ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 31, nº 3 (1981), p. 245-274
<http://www.numdam.org/item?id=AIF_1981__31_3_245_0>

Ann. Inst. Fourier, Grenoble
31,3 (1981), 245-274

ON INFINITE LIE GROUPS

by Alexandre A. Martins RODRIGUES

Introduction.

In this paper we introduce the notion of admissible fibration with relation to an infinitesimal Lie pseudo-group and show that the quotient and the kernel of infinitesimal Lie pseudo-groups by an admissible fibration are again infinitesimal Lie pseudo-groups. In the case of transitive pseudo-groups, every invariant fibration is an admissible fibration. In this sense, our results generalize to intransitive pseudo-groups theorems which are known in the transitive case [11], [12]. We also show that the notion of admissible fibration is the best possible if one wants that both the kernel and the quotient of an infinitesimal Lie pseudo-group by an invariant fibration be again an infinitesimal Lie pseudo-group (theorem 2.3).

Following a basic idea of E. Cartan we say that two infinitesimal Lie pseudo-groups θ_1 and θ_2 are equivalent if there exists a third infinitesimal Lie pseudo-group θ_3 which is at the same time an admissible isomorphic prolongation of θ_1 and θ_2. It has been a crucial point of the theory to prove that this is in fact an equivalence relation. M. Kuranishi has proved that one obtains an equivalence relation if restrictive conditions are imposed on the definition of isomorphic prolongation [5]. We show that Kuranishi's conditions are automatically satisfied for admissible fibrations (theorem 2.2). This allows us to prove that the relation mentioned above is in fact an equivalence relation.

One interesting aspect of our equivalence relation is that it is defined for infinitesimal Lie pseudo-groups globally defined on manifolds. Our notion of admissible fibration also allows to define the quotient of an infinitesimal Lie pseudo-group by a normal infinitesimal Lie pseudo-group.

In § 7 we introduce the notion of abstract Lie pseudo-group. An abstract Lie pseudo-group is an equivalence class of germs of infinitesimal Lie pseudo-groups under the equivalence relation mentioned above. We also introduce the notions of morphism, kernel and quotient of abstract Lie pseudo-groups. In the last paragraph we introduce the category of abstract pseudo-groups of finite type.

The author would like to thank Professors J.L. Koszul and Ngo van Quê for many profitable discussions on the subject of this paper. He is particulary indebted to Ngo van Quê to whom he owes the important theorem 2.3 and the final version of the proof of theorem 4.1.

1. Infinitesimal Lie pseudo-groups.

In this paper all manifolds, vector fields and maps of manifolds into manifolds are supposed to be real analytic.

Let M be a manifold and θ a sheaf of germs of local vector fields defined on M. To simplify our notation we shall write $\xi \in \theta$ to mean that ξ is a section of θ defined on an open set of M. The open set where ξ is defined will be denoted by $U(\xi)$. We shall denote respectively by $T(M)$, $J^k T(M)$, $J^k \theta$, $T_a(M)$ and θ_a the tangent bundle of M, the vector bundle of k-jets of local vector fields of M, the set of k-jets of local sections of θ, the fiber of $T(M)$ and the stalk of θ at the point $a \in M$. When there is no danger of confusion we shall write T, $J^k T$ and T_a instead of $T(M)$, $J^k T(M)$, $T_a(M)$. $j_a^k \xi$ denotes the k-jet of the local vector field ξ and $j^k \xi : U(\xi) \longrightarrow J^k T$ is the mapping which sends $a \in U(\xi)$ into $j_a^k \xi$. If E is a vector bundle over M then we shall write $\underline{E}$ for the sheaf of local sections of E.

DEFINITION 1.1. – *An infinitesimal Lie pseudo-group (ILPG) is a sheaf* θ *of germs of local vector fields of a manifold* M *such that*

1) θ *is a sheaf of Lie algebras, that is each stalk of* θ *is a Lie algebra over* **R** *with respect to the bracket of germs of vector fields.*

2) *There is an integer* k_1 *such that* $J^k\theta$ *is an analytic vector sub-bundle of* $J^k T$ *for all* $k \geqslant k_1$.

3) *There is an integer* k_2 *such that every solution of the system of differential equations* $J^{k_2}\theta \subset J^{k_2}T$ *is a section of* θ. *In other words, if* ξ *is a local vector field defined on the open set* $U \subset M$ *and if* $j_a^{k_2}\xi \in J_a^{k_2}\theta$ *for all* $a \in U$, *then* $\xi \in \theta$.

The least integer verifying both conditions 2) and 3) is called the order of θ and will be denoted by $r(\theta)$.

Assume that $J^0\theta \subset T$ is a vector sub-bundle of T. From condition 1) it follows then that $J^0\theta$ is completely integrable; in this case the maximal integral manifolds of $J^0\theta$ are called the orbits of θ. If $J^0\theta = T$, we shall say that θ is transitive on M.

Frequently we shall meet sheaves θ of germs of vector fields which locally are ILPG. To simplify our language we shall say that θ is locally an ILPG. Precisely,

DEFINITION 1.2. – *A sheaf* θ *of germs of vector fields of the manifold* M *is locally an ILPG if every point* $a \in M$ *has an open neighborhood* U *such that the restriction* $\theta|U$ *of the sheaf* θ *to* U *is an ILPG*.

Let θ_1 and θ_2 be ILPG defined on the manifold M. We shall give a sufficient condition for the sheaf $\theta_1 \cap \theta_2$ to be locally an ILPG. Put $E_1^k = J^k\theta_1$, $E_2^k = J^k\theta_2$ and $H^k = E_1^k \cap E_2^k$. For $k' \geqslant k$, let $\pi^k : J^{k'}T \longrightarrow J^kT$ be the canonical map of k'-jets into k-jets. Clearly, $\pi^k(H^{k+j}) \subset H^k$ for all $k, j \geqslant 0$. For $k, j \geqslant 0$, put $H_j^k = \pi^k(H^{k+j})$; in particular $H_0^k = H^k$. It is also clear that $H_{j'}^k \subset H_j^k$ for $j' \geqslant j$.

THEOREM 1.1. – *If there exist integers* k_0, j_0 *such that* H^k *and* H_j^k *are vector sub-bundles of* J^kT *for all* $k \geqslant k_0$ *and* $j \geqslant j_0$, *then* $\theta_1 \cap \theta_2$ *is locally an ILPG*.

Proof. – Given a vector sub-bundle $E \subset J^k T$ we shall denote by $pE \subset J^{k+1}T$ the first prolongation of E. It is known that if $r \leqslant k$ and $\pi^r(E) \subset J^r T$ is a vector sub-bundle then $\pi^{r+1}(pE) \subset p(\pi^r E)$. Let $k_1 = \sup(r(\theta_1), r(\theta_2))$. Then for $k \geqslant k_1$ we have $E_i^{k+1} \subset pE_i^k$, $i = 1, 2$ and also

$$p(E_1^k \cap E_2^k) = pE_1^k \cap pE_2^k .$$

Hence, for $k \geqslant k_2 = \sup(k_0, k_1)$, $H^{k+1} \subset pH^k$. On the other hand, since

$$\pi^{k+1}(pH^{k+j}) \subset p\pi^k(H^{k+j}) \text{ and } \pi^{k+1}(H^{k+j+1}) \subset \pi^{k+1}(pH^{k+j}),$$

it follows that $\pi^{k+1}(H^{k+j+1}) \subset p\pi^k(H^{k+j})$. Therefore $H_j^{k+1} \subset pH_j^k$ for all $k \geqslant k_2$ and $j \geqslant j_0$. Put $S^k = \bigcap_{j=j_0}^{\infty} H_j^k$, $k \geqslant k_0$. Since $H_{j'}^k \subset H_j^k$ for $j' \geqslant j$, by an argument of dimension, there exists an integer $j_1(k)$ such that $S^k = H_j^k$ for all $j \geqslant j_1(k)$ and $k \geqslant k_0$. In particular S^k, $k \geqslant k_0$, is a vector sub-bundle of $J^k T$. By construction of S^k, the canonical projection $\pi^k : S^{k+1} \longrightarrow S^k$ is surjective. Given an integer $k \geqslant k_2$, we can choose j sufficiently large such that $S^k = H_j^k$ and $S^{k+1} = H_j^{k+1}$. It follows that $S^{k+1} \subset pS^k$ for $k \geqslant k_2$.

By the Cartan-Kuranishi prolongation theorem [5], for every point $a \in M$ there exist an open neighborhood U of a and an integer $k_3 \geqslant k_2$ such that the restriction $S^k | U$ of S^k to U is an involutive system of partial differential equations and $S^{k+1} | U = p(S^k | U)$ for all $k \geqslant k_3$. It is easy to see that a local vector field ξ belongs to $\theta_1 \cap \theta_2 | U$ if and only if ξ is a solution of $S^{k_3} | U$. On the other hand, the theorem of existence of solutions of analytic involutive systems of partial differential equations ensures that every jet of $S^k | U$ is the jet of a solution of S^k for all $k \geqslant k_3$. Hence, $J^k(\theta_1 \cap \theta_2 | U) = S^k | U$ is a vector sub-bundle of $J^k T$ for $k \geqslant k_3$. We have proved that $\theta_1 \cap \theta_2 | U$ is an ILPG of order less or equal to k_3.

For later use, we prove here that for k sufficiently large the integer $j_1(k)$ can be chosen independently of k. Another proof of the same proposition can be found in [3].

PROPOSITION 1.1. – *Preserving the notations of theorem 1.1, there exist integers k_0 and d such that $S^k = H_j^k$ for all $k \geqslant k_0$ and $j \geqslant d$.*

Proof. – We have seen in the proof of theorem 1.1 that there are integers k_0 and d such that $H_d^{k_0} = S_d^{k_0}$ and $H_d^{k_0+1} = S_d^{k_0+1}$. Moreover, given $a \in M$ we can choose k_0 and an open neighborhood U of a such that $S^{k+2} \mid U = p(S^{k+1} \mid U)$ for $k \geqslant k_0$. Since $H_d^{k_0+2} \subset pH_d^{k_0+1}$ it follows that $H_d^{k_0+2} \mid U \subset S^{k_0+2} \mid U$. Since the other inclusion $S^{k_0+2} \subset H_d^{k_0+2}$ also holds true we have $H_d^{k_0+2} \mid U = S^{k_0+2} \mid U$. But $H_d^{k_0+2}$ and S^{k_0+2} are vector bundles, hence $H_d^{k_0+2} = S^{k_0+2}$. Repeating the same argument successively we prove that $S^k = H_d^k$ for all $k \geqslant k_0$.

