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MOEBIUS-INVARIANT ALGEBRAS IN BALLS

by Walter RUDIN ([1])

1. Introduction.

Throughout this paper, n is a positive integer, $\mathbf{C}^n$ is the vector space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers, with hermitian inner product $\langle z,w \rangle = \Sigma\, z_1 \bar{w}_i$, norm $|z| = \langle z,z \rangle^{1/2}$, and corresponding unit ball

$$B = \{z \in \mathbf{C}^n : |z| < 1\}.$$

C(B) denotes the algebra of all continuous (not necessarily bounded) functions $f : B \to \mathbf{C}$; multiplication is of course pointwise. Equipped with the topology of uniform convergence on compact subsets (the so-called compact – open topology), C(B) is a well-known Fréchet algebra. The term *closed* will always refer to this topology, unless something is said to the contrary.

The group of all one-to-one holomorphic maps of B onto B (the group of all *automorphisms* of B) will be denoted by Aut(B). We also call it the *Moebius group* of B; see § 2.1.

(The dimension n is not mentioned in these notations. This simplifies the writing and should cause no confusion.)

The letter $\mathcal{M}$ refers to Moebius – invariance. More specifically, we shall say that Y is an *$\mathcal{M}$-space* or an *$\mathcal{M}$-algebra* in C(B) if Y is a *closed* subspace or subalgebra of C(B) such that the compositions $f \circ \psi$ belong to Y for all $f \in Y$ and all $\psi \in \mathrm{Aut}(B)$.

For example, H(B), the set of all holomorphic functions with domain

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B, is an $\mathscr{M}$-algebra, and so is conj H(B), the set of all f whose complex conjugate $\bar{f}$ belongs to H(B).

Our main result confirms the conjecture [11; p. 287] that these are essentially the only ones:

Main theorem. — *The only Moebius-invariant closed subalgebras of* C(B) *are*

$$\{0\}, \quad \mathbf{C}, \quad H(B), \quad \text{conj}\, H(B), \quad C(B).$$

Here **C** denotes the constant functions. Analogous results, with spaces such as $C(\bar{B})$, $C_0(B)$, C(S), $L^p(S)$ (where S is the sphere that bounds B) in place of C(B) may be found in [1], [2], [3], [10], [11; Chaps. 12, 13]. The present theorem seems to be new even when $n = 1$, i.e., when B is the unit disc U in **C**.

In outline, the proof is as follows:

Let Y be an $\mathscr{M}$-algebra that contains nonconstant functions. This rules out $\{0\}$ and **C**. Let $Y^{\#}$ consist of all *radial* $f \in Y$; recall that f is radial if $f(z) = f(w)$ whenever $|z| = |w|$. A radial function in B may thus be regarded, in a natural way, as being defined on the half-open interval [0,1). There are two cases:

(I) If $Y^{\#}$ fails to separate points on [0,1) or if there is a point in B where the radial derivative of every $f \in Y^{\#} \cap C^{\infty}$ is 0, it will be proved that every $f \in Y$ is $\mathscr{M}$-harmonic (see § 5.1) and that Y is therefore one of $\{0\}$, **C**, H(B), conj H(B).

(II) In the remaining case, a deep approximation theorem due to Stolzenberg [13], [4], [14] leads to the conclusion that $Y = C(B)$.

The work of Berenstein and Zalcman [5], [6], has been very helpful. Although none of their results are used directly, their papers suggested that spherical means might be the right tool to deal with Case (I).

I am very grateful to Jean-Pierre Rosay for discovering two errors in an earlier version of this paper.

2. Preparation.

2.1. The group Aut(B). — As mentioned in the Introduction, this consists of all holomorphic one-to-one maps of B onto B. It is generated by $\mathscr{U}$ — the compact group of all unitary operators on the Hilbert space

$\mathbf{C}^n$ — and by the involutions φ_a, one for every $a \in \mathrm{B}$, given by

(1) $$\varphi_a(z) = \frac{a - \mathrm{P}_a z - \sqrt{1 - |a|^2}\,\mathrm{Q}_a z}{1 - \langle z,a\rangle}$$

where P_a is the orthogonal projection of $\mathbf{C}^n$ onto the subspace generated by a, and $\mathrm{Q}_a z = z - \mathrm{P}_a z$. Chapter 2 of [11] contains a detailed description of Aut (B). We will use the following facts:

(i) $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a^{-1} = \varphi_a$.

(ii) If $\psi \in \mathrm{Aut(B)}$ and $a = \psi(0)$, then $\psi = \varphi_a \mathrm{U}$ for some $\mathrm{U} \in \mathscr{U}$.

(iii) $|\varphi_a(z)| = |\varphi_z(a)|$. Thus $f(\varphi_a(z)) = f(\varphi_z(a))$ if f is radial.

(iv) Formula (1) shows, for $a \in \mathrm{B}$ and $\mathrm{U} \in \mathscr{U}$, that

$$\mathrm{U}\varphi_a\mathrm{U}^{-1} = \varphi_{\mathrm{U}a}.$$

2.2. Radialization. — If $f \in \mathrm{C(B)}$, its *radialization* is the function $f^\#$ defined by

(1) $$f^\# = \int_{\mathscr{U}} f \circ \mathrm{U}\, d\mathrm{U}$$

where $d\mathrm{U}$ denotes Haar measure on $\mathscr{U}$ (normalized so as to have total mass 1) or, equivalently, by

(2) $$f^\#(z) = \int_{\mathrm{S}} f(|z|\zeta)\, d\sigma(\zeta)$$

where σ is the rotation-invariant probability measure on the unit sphere S, and $z \in \mathrm{B}$.

Note that $\mathrm{U} \to f \circ \mathrm{U}$ is a continuous map of the compact group $\mathscr{U}$ into the Fréchet space C(B); the existence of the vector-valued integral (1) is thus assured. Moreover, if Y is an $\mathscr{M}$-space in C(B) and $f \in \mathrm{Y}$, then $f \circ \mathrm{U} \in \mathrm{Y}$ for all $\mathrm{U} \in \mathscr{U}$, and since Y is closed, we conclude that $f^\# \in \mathrm{Y}$.

