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FULLY NONLINEAR SECOND ORDER ELLIPTIC EQUATIONS WITH LARGE ZEROTH ORDER COEFFICIENT

by L. C. EVANS (*) and P.-L. LIONS (**)

1. Introduction.

This paper describes a fairly simple method for proving the classical solvability of certain fully nonlinear second order elliptic equations, provided the coefficient of the zeroth order term is sufficiently large. Briefly, the idea is first to show by an *a priori* estimate that the $C^{2,\alpha}$-norm of a solution cannot lie in a certain interval (C_1, C_2) of the positive real line and, second, to eliminate by a continuation argument the possibility that this norm ever exceeds the constant C_2. (Our technique is thus reminiscent of certain methods for proving global existence in time of solutions to various nonlinear evolution equations with small initial data.)

We begin now the precise statements of our existence theorems by assuming that

$$F : \mathbf{R}^{n^2} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$$

is a given smooth function satisfying the *ellipticity assymption*

$$(1.1)\quad \theta|\xi|^2 \leqslant \frac{\partial F}{\partial p_{ij}}(p,q,r,x)\xi_i\xi_j \quad \text{for all} \quad \xi = (\xi_1,\ldots,\xi_n) \in \mathbf{R}^n,$$

for some real number $\theta > 0$ and all $p \in \mathbf{R}^{n^2}$, $q \in \mathbf{R}^n$, $r \in \mathbf{R}$, $x \in \mathbf{R}^n$. We also

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suppose that there exists a constant M such that

$$|F(0,0,0,x)| \leqslant M \quad \text{for all} \quad x \tag{1.2}$$

and

$$|DF(p,q,r,x)|, \quad |D^2F(p,q,r,x)| \leqslant M \quad \text{for all} \quad p,q,r,x. \tag{1.3}$$

Let us consider first the nonlinear partial differential equation in all of space

$$\lambda u - F(D^2u,Du,u,x) = 0 \quad \text{in} \quad \mathbf{R}^n. \tag{1.4}$$

Our existence theorem is

THEOREM 1. — *Under the above assumptions there exists a constant* λ_0 *such that* (1.4) *has a unique solution*

$$u \in C^{3,\alpha}(\mathbf{R}^n) \quad (\text{for all } 0<\alpha<1)$$

provided

$$\lambda \geqslant \lambda_0. \tag{1.5}$$

The constant λ_0 *depends only on* $n, \theta,$ *and* M.

We prove Theorem 1 in § 3, after first obtaining in § 2 the key estimate described above.

Our method applies also to nonlinear elliptic equations on a bounded domain, provided a restriction ((1.7) below) is placed on F. We consider the equation

$$\begin{cases} \lambda u - F(D^2u,Du,u,x) = 0 & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad 2\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain. Let us now suppose, in addition to (1.1)-(1.3), that

$$F(0,0,0,x) = 0, \qquad x \in \partial\Omega. \tag{1.7}$$

THEOREM 2. — *Under these hypotheses there exists a constant* λ_0 *such that* (1.6) *has a unique solution*

$$u \in C^{3,\alpha}(\bar{\Omega}) \quad (\text{for all } 0<\alpha<1)$$

provided

$$\lambda \geqslant \lambda_0 . \tag{1.10}$$

The constant λ_0 *depends only on* $\Omega, \theta,$ *and* M.

Theorem 2 is proved in § 4.

In § 5 we collect various comments concerning hypothesis (1.7) and also certain extensions of our technique to related problems. The appendix (§ 6) contains some lemmas concerning the standard L^p second order elliptic estimates.

Finally we note that Skrypnik [6] has obtained by a completely different method some results on fully nonlinear elliptic equations (even of higher order) with large zeroth order coefficient. Some other recent papers on fully nonlinear second order elliptic equations are Evans-Friedman [2], P.-L. Lions [5], and Evans [1].

Notation.

$$Du = D_{x_1}, \ldots, D_{x_n})$$
$$D^2u = (D_{x_1x_1}, \ldots, D_{x_ix_j}, \ldots, D_{x_nx_n}).$$

The letter « C » denotes various constants depending only on known quantities.

