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CLASSIFICATION OF NASH MANIFOLDS

by Masahiro SHIOTA

1. Introduction.

In this paper we show when two Nash manifolds are Nash diffeomorphic. A semi-algebraic set in a Euclidean space is called a *Nash manifold* if it is an analytic manifold, and an analytic function on a Nash manifold is called a *Nash function* if the graph is semi-algebraic. We define similarly a Nash mapping, a Nash diffeomorphism, a Nash manifold with boundary, etc. It is natural to ask a question whether any two C^∞ diffeomorphic Nash manifolds are Nash diffeomorphic. The answer is negative. We give a counter-example in Section 5. The reason is that Nash manifolds determine uniquely their "boundary". In consideration of the boundaries, we can classify Nash manifolds by Nash diffeomorphisms as follows. Let M , M_1 , M_2 denote Nash manifolds.

THEOREM 1. — *There exist a compact real non-singular affine algebraic set X , a non-singular algebraic subset Y of X of codimension 1, and a union M' of connected components of $X-Y$ such that M is Nash diffeomorphic to M' and that the closure \bar{M}' of M' is a Nash manifold with boundary Y . Here Y is empty if M is compact.*

In the above we call \bar{M}' a *compactification* of M .

THEOREM 2. — *Let N_1, N_2 be any respective compactifications of M_1, M_2 . Then the following are equivalent.*

- (i) M_1 and M_2 are Nash diffeomorphic.
- (ii) N_1 and N_2 are Nash diffeomorphic.
- (iii) N_1 and N_2 are C^∞ diffeomorphic.

By the h -cobordism theorem [5] we have

COROLLARY 3. — *Assume that M_1 and M_2 are C^∞ diffeomorphic, that the dimension of M_1 is not 3,4 nor 5, and that if $\dim M_1 \geq 6$, for any compact subset A of M_1 there exists a compact subset $A' \supset A$ of M_1 such that $M_1 - A'$ is simply connected. Then M_1 and M_2 are Nash diffeomorphic.*

The correspondence $M \rightarrow$ the compactification of M shows the following.

COROLLARY 4. — *The Nash diffeomorphism classes of all Nash manifolds are in (1-1)-correspondence with the C^∞ diffeomorphism classes of all C^∞ compact manifolds with or without boundary.*

The next corollaries may be useful when we consider Nash manifolds and Nash functions.

COROLLARY 5. — *Let $M_1 \supset M'_1, M_2$ be Nash manifolds and a compact Nash submanifold. Let $f: M_1 \rightarrow M_2$ be a C^∞ mapping such that $f|_{M'_1}$ is a Nash mapping. Then we can approximate f by Nash mappings fixing on M'_1 in the compact-open C^∞ topology.*

COROLLARY 6. — *Assume that M is compact and contained in \mathbb{R}^n . Then there exist Nash functions f_1, \dots, f_p on \mathbb{R}^n such that the common zero point set of f_1, \dots, f_p is M and that $\text{grad } f_1, \dots, \text{grad } f_p$ on M span the normal bundle of M in \mathbb{R}^n .*

2. Preparation.

See [3] for the fundamental properties of semi-algebraic sets.

LEMMA 7. — *Let $M \subset \mathbb{R}^n$ be a Nash manifold. Then there exists a Nash tubular neighborhood U of M in \mathbb{R}^n , (i.e. U is a Nash*

manifold and the orthogonal projection $p : U \rightarrow M$ is a Nash mapping).

Proof. — Let \overline{M} be the Zariski closure of M in \mathbf{R}^n . Let $\text{Sing}(\overline{M})$ denote the set of singular points of \overline{M} . Then $M\text{-Sing}(\overline{M})$ is open and dense in M . Consider the normal bundle

$$N = \{(x, y) \in M \times \mathbf{R}^n \mid y \text{ is a normal vector of } M \text{ at } x \text{ in } \mathbf{R}^n\}.$$

Then clearly N is an analytic manifold. Moreover N is semi-algebraic. The reason is the following. We define the normal bundle \tilde{N} of $\overline{M}\text{-Sing}(\overline{M})$ in the same way. Since \tilde{N} is an algebraic subset of $(\overline{M}\text{-Sing}(\overline{M})) \times \mathbf{R}^n$, $\tilde{N} \cap (M \times \mathbf{R}^n)$ is semi-algebraic. The equality

$$\tilde{N} \cap (M \times \mathbf{R}^n) = N \cap ((M\text{-Sing}(\overline{M})) \times \mathbf{R}^n)$$

and the dense property of $M\text{-Sing}(\overline{M})$ in M imply that N is the topological closure of $\tilde{N} \cap (M \times \mathbf{R}^n)$ in $M \times \mathbf{R}^n$. Hence N is semi-algebraic.

The mapping $q : M \ni (x, y) \rightarrow x + y \in \mathbf{R}^n$ is obviously of Nash class. Let E_1 be the set of critical points of the mapping $q \times q : N \times N \rightarrow \mathbf{R}^n \times \mathbf{R}^n$. Then $(N \times N) - E_1$ contains

$$\Delta_1 = \{(z_1, z_2) \in N \times N \mid z_1 = z_2 = (x, 0)\}.$$

Let E_2 be the set of all points $(z_1, z_2) \in N \times N$ such that $q(z_1) = q(z_2)$. Then E_2 is a closed semi-algebraic subset of $N \times N$ and contains the diagonal

$$\Delta_2 = \{(z_1, z_2) \in N \times N \mid z_1 = z_2\}.$$

Moreover the topological closure $\overline{E_2 - \Delta_2}$ does not intersect with Δ_1 because of the existence of C^∞ tubular neighborhoods of M . Hence $E_1 \cup \overline{(E_2 - \Delta_2)}$ is a closed semi-algebraic subset of $N \times N$ which does not intersect with Δ_1 .

Let φ be a positive continuous function on M defined by

$$\varphi(x) = \text{dist}((x, 0, x, 0), E_1 \cup \overline{(E_2 - \Delta_2)}).$$

It is easy to see that any distance function from a semi-algebraic set is semi-algebraic (i.e. the graph is semi-algebraic). Hence φ is semi-algebraic. Put

$$N' = \{(x, y) \in N \mid 2|y| < \varphi(x)\}.$$

Then N' is an open semi-algebraic subset of N . We want to see that the restriction of q to N' is a Nash diffeomorphism into \mathbf{R}^n . It is trivial that the restriction is an immersion. Assume the existence of points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in N' such that $q(z_1) = q(z_2)$, $z_1 \neq z_2$. Then we have

$$\begin{aligned} & x_1 + y_1 = x_2 + y_2, \\ \left. \begin{aligned} \text{dist}^2((x_1, 0, x_1, 0), (z_1, z_2)) \\ \text{dist}^2((x_2, 0, x_2, 0), (z_1, z_2)) \end{aligned} \right\} &= |x_1 - x_2|^2 + y_1^2 + y_1^2 + y_2^2 \\ & \geq \varphi(x_1)^2, \varphi(x_2)^2, \end{aligned}$$

and
$$2|y_1| < \varphi(x_1), 2|y_2| < \varphi(x_2).$$

It follows that $|x_1 - x_2|^2 = |y_1 - y_2|^2$ and

$$|x_1 - x_2|^2 + y_1^2 + y_2^2 > 4y_1^2, 4y_2^2.$$

Hence $|y_1 - y_2|^2 > y_1^2 + y_2^2$. This is a contradiction. Therefore $q(N')$ is a Nash tubular neighborhood of M in \mathbf{R}^n . The proof is complete.

The following lemma will be used in the proof of Theorem 2, but this may be interesting itself. The case of polynomials on a Euclidean space was treated in Remark 6 in [11].