2.3. The invariant Laplacian $\tilde{\Delta}$. — Let Δ be the ordinary Laplacian on $\mathbf{C}^n$, given by

(1) $$\Delta f = 4 \sum_{i=1}^{n} \mathrm{D}_i \bar{\mathrm{D}}_i f$$

where $D_i = \partial/\partial z_i$, $\bar{D}_i = \partial/\partial \bar{z}_i$. If $f \in C^2(B)$ and $z \in B$, we define

$$(\tilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0). \tag{2}$$

This operator is *invariant* in the sense that it commutes with $\mathrm{Aut}(B)$: if $\psi \in \mathrm{Aut}(B)$ then

$$(\tilde{\Delta} f) \circ \psi = \tilde{\Delta}(f \circ \psi). \tag{3}$$

We note (see Chap. 4 of [11]) that there are other ways to describe $\tilde{\Delta}$, namely

$$(\tilde{\Delta} f)(z) = \lim_{r \to 0} \frac{4n}{r^2} \int_S \{f(\varphi_z(r\zeta)) - f(z)\}\, d\sigma(\zeta) \tag{4}$$

and

$$(\tilde{\Delta} f)(z) = 4(1-|z|^2) \sum_{i,k=1}^{n} (\delta_{ik} - z_i \bar{z}_k)(D_i \bar{D}_k f)(z). \tag{5}$$

When f is *radial*, 2.1 (*iii*) enables us to rewrite (4) in the form

$$\tilde{\Delta} f = \lim_{r \to 0} \frac{4n}{r^2} \int_S \{f \circ \varphi_{r\zeta} - f\}\, d\sigma(\zeta). \tag{6}$$

This shows that $\tilde{\Delta} f \in Y^{\#}$ if $f \in Y^{\#} \cap C^2$ and Y is an $\mathcal{M}$-space.

When f is radial and $f(z) = g(r)$, $r = |z|$, a calculation leads from (5) to

$$(\tilde{\Delta} f)(z) = (1-r^2)^2 g''(r) + (2n-1-r^2)(1-r^2)\frac{1}{r} g'(r). \tag{7}$$

For reasons that will become clear in § 2.6, we note that the change of variables

$$s = \frac{1}{2} \log \frac{1+r}{1-r}, \qquad G(s) = g(r) \tag{8}$$

turns (7) into

$$(\tilde{\Delta} f)(z) = G''(s) + \gamma(s) G'(s) \tag{9}$$

where

(10) $$\gamma(s) = \tanh(s) + (2n-1)\coth(s).$$

The particular form of γ will not be important; what matters is that γ is a continuous function on $(0,\infty)$.

2.4. Smoothing. – Let ν be Lebesgue measure on B, normalized so that $\nu(B) = 1$, and define

(1) $$d\tau(z) = (1-|z|^2)^{-n-1}\,d\nu(z).$$

This measure is $\mathcal{M}$-invariant: $\int_B f\,d\tau = \int_B (f\circ\psi)\,d\tau$ for all $f \in L^1(\tau)$, $\psi \in \mathrm{Aut}(B)$ [11; *p.* 28].

Suppose Y is an $\mathcal{M}$-space in C(B), $f \in Y^{\#}$, $h \geqslant 0$ is a radial C^∞-function with compact support in B, such that $\int h\,d\tau = 1$, and

(2) $$f_h = \int_B h(w) f\circ\varphi_w\,d\tau(w).$$

Then $f_h \in Y$ (for the same reason that was invoked in § 2.2) and f_h converges to f in the topology of C(B) when the support of f shrinks to the center of B. The invariance of τ shows that w can be replaced by Uw in the integral (2); since h and f are radial and since $|\varphi_{Uw}(Uz)| = |\varphi_w(z)|$ (see 2.1 (iv)), f_n is radial. Moreover, it follows from 2.1 (iii) and the invariance of τ that

(3) $$f_h(z) = \int_B h(\varphi_z(w)) f(w)\,d\tau(w),$$

which shows that $f_h \in C^\infty(B)$.

To summarize: $Y^{\#} \cap C^\infty$ *is dense in* $Y^{\#}$. If, in addition, f is bounded (or, more generally, if $f(z)(1-|z|^2)^p$ is bounded in B for some p) then one can define f_h for certain h that do not have compact support, for example for

(4) $$h_m(w) = c_m(1-|w|^2)^m$$

where m is sufficiently large and c_m is chosen so that $\int_B h_m \, d\tau = 1$. Then (3) becomes

(5) $$f_m(z) = c_m(1-|z|^2)^m \int_B \frac{(1-|w|^2)^m}{|1-\langle z,w\rangle|^{2m}} f(w)\, d\tau(w),$$

as in [10], [11; p. 282].

This furnishes *real analytic* approximations to f, letting $m \to \infty$.

2.5. Spherical means. – These are usually defined by specifying the center and the radius of the sphere over which a function is to be averaged [8]. If one ignores the radius and instead specifies a point on the sphere, one obtains a more symmetric object. Accordingly, we note that the average of an $f \in C(B)$ over the sphere with center 0 that passes through the point $w \in B$ is

(1) $$\int_{\mathscr{U}} f(Uw)\, dU = f^{\#}(w)$$

and we define $A_f(z,w)$ to be the corresponding average of the « translate » $f \circ \varphi_z$ of f. Thus

(2) $$A_f(z,w) = \int_{\mathscr{U}} f(\varphi_z Uw)\, dU = (f \circ \varphi_z)^{\#}(w).$$

It is clear that $A_f(z,w)$ is always a radial function of w. If f is itself radial, then the relations 2.1 (iii), (iv) show that

(3) $$A_f(z,w) = A_f(w,z)$$

so that $A_f(z,w)$ is also radial in z.

Parts (ii) and (iii) of the following proposition exhibit another symmetry property, one that does not depend on f being radial.

PROPOSITION. – *If* $f \in C(B)$ *and* $\psi \in \mathrm{Aut}(B)$, *then*

(i) $$A_{f\circ\psi}(z,w) = A_f(\psi(z),w)$$

and, for $z \in B$, $w \in B$, $0 \leqslant r < 1$,

(ii) $$\int_S A_f(\varphi_z(r\zeta),w)\, d\sigma(\zeta) = \int_S A_f(z,\varphi_w(r\zeta))\, d\sigma(\zeta).$$

If $f \in C^2(B)$ *then*

(iii) $$\tilde{\Delta}_z A_f(z,w) = \tilde{\Delta}_w A_f(z,w).$$

The symbol $\tilde{\Delta}_z$ indicates that w is to be heldfixed and that the differentiations are with respect to $z_1, \ldots, z_n$. Likewise for $\tilde{\Delta}_w$.

The differential equation (iii) occurs, for more general symmetric spaces, in [6; p. 613]. Its analogue on R^n is a classical equation of Darboux [8; p. 88]. The proof that is given below is based on 2.3(4), and is quite simple.