2. Admissible fibrations.

A fibered manifold is by definition a triple (M, M', ρ) where $\rho : M \longrightarrow M'$ is a submersion; that is a surjective map whose rank equals the dimension of M'. We shall also say that ρ is a fibration of M over M'. In this paper it will be always assumed that the fibers $M_{a'} = \rho^{-1}(a')$ are connected for all $a' \in M'$. We say that a local diffeomorphism f of the open set $U \subset M$ onto the open set $V \subset M$ is projectable by ρ if there exists a diffeomorphism $f' : \rho(U) \longrightarrow \rho(V)$ such that $\rho \circ f = f' \circ \rho$. A local section $\xi : U \subset M \longrightarrow T(M)$ is projectable by ρ if there exists a local section $\xi' : \rho(U) \longrightarrow T(M')$ such that $d\rho \circ \xi = \xi \circ \rho$ where $d\rho : T(M) \longrightarrow T(M')$ is the extension of ρ to tangent bundles of M and M'. We shall often denote ξ' by $d\rho(\xi)$.

Let (M, M', ρ) be a fibered manifold and denote by Φ the sheaf of germs of local vector fields of M which are projectable by ρ. It is easy to check that Φ is a transitive ILPG of order 1. Let Ω be the sub-sheaf of Φ of germs which project into the germ of the zero vector field of M'; Ω is also an ILPG of order 0. If ξ' is a germ of vector field defined at the point $a' \in M'$ and $\xi \in \Phi$ is a germ of vector field defined at the point $a \in M$, $\rho(a) = a'$, such that $d\rho(\xi) = \xi'$, we say that ξ is a lifting of ξ' at the point a.

The mapping $\rho : M \longrightarrow M'$ extends naturally to a surjective morphism $\rho^k : J^k \Phi \longrightarrow J^k T(M')$ of vector bundles, $k \geqslant 0$. If $\xi \in \Phi_a$ $a \in M$, then, by definition, $\rho^k(j_a^k \xi) = j_a^k, d\rho(\xi))$ where $a' = \rho(a)$. In particular, $\rho^0 = d\rho$. Clearly, $\rho^k \circ \pi^k = \pi^k \circ \rho^{k+j}$,

$k, j \geqslant 0$, where $\pi^k : J^{k+j}T(M) \longrightarrow J^k T(M)$ is the canonical projection.

Let us consider the sheaf $\underline{J^k \Phi}$ of germs of local sections of $J^k \Phi$ and let $\widetilde{\Phi}^k$ be the sub-sheaf of $\underline{J^k \Phi}$ of germs of local sections which are projectable by ρ^k. That is, $\widetilde{\Phi}^k$ is the sheaf of germs of local sections σ of $J^k \Phi$ for which there exists a local section σ' of $J^k T(M')$ such that $\rho^k \circ \sigma = \sigma' \circ \rho$. The projection ρ^k extends naturally to a map $\widetilde{\rho}^k$ of 1-jets of local sections of $\widetilde{\Phi}^k$. Taking local coordinates we can prove easily the proposition below.

PROPOSITION 2.1. – *The following sequences of vector bundles are exact :*

$$0 \longrightarrow J^k(\Omega) \longrightarrow J^k(\Phi) \xrightarrow{\rho^k} J^k T(M') \longrightarrow 0$$

$$0 \longrightarrow J^1(J^k \Omega) \longrightarrow J^1 \widetilde{\Phi}^k \xrightarrow{\widetilde{\rho}^k} J^1(J^k T(M')) \longrightarrow 0 .$$

Assume now we are given an ILPG θ defined over M. We shall say that the fibration (M, M', ρ) is invariant by θ or equivalently that θ is projectable by ρ if θ is a sub-sheaf of Φ. The sheaf θ' defined over M' of germs of projections of germs of θ is called the projection of θ by ρ. Often we shall write $\rho(\theta)$ to denote this sheaf. The kernel of θ by the fibration ρ is by definition the sheaf $\Omega(\theta) = \Omega \cap \theta$. When there is no danger of confusion we shall write simply Ω instead of $\Omega(\theta)$.

Clearly ρ^k maps $J^k \theta$ onto $J^k \theta'$. Let us put $H^k(\rho) = J^k \Omega \cap J^k \theta$. From proposition 2.1 it follows that $H^k(\rho)$ is the kernel of the morphism of vector bundles $\rho^k : J^k \theta \longrightarrow J^k \theta'$. Define as in theorem 1.1, $H_j^k(\rho) = \pi^k(H^{k+j}(\rho))$. Often we shall write H^k and H_j^k instead of $H^k(\rho)$ and $H_j^k(\rho)$.

DEFINITION 2.1. – *A fibration (M, M', ρ) invariant by θ is called admissible with respect to θ if there are integers k_0 and j_0 such that H^k and H_j^k are vector sub-bundles of $J^k \theta$ for $k \geqslant k_0$ and $j \geqslant j_0$.*

From the definition it follows that if (M, M', ρ) is an admissible fibration with respect to θ then $J^k \theta'$ is a vector sub-bundle of $J^k T(M')$ and the following sequence is exact:

$$0 \longrightarrow H^k \longrightarrow J^k\theta \xrightarrow{\rho^k} J^k\theta' \longrightarrow 0$$

for $k \geqslant r(\theta)$, k_0.

THEOREM 2.1. – *Let θ be an ILPG defined over M and let (M, M', ρ) be an admissible fibration with respect to θ. Then the kernel Ω of θ by the fibration ρ is locally an ILPG.*

Proof. – The theorem is an immediate consequence of the definition of admissible fibration and of theorem 1.1.

PROPOSITION 2.2. – *Let θ be an ILPG and let (M, M', ρ) be an invariant fibration with respect to θ. If a, b are two points of the same fiber of M then $\rho^k(J^k_a\theta) = \rho^k(J^k_b\theta)$ for all $k \geqslant r(\theta)$.*

Proof. – Let $\xi_1, \xi_2, \ldots, \xi_s$ be sections of θ defined on a convenient open neighborhood U of a and such that $j^k_x\xi_1, j^k_x\xi_2, \ldots, j^k_x\xi_s$ is a basis of $J^k_x\theta$ for all $x \in U$. Put $\xi'_i = d\rho(\xi_i)$, $i = 1, 2, \ldots, s$ and let θ' be the projection of θ by ρ. Then $\rho^k(j^k_x\xi_i)$, $i = 1, 2, \ldots, s$ is a system of generators of the vector space $\rho^k(J^k_x\theta)$ for all $x \in U$. Hence $j^k_{a'}\xi'_i$, $i = 1, 2, \ldots, s$ is a system of generators of $\rho^k(J^k_x\theta)$ for all $x \in M_{a'} \cap U$ where $a' = \rho(a)$. Therefore, the set of points $x \in M_{a'}$ for which $\rho^k(J^k_x\theta)$ is a given sub-space of $J^k_{a'}\theta'$ is an open set of $M_{a'}$; this implies also that the complement of this set in $M_{a'}$ is the union of open sets. Since by hypothesis $M_{a'}$ is connected it follows that $\rho^k(J^k_a\theta) = \rho^k(J^k_x\theta)$ for all $x \in M_{a'}$.

THEOREM 2.2. – *Let θ be an ILPG defined over M and let (M, M', ρ) be an admissible fibration with respect to θ. Then*

1) *The projection θ' of θ over M′ is locally an ILPG.*

2) *For every $a' \in M'$, for every $a \in M_{a'}$ and for every $\xi' \in \theta'_{a'}$ there is at least one germ $\xi \in \theta_a$ such that $d\rho(\xi) = \xi'$.*

3) *There exist integers k_3 and d satisfying the following property: For every $a' \in M'$, for every integer $r \geqslant k_0$, for every germ $\xi' \in \theta'_{a'}$ which vanishes at order $r + d$ at the point a' and for every $a \in M_{a'}$ there exists at least one lifting $\xi \in \theta_a$ of ξ' which vanishes at order r at the point a.*

Proof. – We shall keep the notations as in proposition 2.1 and definition 2.1. Let us put $E^k = J^k \theta$ and $E'^k = J^k \theta'$. Choose an integer $k_0 \geqslant r(\theta)$ satisfying the condition of definition 2.1. Then E'^k is a vector sub-bundle of $J^k T(M')$ and $0 \longrightarrow H^k \longrightarrow E^k \longrightarrow E'^k \longrightarrow 0$ is an exact sequence for $k \geqslant k_0$. One checks easily that $E'^{k+1} \subset pE'^k$, $k \geqslant k_0$. By the Cartan-Kuranishi prolongation theorem [5], given $a' \in M'_{a'}$ there exist an open neighborhood V' of a' and an integer $k_1 \geqslant k_0$ such that $E'^k \mid V'$ is an involutive system of partial differential equations and $E'^{k+1} \mid V' = p(E'^k \mid V')$ for $k \geqslant k_1$. By construction, every section of θ' is a solution of E'^{k_1}. Conversely, we shall show that every local solution of E'^{k_1} defined in V' is a section of θ'.

Take $x' \in V'$ and let ξ' be a solution of E'^{k_1} defined in the open neighborhood $U' \subset V'$ of x'. By the choice of k_1, $j^k_{a'} \xi' \in E'^k$ for all $k \geqslant k_1$ and $a' \in U'$. Put $U = \rho^{-1}(U')$. To simplify our notation let us write E^k instead of $E^k \mid U$. Let F^k be the set of jets $X \in E^k$ such that $\rho^k(X) = j^k_{b'} \xi'$ where $b' = \rho(\alpha(X)) \in U'$. Clearly $\pi^k(F^{k+1}) \subset F^k$. Assume $k \geqslant k_1$ and take a point $a \in U$; by proposition 2.2 the fiber F^k_a is not empty. Take $X \in F^k_a$; then $F^k_a = X + H^k_a$. We want to show that $F^{k+1} \subset pF^k$, $k \geqslant k_1$. It is enough to show that every section σ of F^{k+1} is also a section of $J^1(F^k) \subset J^{k+1} T(M)$. Put $\tau = j^1 \pi^k \sigma - \sigma$. Since $J^1(F^k) + J^1(H^k) = J^1(F^k + H^k) = J^1(F^k)$ and $j^1 \pi^k \sigma$ is a section of $J^1(F^k)$, it suffices to show that τ is a section of $J^1(H^k)$. But, keeping the notations as in proposition 2.1, $\tilde{\rho}^k \circ \tau = j^1(j^k \xi') - j^{k+1} \xi' = 0$. Hence, by proposition 2.1, τ is a section of $J'(J^k \Omega)$. Consequently τ is also a section of $J^1(J^k \Omega) \cap J^1(E^k) = J^1 H^k$. Hence $F^{k+1} \subset pF^k$.