LEMMA 8. — *Let $M \subset \mathbf{R}^n$ be a Nash manifold closed in \mathbf{R}^n . Let f_1, f_2 be positive proper Nash functions on M . Then there exists a C^∞ diffeomorphism τ of M such that $f_1 \circ \tau$ and f_2 are equal outside a bounded subset of M .*

Proof. — The case where M is compact is trivial. Hence we assume M to be not compact. Let \tilde{f}_1, \tilde{f}_2 be the extension of f_1, f_2 respectively onto a Nash tubular neighborhood U of M defined by $\tilde{f}_i = f_i \circ p$, $i = 1, 2$, where p is the orthogonal projection. Then \tilde{f}_i are Nash functions, since any composition of Nash mappings is of Nash class. We regard $\text{grad } \tilde{f}_i$, $i = 1, 2$ as Nash mappings from U to \mathbf{R}^n also. The restrictions of $\text{grad } \tilde{f}_1$ and $\text{grad } \tilde{f}_2$ to M are vector fields of M . Let the restrictions be denoted by w_1, w_2 respectively. Put

$$B = \{x \in M \mid \langle w_{1x}, w_{2x} \rangle = -|w_{1x}| |w_{2x}|\}.$$

Here \langle , \rangle means the inner product as vectors. Then B is semi-algebraic because of

$$B = M \cap \{x \in U \mid \langle \text{grad } \tilde{f}_1(x), \text{grad } \tilde{f}_2(x) \rangle = -|\text{grad } \tilde{f}_1(x)| |\text{grad } \tilde{f}_2(x)|\}.$$

Obviously B is the set of points x where w_{1x} is zero or w_{2x} is a multiple of $-w_{1x}$ by a real non-negative number.

We will prove by reduction to absurdity that B is bounded. Assume it to be unbounded. As \mathbf{R}^n is Nash diffeomorphic to $S^n - \{a \text{ point } a\}$ by the stereographic projection, we identify them. The germ of B at a is not empty. Hence, considering the germ, we obtain easily an unbounded one-dimensional semi-algebraic set $B' \subset B$ (see [3]). We can assume that B' is a Nash manifold with boundary and Nash diffeomorphic to $[0, \infty)$, because the set of singular points of one-dimensional semi-algebraic set is a semi-algebraic set of dimension 0. Let v be a C^∞ non-singular vector field on B' . Then, by the definition of B , we have

$$vf_1(x) \times vf_2(x) \leq 0 \quad \text{for } x \in B'.$$

On the other hand, any non-constant Nash function defined on $[0, \infty)$ is monotone outside a bounded subset, because the set of critical points is a semi-algebraic set of dimension 0. Hence one of the functions $f_1|_{B'}$ and $f_2|_{B'}$ is monotone decreasing outside a bounded subset. This contradicts the fact that f_1, f_2 are proper and positive.

Let K be a large real number, let φ be a C^∞ function on M such that

$$0 \leq \varphi \leq 1, \quad \varphi = \begin{cases} 0 & \text{for } |x| \leq K^{1/2} \\ 1 & \text{for } |x| \geq (2K)^{1/2}. \end{cases}$$

Put $L = M \cap \{|x| = K^{1/2}\}$, $L' = M \cap \{|x| \geq K^{1/2}\}$,

$$L'' = M \cap \{|x| \geq (2K)^{1/2}\}.$$

For any real $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$, the vector field $w' = c_1 w_1 + c_2 w_2$ is non-singular outside B and satisfies $w'f_1, w'f_2 > 0$ at any point $x \notin B$ such that $c_1 |w_{1x}| = c_2 |w_{2x}|$. Choose K so that $L' \cap B = \emptyset$. Put

$$w = w_1/|w_1| + \varphi w_2/|w_2| \quad \text{on } L'.$$

Then w , w_1 and w_2 are non-singular vector fields on L' . Moreover wf_1 , wf_2 are positive on L' , L'' respectively. It is sufficient to consider the case

$$f_1(x) = x_1^2 + \cdots + x_n^2 \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Since L is a level of f_1 , L is smooth, and w_1 is transversal to L .

On any maximal integral curve of w , f_1 is non-singular and monotone, and the set of values is $[K, \infty)$. Let ψ_t be the local 1 parameter group of transformations of L' defined by w . Then ψ_t is well-defined for $0 \leq t < \infty$. Put

$$\pi'(z, t) = \psi_t(z) \quad \text{for } (z, t) \in L \times [0, \infty).$$

It follows that π' is a diffeomorphism onto L' . The mapping

$$(z, t) \longrightarrow (z, f_1 \circ \pi'(z, t) - K)$$

is a diffeomorphism of $L \times [0, \infty)$. Let $(z, t) \longrightarrow (z, s(z, t))$ be the inverse diffeomorphism. Put

$$\pi(z, t) = \pi'(z, s(z, t)) \quad \text{for } (z, t) \in L \times [0, \infty).$$

Then π is a diffeomorphism from $L \times [0, \infty)$ to L' such that

$$f_1 \circ \pi(z, t) = t + K \quad \text{for } (z, t) \in L \times [0, \infty).$$

By the definition of π and π' we have a positive C^∞ function ρ on L' such that $\pi_* \left(\frac{\partial}{\partial t} \right) = \rho w$.

It follows from $\pi(L \times \{t \geq K\}) = L''$ that

$$\frac{\partial f_2 \circ \pi}{\partial t}(z, t) > 0 \quad \text{for } t \geq K.$$

Hence, for each $x \in L$, the t -function $f_2 \circ \pi(x, t)$ on $[K, \infty)$ is proper and non-singular. Choose real $K' (\geq K)$ so that

$$f_2 \circ \pi(x, t) > K \quad \text{for } (x, t) \in L \times [K', \infty).$$

Then we have a C^∞ function f_3 on $L \times [0, \infty)$ such that $f_3(x, t)$, $0 \leq t < \infty$, is C^∞ regular for each fixed $x \in L$, that $f_3(x, t) = t + K$ in a neighborhood of $L \times 0$ and that $f_3(x, t) = f_2 \circ \pi(x, t)$ for $(x, t) \in L \times [K', \infty)$. It follows that $(x, t) \longrightarrow (x, f_3(x, t) - K)$ is a diffeomorphism of $L \times [0, \infty)$. Let $\pi'' : (x, t) \longrightarrow (x, s'(x, t))$ be the inverse. Then we see that

$$f_2 \circ \pi \circ \pi''(x, t) = t + K \quad \text{if} \quad s'(x, t) \geq K'.$$

Hence

$$f_1 \circ \pi = f_2 \circ \pi \circ \pi'' \quad \text{if} \quad s'(x, t) \geq K'.$$

Since $s'(x, t) = t$ in a neighborhood of $L \times 0$, we can extend $\pi \circ \pi''^{-1} \circ \pi^{-1}$ onto M so that the extension τ is the identity on $M - L'$. Then $f_1 \circ \tau = f_2$ outside a bounded set. Hence Lemma is proved.

3. Proofs of Theorems 1, 2.

For the sake of brevity we assume that M, M_1 and M_2 are connected. We also assume that the manifolds are not compact, because the other case is well-known. Let n' be the dimension of the manifolds. Let $G_{m,m'}$ denote the Grassmann manifold of m -linear subspaces in $\mathbb{R}^{m+m'}$. Put

$$E_{m,m'} = \{(\lambda, x) \in G_{m,m'} \times \mathbb{R}^{m+m'} \mid x \in \lambda\}.$$

Then $G_{m,m'}$ has naturally affine non-singular algebraic structure [7].

Let $\partial M'$ denote $\bar{M}' - M'$ if M' is a manifold contained in \mathbb{R}^n and the usual boundary if M' is a compact manifold with boundary.

Proof of Theorem 1. - (1) First we reduce the problem to the case in which there exist a real compact non-singular algebraic set $X \subset \mathbb{R}^n$ and an algebraic subset Z of X satisfying the following conditions, (this was shown in the proof of Proposition 1 in [9]).

(i) M is a connected component of $X - Z$.