Proof. – Fix f, z, w, choose any $\varphi \in \mathrm{Aut}(B)$ such that $\varphi(0) = z$. Then $\varphi_z = \varphi U'$ for some $U' \in \mathscr{U}$, and (2) becomes

$$A_f(\varphi(0),w) = \int_{\mathscr{U}} (f \circ \varphi)(U'Uw)\, dU = \int_{\mathscr{U}} (f \circ \varphi)(Uw)\, dU.$$

This holds when f is replaced by $f \circ \psi$. Hence

$$A_{f \circ \psi}(\varphi(0),w) = \int_{\mathscr{U}} (f \circ \psi \circ \varphi)(Uw)\, dU = A_f(\psi(\varphi(0)),w).$$

This proves (i).

Next, replace U by U^{-1} in (2) and use 2.1 (iv) to rewrite (2) in the form

(4) $$A_f(r\zeta,w) = \int_{\mathscr{U}} f(U^{-1}\varphi_{Ur\zeta}(w))\, dU.$$

Integrate (4) with respect to $d\sigma(\zeta)$ over S, switch the integrals on the right, note that ζ can then be replaced by $U^{-1}\zeta$ in the (inner) ζ-integral, and that therefore

$$\int_S A_f(r\zeta,w)\, d\sigma(\zeta) = \int_S d\sigma(\zeta) \int_{\mathscr{U}} f(U^{-1}\varphi_{r\zeta}(w)\, dU$$
$$= \int_S A_f(0,\varphi_{r\zeta}(w))\, d\sigma(\zeta).$$

Since A_f is radial in the second variable, $r\zeta$ and w can be interchanged in the last integral, yielding the case $z = 0$ of (ii):

(5) $$\int_S A_f(r\zeta,w)\, d\sigma(\zeta) = \int_S A_f(0,\varphi_w(r\zeta))\, d\sigma(\zeta).$$

But (5) holds with $f \circ \varphi_z$ in place of f. Hence (i) leads from (5) to the general case of (ii).

Now subtract $A_f(z,w)$ from each side of (ii), divide by r^2, and let $r \to 0$. By 2.3(iv) this completes the proof of the Proposition.

One further remark: If $f \in C^2(B)$ is *radial*, and if we put $A_f(z,w) = A^*(s,t)$, where

$$(6) \qquad s = \frac{1}{2}\log\frac{1+|z|}{1-|z|}, \qquad t = \frac{1}{2}\log\frac{1+|w|}{1-|w|}$$

then formula (9) of § 2.3 shows that the differential equation (iii) satisfied by $A_f(z,w)$ takes the form

$$(7) \qquad \frac{\partial^2 A^*}{\partial s^2} + \gamma(s)\frac{\partial A^*}{\partial s} = \frac{\partial^2 A^*}{\partial t^2} + \gamma(t)\frac{\partial A^*}{\partial t}$$

where γ is as in 2.3 (10).

2.6. A uniqueness theorem. – For a concise statement, let us associate to each $f \in C(B)$ and to each $\psi \in \mathrm{Aut}(B)$ the function $\psi f : [0,1) \to \mathbf{C}$ by

$$(1) \qquad (\psi f)(r) = \int_S f(\psi(r\zeta))\, d\sigma(\zeta).$$

THEOREM. – *If* $f \in C^1(B)$ *and if there is one* $a \in (0,1)$ *such that*

$$(2) \qquad (\psi f)(a) = (\psi f)'(a) = 0$$

for every $\psi \in \mathrm{Aut}(B)$, *then* $f = 0$.

This may be regarded as a limiting case of a « two-radius » theorem [5], [6], the two radii being equal.

Proof. – Fix a, $0 < a < 1$, and let X be the set of all $f \in C^1(B)$ that satisfy (2) for every ψ. Then X is an $\mathcal{M}$-invariant closed subspace of $C^1(B)$. As in § 2.2, $f^\# \in X$ whenever $f \in X$.

If $z \in B$ and $\zeta \in S$, then $(\varphi_z f)(r) = A_f(z, r\zeta)$. Hence (2) implies

$$(3) \qquad A_f(z,a\zeta) = \frac{\partial}{\partial r} A_f(z,r\zeta)\big|_{r=a} = 0$$

for all $f \in X$. Apply this to a radial $f \in X$ and change variables as at the end of § 2.5. The resulting function A^* is then a solution of the hyperbolic equation

$$(4) \qquad \frac{\partial^2 A^*}{\partial s^2} - \frac{\partial^2 A^*}{\partial t^2} + \gamma(s)\frac{\partial A^*}{\partial s} - \gamma(t)\frac{\partial A^*}{\partial t} = 0$$

that satisfies the initial conditions

$$(5) \qquad A^*(s,\alpha) = \frac{\partial}{\partial t}A^*(s,t)|_{t=\alpha} = 0$$

for all $s > 0$, where $\alpha = \frac{1}{2}\log\{(1+a)/(1-a)\}$.

A standard uniqueness theorem [7; pp. 310-311] implies now that $A^*(s,t) = 0$ in the region in which $0 < t < \alpha$, $s + t > \alpha$. Going back to our original variables, this says that $A_f(z,w) = 0$ when $0 < |w| < a$ and

$$(6) \qquad \frac{1+|z|}{1-|z|}\cdot\frac{1+|w|}{1-|w|} > \frac{1+a}{1-a}.$$

When $|z| > a$, this holds for arbitrarily small $|w|$; hence

$$f(z) = A_f(z,0) = 0,$$

by continuity.

We have now proved that every $f \in X^{\#}$ vanishes outside the ball aB. Hence (see § 2.4) f can be approximated by *real-analytic* functions $f_i \in X^{\#}$. Each f_i vanishes outside aB, by what we just proved, hence f_i (being real-analytic) vanishes in all of B. Thus $f = 0$.

We conclude that $X^{\#} = \{0\}$.

Finally, if $f \in X$ then $(f \circ \psi)^{\#} \in X^{\#}$ for every $\psi \in \mathrm{Aut}(B)$. Hence $(f \circ \psi)^{\#} = 0$. Thus, for any $z \in B$

$$(7) \qquad f(z) = (f \circ \varphi_z)(0) = (f \circ \varphi_z)^{\#}(0) = 0.$$

2.7. Separation of points. – *If f is a nonconstant function with domain* B *and*

$$X = \{f \circ \psi : \psi \in \mathrm{Aut}(B)\},$$

then X *separates points on* B.

The proof is exactly like that of Proposition 4 in [9]. If X fails to separate, the $\mathcal{M}$-invariance of X shows that X identifies 0 and some $a \neq 0$. Setting $r = |a|$, the $\mathcal{U}$-invariance of X implies then that $g(0) = g(r\zeta)$ for all $g \in \mathrm{X}$, $\zeta \in \mathrm{S}$. The same holds for $g \circ \varphi_w$ when $|w| = r$, and shows that $g(z) = g(0)$ for all $z \in \varphi_w(r\mathrm{S})$, i.e., for all z in the ball $r_1\mathrm{B}$, where $r_1 = 2r/(1+r^2)$. Continuing in this fashion, we see that every member of X is constant, a contradiction.