Put $F^k_j = \pi^k(F^{k+j})$. It is easy to see that if $X \in (F^k_j)_{a'}$ then $(F^k_j)_a = X + (H^k_j)_a$. Hence F^k_j is an affine sub-bundle of $J^k T(M)$ for $k \geqslant k_1$ and $j \geqslant j_0$. Moreover (F^k_j, U, α) is a fibered manifold where α is the map which associates to each jet its source.

Let us consider the intersection $T^k = \bigcap_{j=0}^{\infty} F^k_j$. By an argument of dimension, we can find an integer $j_1(k) \geqslant j_0$ such that $T^k = F^k_j$ for all $k \geqslant k_s$ and $j \geqslant j_1(k)$. On the other hand, given $a \in M$ and $X \in T^k_{a'}$ $T^k_a = X + S^k_a$ where $S^k = \bigcap_{j=0}^{\infty} H^k_j$. From the definition of T^k it follows that $\pi^k : T^{k+1} \longrightarrow T^k$ is surjective; using same argument as in the proof of theorem 1.1 we show that $T^{k+1} \subset pT^k$ for $k \geqslant k_1$.

By the Cartan-Kuranishi prolongation theorem [5], there exists an integer $k_2 \geqslant k_1$ such that T^k is involutive and $T^{k+1} = pT^k$ for $k \geqslant k_2$. Let x be a point of the fiber $M_{x'}$ and let ξ be an analytic solution of T^{k_2} defined in a neighborhood of x. Since $T^{k_2+j} = p^j T^{k_2}$, $j \geqslant 1$, ξ is also a solution of T^{k_2+j} for $j \geqslant j_1$. Since $k_2 \geqslant r(\theta)$ it follows that $\xi \in \theta$; on the other hand, by construction, $j_x^\infty d\rho(\xi) = j_{x'}^\infty \xi'$. Hence the germs of $d\rho(\theta)$ and θ' at the point x' are equal. This shows that $\xi' \in \theta'$. We have proved at the same time that θ' is locally an ILPG and statement 2) of theorem 2.2.

We shall now prove 3). By proposition 1.1 there are integers d and $k_3 \geqslant k_2$ such that $H_j^k = S^k$ for all $k \geqslant k_3$ and $j \geqslant d$. Let $r \geqslant k_3$ be any integer and take a germ $\xi' \in \theta'_a$ such that $j_{a'}^{r+d} \xi' = 0$. Take $X \in T_a^{r+d+1}$ and put $Y' = \pi^{r+d} X$, $Y'' = \pi^r X$. Since $\rho^{r+d} \circ \pi^{r+d} = \pi^{r+d} \circ \rho^{r+d+1}$ and $\pi^{r+d} (j_{a'}^{r+d+1} \xi') = 0$, it follows that $\rho^{r+d}(Y') = 0$. Hence $Y' \in H^{r+d}$ and consequently $Y'' = \pi^r Y \in S^r$. This means that there exists $Y \in H^{r+2d+1}$ such that $Y'' = \pi^r Y$. Let $Z = X - \pi^{r+d+1} Y$; then $Z \in T_a^{r+d+1}$ and $\pi^r Z = 0$. By the theorem of existence of analytic solutions of involutive systems of partial differential equations [5], for k sufficiently large, there exists a solution of T^k defined in a neighborhood of a such that $j_a^{r+d+1} \xi = Z$. Then ξ is a lifting of ξ' at the point a and $j^r \xi = \pi^r Z = 0$.

Remark. – The quotient of an ILPG by an invariant fibration which is not admissible is not always an ILPG. For a counterexample see [4].

The author owes the following theorem to Ngo van Quê.

THEOREM 2.3. – *Let* θ *be an ILPG defined over* M *and let* (M, M', ρ) *be a fibration invariant by* θ. *Let also* Ω *and* θ' *be respectively the kernel and the projection of* θ *by the fibration* ρ. *Assume that* Ω *and* θ' *are ILPG. Then* ρ *is an admissible fibration with respect to* θ.

Proof. – Take an integer $k_0 \geqslant r(\theta)$, $r(\theta')$ and let H^k be the kernel of the surjective morphism of vector bundles $\rho^k : \mathrm{J}^k\theta \longrightarrow \mathrm{J}^k\theta'$. Then h^k is a vector sub-bundle of $\mathrm{J}^k\mathrm{T(M)}$ for all $k \geqslant k_0$. Let us put as before $\mathrm{H}_j^k = \pi^k(\mathrm{H}^{k+j})$, $k \geqslant k_0$, $j \geqslant 0$. The set of points of M where the projection $\pi^k : \mathrm{H}^{k+j} \longrightarrow \mathrm{J}^k\mathrm{T(M)}$ has maximum rank is a dense open set U_j^k, $k \geqslant k_0$, $j \geqslant 0$. The complement of U_j^k in M is an analytic sub-set of M. By Baire's theorem $\mathrm{U} = \bigcap_{k,j} \mathrm{U}_j^k$ is a dense set in M. Take $a \in \mathrm{M}$ and put as before $\mathrm{S}^k = \bigcap_{j=0}^{\infty} \mathrm{H}_j^k$ $k \geqslant k_0$.

For each $k \geqslant k_0$ there exists an integer $j' = j'(k)$ such that $\mathrm{S}_a^k = \pi^k(\mathrm{H}_a^{k+j}) = (\mathrm{H}_j^k)_a$ and $\mathrm{S}_a^{k+1} = (\mathrm{H}_j^{k+1})_a$ for all $j \geqslant j'$. Let g_a^k be the kernel of the linear mapping $\pi^k : \mathrm{S}_a^{k+1} \longrightarrow \mathrm{S}_a^k$. Then, by construction, $0 \longrightarrow g_a \longrightarrow \mathrm{S}_a^{k+1} \xrightarrow{\pi^k} \mathrm{S}_a^k \longrightarrow 0$ is an exact sequence of vector spaces. Restricted to a sufficiently small neighborhood of a, H_j^k and H_j^{k+1} are vector bundles. Moreover $\mathrm{H}_j^{k+1} \subset p\mathrm{H}_j^k$ (see proof of theorem 1.1). Hence $g_a^{k+1} \subset p(g_a^k)$ for all $k \geqslant k_0$. Therefore, by the algebraic prolongation theorem [5], [12], there exists an integer $k_1 \geqslant k_0$ such that g_a^k is an involutive space and such that $g_a^{k+1} = pg_a^k$ for all $k \geqslant k_1$.

Let j_1 be an integer such that $(\mathrm{H}_{j_1}^{k_1})_a = \mathrm{S}_a^{k_1}$ and $(\mathrm{H}_{j_1}^{k_1+1})_a = \mathrm{S}_a^{k_1+1}$. Let V be an open neighborhood of a such that $\mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}$, $\mathrm{H}_{j_1}^{k_1+1} \mid \mathrm{V}$ and $\mathrm{H}_{j_1+1}^{k_1} \mid \mathrm{V}$ are vector bundles. Then, by the choice of j_1, $\pi^{k_1}(\mathrm{H}_{j_1}^{k_1+1} \mid \mathrm{V}) = \mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}$. But $\mathrm{H}_{j_1}^{k_1+1} \mid \mathrm{V} \subset p(\mathrm{H}_{j_1}^{k_1} \mid \mathrm{V})$; hence the mapping $\pi^{k_1} : p(\mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}) \longrightarrow \mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}$ is surjective. By a theorem of Quillen [2], [5], [9], $\mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}$ is an involutive system of linear partial differential equations. Moreover, by the choice of k_1, $(p^s\mathrm{H}_{j_1}^{k_1})_a = \mathrm{S}_a^{k_1+s}$ for all $s \geqslant 0$. Therefore, every analytic solution of $\mathrm{H}_{j_1}^{k_1} \mid \mathrm{V}$ defined in an open connected neighborhood of a is in the kernel Ω. We have shown that $(\mathrm{H}_{j_1}^{k_1})_a = \mathrm{S}_a^{k_1} = \mathrm{J}_a^{k_1}\Omega$. We may assume that we have chosen $k_1 \geqslant r(\Omega)$. Since $\mathrm{J}^{k_1}\Omega$ is a vector bundle, there exists a neighborhood W of a such that $\mathrm{H}_j^{k_1} \mid \mathrm{W} = \mathrm{J}^{k_1}\Omega \mid \mathrm{W}$. We may also assume that we have chosen k_1 and W in such way that

$p^s(J^{k_1}\Omega|W) = J^{k_1+s}\Omega|W$ for $s \geqslant 0$. Since

$$H_{j_1}^{k_1+1}|W \subset p^s(H_{j_1}^{k_1}|W) = J^{k_1+s}\Omega|W \subset H_{j_1}^{k_1+s}|W,$$

it follows that $H_{j_1}^{k_1+s}|W = J^{k_1+s}\Omega|W$ for $s \geqslant 0$. Now we remark that for $j \geqslant j_1$, $H_j^{k_1+s}|W \subset H_{j_1}^{k_1+s}|W = J^{k_1+s}\Omega|W \subset H_j^{k_1+s}|W$. Hence $H_j^{k_1+s}|W = J^{k_1+s}\Omega|W$ for $s \geqslant 0$ and $j \geqslant j_1$. We have proved that $H_j^k|W = J_j^k\Omega|W$ is a vector bundle for $k \geqslant k_1$ and $j \geqslant j_1$. Hence $W \subset U$ and U is open in M.

Take now any point $b \in M$. By the choice of a, $\dim(H_j^k)_b \leqslant \dim(H_j^k)_a = \dim J_a^k\Omega = \dim J_b^k\Omega$, $k \geqslant k_1$, $j \geqslant j_1$. Since $(H_j^k)_b \supset J_b^k\Omega$, we have $(H_j^k)_b = J_b^k\Omega$. Therefore $H_j^k = J^k\Omega$, for $k \geqslant k_1$ and $j \geqslant j_1$. This completes the proof that ρ is an admissible fibration with respect to θ.

Let θ be an ILPG defined on M. We shall say that θ is a homogeneous ILPG if for any two points $a, b \in M$ there exists a local diffeomorphism f of M such that $f(a) = b$ and $df(\theta_a) = \theta_b$.