(ii) For every point $a \in Z$, there exists a smooth rational mapping ζ from X to $\mathbb{R}^{n''}$ for some integer $n'' \leq n'$ such that $\zeta(a) = 0$, that

$$Z \left\{ \begin{array}{c} = \\ \subset \end{array} \right\} \zeta^{-1}(\{(x_1, \dots, x_{n''}) \in \mathbb{R}^{n''} \mid x_1 \dots x_{n''} = 0\}) \left\{ \begin{array}{c} \text{on } U \\ \text{on } X \end{array} \right.$$

where U is a neighborhood of a in X , and that ζ is a submersion on U . In this case we say that Z has only *normal crossings* at a in X .

Proof. — The boundary ∂M is a closed semi-algebraic set in \mathbf{R}^n . By Lemma 6 in [6], there exists a continuous function η on \mathbf{R}^n such that $\eta^{-1}(0) = \partial M$ and that the restriction of η to $\mathbf{R}^n - \partial M$ is of Nash class, (see the remark after Proposition 1 in [9]). Consider the graph of the restriction of $1/\eta$ to M . Then the graph is closed in $\mathbf{R}^n \times \mathbf{R}$ and Nash diffeomorphic to M . Since \mathbf{R}^{n+1} is Nash diffeomorphic to $S^{n+1} - a\ point$ by the Stereographic projection, we can assume that the Zariski closure \bar{M} in \mathbf{R}^n is compact and that ∂M is a point. Let $\lambda : M' \rightarrow \bar{M}$ be the normalization of \bar{M} (see [7]). Then there exists a Nash manifold M'' open in M' such that the restriction of λ to M'' is Nash diffeomorphic onto M and that M'' is a set of non-singular points of M' . It follows that $\partial M'' \subset \lambda^{-1}(\partial M)$ and that M' is compact because so is \bar{M} . Apply Hironaka's desingularization theorem [2] to M' . Then we have a compact non-singular affine algebraic set X of dimension n' and a smooth rational mapping $\mu : X \rightarrow M'$ such that the restriction of μ to $\mu^{-1}(M'')$ is diffeomorphic onto M'' . Moreover we can suppose that $Z = \mu^{-1}(\lambda^{-1}(\partial M))$ has only normal crossings (Main Theorem II in [2]). This means (ii). As $\partial \mu^{-1}(M'') \subset Z$, $\mu^{-1}(M'')$ is a connected component of $X - Z$. Hence we can assume (i).

(2) Let $p : V \rightarrow X$ be the orthogonal projection of a Nash tubular neighborhood V of X in \mathbf{R}^n . Put

$$Z' = Z \cap \bar{M},$$

$$F = \{(x, y) \in X \times \mathbf{R}^n \mid y \text{ is a normal vector of } X \text{ at } x \text{ in } \mathbf{R}^n\}.$$

Then the projection $F \rightarrow X$ shows that F is the normal bundle of X in \mathbf{R}^n . It is easy to see that F is a non-singular algebraic set. Let $F|_Y$ denote $F \cap Y \times \mathbf{R}^n$, the restriction of the bundle to Y , for any subset Y of X .

We want to show the following. There exist a compact non-singular algebraic set Y in M of codimension 1, a connected component M' of $M - Y$, a polynomial mapping $q : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ and open neighborhoods U_1, U_2 of $Y \times 0 \times 0$ in $F|_Y \times \mathbf{R}$ such that,

$$(i) \quad q(x, 0, 0) = x \quad \text{for } x \in Y,$$

$$(ii) \quad \bar{U}_1 \subset U_2,$$

(iii) $q|_{U_1}$ is a diffeomorphism into \mathbf{R}^n whose image contains $M - M'$,

(iv) $q|_{U_2}$ is an immersion whose image contains $\bar{M} - M'$. Hence we can say that $F|_Y \times \mathbf{R}$ and $q(U_1)$ are the normal bundle of Y in \mathbf{R}^n and a « bent » tubular neighborhood respectively.

Proof. — Let α be the ideal of the smooth rational function ring on X consisting of functions which vanish on Z . Let ξ be the square sum of finite generators of α . Then for every point a of Z , there exists an analytic local coordinate system $(x_1, \dots, x_{n'})$ for X centered at a such that $\xi = x_1^2 \dots x_{n'}^2$ in a neighborhood of a for some n' . Put $Y = \xi^{-1}(\epsilon) \cap M$ for sufficiently small $\epsilon > 0$. Here Y is not necessarily algebraic, so we approximate later it by an algebraic set.

For any point $a \in Z'$, consider the set of all connected components of $M \cap$ (a small ball with center at a). Let T be the disjoint union of the set as a runs on Z' . Hence an element c of T means a pair of a point $\sigma_1(c)$ of Z' and a connected set $\sigma_2(c)$ contained in M . Then T has a topological manifold structure such that $\sigma_1 : T \rightarrow X$ is a topological immersion and that $\sigma_2(c) \cap \sigma_2(c') \neq \emptyset$ for close $c, c' \in T$. Let $v_1 : T \rightarrow \mathbf{R}^n$ be a continuous mapping which satisfies the following conditions. For every point c of T , let $(x_1, \dots, x_{n'})$ be an analytic local coordinate system for X centered at $a = \sigma_1(c)$ such that $\sigma_2(c) = \{x_1 > 0, \dots, x_{n'} > 0\}$, $n'' \leq n'$, in a neighborhood of a . Then $v_1(c)$ is a vector tangent to X at a and satisfies

$$v_1(c) x_i > 0 \quad \text{for } 1 \leq i \leq n'',$$

here we regard $v_1(c)$ as a tangent vector of X at a . This means that $v_1(c)$ points at a point of $\sigma_2(c)$. The existence of v_1 is trivial. Moreover we can assume the following, using a C^∞ partition of unity. For every $c \in T$, there exists a C^∞ vector field $v_2(c)$ on a small neighborhood of $a = \sigma_1(c)$ in \mathbf{R}^n such that $v_2(c)_a = v_1(c)$ and that $v_2(c') = v_2(c'')$ on the common domain of definition for any close $c', c'' \in T$.

Put

$$\sigma'_2(c) = p^{-1} \sigma_2(c) \quad \text{for } c \in T.$$

We remark that $p^{-1}(Z)$ has only analytic normal crossings in V (see [2] for the definition) and that $\sigma'_2(c)$ can be regarded as a connected component of $p^{-1}(M) \cap$ (a small ball with center at $\sigma_1(c)$), because we are concerned with only an arbitrarily small neighborhood of Z' . Consider the restrictions of $v_2(c)$ to $\sigma'_2(c)$ for all $c \in T$. Then the restrictions of $v_2(c)$ and $v_2(c')$ to $\sigma'_2(c) \cap \sigma'_2(c')$ are equal for $c, c' \in T$. Hence we have a C^∞ vector field v_3 on (a neighborhood of Z' in \mathbf{R}^n) $\cap p^{-1}(M)$ such that $v_3 = v_2(c)$ on $\sigma'_2(c)$. By the property of v_1, v_3 is transversal to $p^{-1}(Y)$ for any small $\epsilon > 0$ ($Y = \xi^{-1}(\epsilon) \cap M$).

Fix ϵ . Using the integral curves of v_3 , we obtain a C^∞ imbedding q_1 of a neighborhood U_1 of $Y \times 0 \times 0$ in $F|_1 \times \mathbf{R}$ into \mathbf{R}^n such that $q_1(x, y, 0) = x + y$,

$$\frac{\partial q_1}{\partial t}(x, y, t) = v_{3q_1}(x, y, t) \quad \text{for } (x, y, 0), (x, y, t) \in U,$$

and that $q_1(U_1)$ is equal to (a neighborhood of Z' in \mathbf{R}^n) $\cap p^{-1}(M)$. Here U_1 is chosen so that $(U_1, Y \times 0 \times 0)$ is C^∞ diffeomorphic to $(F|_Y \times \mathbf{R}, Y \times 0 \times 0)$. From these arguments it follows that $M - Y$ has two connected components the closure of one of which does not intersect with ∂M . Let the component be written as M' . Then we can assume that $q_1(U_1)$ contains $M - M'$ and hence that (iii). Let q_2 be a C^∞ extension of q_1 to $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$. Then there exists an open neighborhood U_2 of $Y \times 0 \times 0$ in $F|_Y \times \mathbf{R}$ such that (ii) and (iv) are satisfied.