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^n)} = \sup_{\substack{x,y \in \mathbf{R}^n \\ x \neq y}} \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} + \sup_{x \in \mathbf{R}^n} |D^2u(x)|$$
$$+ \sup_{x \in \mathbf{R}^n} |Du(x)| + \sup_{x \in \mathbf{R}^n} |u(x)| ;$$

$\|u\|_{C^{2,\alpha}(\Omega)}$ is similarly defined. We employ the implicit summarion convention throughout.

2. Preliminary estimates.

The goal of this section is our proof (Lemma 2.3) that for $\lambda \geqslant \lambda_0$, λ_0 large enough, then exists an interval (C_1,C_2) in which the $C^{2,\alpha}$-norm of the solution of (1.4) cannot lie. First, however, we must know that the solution and its gradient behave well for large λ; the first two lemmas provide this information.

LEMMA 2.1. — *Suppose that* $v \in C^{2,\alpha}(\mathbf{R}^n)$ *(for some* $0 < \alpha < 1$*) solves the linear elliptic equation*

$$\lambda v - a_{ij}(x)v_{x_i x_j} + b_i(x)v_{x_i} + c(x)v = f(x) \tag{2.1}$$

in $\mathbf{R}^n$, *where*

$$\begin{aligned} &|a_{ij}|, |b_i|, |c|, |f| \leqslant M \\ &a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2 \qquad \text{for all} \qquad x, \xi \in \mathbf{R}^n, \\ &c \geqslant 0 \end{aligned}$$

then

$$\|\lambda v\|_{L^\infty(\mathbf{R}^n)} \leqslant \|f\|_{L^\infty(\mathbf{R}^n)}. \tag{2.2}$$

Proof. — The auxillary function

$$w^\varepsilon(x) = v(x)e^{-\varepsilon|x|^2} \qquad (\varepsilon > 0)$$

solves the p.d.e.

$$\begin{aligned} \lambda w^\varepsilon - a_{ij}w^\varepsilon_{x_i x_j} + b_i w^\varepsilon_{x_i} + cw^\varepsilon \\ = fe^{-\varepsilon|x|^2} - a_{ij}[2\varepsilon x_j v_{x_i} + 2\varepsilon x_i v_{x_j} + 2\varepsilon\,\delta_{ij} - 4\varepsilon^2 x_i x_j]e^{-\varepsilon|x|^2} \\ - b_i[2\varepsilon x_i v]e^{-\varepsilon|x|^2}. \end{aligned}$$

Since $|w^\varepsilon(x)| \to 0$ as $|x| \to \infty$, $|w^\varepsilon|$ attains its maximum at a finite point in $\mathbf{R}^n$. Applying the maximum principle at this point and recalling the inequalities

$$\sqrt{\varepsilon}\,|x|e^{-\varepsilon|x|^2}, \qquad \varepsilon|x|^2 e^{-\varepsilon|x|^2} \leqslant C,$$

we discover

$$\|\lambda w^\varepsilon\|_{L^\infty(\mathbf{R}^n)} \leqslant \|f\|_{L^\infty(\mathbf{R}^n)} + C\sqrt{\varepsilon}(\|Dv\|_{L^\infty(\mathbf{R}^n)} + \|v\|_{L^\infty(\mathbf{R}^n)}).$$

Now send $\varepsilon \to 0$ to obtain (2.2). □

LEMMA 2.2. — *Assume that* $u \in C^{3,\alpha}(\mathbf{R}^n)$ $(0<\alpha<1)$ *solves* (1.4). *Then there exists a constant* C_0 *such that*

$$\|\lambda u\|_{W^{1,\infty}(\mathbf{R}^n)} \leqslant C_0. \tag{2.3}$$

The constant C_0 *is independent of* λ, *provided* λ *is large enough.*

Proof. — We may as well assume

$$\frac{\partial F}{\partial r}(p,q,r,x) \leqslant 0 \qquad \text{for all} \qquad p,\, q,\, r,\, x\,, \tag{2.4}$$

since otherwise we can rewrite (1.4) in the form

$$\lambda' u - F'(D^2u,Du,u,x) = 0 \quad \text{in} \quad \mathbf{R}^n$$

for $F'(p,q,r,x) \equiv F(p,q,r,x) - Mr, \quad \lambda' = \lambda - M$.

