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MAXIMUM MODULUS SETS

by Th. DUCHAMP and E.L. STOUT (*)

Introduction.

Given a domain D in a complex manifold, call a subset $E \subset bD$ a *maximum modulus set* if there is a function $f \in A(D)$ with $|f| = 1$ on E, $|f| < 1$ on $\bar{D} \setminus E$ where A(D) denotes the algebra of functions continuous on $\bar{D}$ and holomorphic on D. In general our domains will be strongly pseudoconvex. In the one dimensional case, there is not much to be said as, *e.g.*, if $D = \Delta$, the open unit disc, then every closed subset of $\mathbf{T} = b\Delta$ is a maximum modulus set. In the higher dimensional case things are more complicated.

There is a close relation between maximum modulus sets and *peak sets*, sets $F \subset bD$ for which there is a function $f \in A(D)$ with $f = 1$ on F, $|f| < 1$ on $\bar{D} \setminus F$. If $f \in A(D)$ has $E \subset bD$ as its maximum modulus set, then for each $\alpha \in \mathbf{T}$ the fiber $E_\alpha = \{p \in E : f(p) = \alpha\}$ is the set on which the function $\frac{1}{2}\,\bar{\alpha}(f + \alpha)$ peaks, provided, of course, that E_α is not empty.

Certain results about maximum modulus sets are clear. If D is strongly pseudoconvex, then as bD can be made strictly convex in a neighborhood of any of its points by a suitable choice of local coordinates, slicing with complex lines shows that no maximum modulus set in bD can contain an open subset of bD. Sibony [31] has shown that if f is a nonconstant function in $A^p(D)$, the algebra of functions with pth order derivatives in A(D), and if $p \geqslant 2$, then the set E on which $|f|$ takes its maximum has finite N-dimensional Hausdorff measure if the

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ambient complex manifold is N-dimensional. This implies that the topological dimension of E does not exceed N. It may be well to recall that Tumanov [36] has constructed in bB_2, B_N the unit ball in $\mathbf{C}^N$, a peak set of Hausdorff dimension 2.5. Tumanov's example is totally disconnected and perfect and so is homeomorphic to the Cantor set; its topological dimension is therefore zero. One of our results is that a maximum modulus set in the boundary of a strongly pseudoconvex domain has topological dimension no more than N.

In most of what follows, we shall be concerned with smooth manifolds. Recall that the set of submanifolds of the boundary of a strongly pseudoconvex domain which are peak sets admits a very concise description. Let D be given by the strongly plurisubharmonic characterizing function Q so that $D = \{Q < 0\}$ and $dQ \neq 0$ on bD. Let $\eta = \eta_Q = d^c Q = i(\bar{\partial} - \partial) Q$. A $\mathcal{C}^1$ submanifold $\Sigma \subset bD$ has the property that each of its compact subsets is a peak set(*) if and only if $\iota^* \eta = 0$ where $\iota : \Sigma \longrightarrow bD$ is the inclusion map [25]. This condition implies that Σ is totally real and of dimension no more than $N-1$. We shall call submanifolds satisfying this condition *interpolation manifolds*. In this connection, see [8].

The one form η_Q is real valued: If we take holomorphic coordinates $z_1, \ldots, z_N$ with underlying real coordinates $z_j = x_j + iy_j$, then

$$\eta_Q = -\sum_{j=1}^{N} \frac{\partial Q}{\partial y_j} dx_j - \frac{\partial Q}{\partial x_j} dy_j .$$

We denote by $T_p^{\mathbf{C}}(bD)$ the kernel of the map $\eta_Q : T_p(bD) \longrightarrow \mathbf{R}$ so that $T_p^{\mathbf{C}}(bD)$ is the maximal complex subspace of $T_p(bD)$. If J is the given complex structure, $T_p^{\mathbf{C}}(bD) = T_p(bD) \cap JT_p(bD)$. Interpolation manifolds admit the alternative description of being those submanifolds Σ of bD that satisfy $T_p(\Sigma) \subset T_p^{\mathbf{C}}(bD)$ for each $p \in \Sigma$.

We prove below that if $\Sigma \subset bD$ is an N-dimensional smooth submanifold which is the maximum modulus set of an

(*) And therefore a peak interpolation set, *i.e.*, if $K \subset \Sigma$ is compact and $g \in \mathcal{C}(K)$, then there is $G \in A(D)$ with $G|K = g$ and $|G(p)| < \|g\|_K$ if $p \in \setminus K$. See [37].

$f \in A^2(D)$, then Σ is necessarily totally real and it must admit a unique foliation by compact interpolation manifolds. There is a semiglobal converse in the real analytic case, and we show by example that there cannot be a global converse. We also prove what amounts to a uniqueness theorem. Roughly put, two A^2 functions with the same smooth N-dimensional manifold as maximum modulus set are analytically related. Under a global polynomial convexity hypotheses, they are polynomially related.

It is important for much of what we do that the form η_Q is a *contact form* in that $\eta_Q \wedge (d\eta_Q)^{N-1}$ is a volume form on bD. This implies the existence of a vector field ξ, the *characteristic vector field associated with* η, characterized by the conditions that $\eta_Q(\xi) = 1$ and $\xi \lrcorner d\eta_Q = 0$, *i.e.*, that $d\eta_Q(\xi, \xi') = 0$ for every choice of vector field ξ' on bD. The forms η_Q and $d\eta_Q$ are invariant under the flow generated by ξ. For contact geometry see [4].

We are indebted to Josip Globevnik who proposed to one of us the study of maximum modulus sets.

1. Necessity.

Fix a strongly pseudoconvex domain D with boundary of class $\mathcal{C}^r$, $2 \leqslant r \leqslant \omega$ in the complex manifold $\mathcal{M}$, $\mathcal{M}$ not necessarily Stein. Set $D = \{Q < 0\}$ with Q defined and of class $\mathcal{C}^r$ and strictly plurisubharmonic on a neighborhood of bD, $dQ \neq 0$ on bD. Let $\Sigma \subset bD$ be a closed submanifold of $\mathcal{M}$ of dimension N, of class $\mathcal{C}^s$, $2 \leqslant s \leqslant \omega$, and assume there to be a function, $f \in A^2(D)$ with $|f| < 1$ on $\bar{D} \backslash \Sigma$, $|f| = 1$ on Σ. Put $\eta = d^c Q = i(\bar{\partial} - \partial) Q$.

The form df has coefficients in $A^1(D)$; our first lemma implies that df does not vanish at any point of Σ.

LEMMA 1.1. – *If* $p \in \Sigma$, *then* $d \log |f| \neq 0$ *at* p.

Proof. – This follows from the Hopf Lemma. The function $\log |f|$ is of class $\mathcal{C}^2$ near the point p, it is subharmonic and negative on D, and it takes the value zero at p.

The lemma implies, since f satisfies the Cauchy-Riemann equations, that not only is $df \neq 0$ at p, but also that $d(f \mid b\mathrm{D}) \neq 0$ at p.

LEMMA 1.2. – *The manifold Σ is totally real.*

Proof. – Fix a point $p \in \Sigma$, and suppose that $\mathrm{T}_p(\Sigma)$ contains a complex line. Choose holomorphic coordinates $z_1, \ldots, z_\mathrm{N}$ near p so that p is the origin, so that $\mathrm{T}_0^{\mathbf{C}}(b\mathrm{D})$ is the complex hyperplane $z_1 = 0$ and so that the complex line $z_1 = z_3 = z_4 = \cdots = z_\mathrm{N} = 0$ is tangent to Σ at the origin.

The function f is of class $\mathrm{A}^2(\mathrm{D})$, so it can be extended to a neighborhood U of 0 in the $(z_1, \ldots, z_\mathrm{N})$ space as a $\mathcal{C}^2$ function and in such a way that $df \neq 0$ on U. Since $df \neq 0$ but $|f|$ takes its maximum at 0 (on $\overline{\mathrm{D}}$) it follows that, at 0, $\dfrac{\partial f}{\partial z_2} = \cdots = \dfrac{\partial f}{\partial z_\mathrm{N}} = 0$ and $\dfrac{\partial f}{\partial z_1} \neq 0$. Consequently if we set $w = \log f = x_1 + ix_2$, then $w, z_2, \ldots, z_\mathrm{N}$ is a set of complex-valued $\mathcal{C}^2$ coordinates in a neighborhood of 0, and Σ is contained in the real hyperplane $x_1 = 0$.

Near 0, the x_1-direction is transverse to $b\mathrm{D}$, so, near 0, $b\mathrm{D}$ can be described by an equation

$$\mathrm{Q}(w, z') = x_1 - \mathrm{F}(x_2, z') = 0$$

where $z' = (z_2, \ldots, z_\mathrm{N})$.

We shall compute the Levi form of $b\mathrm{D}$ with respect to the defining function Q at the point 0. First note that $\partial\bar{\partial} w = \partial\bar{\partial} x_1 = \partial\bar{\partial} x_2 = 0$ on $\overline{\mathrm{D}} \cap \mathrm{U}$ since w is holomorphic on $\mathrm{U} \cap \mathrm{D}$ and of class $\mathcal{C}^2$ on U. For the same reason, $\partial x_2 = -\dfrac{i}{2}\, dw$ and $\bar{\partial} x_2 = \dfrac{i}{2}\, d\bar{w}$ on $\overline{\mathrm{D}} \cap \mathrm{U}$. Therefore, at 0 we have

$$\begin{aligned} 2i\partial\bar{\partial}\mathrm{Q} &= 2i(\partial\bar{\partial} x_1 - \partial\bar{\partial}\mathrm{F}) \\ &= -2i \sum_{j,k=2}^{\mathrm{N}} \frac{\partial^2 \mathrm{F}}{\partial z_j\, \partial \bar{z}_k} - dz_j \wedge d\bar{z}_k - \frac{i}{2} \frac{\partial^2 \mathrm{F}}{\partial x_2^2}\, dw \wedge d\bar{w} \\ &\quad + \sum_{j=2}^{\mathrm{N}} \left(- \frac{\partial^2 \mathrm{F}}{\partial x_2\, \partial \bar{z}_j}\, dw \wedge d\bar{z}_j + \frac{\partial^2 \mathrm{F}}{\partial x_2\, \partial z}\, dz_j \wedge d\bar{w} \right). \end{aligned}$$

Since $T_0^{\mathbf{C}}(b\mathrm{D})$ is in the kernels of dw and $d\bar{w}$, it follows that at 0 the Levi form is the restriction to $T_0^{\mathbf{C}}(b\mathrm{D})$ of the form

$$-2i \sum_{j,k=2}^{\mathrm{N}} \frac{\partial^2 \mathrm{F}}{\partial z_j \, \partial \bar{z}_k}(0)\, dz_j \otimes d\bar{z}_k .$$