We need to approximate Y and q_2 by an algebraic set and a polynomial mapping. Since \bar{M}' is a C^∞ manifold with boundary, we have a C^∞ function χ on X such that χ is C^∞ regular on Y and that the zero set of χ is Y . Approximate χ by a smooth rational function in the C^∞ topology, and consider the zero set. If we use the same notation Y for the set, Y is a compact non-singular algebraic set in M of codimension 1. We have no problem to apply the above argument to this Y , because the old Y can be transformed to the new one by a C^∞ diffeomorphism of \mathbf{R}^n arbitrarily close to the identity. By the equality

$$q_2(x, 0, 0) = x \quad \text{for } x \in Y,$$

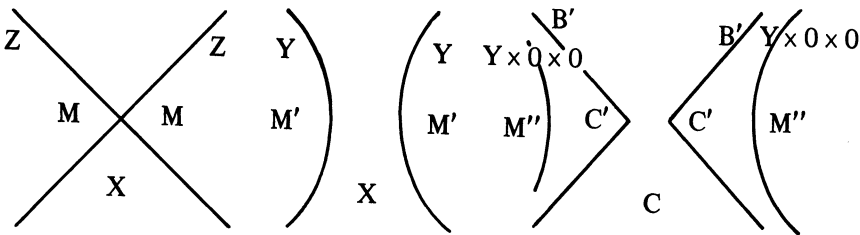
we have polynomial functions ν_1, \dots, ν_k on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and C^∞ mapping $\rho_1, \dots, \rho_k: \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ such that

$$q_2 = \sum_{i=1}^k \nu_i \rho_i + \text{the projection onto the first factor,}$$

and that $\nu_i = 0$ on $Y \times 0 \times 0$. Approximate ρ_i by polynomial mappings ρ'_i in the compact-open C^∞ topology. Then

$$q = \sum_{i=1}^k \nu_i \rho'_i + \text{the projection}$$

is what we wanted. We have to modify U_1, U_2 so that (iii), (iv) remain valid. But this is easy to see, hence we omit it.



(3) By (iii) in (2), q maps diffeomorphically $(q^{-1}(M - M') \cap U_1, Y \times 0 \times 0)$ onto $(M - M', Y)$. The construction of Y and q in (2) shows that $(q^{-1}(M - M') \cap U_1, Y \times 0 \times 0)$ is C^∞ diffeomorphic to $(Y \times (-1, 0], Y \times 0)$. Hence M and M' are C^∞ diffeomorphic. We want to prove that they are Nash diffeomorphic. As it is not easy to prove directly this, we will use an intermediary Nash manifold N which shall be Nash diffeomorphic to M and M' . In (3) we will define a C^∞ manifold M'' whose approximation shall be N .

Let $q' : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be the projection to the first factor. Put $A = F|_Y \times \mathbb{R}$, $S = \text{the critical point set of } q|_{F|_Y \times \mathbb{R}}$,

$$B = \overline{(A \cap q^{-1}(Z)) - S} \quad (\text{where } \overline{\quad} \text{ means the Zariski closure}),$$

$$C = \overline{(A \cap q^{-1}(X)) - S} \quad \text{and} \quad B' = B \cap \bar{U}_1.$$

Then A is a non-singular algebraic set, B and C are algebraic sets of dimension $n' - 1$, n' respectively, and B' is a semi-algebraic set of dimension $n' - 1$. Moreover B has only normal crossings in C at every point of $B \cap U_2$ (see (ii) in (1)), C is non-singular at every point of $C \cap U_2$, and for every point a of B' there exists an algebraic local coordinate system $(x_1, \dots, x_{n'})$ for C centered at a such that

$$B' = \{x_1 = 0, x_2 \geq 0, \dots, x_{n''} \geq 0\} \cup \dots \\ \cup \{x_1 \geq 0, \dots, x_{n''-1} \geq 0, x_{n''} = 0\}$$

in a neighborhood of a for some $n'' \leq n'$ and that

(i) q' maps diffeomorphically $\{x_1 = 0\}, \dots, \{x_{n''} = 0\}$ into Y . We remark that B' is naturally homeomorphic to T in (2). Put

$$C' = q^{-1}(M - \bar{M}') \cap U_1.$$

Then C' is the subdomain of C sandwiched in between B' and $Y \times 0 \times 0$.

We want to find a C^∞ manifold M'' in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and a C^∞ diffeomorphism $\varphi : M'' \rightarrow M$ such that

- (ii) $M'' \supset C'$, $\varphi = q$ on C' , $\bar{M}'' \cap B = \partial M'' = B'$ and $M'' \cap C = C'$.

Proof. — Since q maps $(C' \cup Y \times 0 \times 0, Y \times 0 \times 0)$ diffeomorphically to $(M - M', Y)$, we only have to find a compact C^∞ manifold $M^{(3)}$ with boundary in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and a diffeomorphism $\varphi' : M^{(3)} \rightarrow \bar{M}'$ such that

- (iii) $\partial M^{(3)} = Y \times 0 \times 0$,
- (iv) $q = \varphi'$ on $Y \times 0 \times 0$,
- (v) $M^{(3)} \cap C = Y \times 0 \times 0$, and
- (vi) $M^{(3)} \cup C'$ is a C^∞ manifold.

Let O_ϵ denote the ϵ -neighborhood of $Y \times 0 \times 0$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ for small $\epsilon > 0$. Let χ_i , $i = 1, 2$, be a C^∞ function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$0 \leq \chi_i \leq 1, \quad \chi_i = \begin{cases} 1 & \text{outside } O_{2\epsilon} \\ 0 & \text{in } O_\epsilon \end{cases}$$

and that if $\chi_1(x) \neq 1$ then $\chi_2(x) = 0$. Consider the mapping

$$\varphi'' : O_{3\epsilon} \cap (C - C') \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

defined by

$$\varphi''(z) = (1 - \chi_2(z))(0, z_2, z_3) + \chi_1(z)(q(z) - z_1, 0, 0) + (z_1, 0, 0), \\ z = (z_1, z_2, z_3).$$

Take sufficiently small ϵ . Then, choosing χ_i suitably we see that φ'' is a C^∞ diffeomorphism. It follows that

$$\varphi''((O_{3\epsilon} - O_{2\epsilon}) \cap (C - C')) \subset M' \times 0 \times 0.$$

Put

$$M^{(3)} = (M' - q(O_{3\epsilon} \cap (C - C'))) \times 0 \times 0 \cup \varphi''(O_{3\epsilon} \cap (C - C')).$$

Then $M^{(3)}$ is a compact C^∞ manifold with boundary $Y \times 0 \times 0$ (iii).