Consider now a fibration (M, M', ρ) invariant by θ. By definition, we say that the fibration (M, M', ρ) is homogeneous with relation to θ if given any two points $a, b \in M$ there exists a local diffeomorphism f of M projectable by ρ and such that $f(a) = b$ and $df(\theta_a) = \theta_b$. Denote by $f^k : J^kT(M) \longrightarrow J^kT(M)$ the extension of f to k-jets of local vector fields. If $\pi^k : J^{k+j}T(M) \longrightarrow J^kT(M)$ is the canonical projection then $\pi^k \circ j^{k+j} = f^k \circ \pi^k$. On the other hand, if f satisfies the condition above then $f^k(J_a^k\theta) = J_b^k(\theta)$. Since $f^k(J_a^k\Omega) = J_b^k\Omega$ it follows that $f^k(H_a^k) = H_b^k$. In the same way,

$$f^k((H_j^k)_a) = (f^k \circ \pi^k)(H_a^{k+j}) = (\pi^k \circ f^{k+j})(H_a^{k+j})$$
$$= \pi^k((H^{k+j})_b) = (H_j^k)_b.$$

Hence H^k and H_j^k are vector sub-bundles of $J^kT(M)$ for $k \geqslant 0$ and $j \geqslant 0$ and ρ is an admissible fibration with respect to θ. By theorems 2.1 and 2.2, the projection θ' of θ on M' and the kernel $\Omega(\theta)$ of θ are locally ILPG. It is easy to see that the sheaves θ' and $\Omega(\theta)$ are again homogeneous. This implies that they are ILPG defined respectively on M' and M. We have proved the following corollary to theorems 2.1 and 2.2.

COROLLARY 2.1. – *Let* θ *be an ILPG defined on* M *and let* (M, M', ρ) *be an invariant and homogeneous fibration with respect to* θ. *Then the projection* θ' *of* θ *on* M' *and the kernel* $\Omega(\theta)$ *are homogeneous ILPG.*

Let us consider the special case when θ is transitive on M and let Γ be the Lie pseudo-group obtained by integration of θ [10]. Then Γ operates transitively on M, that is given $a, b \in M$ there exists $f \in \Gamma$ such that $f(a) = b$. Moreover, if (M, M', ρ) is a fibration invariant by θ then every $f \in \Gamma$ is locally projectable by ρ. Hence (M, M', ρ) is homogeneous with relation to θ. This proves the following corollary:

COROLLARY 2.2. – *Let* θ *be a transitive ILPG defined on* M *and let* (M, M', ρ) *be a fibration invariant by* θ. *Then* ρ *is an admissible fibration with respect to* θ *and the projection* θ' *of* θ *on* M' *and the kernel* $\Omega(\theta)$ *are homogeneous ILPG.* θ' *is transitive on* M'.

3. Prolongations of infinitesimal Lie pseudo-groups.

DEFINITION 3.1. – *Let* θ *and* θ' *be ILPG defined on the manifolds* M *and* M'. *A homomorphism of* θ *onto* θ' *is a fibration* $\rho : M \longrightarrow M'$ *which is admissible with respect to* θ *and which is such that* $d\rho(\theta) = \theta'$. *We shall also say in this case that* θ *is a homomorphic prolongation of* θ' *(prolongement meriedrique of E. Cartan). If moreover the kernel* $\Omega(\theta)$ *of* θ *by the fibration* ρ *is the zero sheaf on* M, *we shall say that* ρ *is an isomorphism of* θ *onto* θ', *or equivalently that* θ *is an isomorphic prolongation of* θ' *(prolongement holoedrique of E. Cartan).*

We shall write $\rho : \theta \longrightarrow \theta'$ to indicate that ρ is a homomorphism of θ onto θ'.

Let $\rho : \theta \longrightarrow \theta'$ be a homomorphism and let d be the least integer which satisfies condition 3) of theorem 2.2 with respect to ρ, θ and θ'; we say then that ρ is a homomorphism of order d or that θ is a homomorphic prolongation of order d of θ'.

The following proposition is an immediate consequence of theorem 2.3.

PROPOSITION 3.1. – *Let* θ *be an ILPG defined on* M *and let* (M, M', ρ) *be a fibration invariant by* θ. *If the projection* θ' *of* θ *on* M' *and the kernel* $\Omega(\theta)$ *of* θ *by the fibration* ρ *are ILPG, then* ρ *is an admissible fibration and consequently* $\rho : \theta \longrightarrow \theta'$ *is a homomorphism.*

Remarks. – 1) Let $\rho : \theta \longrightarrow \theta'$ be an isomorphism. Given a point $a \in M$, let L be the filtered Lie algebra of infinite jets of germs of θ at the point a, $L = J_a^\infty \theta$. Put $a' = \rho(a)$ and consider in a similar way $L' = J_{a'}^\infty \theta'$. Then the mapping ρ induces an isomorphism $\rho_* : L \longrightarrow L'$ of Lie algebras which is compatible with the filtrations in the sense that $\rho_*(L^k) \subset L'^k$, $k \geqslant -1$, [7]. Here L^k and L'^k, $k \geqslant -1$ denote the filtrations of L and L'. This means that ρ_* is continuous when L and L' are endowed with the topologies defined by the filtrations. Condition 3) of theorem 2.2 implies then that the inverse isomorphism $(\rho^*)^{-1} : L' \longrightarrow L$ is also continuous. Hence L and L' are isomorphic topological Lie algebras.

2) Let us assume again that $\rho : \theta \longrightarrow \theta'$ is an isomorphism. Take a point $a \in M$ and put $a' = \rho(a)$. For k sufficiently large, let ξ', $\eta' \in \theta'_{a'}$ be two germs such that $j_{a'}^{k+d} \xi' = j_{a'}^{k+d} \eta'$, where d is the order of ρ. Let ξ and η be the unique liftings of ξ' and η' in θ_a. Then $j_a^k \xi = j_a^k \eta$. In fact, since $j_{a'}^{k+d}(\xi' - \eta') = 0$, by condition 3) of theorem 2.2 there is a lifting of $\xi' - \eta'$ in θ_a which vanishes at order k at the point a. Since $\xi - \eta$ is the unique lifting of $\xi' - \eta'$ in θ_a it follows that $j_a^k \xi = j_a^k \eta$. Hence, for k sufficiently large there is a well defined linear surjective map $\mu : J_{a'}^{k+d} \theta' \longrightarrow J_a^k \theta$ such that $\mu(j_{a'}^{k+d} \xi') = j_a^k(\xi)$, $\xi' \in \theta'_a$, where ξ is the lifting of ξ' at the point a.

PROPOSITION 3.2. – *Let* $\rho : \theta \longrightarrow \theta'$ *and* $\rho' : \theta' \longrightarrow \theta''$ *be isomorphisms. Then* $\rho' \circ \rho : \theta \longrightarrow \theta''$ *is an isomorphism.*

Proof. – Obviously the kernel of $\rho' \circ \rho$ is the zero sheaf on M. Hence, by proposition 3.1, $\rho' \circ \rho$ is an isomorphism.

Let F^s denote the principal bundle of frames of order s of the manifold M and let $\beta : F^s \longrightarrow M$ be the canonical projection. Given an analytic diffeomorphism f of an open set $U \subset M$ onto an open set $V \subset M$, we shall denote by $p^s f$ the map defined in $\beta^{-1}(U)$ in the following way: For $X \in \beta^{-1}(U)$, $p^s f(X) = j^k_{\beta(X)} f \circ X$. $p^s f$ is an analytic diffeomorphism of $\beta^{-1}(U)$ onto $\beta^{-1}(V)$. If f and g are two local diffeomorphisms such that $f \circ g$ is defined, then $p^s f \circ p^s g$ is defined and $p^s(f \circ g) = p^s f \circ p^s g$, $p^s(f^{-1}) = (p^s f)^{-1}$.

Consider now a local vector field ξ defined on the open set $U \subset M$. In a sufficiently small neighborhood W of each point of U there exists an analytic family f_t of local diffeomorphisms defined in W, $-\epsilon < t < \epsilon$, such that f_0 is the identical map of W and such that for every $x \in W$, $\xi(x)$ is the tangent vector at the point x to the curve $t \longrightarrow f_t(x)$. We shall denote this tangent vector by $\left[\frac{d}{dt} f_t(x)\right]_{t=0}$. The prolongation $p^s f_t$ is then an one parameter family of analytic diffeomorphism defined in $\beta^{-1}(W)$ such that $p^s f_0$ is the identify map. Using local coordinates we can prove that the tangent vector $\left[\frac{d}{dt}(p^s f_t)\right]_{t=0}$ depends only on ξ and not on the choice of the family f_t. The mapping $z \in \beta^{-1}(W) \longrightarrow \left[\frac{d}{dt}\, p^s f_t\right]_{t=0}$ is then an analytic vector field defined in $\beta^{-1}(W)$ which depends only on ξ. Reasoning similarly for every point $x \in U$, we obtain a well defined analytic vector field over $\beta^{-1}(U)$ which we shall denote by $p^s\xi$; this vector field is called the s-th prolongation of ξ. The mapping $p^s : \xi \in L(U) \longrightarrow p^s\xi \in L(\beta^{-1}(U))$ is an injective homomorphism of the Lie algebra L(U) of all vector fields defined on U into the Lie Algebra $L(\beta^{-1}(U))$ [10]. Clearly, $p^s\xi$ is projectable by $\beta : F^s \longrightarrow M$ and $\beta(p^s\xi) = \xi$. More generally, for $s' \leqslant s$, $p^s\xi$ is projectable by the canonical mapping $\pi^{s'} : F^s \longrightarrow F^{s'}$ and $d\pi^{s'}(p^s\xi) = p^{s'}\xi$.

For each $z \in F^s$ and for each integer $r \geqslant 0$ there is a linear isomorphism $\lambda^s_r(z) : J^{s+r}_{\beta(z)} T(M) \longrightarrow J^r_z(T(F^s))$ such that $\lambda^s_r(j^{s+r}_{\beta(z)}\xi) = j^r_z p^s\xi$. Using local coordinates it can be shown that $\lambda^s_r(z)$ is a well defined injective linear map. Moreover, if

$s' \leqslant s$, $r' \leqslant r$ and $z' = \pi^{s'} z$, then the following diagrams are commutative :

$$\begin{tikzcd} J^{s+r}_{\beta(z)}\, T(M) \arrow[r, "\lambda^s_r(z)"] \arrow[d, "\pi^r"] & J^r_z T(F^s) \arrow[dl, "\beta^r"] \\ J^r_{\beta(z)}\, T(M) & \end{tikzcd} \tag{3.1}$$

$$\begin{tikzcd} J^{s+r}_{\beta(z)}\, T(M) \arrow[r, "\lambda^s_r(z)"] \arrow[d, "\pi^{s+r'}"] & J^r_z(T(F^s)) \arrow[d, "\pi^{r'}"] \\ J^{s+r'}_{\beta(z)}\, T(M) \arrow[r, "\lambda^s_{r'}(z)"] & J^{r'}_z(T(F^s)) \end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd} J^{s+r}_{\beta(z)}\, T(M) \arrow[r, "\lambda^s_r(z)"] \arrow[d, "\pi^{s'+r}"] & J^r_{z'}(T(F^s)) \arrow[d, "(\pi^{s'})^r"] \\ J^{s'+r}_{\beta(z)}\, T(M) \arrow[r, "\lambda^{s'}_r(z')"] & J^r_{z'}(TF^{s'}) \end{tikzcd} \tag{3.3}$$

In diagrams (3.1) and (3.3) β^r and $(\pi^{s'})^r$ denote the extensions of β and $\pi^{s'}$ to the vector bundles or r-jets of local vector fields.