We shall show that this form is degenerate, contradicting the strong pseudoconvexity of $b\mathrm{D}$. To this end, introduce $\mathcal{C}^2$ coordinates $t_1, \ldots, t_{\mathrm{N}}$ in Σ near 0 so that 0 is the origin of the t-coordinate system and so that the z_2-coordinate axis is the (t_1, t_2)-plane (in $T_0(\Sigma)$). We compute the derivatives $\dfrac{\partial^2 \mathrm{F}}{\partial t_j \, \partial t_i}(0)$. These must be zero since $\mathrm{F} = 0$ on Σ. We have

$$\frac{\partial \mathrm{F}}{\partial t_i} = \sum_{\alpha=2}^{2\mathrm{N}} \frac{\partial \mathrm{F}}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial t_i}$$

and so

$$\frac{\partial^2 \mathrm{F}}{\partial t_j \, \partial t_i} = \sum_{\alpha=2}^{\mathrm{N}} \frac{\partial \mathrm{F}}{\partial x_\alpha} \frac{\partial^2 x_\alpha}{\partial t_j \, \partial t_i} + \sum_{\alpha,\beta=2}^{\mathrm{N}} \frac{\partial^2 \mathrm{F}}{\partial x_\beta \, \partial x_\alpha} \frac{\partial x_\beta}{\partial t_j} \frac{\partial x_\alpha}{\partial t_i} .$$

We restrict i and j to the values 1 and 2 (independently) and evaluate at 0. Since $d\mathrm{F}(0) = 0$, the only surviving terms are those in the second sum. But also, as the (t_1, t_2)-plane in $T_0(\Sigma)$ is the z_2-coordinate axis on which x_2 and $x_5, \ldots, x_{2\mathrm{N}}$ all vanish, we get

$$0 = \frac{\partial^2 \mathrm{F}}{\partial t_j \, \partial t_i}(0) = \sum_{\alpha,\beta=3}^{4} \frac{\partial^2 \mathrm{F}}{\partial x_\alpha \, \partial x_\beta}(0) \frac{\partial x_\beta}{\partial t_j}(0) \frac{2 x_\alpha}{\partial t_i}(0) .$$

Thus we have the matrix equation

$$0 = \begin{bmatrix} \dfrac{\partial x_3}{\partial t_j}(0) & \dfrac{\partial x_4}{\partial t_j}(0) \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial^2 \mathrm{F}}{\partial x_3 \, \partial x_3}(0) & \dfrac{\partial^2 \mathrm{F}}{\partial x_3 \, \partial x_4}(0) \\ \dfrac{\partial^2 \mathrm{F}}{\partial x_4 \, \partial x_3}(0) & \dfrac{\partial^2 \mathrm{F}}{\partial x_4 \, \partial x_4}(0) \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial x_3}{\partial x_i}(0) \\ \dfrac{\partial x_4}{\partial x_i}(0) \end{bmatrix} .$$

Since the (t_1, t_2)-plane in $T_0(\Sigma)$ is the same as the (x_3, x_4)-plane, the vectors $\left(\dfrac{\partial x_3}{\partial x_i}(0), \dfrac{\partial x_4}{\partial t_i}(0)\right)$, $i = 1, 2$, are linearly independent. We may conclude that the derivatives $\dfrac{\partial^2 \mathrm{F}}{\partial x_\alpha \, \partial x_\beta}(0)$, α, $\beta = 3, 4$, all vanish. This implies that $\frac{\partial^2 F}{\partial z_2 \, \partial \bar{z}_2}(0)$ vanishes contradicting the strong pseudoconvexity of $b\mathrm{D}$ and proving the lemma.

COROLLARY 1.2. – Let $\dim \Sigma = \mathrm{N}$. *If* $\iota : \Sigma \longrightarrow b\mathrm{D}$ *is the inclusion,* $\iota^* \eta$ *vanishes at no point of* Σ.

Proof. – For $p \in \Sigma$, the kernel of $\eta : \mathrm{T}_p(b\mathrm{D}) \longrightarrow \mathbf{R}$ is the space $\mathrm{T}_p^{\mathbf{C}}(b\mathrm{D})$, an $(\mathrm{N}-1)$-dimensional complex space. This space cannot contain the totally real N-dimensional space $\mathrm{T}_p(\Sigma)$.

COROLLARY 1.3. – *If* $f_\Sigma = f \mid \Sigma$, *then* f_Σ *is a regular map from* Σ *to* $\mathbf{T}$, *i.e.,* $df_\Sigma = 0$ *at no point of* Σ.

Proof. – The Cauchy-Riemann equations, $\bar{\partial} f = 0$, imply that the form df is of type (1,0) on $b\mathrm{D}$. Therefore, since Σ is a totally real submanifold of $\mathbf{C}^{\mathrm{N}}$, df is determined by its restriction, df_Σ, at each point $p \in \Sigma$. We know that df vanishes at no point of Σ, so df_Σ can vanish at no point of Σ.

COROLLARY 1.4. – *On* Σ, *the equation* $\eta_\Sigma \wedge d\eta_\Sigma = 0$ *holds.*

Proof. – Since f_Σ is regular, the closed real form $i d \log f_\Sigma = \frac{i}{f_\Sigma} df_\Sigma$ defines a foliation whose leaves are the compact submanifolds $f_\Sigma^{-1}(\alpha)$ for $\alpha \in \mathbf{T}$. As mentioned in the introduction, $f_\Sigma^{-1}(\alpha)$ is an interpolation manifold and therefore an integral manifold of η. Hence $\ker \eta_\Sigma = \ker i d \log f_\Sigma$ and, since η_Σ and $i d \log f_\Sigma$ are both nowhere zero, there is a function g on Σ with $g \eta_\Sigma = i d \log f_\Sigma$. The result now follows since

$$\eta_\Sigma \wedge d\eta_\Sigma = i g^{-1} d \log f_\Sigma \wedge (i d g^{-1} \wedge d \log f_\Sigma) = 0 .$$

The following theorem summarizes what we have done so far.

THEOREM 1.5. – *Let* D *be a strongly pseudoconvex domain in a complex manifold,* $\mathfrak{M}$, $b\mathrm{D}$ *of class* $\mathcal{C}^r$, $2 \leqslant r \leqslant \infty$. *Let* $\Sigma \subset b\mathrm{D}$ *be a submanifold of* $\mathfrak{M}$ *of dimension* N *and class* $\mathcal{C}^s$, $2 \leqslant s \leqslant \omega$. *If there is* $f \in \mathrm{A}^2(\mathrm{D})$ *with* $|f| < 1$ *on* $\bar{\mathrm{D}} \backslash \Sigma$ *and* $|f| = 1$ *on* Σ, *then the form* η *vanishes at no point of* Σ, $\eta \wedge d\eta = 0$ *on* Σ, *and* Σ *is foliated by compact integral manifolds of the form* η.

For the theory of foliations, see [18,24,33].

Remark 1.6. – We also know that Σ is totally real, lemma 1.2. It is of interest to notice that the condition $\eta \wedge d\eta = 0$ on Σ together with η zerofree on Σ implies total reality: If $X \in T_p(\Sigma)$ satisfies $JX \in T_p(\Sigma)$ and $Y \in T_p(\Sigma)$ satisfies $\eta(Y) \neq 0$, then since η annihilates $T_p^{\mathbf{C}}(bD) = T_p(bD) \cap JT_p(bD)$, we find that $\eta \wedge d\eta(Y, X, JX) = \eta(Y)\, d\eta(X, JX)$. This is supposed to be zero, so as $\eta(Y) \neq 0$, $d\eta(X, JX) = 0$, and as $d\eta = dd^c Q = 2i\partial\bar\partial Q$, strict pseudoconvexity implies that $X = 0$. Thus, Σ is totally real as claimed.

COROLLARY 1.7. – *If* $\dim \mathfrak{M} = 2$, *then* Σ *is torus or a Klein bottle.*

Proof. – If $\mathfrak{M} = \mathbf{C}^2$ and Σ is orientable, the result follows from the fact proved by Wells [38] that the only orientable compact totally real manifold in $\mathbf{C}^2$ is a torus.

The general case follows from the Hopf Index Theorem (or Poincaré-Hopf Theorem) [20]. Since η is nowhere zero on Σ, Σ has a nowhere vanishing vector field and so its Euler characteristic is zero. Thus Σ is a torus or Klein bottle.

Example 1.8. – Denote by B_N the open unit ball in $\mathbf{C}^N$ and, for $r \in (0, \infty)$, let $rB_N = \{rz : z \in B_N\}$. If we define $\varphi_0 : \mathbf{C}^N \longrightarrow \mathbf{C}$ by $\varphi_0(z) = z_1 \ldots z_N$, then $|\varphi_0| = 1$ on $\mathbf{T}^N = \{(e^{i\theta_1}, \ldots, e^{i\theta_N}) : \theta_1, \ldots, \theta_N \text{ real}\}$ and $|\varphi_0| < 1$ on $\sqrt{N}B_N \backslash \mathbf{T}^N$. Thus, the torus $\mathbf{T}^N$ is the maximum modulus set for φ_0. It is totally real and if we define $Q(z) = |z|^2 - N$ so that Q is a strongly plurisubharmonic defining function for $\sqrt{N}B_N$, then the form η is given by

$$\eta = i \sum_{j=1}^{N} z_j\, d\bar z_j - \bar z_j\, dz_j .$$

In terms of the parameters $\theta_1, \ldots, \theta_N$, this is seen to be

$$\eta = 2(d\theta_1 + \cdots + d\theta_N)$$

which vanishes at no point of $\mathbf{T}^N$. On $\mathbf{T}^N$ $d\eta = 0$ and so, *a fortiori,* $\eta \wedge d\eta = 0$. The leaves of the foliation of $\mathbf{T}^N$ defined by η are the fibers $\varphi_0^{-1}(\alpha)$, $|\alpha| = 1$, *i.e.*, cosets of the subgroup $\varphi_0^{-1}(1)$ in $\mathbf{T}^N$. They are connected.

It is worthwhile to pursue this particular example beyond our immediate needs, for it serves to motivate and clarify some of our subsequent work. In particular, we have the following result.

LEMMA 1.9. – *If* $f \in A(\sqrt{N}B_N)$ *satisfies* $|f| = 1$ *on* $\mathbf{T}^N$ *and* $|f| \leqslant 1$ *on* $\sqrt{N}B_N \backslash \mathbf{T}^N$, *then there is a finite Blaschke product b such that* $f = b \circ \varphi_0$, *i.e.*, $f(z) = b(z_1 \ldots z_N)$.