Let $\varphi'^{-1} : \bar{M}' \rightarrow M^{(3)}$ be defined by

$$\varphi'^{-1}(x) = \begin{cases} \varphi''(q^{-1}(x) \cap O_{3\epsilon} \cap (C - C')) & \text{if } x \in q(O_{3\epsilon} \cap (C - C')) \\ (x, 0, 0) & \text{otherwise.} \end{cases}$$

Then φ'^{-1} is a C^∞ diffeomorphism such that $\varphi' = q$ in a neighborhood of $Y \times 0 \times 0$ (iv). From $\varphi''(O_\epsilon \cap (C - C')) = O_\epsilon \cap (C - C')$, (vi) follows. For (v), we modify $M^{(3)}$ as follows. Increasing the dimension n if necessary, we can assume that

$$X \subset \mathbf{R}^{n-1} \times 0, \quad \text{and hence } C, M \subset \mathbf{R}^{n-1} \times 0 \times \mathbf{R}^n \times \mathbf{R}.$$

Let χ_3 be a C^∞ function on $M^{(3)} \cup C'$ such that $\chi_3 = 0$ on $Y \times 0 \times 0 \cup C'$ and > 0 on $M^{(3)} - Y \times 0 \times 0$. Consider

$$\{(x_1, \chi_3(x_1, 0, y, t), y, t) \mid (x_1, 0, y, t) \in M^{(3)}\}$$

in place of $M^{(3)}$. Then (v) is satisfied.

(4) Here we will approximate M'' by a Nash manifold N fixing the "boundary". Let L' be a small open semi-algebraic neighborhood of B' in C , and L be the union of M'' and L' such that \bar{L} is a C^∞ manifold with boundary. This is possible since C is non-singular at every point of B' . Let D' be an open tubular neighborhood of L in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, and D be an open semi-algebraic subset of D' containing L . We can choose M'' , L' and D so that $D \cap C$ is a small neighborhood of \bar{C}' in C and that $D \cap B$ is equal to $L' \cap B$ and that B has only normal crossings in C at every point of $D \cap B$. Let $r : D \rightarrow L$ denote the orthogonal projection. Let $h : D \rightarrow E_{m, n^*}$, $m = 2n - n' + 1$, be defined by

$$h(z) = (h_1(z), h_2(z)) =$$

$$\text{(the normal vector space of } L \text{ at } r(z) \text{ in } \mathbf{R}^{2n+1}, z - r(z))$$

for $z \in D$. Then h is a Nash map on $r^{-1}(L')$, and $h_2^{-1}(0) = L$.

Remark 9. - Let $f : M_1 \rightarrow M_2$ be a C^∞ mapping of Nash manifolds. Then we can approximate f by Nash mappings in the compact-open C^∞ topology (this is announced in [9]).

Proof. — By Proposition 1 in [9] there exist a compact non-singular algebraic set $X_1 \subset \mathbf{R}^{n_1}$, a closed semi-algebraic subset B'_1 of X_1 and a union M'_1 of connected components of $X_1 - B'_1$ such that

(i) M_1 is Nash diffeomorphic to M'_1 ,

(ii) for every point x of B'_1 , there exists an analytic local coordinate system $(x_1, \dots, x_{n'_1})$ for X_1 centered at x such that

$$\begin{aligned} (M'_1, B'_1) = & (\{x_1 > 0, \dots, x_{n'_1} > 0\}, \\ & \{x_1 = 0, x_2 \geq 0, \dots, x_{n'_1} \geq 0\} \cup \dots \\ & \cup \{x_1 \geq 0, \dots, x_{n'_1-1} \geq 0, x_{n'_1} = 0\}) \end{aligned}$$

in a neighborhood of x , for some $n'_1 \leq n_1$. Hence we can say that \bar{M}'_1 is a compact analytic manifold with cornered boundary. We assume $M_1 = M'_1$. It follows that $\partial M_1 = B'_1$.

In the same way as (2), we can construct a compact non-singular algebraic set Y_1 in $M_1 \cap$ (an arbitrarily small neighborhood of ∂M_1) and an analytic imbedding $q'_1 : Y_1 \times [-1, 0] \rightarrow X_1$ such that $q'_1(Y_1 \times 0) = Y_1$ and that the image of q'_1 is an arbitrarily small neighborhood of B'_1 . Put

$$M''_1 = q'_1(Y_1 \times [-1, 0]) \cup M_1.$$

Then M''_1 is a compact analytic manifold with boundary containing \bar{M}_1 , and there exists a C^∞ diffeomorphism π of X_1 arbitrarily close to the identity such that $\pi(M''_1) \subset M_1$.

Let M_2 be contained in \mathbf{R}^{n_2} , and p be the orthogonal projection of a Nash tubular neighborhood of M_2 in \mathbf{R}^{n_2} (Lemma 7). Consider $f \circ \pi$ on M''_1 . Then $f \circ \pi$ is extensible to X_1 and hence to \mathbf{R}^{n_1} as a C^∞ mapping to \mathbf{R}^{n_2} . Let η be an extension, and η' be a polynomial approximation of η . Then $f' = \eta'|_{M_1} : M_1 \rightarrow \mathbf{R}^{n_2}$ is an approximation of $f : M_1 \rightarrow \mathbf{R}^{n_2}$. Since the closure of $\pi(M_1)$ in X_1 is compact, we can assume that $f'(M_1)$ is contained in the Nash tubular neighborhood of M_2 . Hence $p \circ f' : M_1 \rightarrow M_2$ is a Nash approximation of f . Thus Remark is proved.

In many cases we want Nash approximation to be fixed on a given semi-algebraic set. So the following are useful.

LEMMA 10. — For any C^∞ function g on D vanishing on $D \cap B$, there exist C^∞ functions $\alpha_1, \dots, \alpha_\varrho$ and Nash functions $\beta_1, \dots, \beta_\varrho$ on D such that

$$g = \alpha_1 \beta_1 + \dots + \alpha_\varrho \beta_\varrho$$

$$\beta_1 = \dots = \beta_\varrho = 0 \quad \text{on} \quad D \cap B.$$

Proof. — Let \mathfrak{p} be the ideal of the smooth rational function ring on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ consisting of functions which vanish on B . Let $\beta_1, \dots, \beta_\varrho$ be a system of generators of \mathfrak{p} . We want to find $\alpha_1, \dots, \alpha_\varrho$ so that the equality in Lemma is satisfied for these β_i, α_i . By a C^∞ partition of unity we only need to see this locally. This is trivial in a neighborhood of any point of $D - B$.

For any point a of $D \cap B$, there exist smooth rational functions $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_{n''}$ with $k = 2n + 1 - n'$ and for some $n'' \leq n'$ such that $\gamma_1, \dots, \gamma_k$ vanish on C , that $\delta_1 \dots \delta_{n''}$ vanishes on B , that

$$B = \{\gamma_1 = \dots = \gamma_k = \delta_1 \dots \delta_{n''} = 0\}$$

in a neighborhood of a and that

$$\gamma_1 \times \dots \times \gamma_k \times \delta_1 \times \dots \times \delta_{n''} : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^{k+n''}$$

is a submersion in a neighborhood of a , since C is non-singular at a , and B has only normal crossings at a in C .

Hence it is sufficient to prove that if

$$D = \mathbf{R}^{k''} = \{(x_1, \dots, x_{k''})\} \quad \text{and}$$

$$B = \{x_1 = \dots = x_k = x_{k+1} \dots x_{k'} = 0\}$$

with $k \leq k' \leq k''$, and if a C^∞ function g on $\mathbf{R}^{k''}$ vanishes on B , then $g = \alpha_1 x_1 + \dots + \alpha_k x_k + \alpha_{k+1} x_{k+1} \dots x_{k'}$ for some C^∞ functions $\alpha_1, \dots, \alpha_{k+1}$. The case when $k = 0$ is trivial. Hence, considering $g(0, \dots, 0, x_{k+1}, \dots, x_{k''})$, we have a C^∞ function α_{k+1} on $\mathbf{R}^{k''}$ such that $g = \alpha_{k+1} x_{k+1} \dots x_{k'}$ on $0 \times \mathbf{R}^{k''-k}$. This implies that $g - \alpha_{k+1} x_{k+1} \dots x_{k'}$ vanishes on $0 \times \mathbf{R}^{k''-k}$. Then the existence of $\alpha_1, \dots, \alpha_k$ which satisfy $g - \alpha_{k+1} x_{k+1} \dots x_{k'} = \alpha_1 x_1 + \dots + \alpha_k x_k$ is well-known. Hence Lemma follows.