Now u solves the equation

$$\begin{aligned}
\lambda u &- \left[\int_0^1 \frac{\partial F}{\partial P_{ij}}(tD^2u,tDu,tu,x)\,dt\right] u_{x_ix_j} \\
&- \left[\int_0^1 \frac{\partial F}{\partial q_i}(tD^2u,tDu,tu,x)\,dt\right] u_{x_i} \\
&- \left[\int_0^1 \frac{\partial F}{\partial r}(tD^2u,tDu,tu,x)\,dt\right] u \\
&= F(0,0,0,x) \quad \text{in} \quad \mathbf{R}^n .
\end{aligned}$$

Hypotheses (1.1)-(1.3) and (2.4) permit us to invoke Lemma 2.1 and obtain the bound

$$\|\lambda u\|_{L^\infty(\mathbf{R}^n)} \leqslant C .$$

Next let us differentiate (1.4) with respect to $x_k (k=1,2,\ldots,n)$; then we note that $v \equiv u_{x_k}$ solves the linear p.d.e.

$$\begin{aligned}
\lambda v - \frac{\partial F}{\partial P_{ij}}(D^2u,Du,u,x)v_{x_ix_j} &- \frac{\partial F}{\partial q_i}(D^2u,Du,u,x)v_{x_i} \\
- \frac{\partial F}{\partial r}(D^2u,Du,u,x)v &= \frac{\partial F}{\partial x_k}(D^2u,Du,u,x) .
\end{aligned} \tag{2.5}$$

We once more apply Lemma 2.1 to find

$$\|\lambda\, Du\|_{L^\infty(\mathbf{R}^n)} \leqslant C .$$

Next is our main estimate :

LEMMA 2.3. — *Fix some $0 < \alpha < 1$. Then there exists $\lambda_0 > 0$ and constants $0 < C_1 < C_2$, such that if u solves* (1.4),

$$\lambda \geqslant \lambda_0 ,$$

and

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_2,$$

then

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1.$$

Proof. – Choose β so small and p so large that

$$(2.6) \qquad 0 < \beta < \alpha = 1 - \frac{n}{p}\cdot$$

We recall from (2.5) that $v \equiv u_{x_k}$ $(k=1,2,\ldots,n)$ solves the linear elliptic equation

$$(2.7) \quad \frac{\partial F}{\partial p_{ij}}(D^2u,Du,u,x)v_{x_ix_j} + \frac{\partial F}{\partial q_i}(D^2u,Du,u,x)v_{x_i} + \frac{\partial F}{\partial r}(D^2u,Du,u,x)v = \lambda v - \frac{\partial F}{\partial x_k}(D^2u,Du,u,x),$$

the right hand side of which – according to Lemma 2.2 and assumption (1.3) – is bounded on $\mathbf{R}^n$, independently of λ.

Denote by B_1 and B_2 any two concentric closed balls, of radius 1 and 2 respectively. We apply the standard elliptic interior L^p estimates to (2.7) and obtain (see Lemma 6.1 in the appendix):

$$\begin{aligned}\|u_{x_k}\|_{W^{2,p}(B_1)} &= \|v\|_{W^{2,p}(B_1)} \\ &\leqslant C(\|u\|^{N}_{C^{2,\beta}(B_2)} + 1)\left(\left\|\lambda v - \frac{\partial}{\partial x_k}F(\ )\right\|_{L^p(B_2)} + \|v\|_{L^p(B_2)}\right) \\ &\leqslant C(\|u\|^{N}_{C^{2,\beta}(B_2)} + 1) \quad (k=1,2,\ldots,n),\end{aligned}$$

for certain constants C and N (the precise size of N, in particular, is irrelevant).

Then Morrey's theorem and (2.6) imply

$$\|u\|_{C^{2,\alpha}(B_1)} \leqslant C(\|u\|^{N}_{C^{2,\beta}(B_2)} + 1).$$

The constant C does not depend on the location of the balls $B_1 \subset B_2$ in $\mathbf{R}^n$. This estimate therefore implies

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C(\|u\|^{N}_{C^{2,\beta}(\mathbf{R}^n)} + 1).$$

We recall next interpolation inequality

$$\|u\|_{C^{2,\beta}(\mathbf{R}^n)} \leqslant C\|u\|_{C^{2,\alpha}(\mathbf{R}^n)}^{1-\rho}\|u\|_{L^\infty(\mathbf{R}^n)}^{\rho}.$$

(for some $0 < \rho < 1$; cf. Friedman [3]); this gives us the estimate

$$\begin{aligned} (2.10) \qquad \|u\|_{C^{2,\alpha}(\mathbf{R}^n)} &\leqslant C(\|u\|_{C^{2,\alpha}(\mathbf{R}^n)}^{N(1-\rho)}\|u\|_{L^\infty(\mathbf{R}^n)}^{\rho N} + 1) \\ &\leqslant \frac{C(\|u\|_{C^{2,\alpha}(\mathbf{R}^n)}^{N(1-\rho)} + 1)}{\lambda^{\rho N}} \end{aligned}$$

by (2.3). So far the constants C, N, ρ depend only on known quantities and do not depend on λ.