Proof. – We have assumed only that $f \in A(\sqrt{N}B_N)$ but as $|f| = 1$ on $\mathbf{T}^N$, the edge-of-the-wedge theorem [26] implies that f is holomorphic on a neighborhood of $\mathbf{T}^N$. The function f has the Fourier expansion

$$f(e^{i\theta_1}, \ldots, e^{i\theta_N}) = \sum_{j_1, \ldots, j_N \geqslant 0} a_{j_1 \ldots j_N} e^{i(j_1\theta_1 + \cdots + j_N\theta_N)}$$

which converges absolutely and uniformly on $\mathbf{T}^N$. Unless f is constant, the differential df does not vanish on $\mathbf{T}^N$ as follows from Lemma I.1. (Or, more simply in this case, it follows by noting that $\frac{d}{ds} f(e^{is} z_0) \neq 0$ if $z_0 \in \mathbf{T}^N$). Each of the fibers $f^{-1}(\alpha)$, $|\alpha| = 1$, is an integral manifold of the form η, so f is constant on each fiber $\varphi_0^{-1}(\alpha)$, $|\alpha| = 1$. This implies the existence of a function $F : \mathbf{T} \longrightarrow \mathbf{C}$ with $f = F \circ \varphi_0$ on $\mathbf{T}^N$. The function F is clearly continuous. Let it have the Fourier expansion

$$F \sim \sum_{-\infty}^{\infty} A_j e^{ij\theta} .$$

As $F \circ \varphi_0 = f$, a comparison of this Fourier series with that of f shows that $A_j = 0$, $j < 0$, and $a_{j_1} \ldots {}_{j_N} = 0$ unless $j_1 = \cdots = j_N$. The former fact implies that F is the boundary value function of an element of $A(\Delta)$, call it b, and as $|F| = 1$ on $b\Delta$, b must be a finite Blaschke product. The equality $f = b \circ \varphi_0$ holds on $\mathbf{T}^N$, and so it must hold on all of $\sqrt{N}B_N$, as we wished to prove.

Theorem 2.4 of [27] assures us *a priori* that the function f of the lemma is rational and of a particular form, but the present lemma does not seem to follow immediately from the earlier result.

COROLLARY 1.10. – *The union of two disjoint maximum modulus sets need not be a maximum modulus set.*

Proof. – Let Φ be a holomorphic automorphism of $\sqrt{N}B_N$ such that $\Phi(\mathbf{T}^N) \cap \mathbf{T}^N = \emptyset$. If $f \in A(\sqrt{N}B_N)$ has $\Phi(\mathbf{T}^N) \cup \mathbf{T}^N$ as it maximum modulus set, then $f(z) = b(z_1 \ldots z_N)$ as in the lemma, and this implies that $|f| < 1$ on $\sqrt{N}B_N \setminus \mathbf{T}^N$, contradicting $|f| = 1$ on $\Phi(\mathbf{T}^N)$.

We have seen in Theorem 1.6 that necessary conditions for a manifold Σ to be a maximum modulus set are that η vanish nowhere on Σ and that $\eta \wedge d\eta$ vanish identically on Σ. An extreme way in which the latter condition can be satisfied is for $d\eta = 0$ on Σ, *i.e.*, for η to be a closed form on Σ. This occurs in Example 1.9 above. It turns out that by a suitable choice of defining function for the domain, this can always be achieved.

LEMMA 1.11. – *If* D, Σ *and* f *are as in Theorem 1.6, then there is a defining function* $\widetilde{Q}$ *for* D *such that if* $\widetilde{\eta} = d^c\widetilde{Q}$, *then* $\widetilde{\eta}$ *is closed on* Σ.

Proof. – The real forms η and $i d\log f$ have the same kernel on each tangent space $T_p(\Sigma)$, $p \in \Sigma$. Accordingly, there is a zero-free real valued function g on $\mathbf{C}^N$ such that $g\eta_\Sigma = i d\log f_\Sigma$ on Σ. This implies that the forms $\pm g\eta_\Sigma$ are closed. As g is zero-free, we may suppose that g is positive. The function $\widetilde{Q} = gQ$ is a defining function for D, and $d^c\widetilde{Q} = gd^cQ + Qd^cg$. As $Q = 0$ on bD, we see that on Σ, $d^c\widetilde{Q} = g\eta$.

We should remark explicitly that since g is one degree less differentiable than either Q or f, we have lost one degree of differentiability in constructing $\widetilde{Q}$. Also, in general, $\widetilde{Q}$ will not be strongly plurisubharmonic.

That in general η will not be closed on a maximum modulus manifold is illustrated by the following example.

Example 1.12. – Given $\epsilon > 0$, define Q on $\mathbf{C}^2$ by $Q(z) = (1 + \epsilon|z_1 + z_2|^2)(|z|^2 - 2) = h(z)(|z|^2 - 2)$ so that $Q < 0$ defines the ball $\sqrt{2}B_2$. The function Q is strongly plurisubharmonic near $\sqrt{2}B_2$ if ϵ is small. We know that the torus $\mathbf{T}^2$ is a maximum modulus set for $A(\sqrt{2}B_2)$, but the form d^cQ is not closed on $\mathbf{T}^2$.

Write $\mathbf{T}^2 = \{(e^{i\theta_1}, e^{i\theta_2}) : \theta_1, \theta_2 \text{ real}\}$, and compute $dd^c Q$ in terms of θ_1 and θ_2. We have

$$dd^c Q = d\,\{(|z|^2 - 2)\, d^c h + h d^c\, |z|^2\}$$
$$= d(|z|^2 - 2) \wedge d^c h + (|z|^2 - 2)\, dd^c h + dh \wedge d^c\, |z|^2 + h dd^c\, |z|^2 \;.$$

On $\mathbf{T}^2$, $|z|^2 - 2 = 0$ and $dd^c |z|^2 = 0$, so on $\mathbf{T}^2$,

$$dd^c Q = dh \wedge d^c\, |z|^2 \,.$$

We know from Example 1.9 that $d^c |z|^2 = 2(d\theta_1 + d\theta_2)$. Also, on $\mathbf{T}^2$,

$$h(z) = 1 + \epsilon(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 1 + 2\epsilon + \epsilon(z_1 \bar{z}_2 + \bar{z}_1 z_2)$$
$$= 1 + 2\epsilon + 2\epsilon\, i \sin(\theta_1 - \theta_2)\,,$$

so on $\mathbf{T}^2$, $dh = 2\epsilon i \cos(\theta_1 - \theta_2)(d\theta_1 - d\theta_2)$. This means that on $\mathbf{T}^2$, $dd^c Q = 4\epsilon i \cos(\theta_1 - \theta_2)\, d\theta_1 \wedge d\theta_2$. As this is not the zero form, $d^c Q$ is not closed on $\mathbf{T}^2$.

Of course, we have seen in Example 1.9 that if we use the usual defining function $|z|^2 - 2$ for $\sqrt{2}\mathrm{B}_2$, then the associated form η is closed on $\mathbf{T}^2$.

2. Existence.

We can prove, in the real analytic case, a semiglobal converse to Theorem 1.6.

THEOREM 2.1. – *Let* D *be a strongly pseudoconvex domain in the* N*-dimensional complex manifold* $\mathcal{M}$. *Let* $\Sigma \subset b\mathrm{D}$ *be an* N*-dimensional totally real, real analytic closed submanifold of* $\mathcal{M}$ *that admits a real analytic foliation by* $(\mathrm{N}-1)$*-dimensional interpolation manifolds each leaf of which is compact. There is a neighborhood* Ω *of* Σ *in* $\mathcal{M}$ *on which there is defined a holomorphic function* F *such that* $|\mathrm{F}| = 1$ *on* Σ, $|\mathrm{F}| < 1$ *on* $\Omega \cap \bar{\mathrm{D}} \setminus \Sigma$.

Proof. – Let η_Σ be as in section I and recall that η_Σ never vanishes since Σ is totally real and $\eta_\Sigma \wedge d\eta_\Sigma = 0$. Therefore, the form η_Σ defines a global orientation for the normal bundle of the foliation, *i.e.*, the foliation is transversally oriented by η_Σ. We may assume, without loss of generality, that Σ is connected.

As Σ and all the leaves of the foliation are compact, there is a $\mathcal{C}^\infty$ fibration of Σ over the circle whose fibers are the leaves of the foliation [24, B.III.3, p. 136]. Moreover, since the foliation is real analytic, there are real analytic coordinates $(x_\alpha^1, \ldots, x_\alpha^N) : U_\alpha \longrightarrow \mathbf{R}$ with $\Sigma = \bigcup_\alpha U_\alpha$, U_α open in Σ, such that the foliation is locally given by the equation $x_\alpha^1 =$ constant and on the intersection $U_\alpha \cap U_\beta$, x_α^1 is an analytic function of x_β^1 only. Therefore, the locally defined functions x_α^1 descend to the circle to induce a real analytic structure on the circle for which the submersion defining the foliation is analytic. Hence the foliation is given by a real analytic map $f : \Sigma \longrightarrow \mathbf{T}$. The total reality of Σ implies that f extends holomorphically to a neighborhood Ω of Σ in M. Call this extension F.

We shall show that if Ω is small enough, then either $|F|$ or $|1/F|$ is of modulus less that one on $\Omega \cap \bar{D} \backslash \Sigma$. For this purpose, notice that by the Cauchy-Riemann equations, $d \log |F|$ is zero at no point of Σ. Thus, if Ω is a sufficiently small neighborhood of Σ, the form $d \log |F|$ vanishes nowhere in Ω, so $V = \{z \in \Omega : \log |F| = 0\}$ is a real analytic hypersurface in Ω which contains Σ. We may write $\Omega = \Omega^+ \cup V \cup \Omega^-$ with $\Omega^\pm = \{z \in \Omega : \pm \log |F| < 0\}$.

If we choose Ω small enough then it is a complexification of Σ and by [6], each of the fibers $F^{-1}(\alpha)$, $\alpha \in \mathbf{T}$, is a complexification of the fiber $\Sigma_\alpha = f^{-1}(\alpha)$. Moreover, in a sufficiently small neighborhood of Σ_α, the fiber $F^{-1}(\alpha)$ meets $\bar{D}$ only along Σ_α [6]. This means that near Σ, the hypersurface V is fibered by the manifolds $F^{-1}(\alpha)$ and so near Σ, $V \cap \bar{D} = \Sigma$. That is, if we choose Ω small enough, then V is disjoint from $\Omega \cap D$. Thus, $\log |F|$ is of a single sign on D near Σ, and it follows that $|F| < 1$ on $D \cap \Omega$ or else $|F| > 1$ on $D \cap \Omega$.

The theorem is proved.

It is interesting to notice that the function F just constructed plays a certain universal rôle.

LEMMA 2.2. – *If* $G \in A(D)$, $|G| = 1$ *on* Σ *and* $|G| < 1$ *near* Σ *in* D, *then* $G = \varphi \circ F$ *for some function* φ *holomorphic on a neighborhood of* $\mathbf{T}$ *in* $\mathbf{C}$, *unimodular on* $\mathbf{T}$.

Proof. – The function G is necessarily constant on the leaves of the foliation, so there is a function $\varphi : \mathbf{T} \longrightarrow \mathbf{T}$ with $G = \varphi \circ F$ on $\mathbf{T}$. The function φ is easily seen to be real analytic. As $f : \Sigma \longrightarrow \mathbf{T}$ is regular and real analytic, if we fix $\zeta_0 \in \mathbf{T}$ and $z_0 \in f^{-1}(\zeta_0)$, then there is a real analytic map σ from a neighborhood of ζ_0 in $\mathbf{T}$ to Σ with $f(\sigma(\zeta)) = \zeta$. We have then $G(\sigma(\zeta)) = \varphi(F(\sigma(\zeta)))$ whence $\varphi(\zeta) = G(\sigma(\zeta))$: Locally, $\varphi = G \circ \sigma$ and so is real analytic.