LEMMA 11. — *With the same g as in Lemma 10, there exists a Nash function g' on D arbitrarily close to g and vanishing on $D \cap B$.*

Proof. — Using Remark 9, we approximate α_i in Lemma 10 by Nash functions α'_i . Then $g' = \sum_{i=1}^k \alpha'_i \beta_i$ is a Nash approximation of g and vanishes on $B \cap D$.

We continue with the construction of N . By Remark 9 we have a Nash mapping $h'_1 : D \rightarrow G_{m,n'}$, which is an approximation of h_1 . Apply Lemma 11 to h_2 . Then we have a Nash approximation $h'_2 : D \rightarrow \mathbf{R}^{2n+1}$ of h_2 such that $h'_2 = 0$ on $D \cap B$. Let W be a Nash tubular neighborhood of $E_{m,n'}$ in $\mathbf{R}^{n''} \times \mathbf{R}^{2n+1}$ where $G_{m,n'}$ is naturally imbedded in $\mathbf{R}^{n''}$ for some n'' . Let $s : W \rightarrow E_{m,n'}$ be the orthogonal projection. Put

$$h'' = (h'_1, h'_2) = s \circ h' = s \circ (h'_1, h'_2).$$

Then $h'' : D \rightarrow E_{m,n'}$ is a Nash approximation of h , and h''_2 is identical to h_2 on $D \cap B$. Shrinking L and D if necessary, we take this approximation in the uniform C^∞ topology. Put

$$L'' = h''^{-1}(G_{m,n'} \times 0) = h''^{-1}(0).$$

Then there exists a C^∞ diffeomorphism ψ from L'' to L close to the identity such that $\psi = \text{identity}$ on $D \cap B$, because h is transversal to $G_{m,n'} \times 0$ in $E_{m,n'}$. Put $\psi^{-1}(M'') = N$. It follows that L'' is a Nash manifold containing $D \cap B$ and that (\bar{M}'', B') is C^∞ diffeomorphic to (\bar{N}, B') identically on B' . Hence N is the required Nash manifold.

(5) We will prove that M and N are Nash diffeomorphic. Let $\Phi : L \rightarrow X$ be the C^∞ extension of the diffeomorphism $\varphi : M'' \rightarrow M$ to L such that $\Phi(z) = q(z)$ for $z \notin M''$. Let $\Psi : D \rightarrow \mathbf{R}^n$ be a C^∞ extension of $\Phi \circ \psi : L'' \rightarrow X$ to D . Then $\Psi = q$ on $D \cap B$, and $\Psi|_{L''}$ is an immersion. Apply Lemma 11 to $\Psi - q$. Then we obtain a Nash approximation Ψ' of Ψ such that $\Psi' = q$ on $D \cap B$. Compose $\Psi'|_{L''}$ with the orthogonal projection p of a Nash tubular neighborhood of X in \mathbf{R}^n . This is well-defined if we shrink L and D and if the approximation is chosen closely. Then the composed function $\Psi'' : L'' \rightarrow X$ is an approximation of $\Psi|_{L''} = \Phi \circ \psi : L'' \rightarrow X$

such that $\Psi'' = q$ on $D \cap B = L'' \cap B$. Moreover we see $\Psi''(N) = M$ as follows from the facts $\Psi''(B') = q(B') = Z'$, that M is a connected component of $X - Z'$ and that $\Psi''|_N$ is an immersion. It is trivial that $M \cap \Psi''(N)$ is an open subset of M . Assume it to be not closed. Then there exists a convergent sequence of points x_1, x_2, \dots in $\Psi''(N)$ whose limit $x \in M$ is not contained in $\Psi''(N)$. Let z_1, z_2, \dots be points of N such that $\Psi''(z_i) = x_i, i = 1, \dots$. Choosing a subsequence, we can assume that z_1, z_2, \dots converges to $z \in \bar{N}$. Then we have $\Psi''(z) = x$. This is a contradiction. Hence $\Psi''(N) \supset M$. In the same way as above, we see that the set

$$\{x \in M \mid \# \Psi''|_N^{-1}(x) \geq 2\}$$

is empty or equal to M . For any point $x \in M$, if we choose the above approximation closely, this set does not contain x . Hence $\Psi''|_{N \cap \Psi''^{-1}(M)}$ is diffeomorphic onto M . From the same reason it follows that any connected component of $X - Z' - M$ does not contain any point of $\Psi''(N)$, namely that $\Psi''(N) \subset M \cup Z'$. Then $\Psi''(N) \cap Z' = \emptyset$. Hence $\Psi''|_N$ is a Nash diffeomorphism onto M .

(6) Finally we will prove that M' and N are Nash diffeomorphic. For any point $x \in L'' \cap B = D \cap B$, let $C_x^\infty(L'')$ denote the ring of C^∞ function germs at x in L'' . Then the ideal $q_x \subset C_x^\infty(L'')$ of germs vanishing on $L'' \cap B$ is principal because of the normal crossings property of B in $C \cap D$. Moreover we have a polynomial function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ which vanishes on $D \cap B$ and the germ of whose restriction to L'' is a generator of q_x . Choose D so small that the fundamental class of $L'' \cap B$ is mapped to the zero class in $H_{n,-1}(L''; \mathbb{Z}_2)$ by the inclusion map (this is possible since \bar{N} is a compact topological manifold with boundary B' , here we use infinite chain). Then we see easily that the ideal q of $C^\infty(L'')$, the ring of C^∞ functions on L'' , of functions vanishing on $L'' \cap B$ is principal (see Lemma 1 in [12]). Approximate a generator of q by a Nash function ρ by the method of Lemma 11 so that $\rho = 0$ on $L'' \cap B$. Then it follows that ρ generates q . Choose ρ so that $\rho > 0$ on N .

Recall $q' : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, the projection to the first factor. The restriction of q' to B' is homeomorphic onto Y . Let us extend this to \bar{N} so that the extension maps diffeomorphically

N to M' . Let v be the unit normal vector field of Y in X pointing to the interior of M' . Choose small L' . Put

$$\theta(z) = p \circ (q' \circ \psi(z) + \rho(z)v \circ q' \circ \psi(z)) \quad \text{for } z \in \psi^{-1}(L').$$

Here we regard v as a mapping from Y to \mathbb{R}^n , and p is the orthogonal projection of a Nash tubular neighborhood of X in \mathbb{R}^n . This is well-defined because $q'(z) \in Y$ for $z \in L'$. Clearly θ is a Nash mapping.

We can assume that $L' \cap M'' = \{z \in M'' \mid \rho \circ \psi^{-1}(z) < \epsilon\}$ for some $\epsilon > 0$. Let v' be the unit C^∞ vector field on L' the family of whose maximal integral curves consists of $\{q'^{-1}(x) \cap L'\}_{x \in Y}$ and which points into M'' at every point of B' . For any point $a \in B'$, there exists a local analytic coordinate system $z = (z_1, \dots, z_{n'})$ for L' centered at a such that $\rho \circ \psi^{-1}(z) = z_1 \dots z_{n''}$ for some $n'' \leq n'$ and that $M'' = \{z_1 > 0, \dots, z_{n''} > 0\}$ in a neighborhood of a . By (i) in (3), $v'_i z_i > 0, i = 1, \dots, n''$ at a . It follows that $v' \rho \circ \psi^{-1} > 0$ on $L' \cap M''$. Hence v' is transversal to $\{z \in M'' \mid \rho \circ \psi^{-1}(z) = \epsilon'\}$ for some $\epsilon' > 0$. This implies that q' maps $\{z \in M'' \mid \rho \circ \psi^{-1}(z) = \epsilon'\}$ diffeomorphically onto Y . Therefore $\theta|_{N \cap \psi^{-1}(L')}$ is diffeomorphic onto (a C^∞ collar of \bar{M}') $- Y$. The transversality of v' shows also that $(M'' - L', \partial(M'' - L'))$ is diffeomorphic to $(M'' - C', \partial(M'' - C'))$ so that if the diffeomorphism maps a point $z \in \partial(M'' - L')$ to $z' \in \partial(M'' - C')$ we have $q(z) = q(z')$. Hence there exists a diffeomorphism from $(M'' - L', \partial(M'' - L'))$ to (\bar{M}', Y) whose restriction to $\partial(M'' - L')$ is q' . Therefore we extend θ to $\Theta : L'' \rightarrow X$ such that $\Theta|_N$ is a C^∞ diffeomorphism onto M' .

