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RIESZ MEANS FOR THE EIGENFUNCTION EXPANSIONS FOR A CLASS OF HYPOELLIPTIC DIFFERENTIAL OPERATORS

by Giancarlo MAUCERI

1. Introduction.

The problem of studying the asymptotic properties of the spectral functions of differential operators and the related summability of the eigenfunction expansions has received extensive consideration. Elliptic differential operators have been studied by Garding [6], Bergendal [1], Peetre [13], [14], and Hörmander [9]. More recently Clerc [2] has investigated the Riesz summability of the eigenfunction expansion for a biinvariant Laplacian on a compact Lie group, and Metivier [12] has used analysis on nilpotent Lie groups to study the asymptotic behaviour of the spectra of a class of second order hypoelliptic differential operators.

In this paper we study the Riesz summability of the eigenfunction expansions for a class of hypoelliptic differential operators on the Heisenberg group. The Heisenberg group $\mathbf{H}_n$ is the Lie group whose underlying manifold is $\mathbf{R} \times \mathbf{C}^n$, with coordinates $(t,z_1,\ldots,z_n)$, and whose Lie algebra $\mathfrak{h}_n$ is generated by the left invariant vector fields

$$\mathrm{T} = \frac{\partial}{\partial t}, \qquad \mathrm{Z}_j = \frac{\partial}{\partial z_j} + \bar{z}_j\frac{\partial}{\partial t}, \qquad \bar{\mathrm{Z}}_j = \frac{\partial}{\partial \bar{z}_j} + z_j\frac{\partial}{\partial t}, \qquad j = 1,\ldots,n.$$

Let $\mathscr{L} = -\frac{1}{2}\sum_{j=1}^{n} (\mathrm{Z}_j\bar{\mathrm{Z}}_j + \bar{\mathrm{Z}}_j\mathrm{Z}_j)$ be the hypoelliptic « sublaplacian » studied by Folland and Stein in [5]. The operators $i\mathrm{T}$ and $\mathscr{L}$ generate a commutative subalgebra $\mathcal{A}$ of the complexified, universal enveloping algebra $\mathfrak{U}(\mathfrak{h}_n)$, whose spectrum is contained in the subset $\Gamma_n = \{(\lambda,x) : x \geqslant n|\lambda|\}$ of $\mathbf{R}^2$. We consider differential operators on $\mathbf{H}_n$ that can be written in the form $P(iT,\mathscr{L})$, where P is a homogeneous polynomial of degree d in two variables, satisfying the following assumption :

(A) there exists a positive constant C such that $P(\lambda,x) \geqslant Cx^d$ in Γ_n.

Assumption (A) implies that $P(iT,\mathscr{L})$ is a formally nonnegative, hypoelliptic differential operator, homogeneous of degree d, with respect to dilations. Since $P(iT,\mathscr{L})$ is left invariant, via Fourier transform on $\mathbf{H}_n$ it is easily seen that it has a unique self-adjoint extension $\bar{P}$ to $L^2(\mathbf{H}_n)$. Let $E(\lambda)$ denote the spectral resolution of $\bar{P}$, which we normalize so that it is continuous to the left. Then $E(\lambda)$ is a right convolution operator on $\mathbf{H}_n$ with a kernel $e_\lambda \in C^\infty(\mathbf{H}_n)$. Therefore $E(\lambda)f$ can also be defined for distributions f satisfying proper growth conditions, for instance, possibly for $f \in L^p(\mathbf{H}_n)$. It is clear that $E(\lambda)f \to f$ in L^2 norm if f is in $L^2(\mathbf{H}_n)$ and $\lambda \to \infty$. However, in general, for $p \neq 2$, $E(\lambda)f$ fails to converge to f in L^p norm, unless a suitable summation method is applied. In this paper we shall consider Riesz means. For all α, with $\operatorname{Re}\alpha > 0$, consider the operator

$$S_R^\alpha = \int_0^{R-0} \left(1 - \frac{\lambda}{R}\right)^\alpha dE(\lambda)$$

which is given by right convolution with the kernel

$$s_R^\alpha(g) = \int_0^{R-0} \left(1 - \frac{\lambda}{R}\right)^\alpha de_\lambda(g), \qquad g \in \mathbf{H}_n.$$

We have then :

THEOREM 1.1. – *Suppose that $f \in L^p(\mathbf{H}_n)$, $1 \leqslant p \leqslant 2$. Let $Q = 2n+2$ be the homogeneous dimension of $\mathbf{H}_n$. Then :*

(N) $\lim\limits_{R\to\infty} S_R^\alpha f = f$ *in L^p norm, provided* $\operatorname{Re}\alpha > (Q-1)\left[\dfrac{1}{p} - \dfrac{1}{2}\right]$;

(LP) $\lim\limits_{R\to\infty} S_R^\alpha f(g) = f(g)$ *when* $\operatorname{Re}\alpha > (Q-1)/p$, *if g is a Lebesgue point of f in the sense*

$$r^{-Q} \int_{|h|\leqslant r} |f(gh) - f(g)|^p \, dh \to 0 \qquad as \qquad r \to 0;$$

(AE) $\lim_{R \to \infty} S_R^\alpha f(g) = f(g)$ *for almost every* g, *if*

$$\operatorname{Re}\alpha > (Q-1)\left[\frac{2}{p} - 1\right];$$

(L) *if* f *vanishes in a neighborhood of* g *then*

$$S_R^\alpha f(g) = o(R^{[(Q-1)/p - \operatorname{Re}\alpha]/2d})$$

as $R \to \infty$, *provided* $\operatorname{Re}\alpha > (Q-1)/p$.

We also have

THEOREM 1.2. – *Suppose that* $f \in L^p(H_n)$, $2 \leqslant p < \infty$. *Then*

(N*) $\lim_{R \to \infty} S_R^\alpha f = f$ *in* L^p *norm, provided*

$$\operatorname{Re}\alpha > (Q-1)\left[\frac{1}{2} - \frac{1}{p}\right];$$

(LP*) $\lim_{R \to \infty} S_R^\alpha f(g) = f(g)$ *when* $\operatorname{Re}\alpha > (Q-1)/2$ *if* g *is a Lebesgue point for* f;

(AE*) $\lim_{R \to \infty} S_R^\alpha f(g) = f(g)$ *for almost every* g, *if*

$$\operatorname{Re}\alpha > \frac{Q-1}{2}\left[1 - \frac{2}{p}\right];$$

(L*) *if* f *vanishes in a neighborhood of* g *then*

$$S_R^\alpha f(g) = o(R^{[(Q-1)/2 - \operatorname{Re}\alpha]/2d})$$

as $R \to \infty$, *provided* $\operatorname{Re}\alpha > (Q-1)/2$.

The proof of Theorems 1.1 and 1.2 hinges on the following maximal inequality

$$\sup_{R>0} |S_R^\alpha f(g)| \leqslant C\,[M(|f|^p)(g)]^{1/p}$$

which holds for $\operatorname{Re}\alpha > (Q-1)/p$, when $1 \leqslant p \leqslant 2$ and for

$\mathrm{Re}\alpha > (Q-1)/2$, when $2 \leqslant p < \infty$. Here M denotes the Hardy-Littlewood maximal function on $\mathbf{H}_N$, considered as a space of homogeneous type, in the sense of [3]. The maximal inequality is based on estimates for the kernel of the Riesz means s_R^α, obtained adapting to the Heisenberg group a technique used by Peetre [13] to deal with constant coefficient elliptic differential operators on $\mathbf{R}^n$: the kernel s_R^α is decomposed into a sum of two terms such that one has no spectrum at 0 and the other has a smooth Fourier transform. To estimate the second part we use the differential calculus on the dual of $\mathbf{H}_n$ developed in [7].

We proceed now to an outline of the paper.

In section 2 we sketch some basic results of harmonic analysis on $\mathbf{H}_n$, which are relevant to the study of the algebra $\mathcal{A}$ generated by the operators $i\mathrm{T}$ and $\mathcal{L}$.

In section 3 we introduce the differential operators on the dual of $\mathbf{H}_n$ and develop their calculus. The main result here is Theorem 3.2 which allows us to obtain the estimates for the kernel of the Riesz means in section 4.

Section 5 is devoted to the proof of Theorems 1.1 and 1.2.

We conclude in section 6 with a discussion of open problems.