Consider an ILPG θ defined on M and let $p^s\theta$ denote the sheaf of germs of local vector fields ξ of F^s defined in the following way: If ξ is defined in the open set $U \subset F^s$, then for every $z \in U$ there exists a neighborhood V of z and a vector field $\eta \in \theta$ defined in $\beta(V)$ such that $\xi \,|\, V = p^s\eta \,|\, V$. It is easy to show that for every $z \in F^s$, $\lambda^s_r(z)$ induces an isomorphism of vector spaces

$\lambda_r^s(z) : \mathrm{J}^{s+r}_{\beta(z)}\theta \longrightarrow \mathrm{J}^r_z(p^s\theta)$. Hence, if $\mathrm{J}^{s+r}\theta$ is a vector bundle, then $\mathrm{J}^r(p^s\theta)$ is also a vector bundle. It is also a simple matter to show that $p^s\theta$ is an ILPG which is called the s-th normal prolongation of θ. If $r(\theta)$ is the order of θ then $\mathrm{J}^r(p^s\theta)$ is a vector bundle for $r + s \geqslant r(\theta)$. In particular, if $s \geqslant r(\theta)$ then $\mathrm{J}^r(p^s\theta)$ is a vector bundle for all $r \geqslant 0$.

Clearly if $\xi \in p^s\theta$ and $d\beta(\xi) = 0$ then $\xi = 0$. Therefore, by proposition 3.1, $\beta : p^s\theta \longrightarrow \theta$ is an isomorphism. It can be shown that $\beta : p^s\theta \longrightarrow \theta$ is an isomorphism of order s. The preceding discussion can be resumed in the following proposition.

PROPOSITION 3.3. – *The* s-th *normal prolongation* $p^s\theta$ *of* θ *is an isomorphic prolongation of order* s *of* θ. *If* $s \geqslant r(\theta)$ *then* $\mathrm{J}^r(p^s\theta)$ *is a vector sub-bundle of* $\mathrm{J}^r(\mathrm{T}(\mathrm{F^S}))$ *for all* $r \geqslant 0$.

4. The equivalence relation.

DEFINITION 4.1. – *Two ILPG* θ_1 *and* θ_2 *are equivalent if there exists a third ILPG* θ_3 *which is at the same time an isomorphic prolongation of* θ_1 *and* θ_2.

We shall write $\theta_1 \sim \theta_2$ to denote that θ_1 and θ_2 are equivalent. Obviously the relation $\theta_1 \sim \theta_2$ is reflexive and symmetric but it is not at all clear that it is transitive. To prove that it is transitive we shall need the following theorem.

THEOREM 4.1. – *Let* θ, θ_1 *and* θ_2 *be ILPG. Let* $\rho_1 : \theta_1 \longrightarrow \theta$ *be an isomorphism and let* $\rho_2 : \theta_2 \longrightarrow \theta$ *be a homomorphism. Then there exist an ILPG* θ_3, *a homomorphism* $\tau_1 : \theta_3 \longrightarrow \theta_1$ *and an isomorphism* $\tau_2 : \theta_3 \longrightarrow \theta_1$ *such that* $\rho_1 \circ \tau_1 = \rho_2 \circ \tau_2$. *Moreover, if* ρ_2 *is an isomorphism then* τ_2 *is also an isomorphism.*

$$\begin{tikzcd} & \theta_3 \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ \theta_1 \arrow[dr, "\rho_1"'] & & \theta_2 \arrow[dl, "\rho_2"] \\ & \theta & \end{tikzcd}$$

Proof. – Let us assume that θ, θ_1 and θ_2 are defined respectively on the manifolds M, M_1 and M_2 and let d be the order of the homomorphism $\rho_2 : \theta_2 \longrightarrow \theta$. Take an integer $s \geqslant d$ and consider the normal prolongation $p^S\theta_1$ of θ_1 defined on $F^s(M_1) = F_1^s$. Denote by M_3 the submanifold of $F_1^s \times M_2$ of all points (a_1, a_2) such that $(\rho \circ \beta)(a_1) = \rho_2(a_2)$ where $\beta : F_1^s \longrightarrow M_1$ is the canonical fibration. Let $\pi_1 : M_3 \longrightarrow F_1^s$ be the projection onto the first factor and let $\tau_2 : M_3 \longrightarrow M_2$ be the projection onto the second factor. Put $\tau_1 = \beta \circ \pi_1$ and consider the sheaf θ_3 of germs of local vector fields ξ of M_3 which have the following properties: ξ is projectable by π_1 and τ_2 and $d\pi_1(\xi) \in p^s\theta_1$, $d\tau_2(\xi) \in \theta_2$. θ_3 is a sheaf of Lie algebras, every $\xi \in \theta_2$ is projectable by $\rho_1 \circ \tau_1 = \rho_2 \circ \tau_2$ and $d(\rho_1 \circ \tau_1)(\xi) \in \theta$.

We shall prove that $J^{s+r}\theta_3$ is a vector sub-bundle of $J^{s+r}T(M_3)$ for every sufficiently large integer r. Since our argument is of local nature, we may assume that $M = V$, $F^s(M_1) = V \times U_1$, $M_2 = V \times U_2$ and $M_3 = V \times U_1 \times U_2$ where V, U_1, U_2 are open sets of convenient euclidean spaces and that $\rho_1 \circ \beta : V \times U_1 \longrightarrow V$, $\rho_2 : V \times U_2 \longrightarrow V$, $\pi_1 : V \times U_1 \times U_2 \longrightarrow V \times U_1$ and $\tau_2 : V \times U \times U_2 \longrightarrow V \times U_2$ are the natural projections onto the factor spaces. Given a local vector field ξ

$$\begin{tikzcd}
 & V \times U_1 \times U_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\tau_2"] & \\
V \times U_1 \arrow[d, "\beta"'] \arrow[dr, "\rho_1 \circ \beta"] & & V \times U_2 \arrow[dl, "\rho_2"] \\
M_1 \arrow[r, "\rho_1"'] & V &
\end{tikzcd}$$

defined on an open set of V, we shall keep the same notation ξ to denote a lifting of ξ to any one of the product spaces $V \times U_1$, $V \times U_2$, $V \times U_1 \times U_2$.

Let $s \geqslant r(\theta)$ be an integer such that remark 2) below proposition 3.1 applies for θ, θ_2, ρ_2 and $r = s - d$. Consider the pullback $J^s\theta \times U_2$ of the vector bundle $J^s\theta$ by the mapping $\rho_2 : V \times U_2 \longrightarrow V$; $J^s\theta \times U_2$ is a vector bundle over $V \times U_2$ where the projection $J^s\theta \times U_2 \longrightarrow V \times U_2$ is the product $\alpha \times$ identity of $U_2 (\alpha : J^s\theta \longrightarrow V$ is the source mapping). By remark 2) below proposition 3.1, there exists a natural map $J^s\theta \times U_2 \longrightarrow J^r\theta_2$ which is linear in each fiber. Composing with the projection $\pi^0 : J^r\theta_2 \longrightarrow J^0\theta_2$ we get a mapping $\mu : J^s\theta \times U_2 \longrightarrow J^0\theta_2$. Denote by $T_v(V \times U_2)$ the vector bundle of vertical vectors of $V \times U_2$. Composing μ with the projection of $T(V \times U_2)$ onto $T_v(V \times U_2)$ we obtain a morphism of vector bundles $\gamma : J^s\theta \times U_2 \longrightarrow T_v(V \times U_2)$, (in this paragraph we do not assume that morphisms of vector bundles have constant rank). Take $\xi \in \theta$ and let ξ_2 by any lifting of ξ into θ_2. Denote by $j^s\xi : (x, z) \in V \times U_2 \longrightarrow (j^s_x\xi, z) \in J^s\theta \times U_2$ the section of $J^s\theta \times U_2$ defined by ξ. Then we can write $\xi_2 = \xi + \gamma \circ j^s\xi$.

Let now $\xi_3 \in \theta_3$ be a lifting of ξ_2 and put $\eta = d\pi_1(\xi_3)$. Then, $\xi_3 = \eta + \gamma \circ j^s\xi$. Therefore, for every $r \geqslant 0$,

$$j^r\xi_3 = j^r\eta + v \circ j^{s+r}\xi$$

where $v : J^{s+r}\theta \times U_2 \longrightarrow J^r(T_v(V \times U_2))$ is the morphism of vector bundles over $V \times U_2$ obtained from γ by extension to the $(s + r)$-jets of sections of θ.

Denote by $\delta^s_r : J^r(p^s\theta_1) \longrightarrow J^{s+r}(\theta_1)$ the morphism of vector bundles over the mapping $\beta : F^s(M_1) \longrightarrow M_1$ which is, in each fiber, the inverse mapping of

$$\lambda^s_r(z) : J^{s+r}_{\beta(z)}(\theta_1) \longrightarrow J^r_z(p^s\theta_1),\ z \in F^s(M_1),$$

(see paragraph 3). Composing δ^s_r with the extension

$$\rho_1^{s+r} : J^{s+r}\theta \longrightarrow J^{s+r}\theta$$

of ρ_1 to $(s + r)$-jets of local vector fields we get a morphism of vector bundles $\rho_1^{s+r} \circ \delta^s_r : J^r(p^s\theta_1) \longrightarrow J^{s+r}\theta$ which is such that $\rho_1^{s+r} \circ \delta^s_r \circ j^r\eta = j^{s+r}\xi$. This shows that

$$j^r\xi_3 = j^r\eta + (v \circ \rho_1^{s+r} \circ \delta^s_r) \circ j^r\eta .$$

On the other hand, $J^r(p^s\theta_1)$ and $J^r(T_v(V \times U_2))$ can be considered, in a natural way, as vector sub-bundles of $J^rT(V \times U_1 \times U_2)$ whose intersection is the zero vector bundle. Then the last formula says that $J^r\theta_3$ is the graph of the morphism of vector bundles $v \circ \rho_1^{s+r} \circ \delta^s_r : J^r(p^S\theta_1) \longrightarrow J^r(T_v(V \times U_2))$. Let now r_0 be an integer such that $J^r(p^s\theta_1)$ is a vector sub-bundle of $J^r T(F^s(M_1))$ for all $r \geqslant r_0$. Since $J^r\theta_3$ is the graph of a morphism of vector bundles, it follows that $J^r\theta_3$ is a vector sub-bundle of $J^r T(V \times U_1 \times U_2)$ for all $r \geqslant r_0$.