Apply Lemma 11 to $\Theta - q'$, and compose (the approximation mapping $+ q'$) with p . Then we have a Nash approximation $\Theta' : L'' \rightarrow X$ of Θ such that $\Theta' = \Theta$ on $L'' \cap B$. To see that $\Theta'|_N$ is a Nash diffeomorphism onto M' we only need to show the following by the same reason as (5).

- (i) $\Theta'(B') = q'(B') = Y$.
- (ii) M' is a connected component of $X - Y$.
- (iii) $\Theta'|_N$ is an immersion.

(i) and (ii) have been shown already. It is trivial that $\Theta'|_{N-\psi^{-1}(L')}$ is an immersion. Hence we only have to prove the following.

Statement. — Let $\theta_1 : \mathbf{R}^{n'} \rightarrow \mathbf{R}^{n'-1}$ be a submersion, $K \subset \mathbf{R}^{n'}$ be a compact set. Let v_1 be a unit C^∞ vector field on $\mathbf{R}^{n'}$ the family of whose all maximal integral curves consists of $\{\theta_1^{-1}(y)\}_{y \in \mathbf{R}^{n'-1}}$. Put

$$\theta_2(x) = x_1 \dots x_{n''} \quad \text{for } x = (x_1, \dots, x_{n'}) \in \mathbf{R}^{n'}.$$

Assume that $v_1 x_i > 0$, $i = 1, \dots, n'' (\leq n')$. Let (θ'_1, θ'_2) be a C^∞ close approximation of (θ_1, θ_2) such that $\theta'^{-1}_2(0) \supset \theta^{-1}_2(0)$. Then (θ'_1, θ'_2) is an immersion on $\{x_1 > 0, \dots, x_{n''} > 0\} \cap K$.

Proof of Statement. — Since θ'_2 vanishes on

$$\{x_1 = 0\} \cup \dots \cup \{x_{n''} = 0\},$$

there exists a C^∞ function η on $\mathbf{R}^{n'}$ such that $\theta'_2 = \eta \theta_2$. We see easily that η is close to the function 1 (see the statement at p. 268 in [10]). Replacing η by a C^∞ function which is equal to η in a neighborhood of K and is close to 1 in the Whitney C^∞ topology, we can assume that

$$\frac{\partial(\eta x_1)}{\partial x_1} > 0 \quad \text{on } \mathbf{R}^{n'}.$$

Then $\pi : (x_1, \dots, x_{n'}) \rightarrow (\eta x_1, x_2, \dots, x_{n'})$ is a C^∞ diffeomorphism close to the identity. Since $\theta'_2 \circ \pi^{-1} = \theta_2$ and

$$\pi \{x_1 > 0, \dots, x_{n''} > 0\} = \{x_1 > 0, \dots, x_{n''} > 0\},$$

we need to treat only the case where $\theta'_2 = \theta_2$. By the same reason as above, we can assume that θ'_1 is sufficiently close to θ_1 in the Whitney C^∞ topology. Then θ'_1 is a submersion. Let v'_1 be the unit vector field on $\mathbf{R}^{n'}$ the family of whose all maximal integral curves consists of $\{\theta'^{-1}_1(y)\}_{y \in \mathbf{R}^{n'-1}}$ and which is close to v_1 . Then we have $v'_1 x_i > 0$, $i = 1, \dots, n''$. Hence

$$v'_1(x_1 \dots x_{n''}) > 0 \quad \text{on } \{x_1 > 0, \dots, x_{n''} > 0\}.$$

This means that the Jacobian matrix of (θ'_1, θ_2) has the rank n' on $\{x_1 > 0, \dots, x_{n''} > 0\}$. We complete the proofs of Statement and hence of Theorem 1.

Proof of Theorem 2. — Let N_1, N_2 be contained in non-singular algebraic sets $X_1, X_2 \subset \mathbb{R}^n$ respectively so that $\partial N_1 = Y_1$ and $\partial N_2 = Y_2$ are non-singular. The implication (ii) \implies (i) is trivial.

First we will prove (i) \implies (iii). Let φ_1, φ_2 be Nash functions on X_1, X_2 respectively such that $\varphi_i^{-1}(0) = Y_i, \{\varphi_i > 0\} = M_i$ and that φ_i are C^∞ regular at Y_i . The existence of such φ_i follows from the non-singular property of Y_i (see Lemma 1 in [12]) (in fact, we can choose as φ_i polynomial functions). Let Φ_1, Φ_2 be positive proper Nash functions on M_1 defined by

$$\Phi_1 = 1/(\varphi_1|_{M_1}), \quad \Phi_2 = (1/\varphi_2|_{M_2}) \circ \tau,$$

where $\tau : M_1 \longrightarrow M_2$ be a Nash diffeomorphism. Apply Lemma 8 to Φ_1 and Φ_2 . Then there exists a C^∞ diffeomorphism π of M_1 such that Φ_1 and $\Phi_2 \circ \pi$ are equal outside a compact subset of M_1 . Hence we have $\varphi_1 = \varphi_2 \circ \tau \circ \pi$ on (a neighborhood of ∂M_1 in \bar{M}_1) $- \partial M_1$. This means that $\tau \circ \pi$ maps $\{\varphi_1 = \epsilon\}$ to $\{\varphi_2 = \epsilon\}$ for small $\epsilon > 0$. Hence the restriction of $\tau \circ \pi$ on $\{\varphi_1 \geq \epsilon\}$ for small $\epsilon > 0$ is a C^∞ diffeomorphism onto $\{\varphi_2 \geq \epsilon\}$. As $\{\varphi_1 \geq \epsilon\}, \{\varphi_2 \geq \epsilon\}$ are C^∞ diffeomorphic to N_1, N_2 respectively, N_1 and N_2 are C^∞ diffeomorphic.

We prove the inclusion (iii) \implies (ii) in the next general form.

LEMMA 12. — *Let $L_1 \supset L_2, L'_1 \supset L'_2$ be compact Nash manifolds with or without boundary and compact Nash submanifolds. Assume $L_2 = \partial L_1$ if $L_2 \cap \partial L_1 \neq \emptyset$. If there is a C^∞ diffeomorphism from (L_1, L_2) to (L'_1, L'_2) , we can approximate it by Nash one. If the restriction of the given diffeomorphism to L_2 is of Nash class, the approximation can be chosen to take the same image as the diffeomorphism at each point of L_2 .*

Proof. — The idea of the proof is the same as in Proof of Theorem 1, and this proof is easier than that since L_2 and L'_2 are smooth. Hence we give only the sketch. The case where L_1, L'_1 have the boundaries: Consider their doubles L_3, L'_3 , and give them Nash structures [7]. We approximate the natural respective imbeddings of L_1, L'_1 into L_3, L'_3 by Nash mappings. Then we can regard L_1, L'_1 as contained in L_3, L'_3 respectively, and there is a C^∞ diffeomorphism from (L_3, L_1, L_2) to (L'_3, L'_1, L'_2) . If we can

approximate the induced diffeomorphism from $(L_3, \partial L_1 \cup L_2)$ to $(L'_3, \partial L'_1 \cup L'_2)$ by a Nash one, Lemma 12 follows. Hence we can assume that L_1, L'_1 have no boundary. Here we do not necessarily assume that L_2 has the global dimension, namely that the local dimension is constant.