2. Preliminaries.

As general references for harmonic analysis on the Heisenberg group we shall use [5] and [7]. The $2n+1$ – dimensional Heisenberg group $\mathbf{H}_n$ is the manifold $\mathbf{R} \times \mathbf{C}^n$, with multiplication rule

$$(t,z)(t',z') = (t+t'+2\mathrm{Im}\, z.\bar{z}',z+z')$$

where, if z be the vector $(z_1,\ldots,z_n)$, then $z.\bar{z}'$ is $\sum_{j=1}^{n} z_j\bar{z}_j$. Two groups of automorphisms play an important role for the analysis on $\mathbf{H}_n$: the group of « dilations » $\{\gamma_\varepsilon : \varepsilon \in \mathbf{R}_+\}$ and the group of « rotations » $\{\rho_u : u \in \mathrm{U}(n)\}$ acting by

$$\gamma_\varepsilon(t,z) = (\varepsilon^2 t,\varepsilon z)$$
$$\rho_u(t,z) = (t,u.z)$$

where $u.z$ denotes the action of the $n \times n$ unitary matrix u on the vector z. If f is a measurable function on $\mathbf{H}_n$ we shall write $\gamma_\varepsilon.f$ and $\rho_u.f$ for the composite functions $f \circ \gamma_\varepsilon$ and $f \circ \rho_u$. If A is an operator mapping measurable functions on $\mathbf{H}_n$ to measurable functions on $\mathbf{H}_n$, $\gamma_\varepsilon.\mathrm{A}$ and $\rho_u.\mathrm{A}$ shall denote the operators

$$f \to \gamma_\varepsilon^{-1}.(\mathrm{A}(\gamma_\varepsilon.f)) \qquad \text{and} \qquad f \to \rho_u^{-1}.(\mathrm{A}(\rho_u.f)).$$

A measurable function f on $\mathbf{H}_n$ is homogeneous of degree $\zeta (\zeta \in \mathbf{C})$ if $\gamma_\varepsilon.f = \varepsilon^\zeta f$ for all $\varepsilon > 0$; f is $\mathrm{U}(n)$-invariant if $\rho_u f = f$ for all $u \in \mathrm{U}(n)$. Homogeneous and $\mathrm{U}(n)$-invariant operators are similarly defined. We write $|g|$ for the norm of $g = (t,z)$ in $\mathbf{H}_n$, which is defined by the formula

$$|g| = (t^2 + |z|^4)^{1/4}$$

where $|z|$ is the usual length of the vector z. The norm is homogeneous of degree 1 relative to the dilations γ_ε. We define the homogeneous dimension Q of $\mathbf{H}_n$ to be $2n + 2$. This is because $d(\gamma_\varepsilon g) = \varepsilon^{\mathrm{Q}-1} dg$, where dg denotes the Haar measure on $\mathbf{H}_n$ (which coincides with the Lebesgue measure on $\mathbf{R} \times \mathbf{C}^n$).

The operators we consider in this paper form a subset of the algebra $\mathcal{A}$ of all $\mathrm{U}(n)$-invariant differential operators in the enveloping algebra $\mathfrak{U}(\mathfrak{h}_n)$. It is easily seen that the operators $i\mathrm{T}$ and $\mathscr{L}$ are in $\mathcal{A}$. In fact we shall see that they generate $\mathcal{A}$. The operator $\mathscr{L}$, which is hypoelliptic, turns out to play much the same fundamental role on $\mathbf{H}_n$ as the ordinary Laplacian does on $\mathbf{R}^n$.

We next recall the definition of Fourier transform for $\mathbf{H}_n$. In defining it we shall only be concerned with the infinite dimensional representations π_λ, $\lambda \in \mathbf{R}_* = \{\lambda \in \mathbf{R}, \lambda \neq 0\}$. We recall that they can be realized on the same Hilbert space $\mathscr{H}$, which in the Schrödinger realization is $\mathrm{L}^2(\mathbf{R}^n)$. The Fourier transform of a function $f \in \mathrm{L}^1(\mathbf{H}_n)$ is the operator valued function

$$\hat{f}(\lambda) = \int_{\mathbf{H}_n} f(g)\pi_\lambda(g)\, dg.$$

Denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$ and by tr the canonical trace on $\mathscr{B}(\mathscr{H})$. For $\mathrm{S} \in \mathscr{B}(\mathscr{H})$, $1 \leqslant p < \infty$ let

$$|||\mathrm{S}|||_p = (\mathrm{tr}(\mathrm{S}^*\mathrm{S})^{p/2})^{1/p}$$

and let $|||S|||_\infty$ be the usual operator norm of S. Then we have the Plancherel formula

$$\|f\|_2^2 = C_n \int_{\mathbf{R}_*} |||\hat{f}(\lambda)|||_2^2 |\lambda|^n d\lambda$$

$C_n = 2^{n-1}/\pi^{n+1}$, which allows us to extend the Fourier transform as an isometry $\mathscr{I}$ from $L^2(\mathbf{H}_n)$ onto the Hilbert space $\mathfrak{L}^2$ of all operator valued functions $F : \mathbf{R}_* \to \mathscr{B}(\mathscr{H})$ such that

i) $(F(\lambda)\xi,\eta)$ is a measurable function of λ for every $\xi,\eta \in \mathscr{H}$,

ii) $\|F\|_{\mathfrak{L}^2} = (C_n \int_{\mathbf{R}_*} |||\, F(\lambda)\, |||_2^2\, |\lambda|^n\, d\lambda)^{1/2} < \infty$.

More generally, for $1 \leqslant p \leqslant \infty$, let $\mathfrak{L}^p$ denote the Banach space of all weakly measurable operator valued functions $F : \mathbf{R}_* \to \mathscr{B}(\mathscr{H})$ such that

$$\|F\|_{\mathfrak{L}^p} = (C_n \int_{\mathbf{R}_*} |||F(\lambda)|||_p^p |\lambda|^n\, d\lambda)^{1/p} < \infty$$

if $p < \infty$, and

$$\|\, F\,\|_{\mathfrak{L}^p} = \operatorname*{ess\,sup}_{\lambda \in \mathbf{R}_*} |||\, F(\lambda)\, |||_p < \infty$$

if $p = \infty$. Then the following version of the Hausdorff-Young theorem holds [10]. Let $\mathscr{I}^{-1}$ be the inverse Fourier transform, defined on $\mathfrak{L}^1$ by

$$\mathscr{I}^{-1} f(g) = \int_{\mathbf{R}_*} \operatorname{tr}\, (\pi_\lambda^*(g) F(\lambda))\, |\lambda|^n\, d\lambda$$

and then extended to $\mathfrak{L}^p$, $1 < p < 2$, by interpolation. Then $\mathscr{I}^{-1}$ maps $\mathfrak{L}^p$, $1 \leqslant p \leqslant 2$ into $L^q(\mathbf{H}_n)$, $1/p + 1/q = 1$ and

$$\|\, \mathscr{I}^{-1} F\,\|_q \leqslant C_p \|\, F\,\|_{\mathfrak{L}^p}.$$

Every representation π_λ determines a Lie algebra representation $d\pi_\lambda$ of $\mathfrak{h}_n$ as linear maps from the vector subspace $\mathscr{H}_\infty$ of C^∞ vectors in $\mathscr{H}$ into itself, defined by

$$d\pi_\lambda(X)\xi = \left.\frac{d}{ds}\right|_{s=0} \pi_\lambda\, (\exp\, (sX))\xi, \qquad X \in \mathfrak{h}_n,\ \xi \in \mathscr{H}_\infty.$$

This extends uniquely to a representation of the algebra $\mathfrak{U}(\mathfrak{h}_n)$ of left invariant differential operators on $\mathbf{H}_n$ as linear maps from $\mathscr{H}_\infty$ into itself.

The operators $d\pi_\lambda(Z_j)$, $d\pi_\lambda(\bar{Z}_j)$, $j = 1, \ldots, n$ are closable operators on $\mathscr{H}$. Denote by $W_j(\lambda)$, $\bar{W}_j(\lambda)$ their closures respectively. Geller [7] shows that for every $\lambda \in \mathbf{R}_*$ there exists an orthonormal basis $\{E_\alpha^\lambda : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n\}$ such that for $\lambda > 0$:

$$\begin{aligned} W_j(\lambda)E_\alpha^\lambda &= (2|\lambda|(\alpha_j+1))^{1/2}E_{\alpha+e_j}^\lambda \\ \bar{W}_j(\lambda)E_\alpha^\lambda &= (2|\lambda|\alpha_j)^{1/2}E_{\alpha-e_j}^\lambda \end{aligned} \tag{2.1}$$

where $e_j = (0,\ldots,1,\ldots,0) \in \mathbf{N}^n$, with the 1 in the j-th position. For $\lambda < 0$, $W_j(\lambda) = \bar{W}_j(-\lambda)$, and $\bar{W}_j(\lambda) = W_j(-\lambda)$. Let

$$H_j(\lambda) = -\frac{1}{2}(W_j(\lambda)\bar{W}_j(\lambda) + \bar{W}_j(\lambda)W_j(\lambda)), \qquad j = 1, \ldots, n$$

and denote by $H(\lambda)$ the operator $\sum_{j=1}^{n} H_j(\lambda)$, with domain $\mathrm{Dom}(H(\lambda)) = \{\xi \in \mathscr{H} : \sum_\alpha (|\alpha|+n)|(\xi,E_\alpha^\lambda)|^2 < \infty\}$. Then $H(\lambda)$ is the closure of the operator $d\pi_\lambda(\mathscr{L})$. Moreover $H(\lambda)$ is a self-adjoint operator which has a spectral resolution

$$H(\lambda) = \sum_{N=0}^{\infty} (2N+n)|\lambda|P_N^{(n)}(\lambda) \tag{2.2}$$

where $P_N^{(n)}(\lambda)$ is the orthogonal projection on the finite dimensional subspace of $\mathscr{H}$ spanned by the functions $\{E_\alpha^\lambda : |\alpha| = N\}$.

In [7], Geller proves that a function f is $U(n)$-invariant if and only if there exists a function $\mathscr{R}_f : \mathbf{R}_* \times \mathbf{N} \to \mathbf{C}$ such that

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} \mathscr{R}_f(\lambda,N)P_N^{(n)}(\lambda).$$

The Fourier transform of $U(n)$-invariant differential operators can be similarly characterized.