2.8. Real functions. – *If* Y *is an* $\mathcal{M}$*-algebra in* C(B) *that contains a nonconstant real-valued function* f, *then* Y = C(B).

Proof. – Let Y_f be the $\mathcal{M}$-algebra generated by f. Then Y_f is a self-adjoint subalgebra of Y which separates points on B, by § 2.7. The Stone-Weierstrass theorem implies therefore that the restriction of Y_f to any compact $\mathrm{K} \subset \mathrm{B}$ coincides with C(K). Thus $\mathrm{Y}_f = \mathrm{C(B)}$.

3. Unitary invariance.

This section describes some aspects of harmonic analysis in closed subspaces and subalgebras Y of C(B) that are $\mathcal{U}$*-invariant*: If $f \in \mathrm{Y}$ and $\mathrm{U} \in \mathcal{U}$ then $f \circ \mathrm{U} \in \mathrm{Y}$. (See § 2.1.)

These are called $\mathcal{U}$*-spaces* and $\mathcal{U}$*-algebras*, respectively.

3.1. The spaces $\mathrm{H}(p,q)$. For nonnegative integers p and q, we say that $f \in \mathrm{H}(p,q)$ if f is the *restriction to* S of a *homogeneous harmonic polynomial* on $\mathbf{C}^n$ that has total degree p in the variables $z_1, \ldots, z_n$ and total degree q in $\bar{z}_1, \ldots, \bar{z}_n$. The word *harmonic* refers to the ordinary Laplacian. Being harmonic, these polynomials are uniquely determined by their restriction to S.

The $\mathrm{H}(p,q)$'s are pairwise orthogonal in $\mathrm{L}^2(\sigma)$ (σ is defined in § 2.2), they span $\mathrm{L}^2(\sigma)$, and they are $\mathcal{U}$-invariant. In fact, they are *minimally* $\mathcal{U}$-invariant: no proper subspace of $\mathrm{H}(p,q)$, except $\{0\}$, is $\mathcal{U}$-invariant. (See Section 12.2 of [11].)

In the special case $n = 1$, $\mathrm{H}(p,0)$ and $\mathrm{H}(0,q)$ are the one-dimensional spaces (on the unit circle) spanned by $e^{ip\theta}$ and $e^{-iq\theta}$, respectively. The other $\mathrm{H}(p,q)$'s are $\{0\}$. Whenever some later discussion refers to $\mathrm{H}(p,q)$ with $p > 0$ and $q > 0$, it will be tacitly understood that $n > 1$.

The orthogonal projection from $L^2(\sigma)$ onto $H(p,q)$ will be denoted by π_{pq}. These π's *commute* with $\mathscr{U}$: If $f \in L^2(\sigma)$ and $U \in \mathscr{U}$, then

$$\pi(f \circ U) = (\pi f) \circ U.$$

The π_{pq}'s are given by integral kernels ([11], Theorem 12.2.5). Hence they are also continuous from $C(S)$ into $C(S)$.

3.2. The restriction Y_a. – Let Y be a $\mathscr{U}$-space in $C(B)$, fix a, $0 < a < 1$, and define Y_a to be the space of all $f_a \in C(S)$ such that

(1) $$f_a(\zeta) = f(a\zeta) \quad (\zeta \in S)$$

for some $f \in Y$.

Note that f_a is essentially the restriction of f to the sphere aS, except that we take its domain to be S rather than aS.

Since Y_a need not be closed in $C(S)$ (example: $Y = H(B)$) we include the proof of the following proposition. (Otherwise, we could just refer to Theorem 12.3.6 of [11].)

3.3. Proposition. – *Suppose that* Y *is a closed* $\mathscr{U}$*-space in* $C(B)$, *and* $0 < a < 1$. *Then, for any* (p,q), *either*

(i) $H(p,q) \subset Y_a$, *or*

(ii) $H(p,q) \perp Y_a$.

(The symbol $\perp$ refers to orthogonality in $L^2(\sigma)$.)

Proof. – Define $\rho : Y \to C(S)$ by $\rho f = f_a$. Thus $\rho Y = Y_a$.

Assume that (ii) fails for some (p,q), fixed from now on, and write π in place of π_{pq}. The operator

$$\pi\rho : \quad Y \to H(p,q)$$

is linear, continuous, and commutes with $\mathscr{U}$, so that its range is a $\mathscr{U}$-invariant subspace of $H(p,q)$ which (because (ii) fails) is $\neq \{0\}$. The *$\mathscr{U}$-minimality of* $H(p,q)$ shows therefore that

$$\pi\rho Y = H(p,q).$$

The null-space N of $\pi\rho$ is closed in Y, is $\mathscr{U}$-invariant, and

$$\dim(Y/N) = \dim H(p,q) < \infty.$$

Therefore N is complemented in the Fréchet space Y. Moreover, $f \to f \circ U$ is continuous from Y to Y, for every U in the compact group $\mathscr{U}$. These facts imply (see Theorem 5.18 of [12]) that there is a $\mathscr{U}$-invariant space $N' \subset Y$ such that $Y = N \oplus N'$ (direct sum).

We claim that $H(p,q) = \rho N'$.

Note that $\pi\rho : N' \to H(p,q)$ is a bijection. It therefore has an inverse $\Lambda : H(p,q) \to N'$. Since N' is $\mathscr{U}$-invariant, and π and ρ commute with $\mathscr{U}$, so does Λ. The same is true of

$$\pi_{rs}\rho\Lambda : \ H(p,q) \to H(r,s),$$

for all (r,s). By Theorem 12.2.7 of [11], it follows that $\pi_{rs}\rho\Lambda$ annihilates $H(p,q)$ whenever $(r,s) \neq (p,q)$. This implies that

$$\rho\Lambda H(p,q) \perp H(r,s)$$

if $(r,s) \neq (p,q)$. Since $N' = \Lambda H(p,q)$, we conclude that

$$\rho N' \subset H(p,q).$$

Since π is the identity map on $H(p,q)$,

$$\rho N' = \pi\rho N' = H(p,q),$$

and since $\rho N' \subset \rho Y = Y_a$, the proof is complete.

COROLLARY. — *Y_a is dense in $C(S)$ if and only if $H(p,q) \subset Y_a$ for all (p,q).*

3.4. Some facts about $\mathscr{U}$-algebras. — Let Y now be a *$\mathscr{U}$-algebra* in $C(B)$, $0 < a < 1$. The following properties of its restriction Y_a can be found in Sections 12.4 and 12.5 of [11], but it seems preferable to give quick proofs, based on Proposition 3.3, of the few simple facts that will be needed in the present paper.