The function σ extends holomorphically into a neighborhood of $\mathbf{T}$ in $\mathbf{C}$. It follows that near Σ, we have $G = \varphi \circ F$.

In this connection, recall Example 1.8 and Lemma 1.9.

The theorem above is stated in a semiglobal way: We do not assert that there is $F \in A(D)$ with Σ as its maximum modulus set. This is not merely a peculiarity of the proof.

Example 2.3. – Let $D \subset \mathbf{C}^2$ be a strongly pseudoconvex domain that is a union $D = \sqrt{2}B_2 \cup V$ where V is a thin neighborhood of the straight line segment connecting the point $(\sqrt{2}, 0) \in b(\sqrt{2}B_2)$ with the point $\left(2, \frac{1}{2}\right)$. For an explicit construction of such domains, one may consult [21].

Notice that $bD \supset \mathbf{T}^2$ and that near $\mathbf{T}^2$, bD and $b(\sqrt{2}B_2)$ agree. The torus $\mathbf{T}^2$ is foliated by compact interpolation manifolds but there is no $f \in A(D)$ with $\mathbf{T}^2$ as its maximum modulus set. By Lemma 1.9 such an f would be of the form $b(zw)$ for a finite Blaschke product b, and so $\left|f\left(2, \frac{1}{2}\right)\right| = |b(1)| = 1$. As $\left(2, \frac{1}{2}\right) \in D$, the maximum principle shows that f assumes values of modulus greater than one.

A moment's reflection shows that there is also a local version of Theorem 2.1.

THEOREM 2.4. – *Let* $D \subset \mathbf{C}^N$ *be a strongly pseudoconvex domain, let* $\Sigma \subset bD$ *be a closed totally real, real analytic submanifold of an open set* Ω *in* $\mathbf{C}^N$. *Assume* Σ *to admit a real analytic foliation by* (N-1)*-dimensional interpolation manifolds. If* $p \in \Sigma$, *there is a neighborhood* U_p *of* p *in* $\mathbf{C}^N$ *on which there exists a holomorphic function* F_p *with* $|F_p| = 1$ *on* $\Sigma \cap U_p$ *but* $|F_p| < 1$ *on* $U_p \cap (\bar{D} \setminus \Sigma)$.

Proof. – The foliation hypothesis implies that we can choose real analytic coordinates $t_1, \ldots, t_N$ in a neighborhood of p in Σ so that locally the leaves are given by t_1 = constant. If g is a function holomorphic near p and agreeing on Σ with t_1, then $F = e^{ig}$ is holomorphic near p and of modulus one on Σ. The local geometric analysis in the proof of Theorem 2.1 applies, *mutatis mutandis,* to show that in D near p, $|F| < 1$ or else $|F| > 1$, and the result follows.

Thus, for instance, in $b(2B_2)$, the torus $\mathbf{T}_\alpha = (\alpha_1 e^{i\theta_1}, \alpha_2 e^{i\theta_2})$ with $\alpha_1^2 + \alpha_2^2 = 4$ and α_1^2/α_2^2 irrationnal is foliated by interpolation curves that are dense in $\mathbf{T}_\alpha$. Accordingly, $\mathbf{T}_\alpha$ is not a maximum modulus set. Locally, however, it is. Near any point of $\mathbf{T}_\alpha$, a suitable locally defined branch of the function $z^2 w^\gamma$, $\gamma = 2\left(\frac{\alpha_2}{\alpha_1}\right)^2$, takes its maximum modulus along Σ.

Theorem 2.4 is generally applicable in the two-dimensional case.

COROLLARY 2.5. – *If* $\Sigma \subset bB_2$ *is a totally real two-dimensional real analytic submanifold, then locally* Σ *is a maximum modulus set.*

Locally is here understood in the sense of the conclusion of Theorem 2.4.

Proof. – The total reality of Σ implies that $\eta = d^c |z|^2$ vanishes at no point of Σ, and as $\eta \wedge d\eta$ is a 3-form, it is the zero form on Σ. Thus Σ is foliated by integral curves of η, and Theorem 2.4 applies.

We conclude this section by exhibiting an extensive class of examples.

Examples 2.6. – We know that not every manifold of real dimension N can be realized as the maximum modulus set of a function on a domain in an N-dimensional manifold, for we have seen that maximum modulus manifolds can be fibered over the circle. It turns out that this is the only obstruction.

Thus, let Σ be a compact real analytic manifold and let $\tau : \Sigma \longrightarrow \mathbf{T}$ be a real analytic, regular map. Let $\mathcal{M}$ be a Stein complexification of Σ chosen small enough that τ extends holomorphically to the element $F \in \mathcal{O}(\mathcal{M})$, F zero-free. The manifold Σ is totally real in $\mathcal{M}$ and so [23] is the zero set of a nonnegative $\mathcal{C}^\infty$ strongly plurisubharmonic function Q.

Let $\mathcal{M}^- = \{z \in \mathcal{M} : |F(z)| < 1\}$ so that $\Sigma \subset b\,\mathcal{M}^-$. The function $|F|$ is real analytic on $\mathcal{M}$, and $d\,|F| \neq 0$ at each point of Σ. To see this, let $p \in \Sigma$ and choose a real analytic map $\psi : (-1, 1) \longrightarrow \Sigma$ with $\psi(0) = p$ and $\frac{d}{dt}(\tau \circ \psi)$ zero-free. Then ψ extends holomorphically to a neighborhood U of $(-1, 1)$ in $\mathbf{C}$, and $F \circ \psi$ is holomorphic on U. Since $\frac{d}{dz} F \circ \psi$ is zero-free at each point of $(-1, 1)$, it follows that $d\,|F| \neq 0$ at p.

For a suitable small $\epsilon > 0$, the function $Q_\epsilon = Q + \epsilon\,|F|$ is strongly plurisubharmonic on $\mathcal{M}$ – we may need to shrink $\mathcal{M}$ a bit – and dQ_ϵ is not zero on Σ (because dQ is and $d|F|$ is not). By shrinking $\mathcal{M}$ further if necessary, we may suppose that dQ_ϵ is zero-free on $\mathcal{M}$. The domain $D_\epsilon = \{z \in \mathcal{M} : Q_\epsilon < \epsilon\}$ has strongly pseudoconvex, possibly noncompact, $\mathcal{C}^\infty$ boundary. The manifold Σ is contained in bD_ϵ, and the function F satisfies $|F| = 1$ on Σ, $|F(z)| < 1$ on $\bar{D}_\epsilon \backslash \Sigma$.

Since the manifold Σ is totally real, it has a fundamental neighborhood basis consisting of strongly pseudoconvex domains. Let D' be such a domain. Then $D_\epsilon \cap D'$ is a relatively compact domain with $b(D_\epsilon \cap D')$ smooth except at $bD_\epsilon \cap bD'$. It is possible to smooth $b(D_\epsilon \cap D')$ near $bD_\epsilon \cap bD'$ to obtain a smoothly bounded domain $D \subset D_\epsilon \cap D'$ with $bD = bD_\epsilon$ near Σ. (For the details of this kind of smoothing process, see [35]).

The upshot is that D is a strongly pseudoconvex domain in a Stein manifold $\mathcal{M}$, $\dim \mathcal{M} = N = \dim \Sigma$, with $\Sigma \subset bD$, and $F \in \mathcal{O}(\bar{D})$ has Σ as its maximum modulus set.

As a corollary of this construction, it follows that every compact (real analytic) manifold of real dimension $N - 1$ can be realized as an interpolation manifold in the boundary of a strongly pseudoconvex domain an N-dimensional manifold. Given Σ_0 with $\dim_{\mathbf{R}} \Sigma_0 = N - 1$, the manifold $\Sigma = \Sigma_0 \times \mathbf{T}$ fibers over $\mathbf{T}$ with Σ_0 as fiber.

For some related material on interpolation manifolds, see [11].

3. Some interpolation sets.

We can apply the preceding ideas to exhibit certain peak interpolation sets.

Fix a strongly pseudoconvex domain D in a complex manifold $\mathcal{M}$, and suppose that bD admits a real analytic strongly plurisubharmonic defining function Q.

THEOREM 3.1. – *Let ξ be the characteristic vector field of the contact form $\eta = d^c Q$. Let $\varphi : \mathbf{R} \times bD \longrightarrow bD$ be the flow generated by ξ, and let $\Sigma \subset bD$ be a real analytic* $(N-1)$-*dimensional interpolation manifold. If $E \subset \mathbf{R}$ is a closed set of measure zero, then every compact subset of $\varphi(E \times \Sigma)$ is a peak interpolation set for* A(D).

Proof. – Closed countable unions of peak interpolation sets are peak interpolation sets as are closed subsets [34], so it suffices to prove that if $P_0 = (t_0, p_0) \in E \times \Sigma$, then there is a compact neighborhood V_0 of P_0 in $E \times \Sigma$ such that $\varphi(V_0)$ is a peak interpolation set.

Recall that the forms η and $d\eta$ are both invariant under the flow φ. This implies that for any interpolation manifold Σ' and for any choice of t, the manifold $\Sigma'_t = \{\varphi(t, s) : s \in \Sigma'\}$ is an interpolation manifold.

The property $\eta(\xi) = 1$ shows that ξ is everywhere transverse to the spaces $T^{\mathbf{C}}_p(bD)$, and as $T_p(\Sigma) \subset T^{\mathbf{C}}_p(bD)$ for each $p \in bD$, it follows that the map φ is regular from $\mathbf{R} \times \Sigma$ into bD. Thus there is an open set U_0 in $\mathbf{R} \times \Sigma$ containing P_0 that is carried by φ bianalytically onto a (closed) real analytic submanifold $\varphi(U_0)$ of a neighborhood U'_0 of $P'_0 = \varphi(P_0)$.

Let $g : \varphi(U_0) \longrightarrow \mathbf{R}$ be the real analytic map given by $g(\varphi(t, p)) = t$.

LEMMA 3.2. – *The form $\varphi^* d\eta$ is zero.*

We defer the proof of this lemma for the moment.

The lemma implies that $d\eta$ is the zero form on the (immersed) manifold $\varphi(\mathbf{R} \times \Sigma)$, and thus that this manifold is totally real. Consequently, the function g can be extended holomorphically into a neighborhood of U'_0 in the ambient complex manifold. Call the extension G. Our geometric analysis shows that near P'_0 in D, ImG is of a single sign, and so if we set $F = e^{\pm iG}$, then $|F| = 1$ on U'_0 and, if we choose the correct sign, $|F| < 1$ on the part of D near U'_0.