PROPOSITION 2.1. — *The algebra $\mathfrak{A}$ of $U(n)$-invariant operators in $\mathfrak{U}(\mathfrak{h}_n)$ is the polynomial algebra in two variables $\mathbf{C}[iT,\mathscr{L}]$. Given $P = P(iT,\mathscr{L})$ in $\mathfrak{A}$, one has*

$$d\pi_\lambda(P) = \sum_{N=0}^{\infty} P(\lambda, (2N+n)|\lambda|)P_N^{(n)}(\lambda) = P(\lambda I, H(\lambda)) \tag{2.3}$$

for every $\lambda \in \mathbf{R}_*$.

Proof. – Let G be the semidirect product of the groups $U(n)$ and $\mathbf{H}_n$. Then, with the notation of [8, Ch. X], $\mathcal{A}$ is the algebra $D(G/U(n))$ of G-invariant differential operators on $G/U(n) \cong \mathbf{H}_n$. Since the Lie algebra of G is the direct sum of $\mathfrak{h}_n$ and the Lie algebra of $U(n)$ and $\mathrm{Ad}_G(U(n))\mathfrak{h}_n \subset \mathfrak{h}_n$, $\mathcal{A}$ is the image under the Harish-Chandra symmetrization map of the set $I(\mathfrak{h}_n)$ of polynomials on $\mathfrak{h}_n$, which are $\mathrm{Ad}_G(U(n))$-invariant. With our choice of coordinates we can identify $\mathfrak{h}_n$ with $\mathbf{H}_n$ and the adjoint action of $U(n)$ on $\mathfrak{h}_n$ with the action of $U(n)$ on $\mathbf{H}_n$, via the exponential mapping $(t,z) = \exp(tT+zZ+\bar{z}\bar{Z})$. Then a polynomial function $P(tT+zZ+z\bar{Z})$ is in $I(\mathfrak{h}_n)$ if and only if it can be written

$$P = \sum_{k,m} a_{k,m} t^k |z|^{2m}.$$

Since the images of t and $|z|^2$ under the symmetrization map are T and $-\mathcal{L}$, we have proved that $\mathcal{A} = \mathbf{C}[iT,\mathcal{L}]$. To prove (2.3) we only need to observe that $d\pi_\lambda(T) = -i\lambda T$.

COROLLARY 2.2. – *Let* P *be a homogeneous polynomial in two variables of degree* d. *Then the operator* $P(iT,\mathcal{L})$ *is hypoelliptic and formally nonnegative if and only if there exists a positive constant* C *such that* $P(\lambda,x) \geqslant Cx^d$ *in the set*

$$C_n = \{(\lambda,x) \in \mathbf{R}^2 : x = (2N+n)|\lambda|, N \in \mathbf{N}\}.$$

Proof. – The operator $P(iT,\mathcal{L})$ is hypoelliptic if and only if for every irreducible nontrivial unitary representation π of $\mathbf{H}_n$, $d\pi(P(iT,\mathcal{L}))$ has a bounded two sided inverse [15]. There are two classes of irreducible unitary representations of $\mathbf{H}_n$: the family of 1-dimensional representations $\{\pi_\zeta : \zeta \in \mathbf{C}^n\}$ mapping T to 0, Z_j to $\sqrt{-1}\,\zeta_j$ and $\bar{Z}_j$ to $\sqrt{-1}\,\bar{\zeta}_j$; and the family of infinite dimensional representations $\{\pi_\lambda : \lambda \in \mathbf{R}_*\}$ described earlier. Since $d\pi_\zeta(P(iT,\mathcal{L})) = P(0,|\zeta|^2/2)$ and $d\pi_\lambda(P(iT,\mathcal{L})) = P(\lambda I, H(\lambda))$, the Corollary follows easily from (2.3) and the homogeneity of P.

Now consider the Lie algebra $\mathfrak{h}_n$ as an abelian group under addition and let C^k denote the space of all functions that possess continuous and bounded partial derivatives up to the order k, with respect to a fixed linear coordinate system on $\mathfrak{h}_n$. Since $\exp : \mathfrak{h}_n \to \mathbf{H}_n$ is a diffeomorphism, we can also regard C^k as a space of functions on $\mathbf{H}_n$.

COROLLARY 2.3. – *Let* $P(iT,\mathscr{L})$ *be as in Corollary* 2.2, *and denote by* $\bar{P}$ *its closure in* $L^2(\mathbf{H}_n)$. *Then* $\mathrm{Dom}(\bar{P}^m) \subset C^k$, *provided* $2md > 2k + Q/2$.

Proof. – By Corollary 2.2 and the Plancherel formula the operator valued function

$$\lambda \to M(\lambda) = \sum_N [(2N+n)|\lambda|]^{md} P(\lambda, (2N+n)|\lambda|)^{-m} P_N^{(n)}(\lambda)$$

is a Fourier multiplier of $L^2(\mathbf{H}_n)$ [11]. Thus $\mathrm{Dom}(\bar{P}^m) \subset \mathrm{Dom}(\mathscr{L}^{md}) \subset C^k$, provided $2md > 2k + Q/2$ by Corollary 5.16 of [4].

COROLLARY 2.4. – *The spectral function of the operator* $\bar{P}$ *is in* C^∞.

We end this section with a result relating the action of the group of dilations to the Fourier transform. We state it in the setting of $U(n)$-invariant functions, even though it holds more in general.

PROPOSITION 2.5. – *Let* f *be a* $U(n)$-*invariant function in* $L^p(\mathbf{H}_n)$, $1 \leqslant p \leqslant 2$. *Then*

$$(\gamma_\varepsilon . f)\hat{}\,(\lambda) = \varepsilon^{-Q} \sum_{N=0}^{\infty} \mathscr{R}_f(\varepsilon^{-2}\lambda, N) P_N^{(n)}(\lambda).$$

For the proof see [7].

3. Differential calculus on the dual of $\mathbf{H}_n$.

In this section we define differential operators on the dual of $\mathbf{H}_n$ as Fourier transforms of operators of multiplication by polynomials in the coordinate functions on $\mathbf{H}_n$. Given a polynomial $P = P(t,z,\bar{z})$ define a differential operator Δ_P acting on « smooth » functions $F : \mathbf{R}_* \to \mathscr{B}(\mathscr{H})$ as follows. Suppose first that $F = \hat{f}$ for some function f in the Schwartz space $\mathscr{S}(\mathbf{H}_n)$. Then

$$\Delta_P F(\lambda) = (Pf)\hat{}\,(\lambda), \qquad \lambda \in \mathbf{R}_*.$$

Clearly the set of all such operators is an algebra generated by the operators Δ_{it}, Δ_{z_j}, $\Delta_{\bar{z}_j}$, $j = 1,\ldots,n$. Geller [7] has defined these operators in a more general setting, that we presently describe. Let $\mathscr{D}_\lambda$ be the vector space spanned algebraically by the functions $\{E^\lambda_\alpha : \alpha \in \mathbf{N}^n\}$, and denote by $Op_\lambda(\mathscr{H})$ the space of all (possibly unbounded) linear operators mapping $\mathscr{D}_\lambda$ into $\mathscr{H}$. For every function $F : \mathbf{R}_* \to Op_\lambda(\mathscr{H})$ such that $\mathrm{Range}(F(\lambda)) \subset \mathrm{Dom}(W_j(\lambda)) = \mathrm{Dom}(\bar{W}_j(\lambda))$ set

$$\begin{aligned} \delta_j F(\lambda) &= -\frac{1}{2\lambda}[F(\lambda),\bar{W}_j(\lambda)] \\ \bar{\delta}_j F(\lambda) &= \frac{1}{2\lambda}[F(\lambda),W_j(\lambda)] \end{aligned} \tag{3.1}$$

$$\begin{aligned} W_j F(\lambda) &= W_j(\lambda)F(\lambda) \\ \bar{W}_j F(\lambda) &= \bar{W}_j(\lambda)F(\lambda). \end{aligned} \tag{3.2}$$

Here $[A,B]$ denotes the commutator of A and B. Furthermore if for every $\alpha, \beta \in \mathbf{N}^n$ the function $\lambda \to (F(\lambda)E^\lambda_\alpha,E^\lambda_\beta)$ is differentiable, let $D_\lambda F$ be defined by

$$(D_\lambda F(\lambda)E^\lambda_\alpha,E^\lambda_\beta) = \frac{d}{d\lambda}(F(\lambda)E^\lambda_\alpha,E^\lambda_\beta).$$

Then if $F(\lambda) = \hat{f}(\lambda)$ for some $f \in \mathscr{S}(\mathbf{H}_n)$:

$$\Delta_{z_j}F(\lambda) = -\delta_j F(\lambda), \qquad \Delta_{\bar{z}_j}F(\lambda) = \bar{\delta}_j F(\lambda) \tag{3.3}$$

$$\Delta_{it}F(\lambda) = D_\lambda F(\lambda) - \frac{1}{2\lambda}\sum_{j=1}^{n}(W_j\delta_j+\bar{W}_j\bar{\delta}_j)F(\lambda). \tag{3.4}$$

(See [7, Proposition 1.2 and Lemma 2.2]).

We remark that the operators δ_j, $\bar{\delta}_j$ are derivations, i.e. :

$$\delta_j(ST) = (\delta_j S)T + S(\delta_j T)$$

and similarly for $\bar{\delta}_j$, whenever $S, T, ST : \mathbf{R}_* \to Op_\lambda(\mathscr{H})$ are such that $\mathrm{Range}(S(\lambda)) \cup \mathrm{Range}(T(\lambda)) \subset \mathrm{Dom}(W_j(\lambda))$. Euristically it is convenient to think of δ_j and $\bar{\delta}_j$ as derivatives with respect to W_j and $\bar{W}_j$ respectively, in the sense specified by the following lemma.