Now choose

$$C_1 = 2C,$$
$$C_2 = C_1 + 1.$$

Since we have assumed

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_2,$$

(2.10) implies

$$(2.11) \qquad \|u\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant \frac{C(C_2^{N(1-\rho)} + 1)}{\lambda^{\rho N}} \leqslant 2C = C_1$$

for $\lambda \geqslant \lambda_0$, λ_0 large enough. □

3. Proof of Theorem 1.

We suppose now that $0 < \alpha < 1$, λ_0, $0 < C_1 < C_2$ are the constants from Lemma 2.3. We will prove that (1.4) has a solution $u \in C^{2,\alpha}(\mathbf{R}^n)$ whenever $\lambda \geqslant \lambda_0$; and a standard bootstrap argument then implies $u \in C^{3,\gamma}(\mathbf{R}^n)$ for all $0 < \gamma < 1$.

For $0 \leqslant t \leqslant 1$ consider the problems

$$(3.1)_t \qquad \lambda u^t - F_t(D^2u^t, Du^t, u^t, x) = 0 \quad \text{in} \quad \mathbf{R}^n,$$

where

$$(3.2) \qquad F_t(D^2w, Dw, w, x) \equiv (1-t)\Theta\, \Delta w + tF(D^2w, Dw, w, x).$$

Define

$$T \equiv \{t \in [0,1] \mid (3.1)_t \text{ has a solution } u^t,\ \|u^t\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1\}.$$

Obviously $0 \in T$, and $u^0 \equiv 0$. Notice also that standard theory implies the uniqueness of the solutions u^t of $(3.1)_t$ with

$$\|u^t\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1 .$$

It is also evident that T is closed : if $\{t_i\} \subset T$, $t_i \to t_0$, then, since $\|u^{t_i}\|_{C^{3,\alpha}(\mathbf{R}^n)}$ is bounded, we have

$$u^{t_i} \to u^{t_0} \quad \text{in} \quad C^{2,\alpha}_{loc}(\mathbf{R}^n)$$

and

$$\|u^{t_0}\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant \liminf_{i\to\infty} \|u^{t_i}\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1 .$$

Finally we assert that T is relatively open in [0,1]. Once this is proved we can conclude $1 \in T$; that is, (1.4) has a solution. Consider therefore the mapping

$$G(t,u) : [0,1] \times C^{2,\alpha}(\mathbf{R}^n) \to C^{\alpha}(\mathbf{R}^n)$$

defined by

$$G(t,u) \equiv \lambda u - F_t(D^2U,Du,u,x).$$

Clearly G is continuous. Its Frechet derivative in u at any point (t,u) is an isomorphism according to standard theory for linear elliptic equations with Hölder continuous coefficients :

$$G_u(t,u)v \equiv \lambda v - (1-t)\Theta\,\Delta v - t\left[\frac{\partial F}{\partial p_{ij}}(D^2u,Du,u,x)v_{x_ix_j} + \frac{\partial F}{\partial q_i}(D^2u,Du,u,x)v_{x_i} + \frac{\partial F}{\partial r}(D^2u,Du,u,x)v\right].$$

Note also that the mapping

$$(t,u) \to G_u(t,u)$$

is continuous.

Now select any $t_0 \in T \cap (0,1)$. By the implicit function theorem, there exists some $\varepsilon > 0$ and a *continuous* function $v : (t_0-\varepsilon,t_0+\varepsilon) \to C^{2,\alpha}(\mathbf{R}^n)$ so that

$$G(t,v(t)) \equiv G(t_0,u^{t_0}) = 0.$$

Clearly

$$v(t) \equiv u^t$$

solves $(3.1)_t$. Since $\|u^{t_0}\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1$, we have $\|u^{t_0}\|_{C^{2,\alpha}(\mathbf{R}^n)} < C_2$ for $|t-t_0| < \varepsilon', \varepsilon'$ small enough. Then Lemma 2.3 implies

$$\|u^t\|_{C^{2,\alpha}(\mathbf{R}^n)} \leqslant C_1 ;$$

that is, $(t_0-\varepsilon', t_0+\varepsilon') \subset T$.

