\rho(B_\delta(x))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} \leqslant \int_{\rho/2}^{2r} \left(\frac{\gamma(B_\delta(x))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} + AW^\gamma(x_0) \\ &\leqslant \int_{\rho/2}^{2r} \left(\frac{\gamma(B_{2\delta}(x_0))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} + AW^\gamma(x_0) \\ &= A\int_\rho^{4r} \left(\frac{\gamma(B_\delta(x_0))}{\delta^{d-\alpha q}}\right)^{p-1} \frac{d\delta}{\delta} + AW^\gamma(x_0) \leqslant AW^\gamma(x_0) \leqslant A\varepsilon . \end{aligned}$$

But $W^\gamma(x) \leqslant A_1(W^{\gamma_\rho}(x) + W^{\gamma'_\rho}(x))$, so on $B_{\rho/2}(x_0) \cap E$ we have $W^{\gamma_\rho}(x) \geqslant \frac{1}{A_1} W^\gamma(x) - W^{\gamma'_\rho}(x) \geqslant \frac{1}{2A_1} - A\varepsilon \geqslant \frac{1}{4A_1}$ if ρ and ε are small enough. By Proposition 11

$$\mathscr{C}_{\alpha,q}(B_{\rho/2}(x_0)\cap E) \leqslant A\gamma_\rho(1) = A\gamma(B_\rho(x_0)).$$

This proves the theorem.

BIBLIOGRAPHY

[1] D. R. ADAMS and L. I. HEDBERG, Inclusion relations among fine topologies in non-linear potential theory, *Indiana Univ. Math. J.*, to appear.

[2] D. R. ADAMS and N. G. MEYERS, Thinness and Wiener criteria for non-linear potentials, *Indiana Univ. Math. J.*, 22 (1972), 169-197.

[3] T. BAGBY, Quasi topologies and rational approximation, *J. Funct. Anal.*, 10 (1972), 259-268.

[4] A. S. BESICOVITCH, A general form of the covering principle and relative differentiation of additive functions I, II. *Proc. Cambridge Philos. Soc.*, 41 (1945), 103-110, *ibid.*, 42 (1946), 1-10.

[5] M. BRELOT, Sur les ensembles effilés, *Bull. Sci. Math.*, 68 (1944), 12-36.

[6] M. BRELOT, On topologies and boundaries in potential theory, *Lecture Notes in Math.*, 175, Springer Verlag 1971.

[7] A. P. CALDERÓN, Lebesgue spaces of differentiable functions and distributions, *Proc. Symp. Pure Math.*, 4 (1961), 33-49.

[8] L. CARLESON, *Selected problems on exceptional sets*, Van Nostrand, 1967.

[9] G. CHOQUET, Sur les points d'effilement d'un ensemble. Application à l'étude de la capacité, *Ann. Inst. Fourier*, Grenoble, 9 (1959), 91-101.

[10] G. CHOQUET, Convergence vague et suites de potentiels newtoniens, *Bull. Sci. Math.*, 99 (1975), 157-164.

[11] J. A. CLARKSON, Uniformly convex spaces, *Trans. Amer. Math. Soc.*, 40 (1936), 396-414.

[12] O. FROSTMAN, Les points irréguliers dans la théorie du potentiel et le critère de Wiener, *Kungl. Fysiogr. Sällsk. i Lund Förh.*, 9-2 (1939), 1-10.

[13] B. FUGLEDE, Quasi topology and fine topology, *Séminaire de Théorie du Potentiel*, 10 (1965-1966), no. 12.

[14] B. FUGLEDE, The quasi topology associated with a countably additive set function, *Ann. Inst. Fourier*, Grenoble, 21-1 (1971), 123-169.

[15] K. HANSSON, Imbedding theorems of Sobolev type in potential theory, *Math. Scand.*, 45 (1979), 77-102.

[16] V. P. HAVIN, Approximation in the mean by analytic functions, *Dokl. Akad. Nauk SSSR*, 178 (1968), 1025-1028.

[17] L. I. HEDBERG, Non-linear potentials and approximation in the mean by analytic functions, *Math. Z.*, 129 (1972), 299-319.

[18] L. I. HEDBERG, Two approximation problems in function spaces, *Ark. Mat.*, 16 (1978), 51-81.

[19] L. I. HEDBERG, Spectral synthesis and stability in Sobolev spaces, in Euclidean harmonic analysis (Proc., Univ. of Maryland, 1979), *Lecture Notes in Math.*, 779, 73-103, Springer Verlag 1980.

[20] L. I. HEDBERG, Spectral synthesis in Sobolev spaces, and uniqueness of solutions of the Dirichlet problem, *Acta Math.*, 147 (1981), 237-264.

[21] L. I. HEDBERG, On the Dirichlet problem for higher order equations, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund* (Chicago 1981), 620-633. Wadsworth, 1983.

[22] T. KOLSRUD, A uniqueness theorem for higher order elliptic partial differential equations, *Math. Scand.*, 51 (1982), 323-332.

[23] N. S. LANDKOF, *Foundations of modern potential theory*, Nauka, Moscow 1966. (English translation, Springer-Verlag 1972).

[24] V. G. MAZ'JA and V. P. HAVIN, Non-linear potential theory, *Uspehi Mat. Nauk*, 27-6 (1972), 67-138.

[25] N. G. MEYERS, Continuity properties of potentials, *Duke Math. J.*, 42 (1975), 157-166.

[26] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

Manuscrit reçu le 2 août 1982.

Th. H. WOLFF,
Department of Mathematics 253-37
California Institute of Technology
Pasadena, CA 91125 (USA).

L. I. HEDBERG,
Department of Mathematics
University of Stockholm
Box 6701
S-11385 Stockholm (Sweden).