LEMMA 3.1. – *For every* j, $k = 1, \ldots, n$ *we have :*

i) $\delta_j \bar{W}_k = \bar{\delta}_j W_k = 0$,

ii) $\delta_j W_k = \bar{\delta}_j \bar{W}_k = 0$ *if* $j \neq k$,

iii) $\delta_j W_j^\ell = \ell W_j^{\ell-1}$, $\quad \bar{\delta}_j \bar{W}_j = \ell \bar{W}_j^{\ell-1}$, $\quad \ell \in \mathbf{N}$.

Proof. – Since $[Z_j, \bar{Z}_k] = -2i\delta_{jk}T$, $[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = 0$ and $d\pi_\lambda(Z_j) = W_j(\lambda)$, $d\pi_\lambda(\bar{Z}_j) = \bar{W}_j(\lambda)$, $d\pi_\lambda(T) = -i\lambda T$, i) and ii) follow at once, while iii) follows by induction.

In the following, given multiindices α, $\beta \in \mathbf{N}^n$, we shall adopt the notation :

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$$
$$W^\alpha = W_1^{\alpha_1} \ldots W_n^{\alpha_n}, \qquad \bar{W}^\alpha = \bar{W}_1^{\alpha_1} \ldots \bar{W}_n^{\alpha_n}$$
$$\delta^\alpha = \delta_1^{\alpha_1} \ldots \delta_n^{\alpha_n}, \qquad \bar{\delta}^\alpha = \bar{\delta}_1^{\alpha_1} \ldots \bar{\delta}_n^{\alpha_n}$$
$$W.\delta = \sum_{j=1}^{n} W_j \delta_j, \qquad \bar{W}.\bar{\delta} = \sum_{j=1}^{n} \bar{W}_j \bar{\delta}_j.$$

Let Λ denote the operator Δ_P for $P(t,z,\bar{z}) = t^2 + |z|^4$, which is the Fourier transform of the operator of multiplication by the 4-th power of the norm function. A crucial step in the study of the asymptotic behaviour of the kernel of the Riesz means for an hypoelliptic operator $P = P(iT, \mathscr{L})$ is the estimate of the $\mathfrak{L}^p$ norm of the function $\lambda \to \Lambda^N F(P(\lambda I, H(\lambda)))$, $N = 1, 2, \ldots$, for a function $F \in C_c^\infty(\bar{\mathbf{R}}_+)$ in terms of the L^p norm of F and a certain number of its derivatives.

THEOREM 3.2. – *Let* F *be a function in* $C^\infty(\mathbf{R})$, *with support in* $[0,1]$, P *a homogeneous polynomial in two variables of degree* d, *satisfying assumption* (A). *Then for every positive integer* N, *there exists a constant* $C = C(n,N,P)$ *such that for every* p, $1 \leqslant p \leqslant \infty$:

$$\|\Lambda^N(F \circ P)\|_{\mathfrak{L}^p} \leqslant C \sum_{j=1}^{4N} \left(\int_0^1 |F^{(j)}(\rho^d)|^p \rho^n \, d\rho \right)^{1/p}.$$

The proof of Theorem 3.2 rests on the following Proposition.

PROPOSITION 3.2. – *Let* F, P *be as in Theorem* 3.2. *Then*

$\Lambda^N F(P(\lambda I,H(\lambda)))$ *can be written as a linear combination of terms either of the form :*

$$\varepsilon(\mathrm{sign}(\lambda))Q(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))(F^{(m)}\circ P)(\lambda I,H(\lambda)) \tag{3.6}$$

$m = 1,\ldots,4N-1$, *where* Q *is a polynomial in* $n+1$ *variables, and* $\varepsilon(\mathrm{sign}(\lambda)) = \pm 1$, *or of the form :*

$$\begin{aligned}\varepsilon(\mathrm{sign}(\lambda))&R(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))\\ &\times\int_{-\ell}^{k}(F^{(m)}\circ P)(\lambda I,H(\lambda)+2s|\lambda|I)\,\mu(s)\,ds\end{aligned} \tag{3.7}$$

where k,ℓ *are positive integers,* $m = 1,\ldots,4N$, μ *is a bounded function and* R *is a polynomial in* $n+1$ *variables such that*

$$R(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))E^{\lambda}_{\alpha} = 0, \qquad |\alpha| \leqslant \ell - 1. \tag{3.8}$$

Before proceeding to the proof of Proposition 3.2, we show how Theorem 3.2 follows from it. By assumption (A) the set

$$K_1 = \{(\lambda,x_1,\ldots,x_n)\in \mathbf{R}\times\mathbf{R}^n_+ :\ (\lambda,x_1+\ldots+x_n)\in\Gamma_n,\qquad P(\lambda,x_1+\ldots+x_n)\in\mathrm{supp}(F)\}$$

is bounded in $\mathbf{R}\times\mathbf{R}^n_+$. Hence the $\mathfrak{L}^p$ norm of (3.6) can be estimated by $\sup\limits_{K_1}|Q|\,\|F^{(m)}\circ P\|_{\mathfrak{L}^p}$. To estimate $\|F^{(m)}\circ P\|_{\mathfrak{L}^p}$ we prove the following lemma.

LEMMA 3.3. — *Let* G *be a function in* $C^\infty(\mathbf{R}_+)$. *Then there exists a constant* B, *depending on* n *and on the constant* C *of assumption* (A), *such that*

$$\|G\circ P\|_{\mathfrak{L}^p} \leqslant B\left(\int_{\mathbf{R}_+}|G(\rho^d)|^p\rho^n\,d\rho\right)^{1/p}.$$

Proof. — The operator $G\circ P(\lambda I,H(\lambda))$ has the spectral resolution $\sum\limits_{\alpha} G(P(\lambda,(2|\alpha|+n)|\lambda|))P^{(n)}_{\alpha}(\lambda)$, where $P^{(n)}_{\alpha}(\lambda)$ is the orthogonal projection on the space generated by E^{λ}_{α}. Then

$$\|G\circ P\|_{\mathfrak{L}^p} = \left[C_n\int_{\mathbf{R}_*}\sum_{\alpha\in\mathbf{N}^n}|G(P(\lambda,(2|\alpha|+n)|\lambda|))|^p\,|\lambda|^n\,d\lambda\right]^{1/p}.$$

Decomposing the integral over $\mathbf{R}_*$ into two integrals I_+ and I_-, over $\mathbf{R}_+$ and $\mathbf{R}_-$ respectively, and performing the change of variables $\rho^d = \lambda^d P(1,2|\alpha|+n)$ in I_+ and $\rho^d = |\lambda|^d P(-1,2|\alpha|+n)$ in I_-, we obtain

$$(3.9)\quad \|G\circ P\|_{\mathfrak{L}^p} = \left\{ C_n \left[\sum_{N=0}^{\infty} \omega_n(N)(P(1,2N+n)^{-(n+1)/d} + P(-1,2N+n)^{-(n+1)/d}) \right] \int_0^{+\infty} |G(\rho^d)|^p \rho^n \, d\rho \right\}^{1/p},$$

where $\omega_n(N) = \sum_{|\alpha|=N} 1 = O(N^{n-1})$. Since

$$P(\pm 1,2N+n) \geqslant C(2N+n)^d$$

the series in (3.9) converges and the Lemma is proved.

Hence it remains only to estimate the terms of the form (3.7). Let S be any such term and denote by $\prod_\ell$ the orthogonal projection onto the subspace of $\mathscr{H}$ spanned by the vectors E_α^λ, $|\alpha| \geqslant \ell$. Thus, by (3.8), $S\prod_\ell = S$. On the other hand, by an argument similar to the one used in the estimate of (3.6)

$$\left\| S \prod_\ell \right\|_{\mathfrak{L}^p} \leqslant \sup_{K_2} |R| \left[C_n \int_{\mathbf{R}_*} \sum_{|\alpha| \geqslant \ell} \left| \int_{-\ell}^{k} F^{(m)}(P(\lambda,(2(|\alpha|+s)+n)|\lambda|)) \mu(s)\, ds \right|^p |\lambda|^n \, d\lambda \right]^{1/p},$$

where K_2 is a bounded subset of $\mathbf{R} \times \mathbf{R}_+^n$. Now, applying Minkowski's inequality, we see that the right hand side is bounded by

$$C' \int_{-\ell}^{k} \left[\int_{\mathbf{R}_*} \sum_{|\alpha| \geqslant \ell} |F^{(m)}(P(\lambda,(2(|\alpha|+s)+n)|\lambda|))|^p |\lambda|^n \, d\lambda \right]^{1/p} ds.$$

Hence, by the same argument used in Lemma 3.3, we get

$$\|S\|_{\mathfrak{L}^p} \leqslant C \left(\int_{-\ell}^{k} \sum_{N=\ell}^{+\infty} [2(N+s)+n]^{-2} \, ds \right)^{1/p} \left[\int_0^{+\infty} |F^{(m)}(\rho^d)|^p \rho^n \, d\rho \right]^{1/p}.$$

This concludes the proof of Theorem 3.2.

Proof of Proposition 3.2. – We break the proof into several Lemmas.