(i) *If $H(1,0) \subset Y_a$ and $H(0,1) \subset Y_a$ then Y_a is dense in $C(S)$.*

Proof. — Y_a contains ζ_i and $\bar{\zeta}_i$ for $i = 1, \ldots, n$, hence contains all polynomials in these variables, including the constants, since $\Sigma\zeta_i\bar{\zeta}_i = 1$.

To simplify the notation in (ii) and (iii), let us write $u = \zeta_1$, $v = \zeta_2$.

(ii) *If $H(p,q) \subset Y_a$ for some (p,q) with $p > q$ then $H(m,0) \subset Y_a$ for some $m > 0$.*

Proof. – The functions $u^p\bar{v}^q$ and $v^p\bar{u}^q$ are in Y_a, hence so is their product $|uv|^{2q}(uv)^{p-q}$, and this is not orthogonal to $(uv)^{p-q} \in H(m,0)$, $m = 2p - 2q$.

(iii) *If* $H(2,0) \subset Y_a$, $H(0,2) \subset Y_a$, *and* Y_a *separates points on* S, *then* $H(1,0) \subset Y_a$.

Proof. – Every $f \in Y_a$ is in the L^2-closure of the sum of the $H(p,q)$'s that lie in Y_a. If Y_a separates points on S, some $H(p,q) \subset Y_a$ has $p - q$ odd; otherwise $f(\zeta) = f(-\zeta)$ for all $f \in Y_a$. Pick such a pair (p,q). Then Y_a contains

$$uv, \quad \bar{u}^2, \quad \bar{v}^2, \quad u^p\bar{v}^q$$

hence, if $p > q$, also

$$(uv)^{p-1}.\bar{u}^{2p-2}.\bar{v}^{p-q-1}.u^p\bar{v}^q = |u^2v|^{2p-2}u,$$

since $p - q - 1$ is even. If $p < q$, we use

$$(\bar{u}\bar{v})^{q-1}.u^{2q}.v^{q-p-1}.v^p\bar{u}^q = |u|^{4q-2}|v|^{2p-2}u.$$

In either case, we see that Y_a is not orthogonal to $u \in H(1,0)$.

3.5. Lemma. – *Let* X *be a* $\mathcal{U}$*-invariant subalgebra of* C(S) *such that* $H(p,q) \subset X$ *for all* (p,q).

Let $T : X \to C(S)$ *be linear, multiplicative,* $\neq 0$, *and suppose that* T *commutes with* $\mathcal{U}$.

Then there is a $\gamma \in \mathbf{C}$, $\gamma \neq 0$, *such that*

$$Th = \gamma^{p-q}h \tag{1}$$

for all $h \in H(p,q)$.

Proof. – The map $\pi_{rs}T : H(p,q) \to H(r,s)$ is linear and commutes with $\mathcal{U}$, hence (by Theorem 12.2.7 of [11]) is 0 when $(r,s) \neq (p,q)$, and is a multiple of the identity when $(r,s) = (p,q)$. Thus $TH(p,q) \perp H(r,s)$ if $(r,s) \neq (p,q)$. It follows that $TH(p,q) \subset H(p,q)$, and that there are constants c_{pq} such that

$$Th = c_{pq}h \quad \text{if} \quad h \in H(p,q). \tag{2}$$

Since $z_1^p \bar{z}_2^q \in H(p,q)$, the multiplicativity of T shows that

$$c_{p+r,q+s} = c_{pq}c_{rs}. \tag{3}$$

Put $h(\xi) = 1 - n\zeta_1\bar{\zeta}_1$. Since $|z|^2 - nz_1\bar{z}_1$ is harmonic in $\mathbf{C}^n$, $h \in H(1,1)$. Also, $\zeta_1 \in H(1,0)$, $\bar{\zeta}_1 \in H(0,1)$. Hence T, applied to $1 - h = n\zeta_1\bar{\zeta}_1$, yields

$$c_{00} - c_{11}h = c_{01}c_{10}(1-h). \tag{4}$$

By (3), $c_{00} = 1$ and $c_{11} = c_{01}c_{10}$. Hence (4) gives $c_{11} = 1$. Setting $\gamma = c_{10}$, (3) leads now to

$$c_{pq} = (c_{10})^p(c_{01})^q = \gamma_{p-q}. \tag{5}$$

4. $\mathcal{M}$-Algebras in C(B).

4.1. The operators Q and $\bar{\text{Q}}$. — We define these by

$$Q = D_1 - \bar{z}_1 \sum_{i=1}^n \bar{z}_i\bar{D}_i, \qquad \bar{Q} = \bar{D}_1 - z_1 \sum_{i=1}^n z_i D_i \tag{1}$$

where $D_i = \partial/\partial z_i$, $\bar{D}_i = \partial/\partial \bar{z}_i$, as before. These operators are closely related to $\mathcal{M}$-invariance:

(i) *If* Y *is an* $\mathcal{M}$*-space in* C(B) *and* $f \in Y \cap C^1$, *then* $Qf \in Y$ *and* $\bar{Q}f \in Y$.

To see this, put

$$f_\alpha(z) = f\left(\frac{z_1+\alpha}{1+\bar{\alpha}z_1}, \frac{sz_2}{1+\bar{\alpha}z_1}, \dots, \frac{sz_n}{1+\bar{\alpha}z_1}\right)$$

where $\alpha \in \mathbf{C}$, $|\alpha| < 1$, $s = (1-\alpha\bar{\alpha})^{1/2}$ (see § 2.1) and calculate that

$$Qf = \frac{\partial f_\alpha}{\partial \alpha}\Big|_{\alpha=0}, \qquad \bar{Q}f = \frac{\partial f_\alpha}{\partial \bar{\alpha}}\Big|_{\alpha=0}.$$

Writing $\alpha = x + iy$,

$$2\frac{\partial f_\alpha}{\partial \alpha} = \lim_{x\to 0} \frac{1}{x}(f_x - f) - i \lim_{y\to 0} \frac{1}{y}(f_{iy} - f).$$

These quotients lie in Y, and they converge to the respective derivatives in the topology of C(B). Thus $Qf \in Y$. The same argument applies to $\bar{Q}f$.