Assume that L_2 is connected for the sake of brevity. Let $\pi : (L_1, L_2) \rightarrow (L'_1, L'_2)$ be a C^∞ diffeomorphism. If $\pi|_{L_2}$ is not of Nash class, by Remark 9 we approximate $\pi|_{L_2}$ by a Nash diffeomorphism $\pi' : L_2 \rightarrow L'_2$. Choose π' very closely. Then we easily find a C^∞ extension $\pi'' : L_1 \rightarrow L'_1$ of π' such that π'' is an approximation of π . Hence, from the beginning we can assume that $\pi|_{L_2}$ is of Nash class. Let L_1, L'_1 be contained in $\mathbb{R}^n, \mathbb{R}^{n'}$ respectively, and $p : \mathbb{R}^n \times \mathbb{R}^{n'} \rightarrow \mathbb{R}^n, p' : \mathbb{R}^n \times \mathbb{R}^{n'} \rightarrow \mathbb{R}^{n'}$ be the projections. Let $L''_2 \subset \mathbb{R}^{n+n'}$ denote the graph of $\pi|_{L_2}$. Then L''_2 is a Nash manifold such that $p|_{L''_2}, p'|_{L''_2}$ are Nash diffeomorphisms onto L_2, L'_2 respectively. By the normalization of the Zariski closure $\overline{L''_2}$ of L''_2 , there exist a non-singular connected component S_2 of an algebraic set in $\mathbb{R}^{n+n'}$ and a linear mapping φ from $\mathbb{R}^{n+n'}$ to $\mathbb{R}^{n+n'}$ such that $\varphi|_{S_2}$ is diffeomorphic onto L''_2 . Increasing n' if necessary, we construct a C^∞ manifold S_1 in $\mathbb{R}^{n+n'}$ and C^∞ diffeomorphisms $\psi : S_1 \rightarrow L_1, \psi' : S_1 \rightarrow L'_1$ such that $S_2 \subset S_1, \psi = p \circ \varphi$ on $S_2, \psi' = p' \circ \varphi$ on S_2 and $\overline{S_2} \cap S_1 = S_2$. Using $E_{m,m'}$ in the same way as Proof (4) of Theorem 1, we reduce S_1 to a Nash manifold. Then we can find in the same way as Proofs (5), (6) Nash approximations $\Psi : S_1 \rightarrow L_1, \Psi' : S_1 \rightarrow L'_1$ of ψ, ψ' such that $\Psi = \psi, \Psi' = \psi'$ on S_2 . Hence $\Psi' \circ \Psi^{-1} : L_1 \rightarrow L'_1$ is a Nash approximation of π such that $\Psi' \circ \Psi^{-1} = \pi$ on L_2 . Lemma is proved.

4. Proofs of Corollaries.

Proof of Corollary 3. — Let N_1, N_2 be the compactifications of M_1, M_2 respectively. By Theorem 2, we only have to prove that N_1 and N_2 are C^∞ diffeomorphic. Let L be a closed C^∞ collar of N_1 . Put $L' = N_1 - L, L'' = \overline{L'} - L'$. Let $\pi : M_1 \rightarrow M_2$ be a C^∞ diffeomorphism. We see easily that $(N_2 - \pi(L')); \partial N_2,$

$\pi(L'')$) is a C^∞ h -cobordism. On the other hand it follows from the assumption that ∂N_2 is simply connected for $\dim M_1 \geq 6$. Hence, by the h -cobordism theorem $N_2 - \pi(L')$ is diffeomorphic to $\partial N_2 \times [0, 1]$. This means that there exists a homeomorphism $\tau : N_1 \rightarrow N_2$ such that $\tau|_L$ and $\tau|_{L' \cup L''}$ are C^∞ diffeomorphic. It is easy to modify τ to be a C^∞ diffeomorphism. Hence Corollary 3 is proved.

Proof of Corollary 4. – The correspondence is trivially injective by Theorems 1, 2.

Surjectivity: Let N be a compact C^∞ manifold with or without boundary. We need to give to N a Nash manifold structure. If N has the boundary, consider the double N' , and regard N as naturally contained in N' . In the other case, put $N' = N$, $\partial N = \emptyset$. Then, by a Theorem in [1], $(N', \partial N)$ is C^∞ diffeomorphic to a pair (an affine non-singular algebraic set, a non-singular algebraic subset). By this diffeomorphism, we give to N' an algebraic structure. Then, since $N - \partial N$ is a union of connected components of $N' - \partial N$, $N - \partial N$ is a Nash manifold. Obviously N is the compactification of $N - \partial N$. Hence Corollary follows.

Proof of Corollary 5. – Let N_1 be the compactification of M_1 . Obviously we can assume that f is extensible to N_1 and hence to the double of N_1 . Consider a Nash manifold structure on the double and a Nash imbedding of M_1 into it. Then, from the beginning we can assume that M_1 is compact. Let M_2 be contained in \mathbf{R}^n , and q be the orthogonal projection of a Nash tubular neighborhood of M_2 in \mathbf{R}^n . Regard f as a mapping to \mathbf{R}^n . If we can approximate f by a Nash mapping $f' : M_1 \rightarrow \mathbf{R}^n$ so that $f = f'$ on M'_1 , then $q \circ f' : M_1 \rightarrow M_2$ is a required Nash approximation of f . Hence it is sufficient to consider the case of $M_2 = \mathbf{R}$. We regard \mathbf{R} as $S^1 - \{a \text{ point } a\}$. Let $L \subset M_1 \times S^1$ be the graph of f . Put $L' = M_1 \times \{b\}$ where b is a point of S^1 . Then there exists a C^∞ diffeomorphism π of $M_1 \times S^1$ such that

$$\pi(x, b) = (x, f(x)) \quad \text{for } x \in M_1.$$

It follows that $\pi(L') = L$ and that $\pi|_{M'_1 \times \{b\}}$ is of Nash class. Apply Lemma 12 to

$$\pi : (M_1 \times S^1, M'_1 \times \{b\}) \rightarrow (M_1 \times S^1, \pi(M'_1 \times \{b\})).$$

Then we obtain a Nash approximation τ of π such that $\tau = \pi$ on $M_1' \times \{b\}$. For every point $x \in M_1$, put $g(x) = p \circ \tau(x, b)$ where $p : M_1 \times S^1 \rightarrow S^1$ is the projection onto the second factor. Then g is what we want.

Proof of Corollary 6. — This corollary follows from Lemma 12 and the fact that \mathbf{R}^n is Nash diffeomorphic to $S^n - \{\text{a point}\}$ and that (S^n, M) is C^∞ diffeomorphic to (an affine algebraic set, a non-singular algebraic subset) [1].

5. An example.

Let W, W' be compact C^∞ manifold with boundary such that the interiors are C^∞ diffeomorphic, but W and W' are not diffeomorphic (see Theorem 3 in [4]). Let X, X' be the doubles of W, W' respectively. We regard W, W' as naturally contained in X, X' respectively. By a theorem in [1] we can assume that $X, X', \partial W$ and $\partial W'$ are all non-singular algebraic sets in \mathbf{R}^n . Let P, P' be polynomials on \mathbf{R}^n such that

$$P^{-1}(0) = \partial W, \quad P'^{-1}(0) = \partial W'.$$

Put

$$Y = \{(x, y) \in X \times \mathbf{R} \mid yP(x) = 1\},$$

$$Y' = \{(x, y) \in X' \times \mathbf{R} \mid yP'(x) = 1\}.$$

Then Y and Y' are C^∞ diffeomorphic non-singular affine algebraic sets, and their compactifications are the disjoint unions of 2 copies $W + W (\subset X + X)$ and $W' + W' (\subset X' + X')$ respectively. Hence, by Theorem 2, Y and Y' are not Nash diffeomorphic. Here it is not essential that Y, Y' are not connected. In fact we can find connected examples.

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