LEMMA 3.4. — *Let* p *be a positive integer. Then*

$$(W.\delta-\bar{W}.\bar{\delta})^p=\sum_{\alpha,\beta}P_{\alpha,\beta}(W,\bar{W})\delta^\alpha\bar{\delta}^\beta$$

where α,β *are multiindices in* $\mathbf{N}^n$ *such that* $1\leqslant|\alpha+\beta|\leqslant p$ *and* $P_{\alpha,\beta}$ *is a noncommutative monomial in the variables* $W,\bar{W}$, *of degree* α *in* W *and* β *in* $\bar{W}$.

Proof. — By induction on p, applying Lemma 3.1 and Leibniz's formula to the derivatives $\delta_j,\bar{\delta}_j$.

LEMMA 3.5. — *The operator* D_λ *commutes with the operators* $|\lambda|^{1/2}\delta_j$, $|\lambda|^{1/2}\bar{\delta}_j$, $W_j\delta_j$, $\bar{W}_j\bar{\delta}_j$ *for* $j=1,\dots,n$.

Proof. — Let α, $\beta\in\mathbf{N}^n$. By (2.1) the functions

$$|\lambda|^{1/2}((1/2\lambda)W_j(\lambda)E^\lambda_\alpha,E^\lambda_\beta)\qquad\text{and}\qquad|\lambda|^{1/2}((-1/2\lambda)\bar{W}_j(\lambda)E^\lambda_\alpha,E^\lambda_\beta)$$

depend only on sign (λ). Since δ_j, $\bar{\delta}_j$ are the operators of commutation with $(-1/2\lambda)\bar{W}_j(\lambda)$ and $(1/2\lambda)W_j(\lambda)$ respectively, D_λ commutes with $|\lambda|^{1/2}\delta_j$, $|\lambda|^{1/2}\bar{\delta}_j$. The argument for $W_j\delta_j$ and $\bar{W}_j\bar{\delta}_j$ is similar.

LEMMA 3.6. — *Let* p *be a positive integer,* $\gamma\in\mathbf{N}^n$. *Then the operator*

$$\left(D_\lambda-\frac{1}{2\lambda}(W.\delta+\bar{W}.\bar{\delta})\right)^p\delta^\gamma\bar{\delta}^\gamma$$

is a linear combination of operators of the form

$$\lambda^{-p}P_{\alpha,\beta}(W,\bar{W})\delta^{\alpha+\gamma}\bar{\delta}^{\beta+\gamma}\lambda^kD^k_\lambda\tag{3.10}$$

where $\alpha,\beta\in\mathbf{N}^n$, $k\in\mathbf{N}$ *are such that* $|\alpha+\beta|+k\leqslant p$, *and* $P_{\alpha,\beta}$ *is a noncommutative monomial in the variables* $W,\bar{W}$ *of degree* α *in* W, β *in* $\bar{W}$.

Proof. — Let $\mathscr{C}=W.\delta+\bar{W}.\bar{\delta}$. Since by Lemma 3.5 D_λ commutes with $\mathscr{C}$, we have

$$\left(D_\lambda-\frac{1}{2\lambda}\mathscr{C}\right)^p=\sum_{i=0}^p\lambda^{-i}Q_i(\mathscr{C})D^{p-i}_\lambda$$

where $Q_i(\mathscr{C})$ is a polynomial of degree i in the indeterminate $\mathscr{C}$. Hence, by Lemma 3.4 :

$$\left(D_\lambda - \frac{1}{2\lambda}\mathscr{C}\right)^p \delta^\gamma\bar{\delta}^\gamma = \sum_{0\leqslant \ell \leqslant i \leqslant p} C_\ell \mathscr{C}^\ell \lambda^{-i} D_\lambda^{p-i}\delta^\gamma\bar{\delta}^\gamma$$

$$(3.11) \qquad = \lambda^{-p} \sum_{\alpha,\beta,\ell,j} C_\ell P_{\alpha,\beta}(W,\bar{W})\delta^\alpha\bar{\delta}^\beta\lambda^j D_\lambda^j \delta^\gamma\bar{\delta}^\gamma$$

where $P_{\alpha,\beta}$ is as in the thesis and the sum is extended over the set of indices $\alpha,\beta \in \mathbf{N}^n, \ell, j \in \mathbf{N}$ such that $|\alpha+\beta| \leqslant \ell \leqslant p-j \leqslant p$. Now, by Lemma 3.5 :

$$\begin{aligned} D_\lambda^j \delta^\gamma\bar{\delta}^\gamma &= D_\lambda^j |\lambda|^{-|\gamma|}(|\lambda|^{1/2}\delta)^\gamma(|\lambda|^{1/2}\bar{\delta})^\gamma \\ &= \sum_{k=0}^{j} C_{j,k,\gamma}\delta^\gamma\bar{\delta}^\gamma\lambda^{k-j}D_\lambda^k . \end{aligned}$$

This shows that each summand in (3.11) is of the form (3.10).

Now, given a function $G : \mathbf{R}_* \times \mathbf{R}_+ \to \mathbf{C}$, define the forward and backward difference operators of step $2|\lambda|$ as follows :

$$\begin{aligned} D_{+,\lambda}G(\lambda,x) &= (2|\lambda|)^{-1}[G(\lambda,x+2|\lambda|) - G(\lambda,x)] \qquad (\lambda,x) \in \mathbf{R}_* \times \mathbf{R}_+ \\ D_{-,\lambda}G(\lambda,x) &= (2|\lambda|)^{-1}[G(\lambda,x-2|\lambda|) - G(\lambda,x)] \qquad x \geqslant 2|\lambda| . \end{aligned}$$

LEMMA 3.7. — *Let k be a positive integer, $\alpha,\beta \in \mathbf{N}^n$, P a homogeneous polynomial in two variables. Then for any function $F \in C_c^\infty(\bar{\mathbf{R}}_+)$:*

$$(3.12) \quad \lambda^k D_\lambda^k F(P(\lambda I, H(\lambda))) = \sum_{j=1}^{k} Q_j(\lambda I, H(\lambda))(F^{(j)} \circ P)(\lambda I, H(\lambda))$$

where the Q_j's are polynomials. Moreover for $\lambda > 0$

$$(3.13) \quad \delta^\alpha\bar{\delta}^\beta F(P(\lambda I, H(\lambda)))$$

$$= \sum_\gamma C_{\alpha,\beta,\gamma}[W(\lambda)]^{\beta-\alpha+\gamma}[\bar{W}(\lambda)]^\gamma (D_{-,\lambda}^{|\gamma|} D_{+,\lambda}^{|\beta|}(F \circ P))(\lambda I, H(\lambda))$$

where the sum is over the set of multindices $\gamma = (\gamma_1,\ldots,\gamma_n) \in \mathbf{N}^n$ such that $\max(0,\alpha_i-\beta_i) \leqslant \gamma_i \leqslant \alpha_i$, $i = 1, \ldots, n$. For $\lambda < 0$ (3.13) holds interchanging W with $\bar{W}$, α with β in the right hand side.

Proof. – The spectral resolution of the operator $(F \circ P)(\lambda I, H(\lambda))$ is :

$$(F \circ P)(\lambda I, H(\lambda)) = \sum_{N=0}^{+\infty} F(P(\lambda,(2N+n)|\lambda|))P_N^{(n)}(\lambda).$$

Hence

$$\lambda^k D_\lambda^k (F \circ P)(\lambda I, H(\lambda)) = \sum_{N=0}^{+\infty} \left[\lambda^k \frac{d^k}{d\lambda^k} F(P(\lambda,(2N+n)|\lambda|)\right] P_N^{(n)}(\lambda),$$

and (3.12) follows by induction on k. To prove (3.13) we show first that for any function $G : \mathbf{R}_* \times \mathbf{R}_+ \to \mathbf{C}$ one has :

$$\begin{aligned} (3.14) \quad \delta_j G(\lambda I, H(\lambda)) &= \bar{W}_j(\lambda)(D_{-,\lambda} G)(\lambda I, H(\lambda)) \quad &\text{if} \quad \lambda > 0 \\ &= \bar{W}_j(\lambda)(D_{+,\lambda} G)(\lambda I, H(\lambda)) \quad &\text{if} \quad \lambda < 0 \end{aligned}$$

and

$$\begin{aligned} (3.15) \quad \bar{\delta}_j G(\lambda I, H(\lambda)) &= W_j(\lambda)(D_{+,\lambda} G)(\lambda I, H(\lambda)) \quad &\text{if} \quad \lambda > 0 \\ &= W_j(\lambda)(D_{-,\lambda} G)(\lambda I, H(\lambda)) \quad &\text{if} \quad \lambda < 0. \end{aligned}$$

Indeed it is easily seen that

$$\begin{aligned} \bar{\delta}_j P_N^{(n)}(\lambda) &= \bar{W}_j(\lambda)(2\lambda)^{-1}[P_N^{(n)}(\lambda) - P_{N+1}^{(n)}(\lambda)] \quad &\text{if} \quad \lambda > 0 \\ &= \chi_+(N)\bar{W}_j(\lambda)(2\lambda)^{-1}[P_N^{(n)}(\lambda) - P_{N-1}^{(n)}(\lambda)] \quad &\text{if} \quad \lambda < 0 \end{aligned}$$

where χ_+ is the characteristic function of the set of positive integers. Thus for $\lambda > 0$:

$$\delta_j G(\lambda I, H(\lambda)) = \bar{W}_j(\lambda) \sum_{N=0}^{+\infty} G(\lambda,(2N+n)|\lambda|)(2\lambda)^{-1}[P_N^{(n)}(\lambda) - P_{N+1}^{(n)}(\lambda)],$$

while for $\lambda < 0$:

$$\delta_j G(\lambda I, H(\lambda)) = \bar{W}_j(\lambda) \sum_{N=1}^{+\infty} G(\lambda,(2N+n)|\lambda|)(2\lambda)^{-1}[P_N^{(n)}(\lambda) - P_{N-1}^{(n)}(\lambda)].$$

Summation by parts now yields (3.14). The argument for (3.15) is similar. Now (3.13) follows from (3.14) and (3.15) by induction, applying Leibniz's formula and Lemma 3.1.