Suppose next that $f \in Y^{\#} \cap C^{\infty}$. Being radial, f can be written in the form

$$f(z) = g(|z|^2) = g(z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)$$

where g has domain $[0,1)$. It follows from (1) that

$$\begin{aligned} (Qf)(z) &= \bar{z}_1(1-|z|^2)g'(|z|^2), \\ (\bar{Q}f)(z) &= z_1(1-|z|^2)g'(|z|^2). \end{aligned} \tag{3}$$

If we apply Q and $\bar{Q}$ to (2) and (3), we obtain :

(ii) *If* $0 < a < 1$ *and* $g'(a^2) = 0$, *then*

$$(Q^2 f)(a\zeta) = a^2(1-a^2)^2 g''(a^2)\bar{\zeta}_1^2 \tag{4}$$

and

$$(Q^2 f)(a\zeta) = a^2(1-a^2)^2 g''(a^2)\zeta_1^2 \tag{5}$$

for all $\zeta \in S$.

4.2. LEMMA. – *Fix* a, $0 < a < 1$. *For* $\mathcal{M}$*-algebras* Y *in* C(B), *the implications*

$$(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$$

hold among the following properties :

(α) Y_a *is not dense in* C(S).

(β) $\partial f/\partial r = 0$ *on* aS *for every* $f \in Y^{\#} \cap C^{\infty}$.

(γ) $\tilde{\Delta} f = 0$ *on* aS *for every* $f \in Y^{\#} \cap C^{\infty}$.

Proof. – If $\partial f/\partial r \neq 0$ on aS for some $f \in Y^{\#} \cap C^{\infty}$, where $f(z) = g(|z|^2)$ as in § 4.1, then $g'(a^2) \neq 0$. Since Qf and $\bar{Q}f$ are in Y, formulas 4.1(2) and 4.1(3) show that Y_a contains H(1,0) and H(0,1), hence Y_a is dense in C(S), by § 3.4(i). This proves that (α) implies (β).

Suppose next that (β) holds, but that some $f \in Y^{\#} \cap C^{\infty}$ has $\partial^2 f/\partial r^2 \neq 0$ on aS. This will lead to a contradiction.

Write $f(z) = g(|z|^2)$, as before. Then $g'(a^2) = 0$ but $g''(a^2) \neq 0$. Let X be the $\mathcal{M}$-algebra in C(B) generated by f. Then $Q^2 f \in X$, $\bar{Q}^2 f \in X$, so that X_a contains H(0,2) and H(2,0), by 4.1(4), (5). By

§ 2.7, X separates points in B. It follows now from § 3.4(iii) that $H(1,0) \subset X_a$. Hence there is an $h \in X$ such that $h(a\zeta) = \zeta_1$ for all $\zeta \in S$.

The definition of X shows that $X \cap C^\infty$ is dense in X. Hence there are functions $h_i \in X \cap C^\infty$ such that $h_i(a\zeta) \to \zeta_1$ uniformly on S, as $i \to \infty$. Define

$$\text{(1)} \qquad F_i(z) = (h_i Q f)^\#(z) \quad (z \in B, i = 1,2,3,\ldots).$$

By 4.1(2),

$$\text{(2)} \qquad F_i(r\zeta) = r(1-r^2)g'(r^2)\int_S h_i(r\zeta)\bar{\zeta}_1\, d\sigma(\zeta).$$

Now apply $\partial/\partial r$ to both sides of (2) and evaluate at $r = a$. Since $F_i \in X^\# \cap C^\infty \subset Y^\# \cap C^\infty$, and (β) holds, the left side gives 0. Since $g'(a^2) = 0$, we obtain

$$\text{(3)} \qquad 0 = a(1-a^2)g''(a^2)\int_S h_i(a\zeta)\bar{\zeta}_1\, d\sigma(\zeta)$$

for $i = 1,2,3,\ldots$. For large i, the integral is $\neq 0$. Thus $g''(a^2) = 0$, and we have our contradiction.

This proves that (β) implies (γ).

We state one more lemma before we turn to the proof of the main theorem.

4.3. LEMMA. — *Let m be a positive integer, put*

$$\text{(1)} \qquad u_i(z) = z_i^m \quad (i = 1,\ldots,n)$$

and suppose that $f : B \to C$ *satisfies*

$$\text{(2)} \qquad (\tilde{\Delta} f)(z) = 0 \quad \textit{and} \quad \tilde{\Delta}(u_i f)(z) = 0$$

for $1 \leqslant i \leqslant n$, $z \in B$. *Then* $f \in H(B)$.

Proof. — Formula 2.3 (5) shows, after a little computation, that $\tilde{\Delta}(u_i f) - u_i \tilde{\Delta} f$ is equal to

$$\text{(3)} \qquad 4(1-|z|^2)mz_i^{m-1}\left\{\bar{D}_i f - z_i \sum_{k=1}^{n} \bar{z}_k \bar{D}_k f\right\}.$$

The expression in $\{\ldots\}$ is therefore 0, for all i. Setting $w_k = \bar{D}_k f$, this says that the vector $w = (w_1, \ldots, w_n)$ satisfies

$$(4) \qquad w - \langle w,z \rangle z = 0 .$$

Since $|\langle w,z \rangle z| \leqslant |z|^2 |w|$ and $|z|^2 < 1$, this forces $w = 0$, so that the Cauchy-Riemann equations $\bar{D}_k f = 0$ hold for $1 \leqslant k \leqslant n$.

5. Proof of the main theorem.

5.1. – To eliminate the trivial cases $Y = \{0\}$ and $Y = \mathbf{C}$, we assume from now on that Y is an $\mathcal{M}$-algebra in $C(B)$ that contains nonconstant functions. Consider the following four properties which Y may or may not have :

(i) *There exists* $a \neq b$ *in* $(0,1)$ *such that*

$$f^{\#}(a\zeta) = f^{\#}(b\zeta)$$

for every $f \in Y$, $\zeta \in S$.

(ii) *There exists* $a \in (0,1)$ *such that*

$$\partial f / \partial r = 0 \quad on \quad aS$$

for every $f \in Y^{\#} \cap C^{\infty}$.

(iii) *There exists* $a \in (0,1)$ *such that*

$$f^{\#}(a\zeta) = f^{\#}(0)$$

for every $f \in Y$, $\zeta \in S$.

(iv) *Every* $f \in Y$ *is* $\mathcal{M}$-*harmonic in* B, i.e. $(\tilde{\Delta} f)(z) = 0$ *if* $f \in Y$, $z \in B$.

We will prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). If (iv) holds, then every $f \in Y$ satisfies the *invariant mean value property*

$$f(\psi(0)) = \int_S f(\psi(r\zeta))\, d\sigma(\zeta)$$

for all $\psi \in \mathrm{Aut}(B)$, $0 < r < 1$ [11; pp. 43, 52]. Hence $Y^{\#} = \mathbf{C}$, so that (iv) $\Rightarrow$ (i).

These four properties of an $\mathcal{M}$-algebra Y thus turn out to be equivalent.