Theorem 1 is proved. □

4. Proof of Theorem 2.

In proving Theorem 2 we may mimic with obvious modifications the calculations in § 3; the only real difficulty is to modify Lemmas 2.2 and 2.3 to the case that Ω replaces $\mathbf{R}^n$: here the extra hypothesis (1.7) is crucial to our argument.

LEMMA 4.1. – *Assume that* $u \in C^{3,\alpha}(\bar{\Omega})$ $(0<\alpha<1)$ *solves* (1.6). *Then there exists a constant* C_0 *such that*

$$\|\lambda u\|_{W^{1,\infty}(\Omega)} \leqslant C_0 . \tag{4.1}$$

C_0 *is independent of* λ *so long as* λ *is large enough.*

Proof. – As in the proof of Lemma 2.2, we may assume

$$\frac{\partial F}{\partial r}(p,q,r,x) \leqslant 0 \qquad \text{for all} \qquad p, q, r, x .$$

The estimate

$$\|\lambda u\|_{L^\infty(\Omega)} \leqslant C$$

is then immediate from the maximum principle.

We must next prove

$$\lambda |Du|\big|_{\partial\Omega} \leqslant C \tag{4.2}$$

for some constant C. To see this first choose any point $x^* \in \partial\Omega$. As $\partial\Omega$ is smooth and therefore satisfies the uniform exterior sphere condition, we may assume, upon a change of coordinates if necessary, that

$$x^* = (0,0,\dots,R),$$
$$B(0,R) \cap \partial\Omega = \{x^*\}$$

for some fixed $R > 0$.

Consider now the auxillary function

$$v(x) \equiv \frac{\mu}{\lambda}\left(\frac{1}{R^p} - \frac{1}{|x|^p}\right), \tag{4.3}$$

where $\mu, p > 0$ are to be selected. We have

$$v_{x_i} = \frac{\mu}{\lambda} p \frac{x_i}{|x|} p + 2$$

and

$$v_{x_i x_j} = \frac{\mu}{\lambda}\left(\frac{p\,\delta_{ij}}{|x|^{p+2}} - \frac{p(p+2)x_i x_j}{|x|^{p+4}}\right);$$

so that

$$\begin{aligned} F(D^2 v, Dv, v, x) &= \left[\int_0^1 \frac{\partial F}{\partial p_{ij}}(tD^2 v, t\,Dv, tv, x)\,dt\right] v_{x_i x_j} \\ &+ \left[\int_0^1 \frac{\partial F}{\partial g_i}(tD^2 v, t\,Dv, tv, x)\,dt\right] v_{x_i} \\ &+ \left[\int_0^1 \frac{\partial F}{\partial r}(t\,D^2 v, t\,Dv, tv, x)\,dt\right] v + F(0,0,0,x) \\ &\leqslant F(0,0,0,x), \end{aligned}$$

for p large enough. On the other hand since $F(0,0,0,x) = 0$ on $\partial\Omega$, we have

$$|F(0,0,0,x)| \leqslant M|x - x^{**}|$$

where

$$x^{**} \in \partial\Omega \text{ belongs to the segment } \overline{Ox},$$
$$|x^{**}| \geqslant R.$$

But note also that

$$\begin{aligned} \lambda v(x) \geqslant \lambda(v(x) - v(x^{**})) &= \mu\left(\frac{1}{|x^{**}|^p} - \frac{1}{|x|^p}\right) \\ &= \frac{\mu}{|x|^p}\left(\frac{1}{\alpha^p} - 1\right) \quad \text{where } x^{**} = \alpha x, \qquad \frac{R}{\text{dia}(\Omega)} \leqslant \alpha \leqslant 1 \\ &\geqslant \mu C(1-\alpha)|x| = \mu C|x - x^{**}| \end{aligned}$$

for some constant $C > 0$. Hence

$$\lambda v(x) \geqslant F(0,0,0,x) \qquad x \in \Omega \tag{4.5}$$

if μ is large enough. According to (4.4) and (4.5) we have

$$\lambda(v-u)-[F(D^2v,Dv,v,x)-F(D^2u,Du,u,x)] \leqslant 0 \quad \text{in} \quad \Omega.$$