Suppose now that V_0' is a compact neighborhood of P_0 in $E \times \Sigma$, $V_0' \subset U_0$. Under the map F, the set $\varphi(V_0')$ goes onto a closed set, say S, of measure zero in the circle $\mathbf{T}$. The classical theorem of Fatou [15, p. 80] implies the existence of an $h \in A(\Delta)$ with S as its zero set. The function $h \circ F$ is holomorphic on $F^{-1}(\Delta)$, it has continuous boundary values along the part of bD near U_0', and it vanishes on $\varphi(V_0')$. The work of Weinstock [37] implies that compact subsets of $\varphi(V_0')$ are peak interpolation sets for $A(D)$ and the proof is complete, subject only to the verification of the lemma(*).

Proof of the lemma. – Let $\widetilde{X}$ and $\widetilde{Y}$ be local vector fields on $\mathbf{R} \times \Sigma$. They can be expressed as $\widetilde{X} = a \dfrac{\partial}{\partial t} + X$ and $\widetilde{Y} = b \dfrac{\partial}{\partial t} + Y$ where X and Y are vector fields tangent to Σ. As φ is the flow generated by ξ, we have $\varphi_* \dfrac{\partial}{\partial t} = \xi$, so

$$\varphi^* d\eta(\widetilde{X}, \widetilde{Y}) = d\eta(a\xi + \varphi_* X, b\xi + \varphi_* Y).$$

We have $\xi \lrcorner d\eta = 0$, so this is $d\eta(\varphi_* X, \varphi_* Y)$ which is zero since $\varphi_* X$ and $\varphi_* Y$ are tangent to Σ_t. Thus $\varphi_* d\eta = 0$ as claimed.

In the case that $D = B_N \subset \mathbf{C}^N$ and we take the standard defining function $Q(z) = |z|^2 - 1$, the characteristic vector field ξ is given by $\xi_p(f) = \dfrac{\partial}{\partial t}(f(e^{it}p))|_{t=0}$, and the flow φ is given by $\varphi(t, p) = e^{it}p$.

Certain potential extensions of the result just proved come to mind. First, it is reasonable to wonder whether, in the context of the preceding result, the interpolation manifold Σ can be replaced by an arbitrary interpolation set.

Example 3.3. – Let $S \subset [0,1]$ be the usual Cantor middle third set. The set $E = \left\{ \left(\dfrac{1}{\sqrt{2}} e^{i\pi\theta}, \dfrac{1}{\sqrt{2}} e^{i\pi\theta}\right) : \theta \in S \right\} \subset bB_2$ has

(*) There is a slight problem about applying Weinstock's results in the present context, for he is working on domains in $\mathbf{C}^N$. The extension to arbitrary strongly pseudoconvex domains offers only technical resistance. One way to proceed is as follows: Blow the maximal compact subvarieties of D down to points [13 p. 232] to obtain a Stein domain D'. We have that D' and D have the same structure at their boundries. Embed D' as a variety D'' in a strictly convex domain C in $\mathbf{C}^M$ by the imbedding theorem given in [12]. If we use the fact that in this setting, elements of $A(D'')$ extend to elements of $A(C)$ [12] everything follows.

zero linear measure and so is a peak interpolation set [9](*). Since, however, $S + S = \{s + t : s, t \in S\} = [0,2]$, [40, p. 235], the set $\cup\{e^{i\theta} E : \theta \in S\}$ is the circle $\left\{\left(\frac{1}{\sqrt{2}} e^{i\theta}, \frac{1}{\sqrt{2}} e^{i\theta}\right) : \theta \in [0,2\pi]\right\}$ which is not a peak interpolation set.

The second point that arises naturally is this: Might it be possible to replace the flow φ of the theorem by an arbitrary flow subject only to the condition that it be transverse to the complex directions? Such a generalization is not possible because in general, such a flow will not carry interpolation manifolds to interpolation manifolds.

4. Uniqueness.

In this section we shall develop in more detail the matters introduced in Lemmas 1.9 and 2.2.

We fix a strongly pseudoconvex domain D in a complex manifold $\mathfrak{M}$ and in bD a $\mathcal{C}^3$ submanifold Σ of dimension $N = \dim \mathfrak{M}$. Briefly put, the content of this section is that there are very few functions $f \in A^2(D)$ with Σ as maximum modulus set: Any two such functions are necessarily analytically related.

THEOREM 4.1. – *Assume* $D \subset \mathbf{C}^N$ *and that* $\bar{D}$ *is polynomially convex. If* $f, g \in A^2(D)$ *have* Σ *as their maximum modulus set, then there is a polynomial* P *in two complex variables such that* $P(f, g) = 0$ *on* D.

The proof of this will depend on some facts about the cohomology of Runge domains. According to Andreotti and Narasimhan [2], if $W \subset \mathbf{C}^N$ is a Runge domain, then the integral homology groups $H_r(W, \mathbf{Z})$ vanish when $r \geq N$ and $H_{N-1}(W, \mathbf{Z})$ is torsion free. It follows that if, in addition, W is strongly pseudoconvex, the cohomology groups $H^r(W, \mathbf{Z})$ vanish for $r \geq N$(**). Our application of this fact is by way of the following lemma.

(*) The present example is simple enough that it is easy to see directly, without appeal to Davie-Øksendal, that E is a peak interpolation set.

(**) The universal coefficient theorem yields the exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{r-1}(W, \mathbf{Z}), \mathbf{Z}) \longrightarrow H^r(W, \mathbf{Z}) \longrightarrow \mathrm{Hom}(H_r(W, \mathbf{Z}), \mathbf{Z}) \longrightarrow 0.$$

Since $H_{N-1}(W, \mathbf{Z})$ is torsion-free and finitely generated, it is free, and so $\mathrm{Ext}(H_{N-1}(W, \mathbf{Z}), \mathbf{Z}) = 0$. Hence, $\mathrm{Ext}(H_{r-1}(W, \mathbf{Z}), \mathbf{Z}) = \mathrm{Hom}(H_r(W, \mathbf{Z}), \mathbf{Z}) = 0$ for $r \geq N$ whence the result.

LEMMA 4.2. – *No compact* N*-dimensional topological manifold in* $\mathbf{C}^N$ *is polynomially convex.*

Browder's theorem [5] covers the orientable case. The proof given below is simply a rewriting of the earlier proof, using the result of Andreotti and Narasimhan.

Proof. – If Σ is a compact, polynomially convex set, then $\Sigma = \bigcap_{j=1}^{\infty} V_j$ with $V_1 \supset V_2, \ldots,$ each V_j a Runge domain. Accordingly, $H^N(V_j, \mathbf{Z}) = 0$, so by a theorem of Cartan [7] on the continuity of cohomology, $H^N(\Sigma, \mathbf{Z}) = 0$. If Σ is an N-dimensional topological manifold, $H^N(\Sigma, \mathbf{Z}) \neq 0$.

We now take up the proof of the Theorem.

Define $\Phi : \bar{D} \longrightarrow \mathbf{C}^2$ by $\Phi(p) = (f(p), g(p))$. The set $\gamma = \Phi(\Sigma)$ is a compact subset of the torus $\mathbf{T}^2$ with finitely many components.

Set $f_\Sigma = f \mid \Sigma$ and $g_\Sigma = g \mid \Sigma$. By Lemma 1.4, df_Σ and dg_Σ are both zero-free on Σ. Also, at each point p of Σ, they have the same kernel, *viz.*, $T_p(\Sigma) \cap T_p^{\mathbf{C}}(bD)$. Thus, the map $\Phi_\Sigma = \Phi \mid \Sigma$ has real rank constantly one on Σ. By [19, Theorem 13.2], γ is an immersed $\mathcal{C}^1$ curve in $\mathbf{T}^2$. In particular, it has finite length, *i.e.*, one-dimensional Hausdorff measure.

The set γ is not polynomially convex. Assume, for the sake of contradiction, that it is. By the Oka-Weil and Stone-Weierstrass theorems, $\mathcal{R}(\gamma) = \mathcal{C}(\gamma)$ if $\mathcal{R}(\gamma)$ denotes the algebra of functions uniformly approximable on γ by polynomials. Consequently, $\mathcal{R}(\Sigma) \supset \Phi^* \mathcal{C}(\gamma) = \{\varphi \circ \Phi : \varphi \in \mathcal{C}(\gamma)\}$ because, by polynomial convexity and the fact that $\mathcal{O}(\bar{D})$ is dense in $A(D)$, the functions f and g are uniformly approximable on $\bar{D}$ by polynomials. Accordingly, the maximal sets of antisymmetry for the algebra $\mathcal{R}(\Sigma)$ are contained in the fibers $\Phi^{-1}(p)$, $p \in \gamma$. Each of these fibers is contained in a set of the form $f^{-1}(\alpha)$, $\alpha \in \mathbf{T}$, and as this f-fiber is a peak interpolation set, it follows from Bishop's generalized Stone-Weierstrass theorem [3,34] that $\mathcal{R}(\Sigma) = \mathcal{C}(\Sigma)$. This implies that Σ is polynomially convex. However, by the lemma above, $\mathbf{C}^N$ contains no compact polynomially convex N-dimensional manifolds.

As γ is a set of finite length with finitely many components, a theorem of Alexander [1] implies that $\hat{\gamma}$, the polynomially convex hull of γ, has the property that $\hat{\gamma}\backslash\gamma$ is a purely one-dimensional analytic subvariety of $\mathbf{C}^2\backslash\gamma$. Set $E = \hat{\gamma}\backslash\gamma$.

Since $\gamma \subset \mathbf{T}^2$, $E \subset \overline{\Delta}^2$ where $\overline{\Delta}^2$ is the closed unit bidisc in $\mathbf{C}^2$. Let the decomposition of E into global branches be

$$E = (E_1' \cup E_2' \cup \ldots) \cup (E_1'' \cup E_2'' \cup \ldots) = E' \cup E''$$

where each of the branches E_j' meets $b\Delta^2\backslash\mathbf{T}^2$ and none of the branches E_j'' does. If $p = (\alpha, \beta) \in E_j'$ with, say $|\alpha| = 1$, $|\beta| < 1$, then by the maximum principle, $E_j' \subset \{(\alpha, \xi) : |\xi| < 1\}$, and, indeed, E_j' must be this disc. As γ has finite length and contains bE_j', it follows that there are at most finitely many E_j'(*). It follows that there is a polynomial P' of the form

$$P'(z, w) = (z - \alpha_1) \ldots (z - \alpha_r)(w - \beta_1) \ldots (w - \beta_s)$$

that vanishes on E'.

The set E'' is closed in $\overline{\Delta}^2\backslash\mathbf{T}^2$, for if not there is a sequence $\{q_\ell\}_{\ell=1}^\infty$ of points in E'' with $q_\ell \longrightarrow q_0 \in b\overline{\Delta}^2\backslash\mathbf{T}^2$. Each E_j'' is closed in $\mathbf{C}^2\backslash\gamma$ and disjoint from $b\overline{\Delta}^2\backslash\mathbf{T}^2$, so if $q_\ell \in E_{j(\ell)}''$, the sequence $\{j(\ell)\}_{\ell=1}^\infty$ contains infinitely many distinct integers and we reach a contradiction to the local finiteness of the family of global branches of E.