If they hold, then $Y = H(B)$ *or* $Y = \text{conj}\, H(B)$.

If they fail, then $Y = C(B)$.

These assertions will be proved in § 5.5, § 5.7.

5.2. Proof that (i) **implies** (ii). – Assume that (i) holds. Lemma 4.2 shows that (ii) holds if one of Y_a or Y_b fails to be dense in $C(S)$. So let us assume that *both* are dense in $C(S)$.

Pick $h \in Y$ with $h_b = 0$. If $g \in Y$, then $hg \in Y$, and (i) gives

$$\int_S h_a g_a \, d\sigma = (hg)^{\#}(a\zeta) = (hg)^{\#}(b\zeta) = \int_S h_b g_b \, d\sigma = 0 .$$

Since Y_a is dense in $C(S)$, it follows that $h_a = 0$.

Consequently, $f_b \to f_a$ is a one-to-one linear multiplicative map of Y_b onto Y_a that commutes with $\mathcal{U}$. By Lemma 3.5, there is a $\gamma \in \mathbf{C}$, $\gamma \neq 0$, such that

$$h(a\zeta) = \gamma^{p-q} h(b\zeta) \tag{1}$$

whenever $h \in Y$ and $h_b \in H(p,q)$. If $|\gamma|$ were 1, this would imply $h(a\zeta) = h(b\gamma\zeta)$, hence also $f(a\zeta) = f(b\gamma\zeta)$ for all $f \in Y$, contradicting the fact that Y separates points in B. Thus $|\gamma| \neq 1$.

Now suppose $f \in Y^{\#} \cap C^{\infty}$, $f(z) = g(|z|^2)$. Formulas 4.1 (2), (3) show that $(Qf)_b \in H(0,1)$, $(\bar{Q}f)_b \in H(1,0)$. Hence (1), applied to Qf and $\bar{Q}f$ in place of h, yields

$$a(1-a^2)g'(a^2)\bar{\zeta}_1 = \gamma^{-1} b(1-b^2)g'(b^2)\bar{\zeta}_1$$

and

$$a(1-a^2)g'(a^2)\zeta_1 = \gamma b(1-b^2)g'(b^2)\zeta_1 .$$

Since $|\gamma| \neq 1$, we must have $g'(a^2) = g'(b^2) = 0$. Thus $\partial f/\partial r = 0$ on aS and on bS. In particular, (ii) holds.

5.3. Proof that (ii) **implies** (iii). – If (ii) holds, Lemma 4.2 shows that $\tilde{\Delta} f = 0$ on aS, for every $f \in Y^{\#} \cap C^{\infty}$. Fix such an f. Then

$(f \circ \varphi_z)^\# \in Y^\# \cap C^\infty$, so that

(1) $$\tilde{\Delta}((f \circ \varphi_z)^\#) = 0 \quad \text{on} \quad aS$$

for all $z \in B$. Since

(2) $$A_f(z,w) = (f \circ \varphi_z)(w)$$

(see § 2.5), we have $\tilde{\Delta}_w A_f(z,w) = 0$ whenever $z \in B$, $w \in aS$. Proposition 2.5 (iii) implies therefore that

(3) $$\tilde{\Delta}_z A_f(z,w) = 0 \quad (z \in B, w \in aS),$$

i.e., that $z \to A_f(z,w)$ is $\mathcal{M}$-harmonic. But it is also radial (since f is radial; see 2.5(3)), hence is constant, hence equals $A_f(0,w)$. We now conclude from (2) that, for some fixed $w \in aS$,

(4) $$(f \circ \varphi_z)^\#(w) = f^\#(w) = f(w).$$

Also, because (ii) is assumed to hold,

(5) $$\frac{\partial}{\partial r}(f \circ \varphi_z)^\# = 0 \quad \text{on} \quad aS.$$

Theorem 2.6 can now be applied to $f - f(w)$, and leads to the conclusion that every $f \in Y^\# \cap C^\infty$ is constant. Since $Y^\# \cap C^\infty$ is dense in $Y^\#$, it follows that $Y^\# = \mathbf{C}$. Hence (iii) holds.

5.4. Proof that (iii) **implies** (iv). – The assumption is now that

(1) $$f(0) = \int_S f_a \, d\sigma$$

for every $f \in Y$. Since $f \to f(0)$ is multiplicative on Y, the integral in (1) is a multiplicative linear functional on Y_a which is bounded relative to the supremum norm. If Y_a were dense in $C(S)$, then $h \to \int h \, d\sigma$ would therefore be multiplicative on $C(S)$, which it is not. Thus Y_a is not dense in $C(S)$.

Let $g \in Y \cap C^\infty$, put $h = g^\#$. The implication $(\alpha) \Rightarrow (\gamma)$ of Lemma 4.2 shows that $(\tilde{\Delta}h)(a\zeta) = 0$. Since $\tilde{\Delta}h \in Y^\# \cap C^\infty$ (§ 2.3), (1) applies to it and shows that $(\tilde{\Delta}h)(0) = 0$. Hence also $(\tilde{\Delta}g)(0) = 0$. (See 2.3(6).) The same applies to $g \circ \varphi_z$ in place of g, so that $(\tilde{\Delta}g)(z) = 0$ for all $z \in B$.

Every $g \in Y \cap C^\infty$ is thus $\mathcal{M}$-harmonic. In particular, this is true for every $g \in Y^\# \cap C^\infty$. But the constants are the only radial $\mathcal{M}$-harmonic functions in B. Since $Y^\# \cap C^\infty$ is dense in $Y^\#$, we conclude that $Y^\# = \mathbf{C}$.

If now $f \in Y$, and $\psi \in \mathrm{Aut}(B)$, it follows that

$$(f \circ \psi)^\#(z) = (f \circ \psi)^\#(0) = f(\psi(0)) \tag{2}$$

for all $z \in B$. This says that f has the invariant mean value property (see § 5.1), and therefore f is $\mathcal{M}$-harmonic [11; *p.* 52].

5.5 $\mathcal{M}$-Algebras of $\mathcal{M}$-harmonic functions. — The assumption is now that every $f \in Y$ is $\mathcal{M}$-harmonic. The desired conclusion is that then $Y = H(B)$ or $Y = \mathrm{conj}\, H(B)$. The proof uses the following four observations.

(1) *It is enough to prove that* $Y \subset H(B)$ *or* $Y \subset \mathrm{conj}\, H(B)$. For if an $\mathcal{M}$-space Y satisfies one of these inclusions and if Y contains nonconstant functions (which is now our standing assumption) then equality actually holds [11; p. 287].