One shows easily that θ_3 satisfies the second condition of definition 1.1. To complete the proof it remains to show that all fibrations into consideration are admissible fibrations. Let us remark that the kernel $\Omega(\theta_3)$ of θ_3 by the fibration τ_2 is an ILPG. In fact, a germ ξ_3 of a local vector field of M_3 belongs to $\Omega(\theta_3)$ if and only if it is projectable by π_1 and τ_2 and $\tau_2(\xi_3) = 0$ and $d\pi_1(\xi_3) \in p^s(\Omega(\theta_1))$. Since $p^s\Omega(\theta_1)$ and the sheaf of germs of the zero vector field on M_2 are ILPG, by the same argument we used to prove that θ_3 is an ILPG, it follows that $\Omega(\theta_3)$ is an ILPG for s sufficiently large. Hence, by theorem 2.3, τ_2 is an admissible fibration. Since the kernel of θ_3 by the fibration $\rho_2 \circ \tau_2$ is also $\Omega(\theta_3)$, by the same theorem $\rho_2 \circ \tau_2$ is an admissible fibration. Repeating the argument we can prove that all other fibrations we have considered are admissible fibrations. Finally let us remark that $\pi_1 : \theta_3 \longrightarrow p^s\theta_1$ is an isomorphism. In fact, if $\xi_3 \in \theta_3$ and $d\pi_1(\xi_3) = 0$ then $d\rho_2(d\tau_2(\xi_3) = 0)$; since $\rho_2 : \theta_2 \longrightarrow \theta_1$ is an isomorphism this implies that $d\tau_2(\xi_3) = 0$. Hence $\xi_3 = 0$. Consequently $\tau_1 = \beta \circ \pi_1$ is also an isomorphism of θ_3 onto θ_1. In the same way, if $\rho_1 : \theta_1 \longrightarrow \theta$ is an isomorphism then τ_2 is also an isomorphism.

PROPOSITION 4.1. – *The relation* $\theta_1 \sim \theta_2$ *is an equivalence relation.*

Proof. – Clearly only the transitivity of the relation needs a proof. Assume then that $\theta_1 \sim \theta_2$ and $\theta_2 \sim \theta_3$. By definition, there are ILPG $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ and isomorphisms $\rho_1 : \widetilde{\theta}_1 \longrightarrow \theta_1$, $\rho_2 : \widetilde{\theta}_2 \longrightarrow \theta_2$, $\tau_2 : \widetilde{\theta}_2 \longrightarrow \theta_2$, $\tau_3 : \widetilde{\theta}_2 \longrightarrow \theta_3$. By theorem 4.1, there exist

$$\begin{tikzcd}
 & & \widetilde{\theta}_3 \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & & \\
 & \widetilde{\theta}_1 \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & & \widetilde{\theta}_2 \arrow[dl, "\tau_2"'] \arrow[dr, "\tau_3"] & \\
\widetilde{\theta}_1 & & \widetilde{\theta}_2 & & \widetilde{\theta}_3
\end{tikzcd}$$

an ILPG $\widetilde{\theta}_3$ and isomorphisms $\mu_1 : \widetilde{\theta} \longrightarrow \widetilde{\theta}_1$, $\mu_2 : \widetilde{\theta}_3 \longrightarrow \widetilde{\theta}_2$. Then $\rho_1 \circ \mu_1$ and $\tau_3 \circ \mu_2$ are isomorphisms and consequently $\theta_1 \sim \theta_3$.

5. Quotient by a normal sub-pseudo-group.

We say that a sub-ILPG θ_0 of θ is normal in θ if $[\theta, \theta_0] \subset \theta_0$. The aim of this paragraph is to show that there exists a quotient of θ by θ_0.

DEFINITION 5.1. – *Let θ_0 be a normal sub-ILPG of θ. An ILPG θ' is a quotient of θ by θ_0 if there exist ILPGs $\widetilde{\theta}$ and $\widetilde{\theta}_0 \subset \widetilde{\theta}$, an isomorphism $\rho : \widetilde{\theta} \longrightarrow \theta$ such that $\rho(\widetilde{\theta}_0) = \theta_0$ and a homomorphism $\tau : \widetilde{\theta} \longrightarrow \theta'$ whose kernel is $\widetilde{\theta}_0$.*

PROPOSITION 5.1. – *Let θ be an ILPG defined over the manifold* M *and let* $(\mathrm{M}, \mathrm{M}'\rho)$ *be any fibration of* M. *Let* $\mathrm{T}_v(\mathrm{M}) \subset \mathrm{T}(\mathrm{M})$ *be the sub-bundle of* T(M) *of vertical tangent vectors. If there exists a normal sub-ILPG θ_0 of θ such that* $\mathrm{J}^0 \theta_0 = \mathrm{T}_v(\mathrm{M})$ *then θ is projectable by ρ.*

Proof. – Let Ω be the sheaf of vertical tangent vectors of M. By hypothesis $\theta_0 \subset \Omega$. Let $\mathrm{N}(\theta_0)$ and $\mathrm{N}(\Omega)$ denote the normalizers of θ_0 and Ω in the sheaf of all germs of vector fields of M. Then $\mathrm{N}(\theta_0) \subset \mathrm{N}(\Omega)$. To prove this inclusion we remark that if $\eta \in \Omega$ then, locally, we can write $\eta = \sum_{i=1}^{k} f^i \xi_i$ where $\xi_i \in \theta_0$ and f_i are real functions defined on an open set of M. Hence, for any $\xi \in \mathrm{N}(\theta_0)$, $[\xi, \eta] = \sum_{i=1}^{k} (\xi f^i)\, \xi_i + \sum_{i=1}^{k} f^i [\xi, \xi_i]$. Since $[\xi, \xi_i] \in \theta_0$, it follows that $[\xi, \eta] \in \Omega$. Therefore $\theta \subset \mathrm{N}(\theta_0) \subset \mathrm{N}(\Omega)$. Using local coordinates we can show easily that $\mathrm{N}(\Omega)$ is the sheaf Φ of all germs of local vector fields of M which are projectable by ρ. Since $\theta \subset \Phi$, θ is projectable by ρ.

Let χ be the sheaf of all germs of local vector fields of M. We have seen that there is a natural morphism of vector bundles $\delta_r^s : \mathrm{J}^r(p^s \chi) \longrightarrow \mathrm{J}^{r+s}\chi$ defined over the projection $\beta : \mathrm{F}^s \longrightarrow \mathrm{M}$ which induces an isomorphism of vector spaces in each fiber.

PROPOSITION 5.2. – *Let* $E \subset J^r(p^s\chi)$ *and* $F \subset T^{r+s}(M)$ *be two vector sub-bundles and assume that* $\delta^r_s | E : E \longrightarrow F$ *is a surjective morphism of vector bundles over the projection* $\beta : F^s \longrightarrow M$. *Then* $pE \subset J^{r+1}(p^s\chi)$ *and* $\delta^s_r | pE : pE \longrightarrow pF$ *is a surjective mapping.*

Proof. – Using local coordinates we can show easily that $pJ^r(p^s\chi) = J^{r+1}(p^s\chi)$. Hence, $pE \subset J^{r+1}(p^s\chi)$. Take $X \in pE$ and let $\sigma \in \underline{pE}$ be a local section projectable by δ^s_r and such that $\sigma(a) = X$ where $a = \alpha(\chi) \in F^s$. Then there exists a local section $\sigma' \in \underline{J^{s+r+1}\,T(M)}$ such that $\sigma' = \delta^s_r \circ \sigma$. We shall prove that $\sigma' \in \underline{pF}$. By hypothesis $\pi^r \circ \sigma \in \underline{E}$ and $D\sigma \in \underline{T^*(F^s) \otimes E}$ where D denotes the Spencer operator [8]. Locally, there is a finite number of vector fields ξ_i such that $\sigma = \sum_i (f^i \circ \beta)\, j^{r+1} p^s \xi_i$. Consequently, $D\sigma = \sum_i (df^i \circ d\beta) \otimes j^r p^s \xi_i$. Since $D\sigma \in \underline{T^*(F^s) \otimes E}$ it follows that $\sum_i (df^i \circ j^{r+s}\xi_i) \in \underline{T^*(M) \otimes F}$. Hence $D\sigma' \in \underline{T^*(M) \otimes F}$. On the other hand

$$\pi^{r+s} \circ \sigma' = \pi^{r+s} \circ \delta^s_{r+1} \circ \sigma = \delta^s_r \circ \pi^r \circ \sigma\,.$$

Therefore $\pi^{r+s} \circ \sigma' \in \underline{pE}$. This proves that $\delta^s_{r+1}(\chi) \in pF$. We have shown that $\delta^s_{r+1}(pE) \subset pF$. The other inclusion can be proved in the same way.

THEOREM 5.1. – *Let* θ *be an ILPG defined on the manifold* M *and let* $\theta_0 \subset \theta$ *be a normal sub-ILPG. Then, for every point* $a \in M$, *one can find a neighborhood* V *of* a *such that there exists a quotient of* $\theta | V$ *by* $\theta_0 | V$.

Proof. – Choose an open neighborhood W of a and an integer $s \geqslant r(\theta_0)$ such that $pJ^k(\theta_0 | W) = J^{k+1}(\theta_0 | W)$ for $k \geqslant s$. We have shown that $J^k(p^s(\theta_0 | W))$ is a vector sub-bundle of $J^k\,T(F^s(W))$ for $k \geqslant 0$. In particular $J^0(p^s(\theta_0 | W))$ is a completely integrable vector sub-bundle of $T(F^s(W))$. Take a point $b \in F^s(W)$. There exist an open neighborhood U of b and a fibered manifold (U, U', τ) whose fibers are the maximal integral manifolds of $J^0(p^s\theta_0 | U)$. Put $V = \beta(U)$. To simplify our notation we shall denote the restrictions $\theta | V$ and $p^s\theta | U$ by θ and $p^s\theta$. It is a simple matter to prove that a local vector field $\xi \in p^s\theta$ belongs to $p^s\theta_0$ if and only if $j^0\xi \in \underline{J^0(p^s\theta_0)}$.