LEMMA 3.8. — *Let* Φ *be a function in* $C^\infty(\mathbf{R}^{n+1})$. *Then*

$$\Lambda^N\Phi(\lambda I,H_1(\lambda),H_2(\lambda),\ldots,H_n(\lambda)) = \Phi_N(\lambda I,H_1(\lambda),H_2(\lambda),\ldots,H_n(\lambda)),$$

with $\Phi_N \in C^\infty(\mathbf{R}^{n+1})$.

Proof. — By a straightforward induction argument we can reduce matters to prove that :

$$\delta_j\bar{\delta}_j\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)) = \Psi_1(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))$$

and

$$\left[D_\lambda - \frac{1}{2\lambda}(W\cdot\delta+\bar{W}\cdot\bar{\delta})\right]\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)) = \Psi_2(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)),$$

with $\Psi_1, \Psi_2 \in C^\infty(\mathbf{R}^{n+1})$. Let $P_\alpha^{(n)}(\lambda)$ be the orthogonal projection on the space spanned by $E_\alpha^\lambda, \alpha \in \mathbf{N}^n$. Thus

$$\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)) = \sum_{\alpha\in\mathbf{N}^n} \Phi(\lambda I,(2\alpha_1+1)|\lambda|,\ldots,(2\alpha_n+1)|\lambda|P_\alpha^{(n)}(\lambda).$$

As in the proof of Lemma 3.7 one can show that for $\lambda > 0$:

$$\delta_j\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)) = \bar{W}_j(\lambda)(D_{-,\lambda,j}\Phi)(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))$$
$$\bar{\delta}_j\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda)) = W_j(\lambda)(D_{+,\lambda,j}\Phi)(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))$$

where $D_{-,\lambda,j}$ and $D_{+,\lambda,j}$ are respectively the backward and forward difference operators of step $2|\lambda|$, acting on the variable $H_j(\lambda)$. For $\lambda < 0$ one has the same formulas, except for the interchanging of $D_{-,\lambda,j}$ and $D_{+,\lambda,j}$. Thus for $\lambda > 0$:

$$\begin{aligned}(3.16)\quad &\delta_j\bar{\delta}_j\Phi(\lambda I,H_1(\lambda),\ldots,H_n(\lambda))\\ &= (D_{+,\lambda,j}\Phi) + W_j(\lambda)\bar{W}_j(\lambda)(D_{-,\lambda,j}D_{+,\lambda,j}\Phi)\\ &= (D_{+,\lambda,j}\Phi) + (H_j(\lambda)-\lambda I)(D_{-,\lambda,j}D_{+,\lambda,j}\Phi),\end{aligned}$$

while for $\lambda < 0$ (3.16) holds interchanging $D_{+,\lambda,j}$ and $D_{-,\lambda,j}$. Let $x = (x_1,\ldots,x_n)$ and denote by e_j the unit vector in the direction of the x_j axis. Applying Taylor's theorem to (3.16) we obtain the following expression for the function Ψ_1, valid for all $\lambda \in \mathbf{R}$:

$$\begin{aligned}\Psi_1(\lambda,x) = \int_0^1 (\partial_{x_j}\Phi)(\lambda,x+2\lambda s e_j)\,ds + (x_j-\lambda)\int_0^1 (1-s)[(\partial_{x_j}^2\Phi)(\lambda,x+2\lambda s e_j)\\ + (\partial_{x_j}^2\Phi)(\lambda,x-2\lambda s e_j)ds.\end{aligned}$$

Hence $\Psi_1 \in C^\infty(\mathbf{R}^{n+1})$. The argument for Ψ_2 is similar. We omit the details.

We are now in a position to conclude the proof of Proposition 3.2. By (3.3), (3.4) and Lemma 3.6, the operator Λ^N is a linear combination of operators of the form

$$\lambda^{-p} P_{\alpha,\beta}(W,\bar{W})\delta^{\alpha+\gamma}\bar{\delta}^{\beta+\gamma}\lambda^k D_\lambda^k \tag{3.17}$$

where $p, k \in \mathbf{N}$, $\alpha, \beta, \gamma \in \mathbf{N}^n$ are such that $p + |\gamma| = 2N$, $|\alpha+\beta| + k \leqslant p$ and $P_{\alpha,\beta}$ is as in the statement of Lemma 3.6. Applying one of the operators in (3.17) to the function $(F \circ P)(\lambda I, H(\lambda))$, by Lemma 3.7 we obtain a linear combination

$$\lambda^{-p} \sum_{\varepsilon, j} A_{\alpha,\beta,\varepsilon} B_{\beta,\gamma,\varepsilon,j} \tag{3.18}$$

$j = 1, \ldots, k$, $\varepsilon \in \mathbf{N}^n$, $\max(0, \alpha_i - \beta_i) \leqslant \varepsilon_i \leqslant \alpha_i + \gamma_i$, where

$$\begin{aligned} A_{\alpha,\beta,\varepsilon} &= P_{\alpha,\beta}(W,\bar{W}) W^{\beta-\alpha+\varepsilon} \bar{W}^{\varepsilon} \quad \text{for} \quad \lambda > 0 \\ &= P_{\alpha,\beta}(\bar{W},W) \bar{W}^{\beta-\alpha+\varepsilon} W^{\varepsilon} \quad \text{for} \quad \lambda < 0 \\ B_{\beta,\gamma,\varepsilon,j} &= [D_{-,\lambda}^{|\varepsilon|} D_{+,\lambda}^{|\beta+\gamma|}(Q_j(F^{(j)} \circ P))] \quad (\lambda I, H(\lambda)) \end{aligned}$$

and the Q_j are polynomials. Since $A_{\alpha,\beta,\varepsilon}$ is a monomial of degree $\beta + \varepsilon$ both in W and $\bar{W}$, it is easily seen by induction that $A_{\alpha,\beta,\varepsilon}$ is a polynomial of degree $|\beta+\varepsilon|$ in the variables $(\lambda I, H_1(\lambda), \ldots, H_n(\lambda))$. Moreover by (2.1) if $\alpha_j \leqslant \varepsilon_j - 1$, $\bar{W}^\varepsilon(\lambda)E_\alpha^\lambda = 0$ for $\lambda > 0$ and $W^\varepsilon(\lambda)E_\alpha^\lambda = 0$ for $\lambda < 0$. Hence $A_{\alpha,\beta,\varepsilon}E_\alpha^\lambda = 0$ for $|\alpha| \leqslant |\varepsilon| - 1$. On the other hand, by Taylor's theorem :

$$B_{\beta,\gamma,\varepsilon,j} = \int_{-|\varepsilon|}^{|\beta+\gamma|} [\partial_2^{|\beta+\gamma+\varepsilon|}(Q_j(F^{(j)} \circ P))](\lambda I, H(\lambda) + 2u|\lambda| I)\nu(u)\, du$$

where ν is a continuous function in $[-|\varepsilon|, |\beta+\gamma|]$ which depends on $|\varepsilon|$, $|\beta+\gamma|$. Hence $B_{\beta,\gamma,\varepsilon,j}$ is linear combination of terms of the form

$$\int_{-|\varepsilon|}^{|\beta+\gamma|} [T(F^{(\ell)} \circ P)](\lambda I, H(\lambda) + 2u|\lambda| I)\nu(u)\, du \tag{3.19}$$

where T is a polynomial and $\ell = j, \ldots, j + |\beta+\gamma+\varepsilon|$. Expanding the integrand in (3.19) in Taylor series centered at $H(\lambda)$ up to the order p, we can write (3.19) as a linear combination of terms of the form

$$|\lambda|^h[S(F^{(m)} \circ P)](\lambda I,H(\lambda)) \tag{3.20}$$

where S is a polynomial, $h = 0, \ldots, p-1, m = j, \ldots, j + |\beta+\gamma+\varepsilon| + p - 1$, plus an error term

$$|\lambda|^p \int_{-|\varepsilon|}^{|\beta+\gamma|} u^p \int_0^1 (1-s)^{p-1}[S(F^{(m)} \circ P)](\lambda I,H(\lambda)+2us|\lambda|I)\, ds\,\nu(u)\, du \tag{3.21}$$

$m = j, \ldots, j + p - 1 + |\beta+\gamma+\varepsilon|$. Thus each summand in (3.18) can be expanded into a sum of

$$\lambda^{-p}|\lambda|^h A_{\alpha,\beta,\varepsilon} S(\lambda I,H(\lambda))(F^{(m)} \circ P)(\lambda I,H(\lambda)) \tag{3.22}$$

plus the contribution

$$(\operatorname{sign}(\lambda))^p A_{\alpha,\beta,\varepsilon} \int_{-|\varepsilon|}^{|\beta+\gamma|} [S(F^{(m)} \circ P)](\lambda I,H(\lambda)+2v|\lambda|I)\mu(v)\, dv \tag{3.23}$$

coming from the error term (3.21), which contains only positive powers of λ. Now observe that by Lemma 3.8 the terms containing negative powers of λ in the coefficient $\lambda^{-p}|\lambda|^h A_{\alpha,\beta,\varepsilon} S$ of (3.22) must add up to zero. Thus we can neglect them. The remaining terms are of the form (3.6). The error term (3.23) is easily seen to be of the form (3.7). This concludes the proof of Proposition 3.2.