The maximum principle therefore implies

$$u \leqslant v \quad \text{in}\, \bar{\Omega}.$$

Since $u(x^*) = v(x^*) = 0$, we have

$$\frac{\partial u(x^*)}{\partial n} \geqslant \frac{\partial v(x^*)}{\partial n} \geqslant -\frac{C}{\lambda}\cdot$$

A similar argument provides an upper bound. This proves (4.2).

The interior bound on Du is easy now. We differentiate (1.6) with respect to x_k $(k=1,2,\ldots,n)$:

$$\lambda u_{x_k} - \left(\frac{\partial F}{\partial p_{ij}} u_{x_k x_i x_j} + \frac{\partial F}{\partial q_i} u_{x_k x_i} + \frac{\partial F}{\partial r} u_{x_k}\right) = F_{x_k}.$$

Should $\pm u_{x_k}$ attains its maximum at some point $x_0 \in \Omega$, we have

$$\pm \lambda u_{x_k}(x_0) \leqslant \pm F_{x_k}(D^2u(x_0), Du(x_0), u(x_0), x_0) \leqslant M;$$

and should the maximum occur on $\partial\Omega$, we recall (4.2).

LEMMA 4.2. – *Fix some* $0 < \alpha < 1$. *Then there exists* $\lambda_0 > 0$ *and constants* $0 < C_1 < C_2$ *such that if* u *solves* (1.6),

$$\lambda \geqslant \lambda_0,$$

and

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C_2,$$

then

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C_1.$$

Proof. – As in the proof of Lemma 2.3 choose β and p so that

$$0 < \beta < \alpha = 1 - \frac{n}{p}\cdot$$

According to Lemma 4.1 and Lemma 6.2 in the appendix we have

$$\|u\|_{W^{3,p}(\Omega)} \leqslant C(\|u\|^{N}_{C^{2,\beta}(\Omega)} + 1)$$

for some constants C and N. This estimate and a calculation almost precisely like that in the proof of Lemma 2.3 imply the result. □

5. Comments and extensions.

a) Hypothesis (1.7).

A review of § 3 and § 4 makes it clear that the estimate Lemma 4.2 provides is crucial for our technique; for if the right hand side of (2.7) becomes unbounded with large λ we cannot then select λ_0 large enough to obtain (2.11). Lemma 4.2 in turn depends on the assumption (1.7) (*i.e.* « $F(0,0,0,x) = 0$ *on* $\partial\Omega$ ») as the following example shows : Consider the problem

$$\begin{cases} \lambda u - u'' = 1 & \text{on} \quad (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Then

$$u(x) = \frac{1}{\lambda}\left[1 - \left(\frac{1-e^{-\sqrt{\lambda}}}{e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}}\right)e^{\sqrt{\lambda x}} - \left(\frac{e^{\sqrt{\lambda}}-1}{e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}}\right)e^{-\sqrt{\lambda x}}\right]$$

so that

$$\lambda u'(0) \sim C\sqrt{\lambda} \quad \text{for large} \quad \lambda.$$

In this case Lemma 4.1 fails, as do its obvious modifications (e.g. replacing the L^∞ with L^p norms).

b) Neumann boundary conditions.

Consider the p.d.e.

$$(5.1) \qquad \begin{aligned} &\lambda u - F(D^2u, Du, u, x) = 0 \quad \text{in} \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

when Ω is now assumed to be a smooth bounded, *convex* domain in $\mathbf{R}^n$

and $\frac{\text{« } \partial \text{ »}}{\partial n}$ denotes the outward normal derivative. We claim that (5.1) admits a unique solution assuming that λ is large enough and F satisfies hypotheses (1.1)-(1.3); assumption (1.7) is not needed here.