By proposition 5.1, $p^s\theta$ is projectable by τ. We shall prove that τ is an admissible fibration with relation to $p^s\theta$. It is clear that $H^0(p^s\theta, \tau) = J^0 p^s\theta_0$. We shall show by induction on k that $H^k(p^s\theta, \tau) = J^k p^s \partial_0$ for all $k \geqslant 0$. Assume the statment true for an integer k. By definition of $H^{k+1}(p^s\theta, \tau)$ it is clear that $J^{k+1} p^s \theta_0 \subset H^{k+1}(p^s\theta, \tau)$. On the other hand

$$H^{k+1}(p^s\theta, \tau) \subset pH^k(p^s\theta, \tau) = pJ^k p^s\theta_0 .$$

By our choice of s and by proposition 5.2, $pJ^k p^s\theta_0 = J^{k+1} p^s\theta_0$. Hence $H^{k+1}(p^s\theta, \tau) = J^{k+1} p^s\theta_0$. Since $\pi^k : J^{k+1} p^s\theta_0 \longrightarrow J^k p^s\theta_0$ is surjective, it follows that $H_j^k(p^s\theta, \tau) = J^k p^s\theta_0$ for all $k, j \geqslant 0$. Therefore τ is an admissible fibration with relation to $p^s\theta$. By construction, the kernel of $p^s\theta$ by the fibration τ is $p^s\theta_0$. By theorem 2.2, we can choose U and U$'$ such that the projection θ' of θ by τ is an ILPG. Then θ' is a quotient of θ by θ_0.

6. Fine homomorphisms.

Let θ be an ILPG defined on M and let (M, M', ρ) be a fibration invariant by θ.

DEFINITION 6.1. – *The fibration ρ is fine with relation to θ if the kernel Ω of θ by ρ is an ILPG such that :*

1) $J^k\Omega$ *is a vector sub-bundle of* $J^k T$ *for all* $k \geqslant 0$.
2) $H^k(\theta, \rho) = J^k\Omega$ *for all* $k \geqslant 0$.

It is clear that any fibration which is fine with relation to θ is admissible with relation to θ. Consequently any fibration ρ which is fine with relation to θ is a homomorphism $\rho : \theta \longrightarrow \theta'$ of θ onto the projection θ' of θ. We shall say in this case that ρ is a fine homomorphism. The homomorphism $\tau : \theta \longrightarrow \theta'$ defined in § 5 is fine.

PROPOSITION 6.1. – *Let θ be an ILPG defined on* M *and let $\rho : \theta \longrightarrow \theta'$ be a fine homomorphism. Let Ω be the kernel of θ by ρ and let θ_0 be an infinitesimal Lie pseudo-group such that $\Omega \subset \theta_0 \subset \theta$. Then the projection θ'_0 of θ_0 is locally an ILPG. Moreover, on a suitable neighborhood of each point of* M, *$\rho : \theta_0 \longrightarrow \theta'_0$ is a fine homomorphism.*

Proof. – It suffices to prove that ρ is an admissible fibration with relation to θ_0. By hypothesis $J^k\Omega \subset H^k(\theta_0, \rho) \subset H^k(\theta, \rho)$. Since ρ is a fine homomorphism, $J^k\Omega = H^k(\theta;\rho)$, $k \geqslant 0$. Hence $H^k(\theta_0, \rho) = J^k\Omega$ for $k \geqslant 0$. This proves that ρ is an admissible fibration with relation to θ_0.

PROPOSITION 6.2. – *Let θ and θ' be ILPGs defined on the manifolds* M *and* M′ *and let $\rho : \theta \longrightarrow \theta'$ be a fine homomorphism. Given an ILPG $\theta'_0 \subset \theta'$, let $\theta_0 = \rho^{-1}(\theta'_0)$ be the sheaf of germs of local sections of θ whose projections are in θ'_0. Then θ_0 is locally an ILPG and $\rho : \theta_0 \longrightarrow \theta'_0$ is locally a fine homomorphism.*

Proof. – The proof of proposition 6.2 is similar to the proof of proposition 6.1 and will be omitted.

Remark. – Let $\rho : \theta \longrightarrow \theta'$ be a fine homomorphism and let $\rho' : \theta' \longrightarrow \theta''$ be a homomorphism. It follows from propositions 6.2 and 1.1 and theorem 2.3 that $\rho' \circ \rho : \theta \longrightarrow \theta''$ is a homomorphism.

PROPOSITION 6.3. – *Let θ and θ' be ILPGs defined on the manifolds* M *and* M′ *and let $\rho : \theta \longrightarrow \theta'$ be a homomorphism. Given points $a \in$ M and $a' = \rho(a) \in$ M′ and restricting, if necessary, θ and θ' to convenient neighborhoods of a and a', there exist a normal prolongation $p^s\theta$ of θ, an ILPG θ'', a fine homomorphism $\tau : p^s\theta \longrightarrow \theta''$ and an isomorphism $\rho' : \theta'' \longrightarrow \theta'$ such that $\rho' \circ \tau = \rho \circ \beta$.*

$$\begin{tikzcd}
\theta \arrow[d, "\rho"'] & p^s\theta \arrow[l, "\beta"'] \arrow[d, "\tau"] \\
\theta' & \theta'' \arrow[l, "\rho'"]
\end{tikzcd}$$

Proof. – Let θ_0 be the kernel of $\rho : \theta \longrightarrow \theta'$. Then θ_0 is locally an ILPG and is normal in θ. By theorem 5.1 we can find an open neighborhood V of a, a normal prolongation $p^s(\theta \mid V)$ of $\theta \mid V$ and a fine homomorphism $\tau : p^s(\theta \mid V) \longrightarrow \theta''$ whose kernel is $p^s(\theta_0 \mid V)$. To simplify our notation we shall assume that $V = M$. Denote respectively, by $\widetilde{M}$ and M'' the manifolds where $p^s\theta$ and θ'' are defined. Each fiber of $\tau : \widetilde{M} \longrightarrow M''$ is an orbit of $p^s\theta_0$, therefore it is the inverse image by the projection β of an orbit of θ_0. Since the orbits of θ_0 are contained in the fibers of $\rho : M \longrightarrow M'$ it follows that each fiber of $\tau : \widetilde{M} \longrightarrow M'$ is contained in the inverse image by β of a fiber of $\rho : M \longrightarrow M'$. Hence there exists a fibration $\rho' : M'' \longrightarrow M'$ such that $\rho' \circ \tau = \rho \circ \beta$. It is easy to show that θ'' is projectable by ρ'. The image θ'' by ρ' is θ' and the kernel of $\rho' : \theta'' \longrightarrow \theta'$ is zero. Hence $\rho' : \theta'' \longrightarrow \theta'$ is an isomorphism. By the previous remark $\rho' \circ \tau$ is a homomorphism.

7. Abstract pseudo-groups.

Given two ILPGs θ_1 and θ_2 defined on the manifold M and a point $a \in M$, we shall say that θ_1 and θ_2 define the same germ of ILPG at the point a if there exists an open neighborhood U of a such that $\theta_1 | U = \theta_2 | U$. We shall denote the germ of ILPG defined by θ_1 at the point $a \in M$ by (θ_1, a) or simply by θ_1 when there is no danger of confusion.

PROPOSITION 7.1. – *Let θ_1 and θ_2 be two ILPG defined on the manifold* M *and let $a \in M$ be a point of* M. *If $J_a^\infty\theta_1 = J_a^\infty\theta_2$ then there exists a neighborhood* U *of a such that $\theta_1 | U = \theta_2 | U$.*

Proof. – Choose an integer $k \geq r(\theta_1)$, $r(\theta_2)$ and let $\xi_i \in \theta_1$, $i = 1, 2, \dots, s$, be vector fields defined on an open neighborhood of a such that $j_a^k \xi_1, \dots, j_a^k \xi_1$ is a basis of $J_a^k \theta_1 = J_a^k \theta_2$. Then, there exists an open neighborhood U of a such that for $x \in U$ $j_x^k \xi_1, \dots, j_x^k \xi_1$ is a basis of $J_x^k \theta_1$ and also of $J_x^k \theta_2$. Hence $J^k \theta_1 | U = J^k \theta_2 | U$. Therefore $\theta_1 | U = \theta_2 | U$.

Proposition 7.1 shows that the germ of an analytic ILPG θ at the point $a \in M$ is determined by the sub-Lie algebra θ_a of the Lie algebra $X(M)_a$ of germs of local analytic vector fields of M at the point a. However, it is an open problem of the theory to determine which sub-algebras of $X(M)_a$ are the fiber at the point a of an ILPG defined in a neighborhood of a.

We can repeat for the germs of ILPG the definition of admissible fibration, homomorphism, isomorphism kernel of a homomorphism and quotient by the germ of a normal sub-ILPG. In particular, repeating the construction of § 5 we can define an equivalence relation in the set of all germs of ILPG.

By definition, an abstract pseudo-group (APG) is an equivalence class of germs of ILPG under the equivalence relation described above. Let $\mathcal{C}$ denote the class of APG. If $\mathcal{G} \in \mathcal{C}$ is an APG and (θ, a) is a representative of $\mathcal{G}$ defined on the manifold M, we say that θ is an infinitesimal action of $\mathcal{G}$ on M.

It is known [7] that abstract pseudo-groups which have a transitive representative are in one-to-one correspondance with classes of isomorphic transitive topological Lie algebras.

To define morphisms of APG we proceed in the following way. Let $\mathcal{G}$, $\mathcal{H} \in \mathcal{C}$ be two APG and consider two pairs of representatives (θ_1, a_1), (θ_2, a_2), (θ'_1, a'_1), (θ'_2, a'_2) of $\mathcal{G}$ and $\mathcal{H}$ respectively. Given homomorphic prolongations

$$\rho_1 : (\theta_1, a_1) \longrightarrow (\theta'_1, a'_1), \quad \rho_2(\theta_2, a_2) \longrightarrow (\theta'_2, a'_2),$$

we shall say that ρ_1 is equivalent to ρ_2 $(\rho_1 \sim \rho_2)$ if there exist representatives $(\theta_3, a_3) \in \mathcal{G}$, $(\theta'_3, a'_3) \in \mathcal{H}$, a homomorphic prolongation $\rho_3 : (\theta_3, a_3) \longrightarrow (\theta'_3, a'_3)$ and isomorphic prolongations $\tau_i : (\theta_3, a_3) \longrightarrow (\theta_i, a_i)$, $\tau'_i : (\theta'_3, a'_3) \longrightarrow (\theta'_i, a'_i)$, $i = 1, 2$ such that $\rho_i \circ \tau_i = \tau_i \circ \rho_i$, $i = 1, 2$.

$$\begin{tikzcd}
 & \theta_3 \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] \arrow[dd, "\rho_3"] & \\
\theta_1 \arrow[dd, "\rho_1"'] & & \theta_2 \arrow[dd, "\rho_2"] \\
 & \theta'_3 \arrow[dl, "\tau'_1"] \arrow[dr, "\tau'_2"'] & \\
\theta'_1 & & \theta'_2
\end{tikzcd}$$

PROPOSITION 7.2. – *The relation* $\rho_1 \sim \rho_2$ *is an equivalence relation.*