4. Estimates for the kernel of the Riesz means.

In this section we shall estimate the L^p norm, $1 \leqslant p \leqslant \infty$, of the kernel of the Riesz means outside a ball of radius r in $\mathbf{H}_n$.

PROPOSITION 4.1. – *If* $2 \leqslant p \leqslant \infty$, *and* $p^{-1} + q^{-1} = 1$:

$$\left(\int_{|g| \geqslant r} |s_R^\alpha(g)|^p\, dg\right)^{1/p} \leqslant C_{\alpha,p} \frac{R^{Q/2dq}}{1 + (R^{1/2d} r)^{\operatorname{Re}\alpha + 1/q}}. \tag{4.1}$$

If $1 \leqslant p \leqslant 2$, $p^{-1} + q^{-1} = 1$ *and* $\operatorname{Re}\alpha > (Q-1)/2$:

$$\left(\int_{|g| \geqslant r} |s_R^\alpha(g)|^p\, dg\right)^{1/p} \leqslant C_{\alpha,p} \frac{R^{Q/2d}}{1 + (R^{1/2d} r)^{\operatorname{Re}\alpha + 1/2 - Q(1/p - 1/2)}}. \tag{4.2}$$

Here $C_{\alpha,p}$ denotes a constant that grows at most polynomially in $\operatorname{Im}\alpha$, when $\operatorname{Re}\alpha$ is bounded from above.

Proof. – Since the differential operator $P(iT,\mathscr{L})$ is homogeneous of order $2d$, it follows from Proposition 2.3 that $s^\alpha_R = R^{Q/2d}(s^\alpha_1 \cdot \gamma_{R^{1/2d}})$. Hence we only need to prove (4.1) and (4.2) for $R = 1$.

Now, by the Hausdorff-Young theorem :

$$\left(\int_{|g|\geqslant r} |s^\alpha_1(g)|^p \, dg\right)^{1/p} \leqslant C\|(s^\alpha_1)^\wedge\|_{\mathfrak{L}^q}$$

for $2 \leqslant p \leqslant \infty$. By Lemma 3.3

$$\|(s^\alpha_1)^\wedge\|_{\mathfrak{L}^q} = \|(I-P)^\alpha_+\|_{\mathfrak{L}^q} \leqslant C'\left(\int_0^1 |1-\rho^d|^{\operatorname{Re}\alpha q}\rho^n \, d\rho\right)^{1/q}$$
$$\leqslant C''.$$

Hence (4.1) is trivial for $r \leqslant 1$. To prove it for $r > 1$ we decompose s^α_1 into a sum of two terms such that one has no spectrum at 0 and the other has a smooth Fourier transform.

LEMMA 4.2. – *Let* N *be an integer such that* $4N > \operatorname{Re}\alpha + 1/q$. *For every* t, $0 < t < 1$, *there exists two functions* σ^α_t, τ^α_t *such that*

i) $s^\alpha_1 = \sigma^\alpha_t + \tau^\alpha_t$;

ii) $\|(\sigma^\alpha_t)^\wedge\|_{\mathfrak{L}^q} \leqslant C\, t^{\operatorname{Re}\alpha + 1/q}$;

iii) $\|\Lambda^N(\tau^\alpha_t)^\wedge\|_{\mathfrak{L}^q} \leqslant C_{\alpha,q}\, t^{\operatorname{Re}\alpha + 1/q - 4N}$.

Proof. – Let ω be a smooth function of one variable such that

$$\omega(x) = 1 \quad \text{if} \quad x < 1 - t$$
$$\omega(x) = 0 \quad \text{if} \quad x > 1 - t/2$$
$$|\omega^{(m)}(x)| \leqslant Ct^{-m} \quad \text{if} \quad 1 - t < x < 1 - t/2, \qquad m \leqslant 4N.$$

Define $(\sigma^\alpha_t)^\wedge(\lambda) = (1-(\omega \circ P)(\lambda I, H(\lambda)) \quad (s^\alpha_1)^\wedge(\lambda)$ and $(\tau^\alpha_t)^\wedge(\lambda) = (\omega \circ P)(\lambda I, H(\lambda))\ (s^\alpha_1)^\wedge(\lambda)$. Now i) is obvious and ii) follows from the estimate

$$\|(\sigma^\alpha_t)^\wedge\|_{\mathfrak{L}^q} = \|(I-\omega \circ P)(I-P)^\alpha_+\|_{\mathfrak{L}^q}$$
$$\leqslant C\left(\int_0^1 |1-\omega(\rho^d)|^q|1-\rho^d|^{\operatorname{Re}\alpha q}\rho^n \, d\rho\right)^{1/q}$$
$$\leqslant C\, t^{\operatorname{Re}\alpha + 1/q}.$$

To prove iii) observe that $(\tau_t^\alpha)\hat{}(\lambda) = (F \circ P)(\lambda I, H(\lambda))$ where $F(x) = \omega(x)(1-x)_+^\alpha$ is a function in $C^\infty(\bar{\mathbf{R}}_+)$ with support in $[0,1]$. Since $\mathrm{Re}\,\alpha + 1/q < 4N$ it is easily seen that for $j = 1, \ldots, 4N$:

$$\left(\int_0^1 |F^{(j)}(\rho^d)|^q \rho^n \, d\rho\right)^{1/q} \leqslant C_{\alpha,q}\, t^{\mathrm{Re}\,\alpha + 1/q - 4N}.$$

Hence iii) follows by Theorem 3.2. The proof of the Lemma is now complete.

Applying the Lemma, the triangle inequality and Hausdorff-Young's theorem, we obtain for $r > 1$:

$$\begin{aligned}\left(\int_{|g|\geqslant r} |s_1^\alpha(g)|^p \, dg\right)^{1/p} &\leqslant \|\sigma_t^\alpha\|_p + r^{-4N}\| \, |g|^{4N}\tau_t^\alpha\|_p \\ &\leqslant C(\|(\sigma_t^\alpha)\hat{}\|_{\mathfrak{L}^q} + r^{-4N}\|\Lambda^N(\tau_t^\alpha)\hat{}\|_{\mathfrak{L}^q}) \\ &\leqslant C_{\alpha,p}(t^{\mathrm{Re}\,\alpha+1/q} + r^{-4N}t^{\mathrm{Re}\,\alpha+1/q-4N}).\end{aligned}$$

Taking $t = r^{-1}$ we obtain (4.1) also for $r > 1$. To prove (4.2) consider a partition of $\mathbf{H}_n$ into dyadic annuli $A_i = \{g \in \mathbf{H}_n : 2^i r \leqslant |g| < 2^{i+1}r\}$. Then for $1 \leqslant p \leqslant 2$, applying Holder's inequality we obtain :

$$\begin{aligned}\left(\int_{|g|\geqslant r} |s_1^\alpha(g)|^p \, dg\right)^{1/p} &= \sum_{i=0}^{\infty}\left(\int_{A_i} |s_1^\alpha(g)|^p \, dg\right)^{1/p} \\ &\leqslant \sum_{i=0}^{\infty}\left(\int_{A_i} |s_1^\alpha(g)|^2 \, dg\right)^{1/2}\left(\int_{A_i} dg\right)^{(2-p)/2p}.\end{aligned}$$

Since $\int_{A_i} dg \leqslant C(2^i r)^Q$, using (4.1) we can estimate the last term with :

$$C_{\alpha,p}\sum_{i=0}^{\infty}\frac{2^i r^{Q(1/p-1/2)}}{1 + (2^i r)^{\mathrm{Re}\,\alpha+1/2}} \leqslant C_{\alpha,p}\frac{1}{1 + r^{\mathrm{Re}\,\alpha+1/2-Q(1/p-1/2)}}$$

provided $\mathrm{Re}\,\alpha > (Q-1)/2$.

5. Proof of theorem 1.1.

We consider first the convergence of the Riesz means for smooth functions with compact support.

LEMMA 5.1. – *Let* $f \in C_c^\infty(\mathbf{H}_n)$, $\mathrm{Re}\,\alpha \geqslant 0$. *Then* $S_R^\alpha f \to f$ *in the* *topology of* $C^\infty(\mathbf{H}_n)$ *as* $R \to \infty$. *If* f *vanishes in a neighborhood of* $g \in \mathbf{H}_n$, *then* $S_R^\alpha f = 0(R^{-s})$ *for every* $s > 0$.

Proof. – Since $[P(iT,\mathscr{L})]^N S_R^\alpha f = S_R^\alpha [P(iT,\mathscr{L})]^N f$ is bounded in L^2 when $R \to \infty$ for every N, it follows from the hypoellipticity of $P(iT,\mathscr{L})$ that $S_R^\alpha f$ belongs to a bounded, thus compact, subset of $C^\infty(\mathbf{H}_n)$. Hence $S_R^\alpha f \to f$ in $C^\infty(\mathbf{H}_n)$, because $S_R^\alpha f \to f$ in $L^2(\mathbf{H}_n)$.