Thus E'' is purely one dimensional subvariety of Δ^2 with $bE'' \subset \mathbf{T}^2$. A theorem of Tornehave [30] yields a polynomial P'' with E'' the part of the zero set of P'' contained in Δ^2. The product $P = P'P''$ is a polynomial vanishing on $\hat{\gamma}\backslash\gamma$.

If $P \equiv 0$ on γ, we are done, for then $P \circ \Phi$ is an element of $A^1(D)$ that vanishes on the N-dimensional totally real manifold $\Sigma \subset bD$. A theorem of Pinchuk [22] implies that $P \circ \Phi$ vanishes identically on D, and the theorem is proved.

It is not obvious that P vanishes identically on γ. It is clear, though, that P vanishes on bE. This is a closed subset of γ, and it must have positive length: If the one-dimensional Hausdorff measure of $\overline{E}\backslash E$ is zero, then by [27, Cor. 4.2], there is $h \in A(\Delta^2)$ with $h = 0$ on $\overline{E}\backslash E$ and h vanishing nowhere else on $\overline{\Delta}^2$. The function h violates the maximum principle on the variety E.

(*) In fact, there are no E_j', but that is irrelevant for our purposes.

If $p \in \gamma$, then the fiber $\Phi^{-1}(p)$ is an $(N-1)$-dimensional submanifold of Σ and so has positive $(N-1)$-dimensional Hausdorff measure. As Φ is of class $\mathcal{C}^1$, an inequality in the theory of Hausdorff measures [10, 2.10.25] implies that $\Phi^{-1}(\overline{E}\backslash E)$ has positive (N-dimensional) measure. As $P \circ \Phi$ vanishes on $\Phi^{-1}(\overline{E}\backslash E)$, a strengthened version of Pinchuk's theorem given in [28, 17] implies that $P \circ \Phi$ vanishes identically, so the theorem is proved.

The preceding theorem hypothesizes the global condition of polynomial convexity, and the polynomial convexity was involved at several turns in the proof. It is easy to give an example showing the necessity of this hypothesis.

Example 4.3. – Let φ be a function holomorphic on a neighborhood of the unit circle $\mathbf{T}$, of modulus one on $\mathbf{T}$, and transcendental. (For example, let $\widetilde{\varphi}$ be a conformal map of the annulus $R_{1/2} = \{\zeta \in \mathbf{C} : 1/2 < |\zeta| < 1\}$ onto an annular domain G bounded by $|\zeta| = 1$ and a simple closed nonalgebraic curve J in the unit disc with $\widetilde{\varphi}(\mathbf{T}) = \mathbf{T}$. By the Schwarz reflection principle, $\widetilde{\varphi}$ continues analytically to a function φ holomorphic on a neighborhood of $\mathbf{T}$). Let $D \subset \sqrt{2}B_2$ be a strongly pseudoconvex domain with $\mathbf{T}^2 \subset bD$ and so small that if $f(z, w) = zw$, then $f(D) \subset R_{1/2}$. Then f and $g = \varphi \circ f$ both lie on $\mathcal{O}(\overline{D})$ and both have the torus $\mathbf{T}^2$ as their maximum modulus set. Since φ is not algebraic, f and g cannot be polynomially related.

However, even in the general case, we have analytic dependence near Σ as our next result shows.

THEOREM 4.4. – *Let* D *be a strongly pseudoconvex domain in the complex manifold* $\mathfrak{M}$, bD *of class* $\mathcal{C}^r$, $3 \leqslant r \leqslant \infty$. *Let* $\Sigma \subset bD$ *be a closed submanifold of dimension* $N = \dim \mathfrak{M}$, Σ *of class* $\mathcal{C}^3$, *that is the maximum modulus set of the functions* $f, g \in A^2(D)$. *If* $\Phi = (f, g) : \overline{D} \longrightarrow \mathbf{C}^2$, *then there is a neighborhood* V *of* $\Phi(\Sigma)$ *in* $\mathbf{C}^2$ *and on* V *a nonconstant holomorphic function* h *such that* $h \circ \Phi$ *vanishes on* D *near* Σ.

The proof of this theorem depends on finding certain analytic discs that abut Σ at prescribed points.

LEMMA 4.5. – *If* $\Sigma_0 \subset bD$ *is a* $\mathcal{C}^k$, $3 \leqslant k \leqslant r$, *totally real submanifold of dimension* N *in an open subset of* bD, *and if*

$p \in \Sigma_0$, *then there is a holomorphic map* $\varphi : \Delta \longrightarrow \mathrm{D}$ *such that* $\varphi^{(k-2)} \in \mathrm{Lip}_\alpha(\mathbf{T})$ *for an* $\alpha \in (0,1)$ *and such that* $\varphi(1) = p$ *and* $\varphi(e^{i\theta}) \in \Sigma_0$ *if* $|\theta| < \frac{\pi}{2}$.

This lemma is not essentially new; we relegate its discussion to Appendix B.

Proof of the theorem. – The manifold Σ is totally real, so we may invoke Lemma 4.5: If $p \in \Sigma$, there is a holomorphic map $\varphi : \Delta \longrightarrow \mathrm{D}$ with continuous boundary values such that $\varphi(1) = p$ and $\varphi(e^{i\theta}) \in b\mathrm{D}$ for $|\theta| < \frac{\pi}{2}$. The functions $\widetilde{f} = f \circ \varphi$ and $\widetilde{g} = g \circ \varphi$ lie in $\mathrm{A}(\Delta)$ and have boundary values of modulus one along the arc $\Lambda = \left\{ e^{i\theta} : |\theta| < \frac{\pi}{2} \right\}$. Accordingly, they continue analytically across Λ. Let W' be a connected open set in $\mathbf{C}$ containing Λ and into which $\widetilde{f}$ and $\widetilde{g}$ both continue analytically. Neither $d\widetilde{f}$ nor $d\widetilde{g}$ vanishes at any point of Λ. Thus, if W is a small disc in $\mathbf{C}$ centered at 1, $\mathrm{W} \subset \mathrm{W}'$, then the map $\widetilde{\Phi} : \mathrm{W} \longrightarrow \mathbf{C}^2$ given by $\widetilde{\Phi} = (\widetilde{f}, \widetilde{g})$ carries W onto a complex submanifold of a neighborhood V_p of $\widetilde{\Phi}(1) = \Phi(p)$. By choosing $\mathrm{F} \in \mathcal{O}(\mathrm{V}_p)$ a defining function for $\widetilde{\Phi}(\mathrm{W})$, we have $\mathrm{F} \circ \widetilde{\Phi} = 0$ whence $\mathrm{F} \circ \Phi$ vanishes on $\varphi(\Lambda)$. It follows from the functional dependence of $f|\Sigma$ and $g|\Sigma$ that a neighborhood of p in Σ is mapped into $\widetilde{\Phi}(\Lambda)$.

We have shown then that the set $\Phi(\Sigma)$ has the structure of a one-dimensional real analytic set and, moreover, that it is locally a finite union of analytic arcs. It follows [7, pp. 88 and 95] that $\Phi(\Sigma)$ is a coherent real analytic set, and consequently [39], $\Phi(\Sigma)$ has a complexification in $\mathbf{C}^2$: There is a complex variety E in a neighborhood V_0 of $\Phi(\Sigma)$, $\dim \mathrm{E} = 1$, that is a complexification of $\Phi(\mathrm{E})$. In particular, $\mathrm{E} \cap \mathbf{T}^2 = \phi(\Sigma)$.

We may take the neighborhood V_0 to be a connected domain of holomorphy, say a product of two annuli, so that there is certainly a function $h \in \mathcal{O}(\mathrm{V}_0)$, h not identically zero, with $h = 0$ on E. Then $h \circ \Phi$ vanishes on Σ, and so on each component of $\Phi^{-1}(\mathrm{V}_0)$ that meets Σ, for by Pinchuk [22] Σ is a uniqueness set.

The theorem is proved.

It is possible to refine the analytic dependence of the last theorem to algebraic dependence if we impose certain additional geometric conditions. Our treatment will be based on a theorem contained in [32]. (See also [32a].)

Let $\mathcal{M}$ be a complex manifold. It is possible to assign to each nonconstant $f \in \mathcal{O}(\mathcal{M})$ a Riemann surface R_f, a surjective holomorphic map $\sigma_f : \mathcal{M} \longrightarrow R_f$ and a function $F \in \mathcal{O}(R_f)$ so that

(1) $f = F \circ \sigma_f$.

(2) The induced map $\sigma_{f*} : \pi_1(\mathcal{M}) \longrightarrow \pi_1(R_f)$ of fundamental groups is surjective.

(3) Given $g \in \mathcal{O}(\mathcal{M})$ with $df \wedge dg = 0$, there is a biholomorphic map $\chi : R_f \longrightarrow R_g$ with $\sigma_g = \chi \circ \sigma_f$.

Consider now $\mathcal{M}$, D, Σ, f, g, and Φ as in Theorem 4.4, and assume Σ connected. The nowhere vanishing form η_Σ defines a codimension one foliation of Σ by compact manifolds. Fix a smooth simple closed curve γ in Σ that is transverse to the leaves and meets each leaf exactly once.

Denote by $[\gamma] \in H_1(\Sigma, \mathbf{Z})$ the homology class of γ, and let $[\gamma]_D$ be the image of $[\gamma]$ in $H_1(\bar{D}, \mathbf{Z}) \cong H_1(D, \mathbf{Z})$ under the map of homology induced by the inclusion of Σ in $\bar{D}$. The homology class $[\gamma]$ is never trivial, but $[\gamma]_D$ may or may not be trivial.

THEOREM 4.6. – *If* $[\gamma]_D$ *is trivial, then there is a nonzero polynomial* P *in two variables with* $P(f, g) = 0$. *Moreover, if the curve* γ *is homotopically trivial in* $\bar{D}$, *then there is* $\varphi \in A(D)$ *with* Σ *as its maximum modulus set and with the further property that if* $h \in A^2(D)$ *has* Σ *as its maximum modulus set, then* $h = B \circ \varphi$ *for some finite Blaschke product.*

Remark. – The function φ is essentially unique, for if ψ has the same property, it is clear that $\psi = \chi \circ \varphi$, χ an automorphism of the disc.

Proof. – Assume $[\gamma]_D$ trivial. By the preceding theorem, we know that $df \wedge dg = 0$ near Σ, and so, as D is connected, $df \wedge dg = 0$ throughout D. The result quoted above yields a Riemann surface R, holomorphic functions F, $G \in \mathcal{O}(R)$ and a surjective holomorphic map $\sigma : D \longrightarrow R$ with $f = F \circ \sigma$, $g = G \circ \sigma$.