Indeed it suffices to obtain the bound

$$\|\lambda u\|_{W^{1,\infty}(\Omega)} \leqslant C, \tag{5.2}$$

for C independent of λ, λ large enough. According to Hopf's maximum principle $|u|$ must attain its maximum at some point of Ω, where as before

$$\|\lambda u\|_{L^\infty(\Omega)} \leqslant C.$$

Next a straightforward calculation shows us that

$$v \equiv |Du|^2$$

solves

$$\begin{aligned} 2\lambda v - \frac{\partial F(\ )}{\partial p_{ij}} v_{x_i x_j} - \frac{\partial F(\ )}{\partial q_i} v_{x_i} - 2\frac{\partial F(\ )}{\partial r} v \\ = \frac{\partial F(\ )}{\partial x_k} u_{x_k} - 2\frac{\partial F(\ )}{\partial p_{ij}} u_{x_k x_i} u_{x_k x_j} \quad \text{in} \quad \Omega. \end{aligned} \tag{5.3}$$

If v attains its maximum in Ω, the maximum principle gives the desired estimate

$$\|\lambda v\|_{L^\infty(\Omega)} \leqslant C. \tag{5.3}$$

On the other hand Lemma 1.1 in P.-L. Lions [6] implies

$$\frac{\partial v}{\partial n} \leqslant 0 \quad \text{on} \quad \partial\Omega$$

(the convexity of Ω is used here). The Hopf maximum principle therefore eliminates the possibility that v attains its maximum only on $\partial\Omega$.

This proves the estimate (5.2) and – as noted – the remainder of the existence proof for (5.1) follows as in Lemma 2.3 and § 3. □

6. Appendix : the dependence of L^p estimates upon the second-order coefficients.

In § 2 we made reference to the following estimate concerning the dependence of the standard L^p elliptic estimates on the C^β-norm of the second order coefficients :

LEMMA 6.1. — *Let* B_1 *and* B_2 *be two concentric closed balls in* $\mathbf{R}^n$, *of radius* 1 *and* 2, *respectively. Assume that* $v \in C^{2,\alpha}(B_2)$ *solves the linear equation*

$$- a_{ij}(x)v_{x_ix_j} + b_i(x)v_{x_i} + c(x)v = f \tag{6.1}$$

in B_2, *where*

$$\begin{cases} |a_{ij}|, |b_i|, |c| \leqslant M \\ a_{ij}(x)\xi_i\xi_i \geqslant \theta|\xi|^2 \quad \text{for all} \quad x, \xi \in \mathbf{R}^n \end{cases} \tag{6.2}$$

and

$$a_{ij} \in C^\beta(B_2) \quad \text{for some} \quad 0 < \beta < 1 .$$

Then for each $1 < p < \infty$ *there exist constants* C *and* N, *depending only on* M, θ, p, *and* n *such that*

$$\|v\|_{W^{2,p}(B_1)} \leqslant C(\|a_{ij}\|^N_{C^\beta(B_2)} + 1)(\|f\|_{L^p(B_2)} + \|v\|_{L^p(B_2)} + 1) . \tag{6.3}$$

Proof. — The bound (6.3) is a standard consequence of linear L^p theory, except for the stated dependence on the C^β-norm of the a_{ij}.

Briefly then, let us first note that a solution $\hat{v}$ of

$$\begin{cases} L\hat{v} = \hat{f} \quad \text{in} \quad B(R) \\ \hat{v} = 0 \quad \text{near} \quad \partial B(R) \end{cases}$$

(L denoting the operator in (6.1) and B(R) some ball of radius R) satisfies the bound

$$\|D^2\hat{v}\|_{L^p(B(R))} \leqslant C(\|\hat{f}\|_{L^p(B(R))} + \|\hat{v}\|_{W^{1,p}(B(R))}), \tag{6.4}$$

provided

$$R^\beta\|a_{ij}\|_{C^\beta(\overline{B(R)})} = \varepsilon' , \tag{6.5}$$

for some small, but fixed constant ε'. (Proof : a standard perturbation of coefficients (cf. Ladyžewskaja and Ural'ceva [4, p. 190-193]) reduces (6.4) to the known estimate for Δ.)