Assume now that f vanishes in a neighborhood of $g \in \mathbf{H}_n$. Then for every positive integral m:

$$S_R^\alpha f(g) = S_R^\alpha f(g) - \sum_{k=0}^{m-1} C_k^\alpha (-R)^{-k}[P(iT,\mathscr{L})]^k f(g).$$

With the aid of the spectral resolution of the closure $\bar{P}$ of $P(iT,\mathscr{L})$ we see that for every $\ell \in \mathbf{N}$

$$\|\bar{P}^\ell (S_R^\alpha f - \sum_{k=0}^{m-1} C_k^\alpha (-R)^{-k} \bar{P}^k f)\|_2 \leqslant CR^{-m} \|\bar{P}^{\ell+m} f\|_2$$

when R is large. Hence $S_R^\alpha f - \sum_k C_k^\alpha (-R)^{-k} \bar{P}^k f$ is $0(R^{-m})$ in $\mathrm{Dom}(\bar{P}^\ell)$. Taking $l > Q/4d$ we obtain that $S_R^\alpha f(g) = 0(R^{-m})$, by Corollary 2.3.

Proof of (N). – By estimate (4.2) with $p = 1$, the kernel of the Riesz means s_R^α is in $L^1(\mathbf{H}_n)$ uniformly with respect to R, provided $\mathrm{Re}\,\alpha > (Q-1)/2$. Hence

$$\|S_R^\alpha f\|_1 \leqslant C_{\alpha,1} \|f\|_1$$

when $\mathrm{Re}\,\alpha > (Q-1)/2$. Since $\|S_R^0 f\|_2 \leqslant \|f\|_2$, by Stein's interpolation theorem we have

$$\|S_R^\alpha f\|_p \leqslant C_{\alpha,p} \|f\|_p, \qquad \mathrm{Re}\,\alpha > (Q-1)\left[\frac{1}{p} - \frac{1}{2}\right].$$

Hence (N) follows by a density argument. We also note that (N*) is a consequence of the self-adjointness of S_R^α.

LEMMA 5.2. – *Assume that* $f \in L^p(\mathbf{H}_n)$, $1 \leqslant p \leqslant 2$. *Thus for* $\mathrm{Re}\,\alpha > (Q-1)/p$ *we have*

$$\sup_{R>0} |S_R^\alpha f(g)| \leqslant C_{\alpha,p} M_p(f)(g) \tag{5.1}$$

where $M_p(f)(g) = \left(\sup_{r>0} r^{-Q} \int_{|h|\leqslant r} |f(gh)|^p\, dh\right)^{1/p}$ *is the Hardy-Littlewood maximal function of power* p.

Proof. – Since both S_R^α and M_p are operators commuting with left translations, it suffices to prove (5.1) for $g = e = (0,0)$. Let $f \in L^p(\mathbf{H}_n)$, $1 \leqslant p \leqslant 2$, be a function vanishing in $\{h : |h| \leqslant r\}$. Then by Proposition 4.1 :

$$|S_R^\alpha f(e)| \leqslant C \frac{R^{Q/2dp}}{1 + (R^{1/2d} r)^{\mathrm{Re}\alpha + 1/p}} \|f\|_p. \tag{5.2}$$

Next, given any function $f \in L^p(\mathbf{H}_n)$, write $f_\nu(g) = f(g)$ if $2^\nu \leqslant |g| \leqslant 2^{\nu+1}$, $\nu \in \mathbf{Z}$; $f_\nu(g) = 0$ otherwise. Hence :

$$\begin{aligned}|S_R f(e)| &\leqslant \sum_{\nu=-\infty}^{+\infty} |S_R f_\nu(e)| \\ &\leqslant C \sum_{\nu=-\infty}^{+\infty} \frac{R^{Q/2d}}{1 + (R^{1/2d} 2^\nu)^{\mathrm{Re}\alpha + 1/p}} \|f_\nu\|_p.\end{aligned}$$

Since $\|f_\nu\|_p \leqslant 2^{(\nu+1)Q/p} M_p(f)(e)$ the above sum can be estimated by :

$$C M_p(f)(e) \sum_{\nu=-\infty}^{+\infty} \frac{(R^{1/2d} 2^\nu)^{Q/p}}{1 + (R^{1/2d} 2^\nu)^{\mathrm{Re}\alpha + 1/p}}.$$

If $\mathrm{Re}\alpha > (Q-1)/p$ the series converges to a bounded function of R, because it is locally bounded and it does not change if R is replaced by $2^{2dm} R$. This proves (5.1).

Proof of (LP). – We may assume that $f(g) = 0$. If g is a Lebesgue point for f we can approximate f by functions in $C_c^\infty(\mathbf{H}_n)$, vanishing in a neighborhood of g, in the norm $M_p(f)(g)$. Since for such functions h, $S_R^\alpha h(g) \to h(g)$ regardless of the value of α, we obtain (LP) via the maximal inequality.

Proof of (AE). – To prove (AE) we establish first the following inequality

$$\left\| \sup_{R>0} |S_R^\alpha f| \right\|_p \leqslant A_{\alpha,p} \|f\|_p, \qquad \mathrm{Re}\alpha > (Q-1)[(2/p)-1] \tag{5.3}$$

for $1 < p \leqslant 2$. Indeed it follows from (5.1) with $p = 1$, and the Hardy-Littlewood maximal theorem, that for $\mathrm{Re}\alpha > Q - 1$:

$$(5.4) \qquad \left\| \sup_{R>0} |S_R^\alpha f| \right\|_p \leqslant B_{\alpha,p} \|f\|_p, \qquad \text{for} \qquad 1 < p \leqslant 2.$$

By a classical result of Kaczmarcz on orthogonal expansions, extended to the context of continuous eigenfunction expansions by Peetre [14] (see also [9]) :

$$\left\| \sup_{R>0} |S_R^\alpha f| \right\| \leqslant C_\alpha \left\{ \|f\|_2 + \left\| \sup_{R>0} |S_R^{\alpha_0} f| \right\|_2 \right\}$$

for every α, $\mathrm{Re}\alpha > 0$ and fixed α_0. Taking $\mathrm{Re}\alpha_0 > Q - 1$ we obtain for all α, $\mathrm{Re}\alpha > 0$:

$$(5.5) \qquad \left\| \sup_{R>0} |S_R^\alpha f| \right\|_2 \leqslant D_\alpha \|f\|_2 .$$

Using (5.4), (5.5) and the interpolation theorem of Stein as in [9] we obtain (5.3). Since $S_R^\alpha f \to f$ uniformly if $f \in C_c^\infty(\mathbf{H}_n)$, a dense subset of $L^p(\mathbf{H}_n)$, (AE) follows immediately from (5.3) for $1 < p \leqslant 2$. For $p = 1$ it is already a consequence of (LP).

Proof of (L). – It is a straightforward consequence of (5.2) and of Lemma 5.1.

The proof of Theorem 1.2 is completely analogous. We only need to remark that when $2 \leqslant p < \infty$, by (4.2) the maximal inequality (5.1) holds for $\mathrm{Re}\alpha > (Q-1)/2$.

6. Open problems.

We would like to conclude this paper by briefly discussing two open problems.

One open question is whether Theorems 1.1 and 1.2 are valid in the case of a general hypoelliptic, formally nonnegative differential operator in $\mathfrak{U}(\mathbf{H}_n)$. In this paper we restricted our consideration to a particular class of operators contained in the algebra $\mathcal{A}$ generated by iT and $\mathcal{L}$. There are two reasons why we did so. The first one is that operators in the class $\mathcal{A}$ have symbols which are scalar valued functions of $(\lambda,N) \in \mathbf{R}_* \times \mathbf{N}$. One cannot expect in the general case of operator valued symbols to be able to estimate norms in the $\mathfrak{L}^p$ spaces. Even so we did not consider general hypoelliptic operators in $\mathcal{A}$. As we proved in Corollary 2.2, an operator in $\mathcal{A}$ is hypoelliptic and formally nonnegative if and only if its symbol P satisfies assumption (A) in the proper subset C_n of Γ_n. However in order to obtain the estimate of Theorem 3.2, we had to consider the error terms (3.7) in Proposition 3.2, which depend also on the values of the polynomial P on the set Γ_n. This is a substantial difficulty in trying to extend our results to any homogeneous, formally nonnegative hypoelliptic operator in $\mathcal{A}$.

Another interesting problem is whether the results of Theorem 1.1 could be sharpened for some operators in $\mathcal{A}$. Comparing our results with those obtained by Peetre [13] for constant coefficient elliptic operators, we see that the statements of Theorems 1.1 and 1.2 can be obtained from the corresponding statements of Theorems 2.1 and 2.2, replacing the Euclidean dimension n by the homogeneous dimension Q and the order of the operator by the degree of homogeneity $2d$. Now it is known that in $\mathbf{R}^n$ sharper results can be obtained for the Laplace operator Δ, or more generally for operators such that the unit ball defined by the principal symbol is strictly convex. For instance, if s_R^α is the kernel of the Riesz means for $-\Delta$, the estimate

$$|s_R^\alpha(x)| \leqslant C \frac{R^{n/2}}{1 + (R^{1/2}|x|)^{\operatorname{Re}\alpha + (n-1)/2 + 1}}$$

can be obtained by explicit Fourier inversion.

It would be interesting to know whether the corresponding estimate

$$|s_R^\alpha(g)| \leqslant C \frac{R^{Q/2}}{1 + (R^{1/2}|g|)^{\operatorname{Re}\alpha + (Q-1)/2 + 1}}$$

holds for the sublaplacian $\mathscr{L}$ on the Heisenberg group. In that case the results (LP), (AE), (L) of Theorem 1.1 and the maximal inequality (5.1) could be correspondingly sharpened.

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