(2) *If* $0 < a < 1$ *and if two $\mathcal{M}$-harmonic functions with domain* B *coincide on* aS, *then they coincide on* B. Indeed, the maximum principle satisfied by $\mathcal{M}$-harmonic functions [11; p. 55] shows that they coincide in aB, hence they coincide on all of B since they are real — analytic [11; p. 52].

(3) *If* f_a *is real-valued for some* $f \in Y$, *then* $f(z)$ *is real for all* $z \in B$. This follows from (2) if the maximum principle is applied to the imaginary part of f.

(4) *There is some* $(p,q) \neq (0,0)$ *such that* $H(p,q) \subset Y_a$. Otherwise, $H(p,q) \perp Y_a$ for all $(p,q) \neq 0$ (Proposition 3.3), so that $Y_a \subset H(0,0) = \mathbf{C}$. This implies $Y \subset \mathbf{C}$, by (2). But this trivial case has been excluded.

Suppose now that $p > q$ for some $H(p,q) \subset Y_a$. By 3.4 (ii), $H(m,0) \subset Y_a$ for some $m > 0$. Hence there are functions $h_i \in Y$ $(i=1,\ldots,n)$ and that $h_i(a\zeta) = (a\zeta_i)^m$. By (2), this implies that $h_i(z) = z_i^m$ in B. This enables us to conclude from Lemma 4.3 that $Y \subset H(B)$.

Similarly, if $p < q$ for some $H(p,q) \subset Y_a$, we conclude that $Y \subset \operatorname{conj} H(B)$.

Finally, if neither of these two cases occurs, then $H(p,p) \subset Y_a$ for some $p > 0$. Writing $u = \zeta_1$, $v = \zeta_2$, as in § 3.4, Y_a contains $u^p\bar{v}^p$ and $v^p\bar{u}^p$, hence also their product $|uv|^{2p}$. By (3), there is a real-valued $f \in Y$ with $f_a = |uv|^{2p}$. Hence f is not constant, and § 2.8 implies that $Y = C(B)$, contradicting the assumption that every member of Y is $\mathcal{M}$-harmonic. Thus Y_a contains no $H(p,p)$ with $p > 0$, so that one of the preceding two cases must occur, by (4).

Because of (1), this completes the proof of one half of the main theorem. The second half uses the separation lemma which is proved next, although it is quite elementary.

5.6. Lemma. — *Let Φ be a collection of continuous maps from a compact space* K *into some Hausdorff space. If*

(*a*) *Φ separates points on* K, *and*

(*b*) *to every $x \in K$ corresponds a $g_x \in \Phi$ which is one-to-one in some neighborhood V_x of x,*

then some finite subcollection of Φ satisfies (*a*).

Proof. — Pick $p = (x,y) \in K^2$, where $K^2 = K \times K$. If $x \neq y$, there is an $f_p \in \Phi$ with $f_p(x) \neq f_p(y)$. By continuity, p has a neighborhood W_p in K^2 such that $f_p(\xi) \neq f_p(\eta)$ for all $(\xi,\eta) \in W_p$.

If $x = y$, choose g_x, V_x as in (*b*), and put $f_p = g_x$, $W_p = V_x^2$. Then W_p is a neighborhood of p in K^2 such that $f_p(\xi) \neq f_p(\eta)$ for all $(\xi,\eta) \in W_p$ that have $\xi \neq \eta$.

By compactness, finitely many $W_p's$ cover K^2. The corresponding $f_p's$ separate points on K.

5.7. Proof that $Y = C(B)$ in the remaining case. — We assume now that Y fails to have properties 5.1 (i), (ii), (iii). Fix $\zeta \in S$. Since $Y^\# \cap C^\infty$ is dense in $Y^\#$, the failure of (i) and (iii) shows that the functions

$$(*) \qquad t \to f(t\zeta), \qquad f \in Y^\# \cap C^\infty$$

separate points on $[0,1)$. The failure of (ii) shows that to every $t \in (0,1)$ corresponds some function (*) that is one-to-one in some neighborhood of t. Since nonconstant radial functions are not $\mathcal{M}$-harmonic, some $f \in Y^{\#} \cap C^{\infty}$ has $(\tilde{\Delta} f)(z) \neq 0$ at some $z \in B$. Setting $h = (f \circ \varphi_z)^{\#}$, $h \in Y^{\#} \cap C^{\infty}$, and $(\tilde{\Delta} h)(0) \neq 0$. The corresponding function (*) has its second derivative $\neq 0$ at $t = 0$, hence is one-to-one on $[0,\delta)$ for some $\delta > 0$.

We can now apply Lemma 5.6 and conclude that to every $r \in (0,1)$ corresponds a finite set of functions $f_1, \ldots, f_N \in Y^{\#} \cap C^{\infty}$ that separates points on the compact interval $[0,r]$. If

$$\Gamma(t) = (f_1(t), \ldots, f_N(t)) \quad (0 \leqslant t \leqslant r)$$

then Γ is a smooth arc in $\mathbf{C}^N$. A theorem of Stolzenberg ([13], [4], [14; Chap. 6]) asserts that the polynomials in $z_1, \ldots, z_N$ are dense in $C(\Gamma)$. The polynomials in $f_1, \ldots, f_N$ are thus dense in $C([0,r])$. This implies that $Y^{\#}$ (restricted to the set of all $t\zeta$, $0 \leqslant t < 1$) is equal to $C([0,1))$. In particular, $Y^{\#}$ contains nonconstant real functions. Hence $Y = C(B)$, by § 2.8, and the proof is complete.

5.8. Remark. – In contrast with $\mathcal{M}$-algebras, there is a huge collection of *$\mathcal{M}$-spaces* in $C(B)$, for every dimension $n \geqslant 1$. The ones that are easiest to describe are the eigenspaces X_λ of the invariant Laplacian $\tilde{\Delta}$, one for every $\lambda \in \mathbf{C}$: $f \in X_\lambda$ if and only if $\tilde{\Delta} f = \lambda f$. These spaces are closed in $C(B)$ [11: p. 52]; for each λ, $(X_\lambda)^{\#}$ is a one-dimensional space [11; p. 50]; to every $a \in (0,1)$ correspond infinitely many λ such that $(X_\lambda)^{\#}$ identifies 0 and $a\zeta$ [11; *p.* 58]. The same proof shows that if $0 < a < b < 1$, then $(X_\lambda)^{\#}$ identifies a and b, for infinitely many λ, and that to every $a \in (0,1)$ correspond infinitely many λ such that $\partial f / \partial r = 0$ on aS for all $f \in (X_\lambda)^{\#}$.

In the context of $\mathcal{M}$-spaces, properties 5.1 (i), (ii), (iii), and (iv) are thus not equivalent.

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