Proof. – Clearly only the transitivity of the relation needs a proof. To prove that the relation is transitive it suffices to prove the following: given representatives $(\theta_i, a_i) \in \mathcal{G}$, $(\theta'_i, a'_i) \in \mathcal{H}$, $i = 1, 2, 3$ and homomorphic prolongations $\rho_i : (\theta_i, a_i) \longrightarrow (\theta'_i, a'_i)$, $i = 1, 2, 3$, and isomorphic prolongations $\tau_i : (\theta_i, a_i) \longrightarrow (\theta_1, a_1)$,

$$\tau'_i : (\theta'_i, a'_i) \longrightarrow (\theta'_1, a'_1), \ i = 2, 3,$$

such that $\rho_1 \circ \tau_i = \tau'_i \circ \rho_i$, $i = 2, 3$, then there exist representatives $(\widetilde{\theta}, \widetilde{a}) \in \mathcal{G}$, $(\widetilde{\theta}', \widetilde{a}') \in \mathcal{H}$, a homomorphic prolongation

$$\widetilde{\rho} : (\widetilde{\theta}, \widetilde{a}) \longrightarrow (\widetilde{\theta}', \widetilde{a}')$$

and isomorphic prolongations

$$\widetilde{\tau}_i : (\widetilde{\theta}, \widetilde{a}) \longrightarrow (\theta_i, a_i), \ \widetilde{\tau}'_i : (\widetilde{\theta}', \widetilde{a}') \longrightarrow (\theta'_i, a'_i), \ i = 2, 3$$

such that $\rho_i \circ \widetilde{\tau}_i = \widetilde{\tau}'_i \circ \widetilde{\rho}$, $i = 2, 3$.

$$\begin{tikzcd}
 & & \widetilde{\theta} \arrow[d, "\widetilde{\rho}"] \arrow[ddll, "\widetilde{\tau}_2"'] \arrow[ddrr, "\widetilde{\tau}_3"] & & \\
 & & \widetilde{\theta}' \arrow[dddll, "\widetilde{\tau}'_2"'] \arrow[dddrr, "\widetilde{\tau}'_3"] & & \\
\theta_2 \arrow[drr, "\tau_2"] \arrow[dd, "\rho_2"'] & & & & \theta_3 \arrow[dll, "\tau_3"'] \arrow[dd, "\rho_3"] \\
 & & \theta_1 \arrow[dd, "\rho_1"] & & \\
\theta'_2 \arrow[drr, "\tau'_2"] & & & & \theta'_3 \arrow[dll, "\tau'_3"'] \\
 & & \theta'_1 & &
\end{tikzcd}$$

To construct $(\widetilde{\theta}, \widetilde{a})$ and $(\widetilde{\theta}', \widetilde{a}')$ we proceed as follows. Let (Ω_i, a_i) be the kernels of (θ_i, a_i) by the fibration ρ_i, $i = 1, 2, 3$; then $\tau_i((\Omega_i, a_i)) = (\Omega_1, a_1)$, $i = 2, 3$. Repeating the argument of theorem 4.1, we can construct germs of ILPG_s, $(\overline{\theta}, \overline{a})$ and $(\overline{\Omega}, \overline{a})$ and isomorphic prolongations $\overline{\tau}_i : (\overline{\theta}, \overline{a}) \longrightarrow (\theta_i, a_i)$, $i = 2, 3$, such that $\tau_2 \circ \overline{\tau}_2 = \tau_3 \circ \overline{\tau}_3$ and $\overline{\tau}_i(\overline{\Omega}, \overline{a}) = (\Omega_i, a_i)$, $i = 2, 3$.

By proposition 6.3 there exist a normal prolongation $(\widetilde{\theta}, \widetilde{a})$ of $(\overline{\theta}, \overline{a})$, a germ of ILPG, $(\widetilde{\theta}', \widetilde{a}')$, a fine homomorphism $\widetilde{\rho} : (\widetilde{\theta}, \widetilde{a}) \longrightarrow (\widetilde{\theta}', a')$ and isomorphic prolongations $\widetilde{\tau}_i' : (\widetilde{\theta}', \widetilde{a}') \longrightarrow \widetilde{\theta}_i'$ $i = 2,3$ such that $\rho_i \circ \widetilde{\tau}_i = \widetilde{\tau}_i \circ \widetilde{\rho}$, where $\widetilde{\tau}_i = \overline{\tau}_i \circ \beta$, $i = 2,3$.

DEFINITION 7.1. – *Given two APG* $\mathfrak{G}$, $\mathcal{H} \in \mathcal{C}$ *a homomorphism* $h : \mathfrak{G} \longrightarrow \mathcal{H}$ *is an equivalence class of homomorphic prolongations* $\rho : (\theta, a) \longrightarrow (\theta', a')$ *where* $(\theta, a) \in \mathfrak{G}$ *and* $(\theta', a') \in \mathcal{H}$.

Remark. – By theorem 4.1 any isomorphic prolongation $\rho : (\theta, a) \longrightarrow (\theta', a')$ is equivalent to the identical isomorphism $\mathrm{id} : (\theta', a') \longrightarrow (\theta', a')$.

Let $\mathfrak{G}$ and $\mathcal{H}$ be two APG. An imbedding of $\mathfrak{G}$ into $\mathcal{H}$ is an equivalence class of pairs (ψ, a), (θ, a) such that $(\psi, a) \in \mathfrak{G}$, $(\theta, a) \in \mathcal{H}$ and $(\psi, a) \subset (\theta, a)$. Two pairs (ψ_1, a_1), (θ_1, a_1) and (ψ_2, a_2), (θ_2, a_2) are considered equivalent if there exists a third pair (ψ_3, a_3), (θ_3, a_3) satisfying same conditions and isomorphic prolongations $\tau_1 : (\theta_3, a_3) \longrightarrow (\theta_i, a_i)$ such that $\tau_i(\psi_3) = \psi_i$, $i = 1,2$. Repeating the argument of proposition 7.1 we can prove easily that this relation is in fact an equivalence relation.

DEFINITION 7.2. – *A sub-APG of an APG* $\mathcal{H}$ *is an APG* $\mathfrak{G}$ *together with an imbedding of* $\mathfrak{G}$ *into* $\mathcal{H}$.

Let $\mathcal{H}$ and $\mathfrak{G}$ be APG and let $h : \mathcal{H} \longrightarrow \mathfrak{G}$ be a homomorphism. Take two representatives $\rho_1 : (\theta_1, a_1) \longrightarrow (\theta_1', a_1')$, $\rho_2 : (\theta_2, a_2) \longrightarrow (\theta_2', a_2')$ of h and let (Ω_1, a_1), (Ω_2, a_2) be the kernels of ρ_1 and ρ_2. From the definition of the equivalence relation $\rho_1 \sim \rho_2$ it follows immediatly that $(\Omega_1, a_1) \sim (\Omega_2, a_2)$. Hence the homomorphism h determines an APG $\mathcal{K}$ which is represented by the kernel (Ω_1, a) of the representant ρ_1 of h. Moreover h determines also an imbedding of $\mathcal{K}$ into $\mathcal{H}$. The sub-APG $\mathcal{K}$ of $\mathcal{H}$ is by definition the kernel of the homomorphism h.

Let $\mathfrak{G}$ be a sub-APG of $\mathcal{H}$ and assume that $\mathfrak{G}$ is normal in $\mathcal{H}$. This means that any pair (ψ, a), (θ, a) which represents the sub-APG $\mathfrak{G}$ of $\mathcal{H}$ is such that ψ is a normal sub-ILPG of θ.

PROPOSITION 7.2. – *Under the hypothesis above there exist a unique APG $\mathcal{Q}$ and a unique homomorphism $h : \mathcal{H} \longrightarrow \mathcal{Q}$ whose kernel is the sub-APG $\mathcal{G}$ of $\mathcal{H}$.*

Proof. – The existence of $\mathcal{Q}$ follows from theorem 5.1. The proof of the uniquess of $\mathcal{Q}$ and h is similar to the proof of proposition 7.1 and will be omitted.

By definition $\mathcal{Q}$ is the quotient of $\mathcal{H}$ by the normal sub-APG $\mathcal{G}$.

8. Abstract pseudo-groups of finite type.

PROPOSITION 8.1. – *Let θ be an ILPG defined on the manifold M and let a M be any point. The following propositions are equivalent :*

1) *θ_a is a finite dimensional Lie algebra.*

2) *There exists an integer k_0 such that $\pi^k : J^{k+1}\theta \longrightarrow J^k\theta$ is bijective for all $k \geqslant k_0$.*

3) *There exists an open neighborhood U of a and a finite dimensional Lie algebra L of vector fields defined in U such that $\theta \mid U$ is the sheaf of germs of vector fields of L.*

Proof. – The proof is easy and well be omitted.

DEFINITION 8.1. – *An ILPG θ is of finite type if it satisfies any one the equivalent conditions 1), 2) or 3) of proposition 8.1.*

We remark that if θ is an ILPG of finite type then $\dim \theta_a$ does not depend on the point $a \in M$.

PROPOSITION 8.2. – *Let L be a finite dimensional Lie algebra of vector fields defined on the manifold M and let θ be the sheaf of germs of vector fields of L. Then θ is locally an ILPG.*

Proof. – Take a point $a \in M$. There exists an integer k_0 depending on a such that the mapping $\xi \in L \longrightarrow j_a^k \xi \in J_a^k \theta$ is a linear isomorphism for all $k \geqslant k_0$. Consequently there is a neighborhood U of a such that $E^k = J^i(\theta \mid U)$ is a vector sub-bundle of $J^k T$ and $\pi^k : E^{k+1} \longrightarrow E^k$ is bijective for all $k \geqslant k_0$. Clearly $E^{k+1} \subset pE^k$. Therefore, restricting if necessary the neighborhood U, there exists an integer $k_1 \geqslant k_0$ such that $E^{k+1} = pE^k$ for all $k \geqslant k_1$. If ξ is a solution of E^{k_1} defined in the open connected neighborhood V of $b \in U$, then $j_b^k \xi \in E^k$ for $k \geqslant k_1$. Let $\eta \in L$ be a vector field such that $j_b^{k_1} \xi = j_b^{k_1} \eta$. Since $\pi^k : E^{k+1} \longrightarrow E^k$ is bijective it follows that $j_b^k \xi = j_b^k \eta$ for $k \geqslant 0$. Hence $\xi = \eta \mid V$ and $\xi \in \theta$. Therefore $\theta \mid U$ is an ILPG. By proposition 8.1 $\theta \mid U$ is of finite type.

It follows from proposition 8.2 and Theorem 2.3 that any fibration invariant by an ILPG of finite type is locally an admissible fibration. Moreover any germ of ILP which is equivalent to a germ of ILPG of finite type is itself of finite type. Hence we can define APG of finite type. The class of APG of finite type and morphisms of APG of finite type is a category $\mathcal{C}_0 \subset \mathcal{C}$.

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Manuscrit reçu le 27 août 1980.

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