The map σ induces a surjection of fundamental groups, so $\pi_1(R)$ is finitely generated. We may, therefore, assume that R is a domain in a compact Riemann surface R^* with bR a finite set of points together with a finite family of analytic simple closed curves.

LEMMA 4.7. – *The map* σ *extends continuously to* $D \cup \Sigma$.

Proof. – By connectedness, all the limit points of σ at γ lie in a single component, S, of bR.

The component S is not a point, for if it were, then by the Riemann removable singularity theorem F would continue holomorphically through the point and from $f = F \circ \sigma$ would follow that f is constant on γ.

Put a smooth metric on $\mathfrak{M}$ and, with respect to this metric, let n_p denote the unit inner normal to bD at $p \in bD$. Let $\tau_0 : \mathbf{T} \longrightarrow \gamma$ be a diffeomorphism, and define

$$\tau : \mathbf{T} \times [0, \epsilon_0] \longrightarrow D \cup \gamma$$

by $\tau(e^{i\theta}, t) = \tau_0(e^{i\theta}) + tn_{\tau_0(e^{i\theta})}$. If $\epsilon_0 > 0$ is small enough, then τ carries $\mathbf{T} \times [0, \epsilon_0]$ diffeomorphically onto a thin ribbon W' in $D \cup \gamma$ abutting γ. As $f : \gamma \longrightarrow \mathbf{T}$ is a covering map, $f \circ \tau$ will carry an annular domain A in $\mathbf{T} \times [0, \epsilon_0]$ as a λ-sheeted cover onto an annular domain $A'' = \{z \in \mathbf{C} : r_0 < |z| < 1\}$ in Δ. As $f = F \circ \sigma$, it follows that $\sigma \circ \tau$ takes A onto an annular region A' in R as a λ'-sheeted cover, and F carries A' onto A'' as a λ''-sheeted cover, $\lambda'\lambda'' = \lambda$. One boundary component of A' is S, and we see that $|F| \longrightarrow 1$ at S.

Consequently, as S is an analytic curve, F continues analytically across S – this follows from the edge-of-the wedge theorem [26]. In the same way G continues across S.

Since F continues analytically across S and $dF \neq 0$ on S, we can use local inverses of F together with the equation $F \circ \sigma = f$ to see that, in fact, σ extends continuously to $D \cup \Sigma$, as we wished to show.

Notice that since $f : \gamma \longrightarrow \mathbf{T}$ is a covering map, the map $\sigma : \gamma \longrightarrow S$ is also a covering map of degree λ'.

The map σ induces a map of homology σ_* :

$$H_1(D \cup \gamma, \mathbf{Z}) \longrightarrow H_1(R \cup S, \mathbf{Z}),$$

and as $[\gamma]_D = 0$, it follows that $\sigma_*([\gamma]_D) = 0$. Since $H_1(R \cup S, \mathbf{Z})$ is free abelian, this implies that the homology class [S] is zero since $\sigma_*([\gamma]_D) = \lambda'[S]$ is zero.

The exact homology sequence of the pair $(R \cup S, S)$ includes the segment $H_2(R \cup S, S, \mathbf{Z}) \longrightarrow H_1(S, \mathbf{Z}) \xrightarrow{\iota_*} H_1(R \cup S, \mathbf{Z})$. As $H_1(S, \mathbf{Z}) = \mathbf{Z}$ and $\iota_* = 0$ because [S] is trivial, it follows that $H_2(R \cup S, S, \mathbf{Z}) \neq 0$. This group is isomorphic to $H_2((R \cup S)/S, \mathbf{Z})$ if $(R \cup S)/S$ denotes the space obtained from $R \cup S$ by collapsing S to a point. However, $(R \cup S)/S$ is a 2-dimensional manifold which must be closed as it has nonzero second integral homology.

We now have that bR consists of a single component, S, and we see that we may take for the surface R^* the double of R with anticonformal involution $\mu : R^* \longrightarrow R^*$ that has S as its fixed point set. As F and G take modulus one along S, they continue meromorphically to R^* by the Schwarz reflection principle. Since compact Riemann surfaces are algebraic, there is a polynomial relation $P(F, G) = 0$ whence $P(F \circ \sigma, \; G \circ \sigma) = 0$, so that $P(f, g) = 0$ as claimed.

This proves the first part of the theorem (subject to the verification of Lemma 4.5).

Let us now take up the more restrictive hypothesis that the curve γ be homotopically trivial in $\bar{D}$ (and hence, in $D \cup \gamma$). In this case the boundary component S considered above is homotopically trivial in $R \cup S$, and this implies that R is itself simply connected. Thus, we may assume $R = \Delta$ so that R^* is the Riemann sphere.

Suppose given $h \in A^2(D)$ with Σ as its maximum modulus set. We have that $h = B \circ \sigma$ for some $B \in \mathcal{O}(\Delta)$, and the function B extends holomorphically across $b\Delta$. As $|B| = 1$ on $b\Delta$, B is a finite Blaschke product.

The one remaining point is to see that $\sigma \in A(D)$. We know that $\sigma \in \mathcal{O}(D)$ and that it extends continuously to $D \cup \Sigma$. Choose an $h \in A^2(D)$ with Σ as its maximum modulus set and write $h = B \circ \sigma$. Since for each $\zeta \in \Delta$, the fiber $B^{-1}(\zeta)$ consists of only finitely many points, σ extends continuously to $\bar{D}$.

Remark. – If we write $B^{-1} \circ h = \sigma$, a local equation valid for a suitable branch of B^{-1}, we see that, in fact, σ has $\mathcal{C}^2$ boundary values off $\sigma^{-1}(\{\zeta \in \Delta : B'(\zeta)) = 0\})$. In particular, σ has $\mathcal{C}^2$ boundary values near Σ in $\overline{D}$.

5. The dimension of maximum modulus sets.

The context of this section is somewhat more general than that of the preceding sections in that we relax all smoothness hypotheses. Our result is this :

THEOREM 5.1. – *If* D *is a strongly pseudoconvex domain in the* N*-dimensional complex manifold* $\mathfrak{M}$ *and if* $E \subset bD$ *is the maximum modulus set of an* $f \in A(D)$, *then* $\dim E \leqslant N$.

Notice that we do not assume any smoothness on E, and we do not assume the continuity of the derivatives of f at bD.

The dimension of E is understood in the topological sense. See [16] for dimension theory.

COROLLARY 5.2. – *The dimension of a peak set in* bD *does not exceed* N.

Rudin [25] conjectured that a peak set has dimension no more that $N-1$.

The proof of the theorem requires some lemmas. Fix a function $f_0 \in A(D)$ with $|f_0| = 1$ on E and $|f_0| < 1$ on $\overline{D}\backslash E$. Fix also a domain $\widetilde{D}$, $D \subset\subset \widetilde{D} \subset \mathfrak{M}$ with $\mathcal{O}(\widetilde{D})$ dense in $A(D)$. For a subset S of the circle $\mathbf{T}$, let $E_S = f_0^{-1}(S) \subset E$.

LEMMA 5.3. – *If* S *is a proper closed subset of* $\mathbf{T}$, *then* E_S *is* $\mathcal{O}(\widetilde{D})$ *convex, and* $\mathcal{O}(\widetilde{D}) \mid E$ *is dense in* $\mathcal{C}(E)$.

Proof. – Denote by $\mathfrak{A}_S$ the closure of the algebra $\mathcal{O}(\widetilde{D}) \mid E_S$ in $\mathcal{C}(E_S)$. We have $f_0 \mid E \in \mathfrak{A}_S$, so since $f_0(E_S) \subset S$ is a proper subset of $\mathbf{T}$, it follows that $\mathcal{R}(f_0(E_S)) = \mathcal{C}(f_0(E_S))$. Thus, $\mathfrak{A}_S \supset \{\varphi \circ f_0 : \varphi \in \mathcal{C}(f_0(E))\}$, and thus the maximal sets of antisymmetry for $\mathfrak{A}_S$ are contained in the fibers $E_{\{\zeta\}} = f_0^{-1}(\zeta)$. However, for $\zeta_0 \in S$ the fiber $E_{\{\zeta_0\}}$, if not empty, is the zero set of a function in $A(D)$, *viz.*, the function $f - \zeta_0$, and so is a peak interpolation set for $A(D)$ [37]. Thus $A(D) \mid E_{\zeta_0} = \mathcal{C}(E_{\zeta_0})$, and Bishop's generalized Stone-Weierstrass theorem implies that the subalgebra of $\mathcal{C}(E_S)$ generated by $A(D) \mid E_S$ is dense in $\mathcal{C}(E_S)$. As $\mathcal{O}(\widetilde{D})$ is dense in $A(D)$, we have $\mathcal{A}_S = \mathcal{C}(E)$ as claimed.

In the same way, we see that if $\widetilde{\mathcal{A}}_T$ is the subalgebra of $\mathcal{C}(E) = \mathcal{C}(E_T)$ generated by $\mathcal{O}(\widetilde{D}) \mid E$ and the reciprocal of $f_0 \mid E$, then $\widetilde{\mathcal{A}}_T = \mathcal{C}(E)$. (Of course, $\widetilde{\mathcal{A}}_T = \mathcal{A}_T$ if $f_0(E)$ is a proper subset of $\mathbf{T}$.)

In particular if $D \subset \mathbf{C}^N$ and $\bar{D}$ is polynomially convex, say a ball or a convex domain, then E is rationnally convex, and $\mathcal{R}(E) = \mathcal{C}(E)$. In general, of course, a maximum modulus set, even in bB_N, will not be polynomially convex.

LEMMA 5.4. – *If* X *is a compact metric space with the property that* $X = \bigcup_{j=1}^{\infty} Y_j$, *each* Y_j *a closed subset such that* $\mathcal{C}(Y_j)$ *admits a system of* N *generators, then* $\dim X \leqslant N$.

Proof. – If $E \subset Y_j$ is closed, then $\mathcal{C}(E)$ is generated by N functions, say $f_1, \ldots, f_N$. The set

$$E^\dagger = \{(f_1(x), \ldots, f_N(x)) : x \in E\}$$

is polynomially convex so $E^\dagger = \bigcap_{k=1}^{\infty} W_k$, W_k a strongly pseudoconvex Runge domain, $W_1 \supset W_2 \supset \ldots$ Thus, $H^N(E, \mathbf{Z}) = 0$.

Consequently, for each closed set $E \subset Y_j$, the map $H^N(Y_j, \mathbf{Z}) \longrightarrow H^N(E, \mathbf{Z})$ dual to the inclusion $E \subset Y_j$ is surjective. If follows that $\dim Y_j \leqslant N$ [16, p. 151]. The Sum Theorem [16, p. 30] implies $\dim X \leqslant N$, and the lemma is proved.

Of course if we replace $\mathcal{C}(E)$ by the algebra $\mathcal{C}_{\mathbf{R}}(E)$ of real-valued continuous functions, the corresponding statement is trivial, granted the Sum Theorem.

Proof of the Theorem. – Let $p \in E$, and let $W_p \subset \widetilde{D}$ be a neighborhood that is mapped biholomorphically onto a ball in $\mathbf{C}^N$ by the coordinates $z_1, \ldots, z_N$.

By Lemma 5.3, there is an open subset E_p of E with $\bar{E}_p \subset W_p$ and with $\mathcal{O}(\widetilde{D}) \mid \bar{E}_p$ dense in $\mathcal{C}(\bar{E}_p)$. As $W_p \subset \widetilde{D}$, $\mathcal{O}(W_p) \mid \bar{E}_p$ is, *a fortiori*, dense in $\mathcal{C}(\overline{E}_p)$, so $\mathcal{C}(\overline{E}_p)$ is generated by $z_1, \ldots, z_N$. By Lemma 5.4 the dimension of E cannot exceed N.

Appendix A.
Intersections of strongly pseudoconvex domains.

We now revert to the proof of Lemma 1.2 to show that the method used there yields the following result.

THEOREM. – *If* D_1 *and* D_2 *are disjoint strongly pseudoconvex domains in a complex manifold* $\mathfrak{M}$ *such that* $\bar{D}_1 \cap \bar{D}_2 = \Sigma$, Σ *a* $\mathcal{C}^2$ *submanifold of* $\mathfrak{M}$ *then* Σ *is totally real.*

This settles a question left open in [35]. We refer to that paper for a discussion of this and related matters.

Proof. – Let $p \in \Sigma$ and suppose Σ is not totally real at p. Choose holomorphic coordinates $z_j = x_{2j-1} + ix_{2j}$ near p so that p is the origin of the z-coordinate system, so that $T_p^{\mathbf{C}}(bD_1) = T_p^{\mathbf{C}}(bD_2)$ is the complex subspace $z_1 = 0$ and so that the x_1-direction is normal at 0 to bD_1 (and so to bD_2). Thus, replacing z_1 by $-z_1$ if necessary, we may assume that near 0, D_1 is given by $x_1 < F(x_2, z')$, $z' = (z_2, \ldots, z_N)$ and D_2 is given by $x_1 > G(x_2, z')$ where $F \leqslant G$. Finally, as Σ is assumed not to be totally real at 0, we can assume that the line $z_1 = z_3 = \cdots = z_N = 0$ is tangent to Σ at 0.

As in the proof of Lemma 1.2, choose coordinates $t_1, \ldots, t_k$ in Σ near 0, $k = \dim \Sigma$, so that the (t_1, t_2)-space in $T_0(\Sigma)$ is the same as the z_2-space.

Since $\Sigma \subset bD_1 \cap bD_2$, we have $F = G$ on Σ, so far all i, j, $\dfrac{\partial^2 F}{\partial t_i \partial t_j} = \dfrac{\partial^2 G}{\partial t_i \partial t_j}$ at each point of Σ.

But also, we again have

$$\frac{\partial^2 F}{\partial t_j \partial t_i} = \sum_{\alpha=2}^{N} \frac{\partial F}{\partial x_\alpha} \frac{\partial^2 x_\alpha}{\partial t_j \partial t_i} + \sum_{\alpha,\beta=2}^{N} \frac{\partial^2 F}{\partial x_\beta \partial x_\alpha} \frac{\partial x_\beta}{\partial t_j} \frac{\partial x_\alpha}{\partial t_i}$$

with a corresponding equation involving the derivatives of G. Restrict i and j to the values 1 and 2 and evaluate at 0 to get

$$\frac{\partial^2 F}{\partial t_j\,\partial t_i}(0) = \sum_{\alpha,\beta=3}^{4} \frac{\partial^2 F}{\partial x_\alpha\,\partial x_\beta}(0)\,\frac{\partial x_\beta}{\partial t_j}(0)\,\frac{\partial x_\alpha}{\partial t_i}(0)$$

and

$$\frac{\partial^2 G}{\partial t_j\,\partial t_i}(0) = \sum_{\alpha,\beta=3}^{4} \frac{\partial^2 G}{\partial x_\alpha\,\partial x_\beta}(0)\,\frac{\partial x_\beta}{\partial t_j}(0)\,\frac{\partial x_\alpha}{\partial t_i}(0).$$

As the t-derivatives of F and G agree, and as the matrix $\left(\frac{\partial x_\alpha}{\partial t_i}\right)_{\substack{i=1,2\\ \alpha=3,4}}$ is nonsingular, it follows that $\frac{\partial^2 G}{\partial x_\alpha\,\partial x_\beta}(0) = \frac{\partial^2 F}{\partial x_\alpha\,\partial x_\beta}(0)$, $\alpha,\beta = 3,4$. Thus, $\frac{\partial^2 F}{\partial z_2\,\partial \bar{z}_2}(0) = \frac{\partial^2 G}{\partial z_2\,\partial \bar{z}_2}(0)$.

By strict pseudoconvexity, $\frac{\partial^2 F}{\partial z_2\,\partial \bar{z}_2}(0) < 0$ and $\frac{\partial^2 G}{\partial z_2\,\partial \bar{z}_2}(0) > 0$. This is a contradiction.

Thus, Σ is totally real.

Appendix B.
Proof of the Lemma 4.5.

Lemma 4.5 is essentially contained in Pinchuk's work [22]. The proof given below follows Pinchuk closely but uses the splendid innovation of Hill and Taiani [14] of basing this kind of result on the implicit function theorem.

Given $p \in \Sigma_0$, choose holomorphic coordinates $z_1, \ldots, z_N$ on $\mathbf{C}^N$ so that p is the origin, so that

$$T_0(\Sigma_0) = \mathbf{R}^N = \{(x_1, \ldots, x_N) \in \mathbf{C}^N : x_1, \ldots, x_N \in \mathbf{R}\}$$

and so that near p, D is described by

$$D = \{y_N < G(x_1, \ldots, x_N, y_1, \ldots, y_{N-1})\}$$

for some function G of class C^r with $dG(0) = 0$. Near 0 the manifold Σ_0 can be represented as a graph

$$\Sigma_0 = \{x + ih(x) : x \in \mathbf{R}^N\}$$

for some $\mathcal{C}^k$ function $h : \mathbf{R}^N \longrightarrow \mathbf{R}^N$ with $h(0) = 0$, $dh(0) = 0$.

Denote by $\mathcal{C}^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$ the set of all $\mathbf{R}^N$-valued functions on $\mathbf{T}$ whose k^{th}-derivatives satisfy a Hölder condition of order α,

$\alpha \in (0,1)$, and denote by $\mathcal{C}_1^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$ the subspace consisting of those functions taking the value 0 at $1 \in \mathbf{T}$.

Define an operator T by the condition that for a function f on the unit circle, Tf is the (boundary value of the) conjugate Poisson integral of f. Thus, $f + iTf$ is the boundary value of a function holomorphic on the unit disc and real at the origin. It is clear that we can let T act on $\mathbf{R}^N$-valued functions. A theorem of Privaloff (see [14]) shows that T acts as a bounded linear operator from $\mathcal{C}^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$ to itself.

The map $f \longrightarrow f(1)$, call it E, is a continuous linear map from $\mathcal{C}^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$ onto $\mathbf{R}^N$, so if we regard $\mathbf{R}^N$ as a subspace of $\mathcal{C}^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$, we may define $S : \mathcal{C}_1^{k,\alpha}(\mathbf{T}, \mathbf{R}^N) \longrightarrow \mathcal{C}_1^{k,\alpha}(\mathbf{T}, \mathbf{R}^N)$ by $S(f) = T(f) - E(T(f))$. Also, as h is of class $\mathcal{C}^k$, we can define a map $H : \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N) \longrightarrow \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ by $Hf = h \circ f$. Lemma 5.1 of [14] shows that H is of class $\mathcal{C}^1$.

Fix a $\mathcal{C}^\infty$ function $u_N : \mathbf{T} \longrightarrow \mathbf{R}$ with $u_N(e^{i\theta}) = 0$ if $|\theta| < \frac{\pi}{2}$, $u(e^{i\theta}) < 0$ if $\theta = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and define $u \in \mathcal{C}^\infty(\mathbf{T}, \mathbf{R}^N)$ by $u(e^{i\theta}) = (0, \ldots, 0, u_N(e^{i\theta}))$.

For $\xi \in \mathbf{R}$, define $F_\xi : \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N) \longrightarrow \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ by

$$\begin{aligned} F_\xi(x) &= x - S(h \circ x) - \xi S u \\ &= (I - SH)x - \xi S u . \end{aligned}$$

If $x \in \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ is a zero of F_ξ, $\xi \neq 0$, that has small norm, then $x + i(h \circ x + \xi u)$ is the boundary value of a function $\varphi : \bar{\Delta} \longrightarrow \mathbf{C}^N$ holomorphic on Δ and of class $\mathcal{C}^{k-2,\alpha}$ on $\bar{\Delta}$ such that $\varphi(e^{i\theta}) \in bD$ when $\theta < \frac{\pi}{2}$ and $\varphi(1) = 0$. Moreover, $\varphi(\zeta) \in D$ when $\zeta \in \Delta$ is near the right half on $\mathbf{T}$. (Note that φ is not constant, for the only constant in $\mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ is zero, and zero is not a solution of $F_\xi = 0$ for $\xi \neq 0$).

Thus, we have to find small solutions of $F_\xi = 0$ for $\xi \neq 0$. To do this, notice first that as S is linear and H of class $\mathcal{C}^1$, F_ξ is of class $\mathcal{C}^1$. Compute the derivative DF_ξ. We have $DF_\xi(x) = (I - S \circ DH)x$ for all x. By referring to [14, proof of Lemma 5.1] or by computing directly, we find that the differential DH is the linear transformation $\mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N) \longrightarrow \mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ effected by the matrix

$$\begin{bmatrix} D_1 h_1 \circ x & \dots & D_N h_1 \circ x \\ \vdots & & \\ D_1 h_N \circ x & \dots & D_N h_N \circ x \end{bmatrix}$$

so that for x and y in $\mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$,

$$DH(x)(y) = \left(\sum_{i=1}^{N} ((D_i h_1) \circ x) y_i, \dots, \sum_{i=1}^{N} ((D_i h_N) \circ x) y_i \right).$$

Since $dh(0) = 0$, it follows that $DH(0) = 0$, so by continuity, if $\|x\|_{k-2,\alpha} < \delta$ then $\|D(S \circ H)(x)\|_{k-2,\alpha} = \|S \circ DH(x)\| < \frac{1}{2}$ ·

According to [29 Corollary 1.19], the function

$$F_0 - F_0(x) = x - S \circ H(x)$$

carries the ball of radius δ_0 in $\mathcal{C}_1^{k-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ onto a set containing the ball of radius $\frac{\delta_0}{2}$. For small $\xi \in \mathbf{R}$, $-\xi Su$ is in this ball, so for small $\xi \in \mathbf{R} \setminus \{0\}$ there are solutions x of $F_\xi(x) = 0$ of small $\mathcal{C}_1^{k-2,\alpha}$-norm.

The lemma is proved.

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