Now B_1 can be covered by $K \equiv C\left[\|a_{ij}\|_{C^\beta}^{\frac{n}{\beta}}+1\right]$ balls B_k of radius $\frac{R}{2}$, R satisfying (6.5). We choose cutoff functions ζ_k so that

$$(6.6)\qquad \begin{cases} \zeta_k = 1 \text{ on } B_k \\ \zeta_k = 0 \text{ near } \partial 2B_k \ (2B_k \equiv \text{ball concentric with } B_k \text{ and with radius } R) \end{cases}$$

$$|D\zeta_k| \leqslant \frac{C}{R}, \quad |D^2\zeta_k| \leqslant \frac{C}{R^2}$$

and set

$$(6.7)\qquad n_k = \zeta_k\left(\sum_{\ell=1}^{k}\zeta_\ell\right)^{-1}$$

to obtain a partition of unity on B_1. Define

$$(6.8)\qquad \hat{v}_k \equiv \eta_k v \quad \text{on} \quad 2B_k.$$

We have

$$(6.9)\qquad L\hat{v}_k = \eta_k f - a_{ij}[2v_{x_i}\eta_{kx_j}+v\eta_{kx_ix_j}] + b_i v\eta_{kx_i} \equiv \hat{f}_k.$$

Then (6.4) implies

$$(6.10)\qquad \|D^2v\|_{L^p(B_1)} \leqslant \sum_{k=1}^{K}\|D^2\hat{v}_k\|_{L^p(2B_k)} \leqslant \frac{CK}{R^2}\,|\,\|f\|_{L^p(B_2)} + \|v\|_{W^{1,p}(B_{3/2})}|\,.$$

Similarly

$$\|v\|_{W^{1,p}(B_{3/2})} \leqslant \frac{CK}{R^2}\,|\,\|f\|_{L^p(B_2)} + \|v\|_{L^p(B_2)}.$$

The last two estimates, (6.5), and the definition of K give us (6.3). □

For the proof of Lemma 4.2 we need

LEMMA 6.2. — *Suppose that* $u \in C^{3,\gamma}(\bar{\Omega})$ *for some* $0 < \gamma < 1$ *solves*

$$\begin{cases} F(D^2u,Du,u,x) = f(x) \ \ in\ \Omega \\ u = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{6.11}$$

for some $f \in W^{1,p}(\Omega)$. Then for each $1 < p < \infty$ and $0 < \beta < 1$ there exist constants C and N, depending only on M, θ, p, and Ω, such that

$$\|u\|_{W^{3,p}(\Omega)} \leqslant C(\|u\|^{N}_{C^{2,\beta}(\Omega)} + 1)\|f\|_{W^{1,p}(\Omega)}. \tag{6.12}$$

Proof. — Differentiating (6.11) we note that $v = u_\xi$ (the derivative of u in an arbitrary direction ξ) satisfies

$$\frac{\partial F}{\partial p_{ij}}(D^2u,Du,u,x)v_{x_ix_j} + \frac{\partial F}{\partial q_i}(D^2u,Du,u,x)v_{x_i}$$
$$+ \frac{\partial F}{\partial r}(D^2u,Du,u,x)v = f_\xi - \frac{\partial F}{\partial \xi}(D^2u,Du,u,x);$$

the right hand side of this expression belongs to $L^p(\Omega)$. Now cover $\bar{\Omega}$ with $K = C\left[\left\|\dfrac{\partial F(\ )}{\partial p_{ij}}\right\|^{\frac{n}{\beta}}_{C^\beta(\Omega)} + 1\right]$ balls B_k of radius $\dfrac{R}{2}$, for R defined by

$$R^\beta \left\|\frac{\partial F(\ )}{\partial p_{ij}}\right\|_{C^\beta(\bar{\Omega})} = \varepsilon',$$

ε' from (6.5); we may assume that those balls B_k which intersect $\partial\Omega$ are in fact centered at a point belonging to $\partial\Omega$.

Define ζ_k, η_k, $\hat{v}_k$ by (6.6)-(6.8).

Now if $B_k \subset \Omega$ for any given $k = 1,2,\ldots,k$ we recall estimate (6.4) for $\hat{v} = \hat{v}_k$. If $B_k \cap \partial\Omega \neq \varnothing$, we transform coordinates to the case that $\partial\Omega \cap B_k \subset \{x_n = 0\}$, reflect $\hat{v}_k$ across the x_n plane (assuming $\hat{v}_k = 0$ on $\{x_n = 0\}$), and again apply (6.4). This method yields a bound on $\|u_\xi\|_{W^{2,p}(B_k)}$ for $\xi = x_1, \ldots, x_{n-1}$. The remaining derivative $u_{x_nx_nx_n}$ we estimate using equation (6.13) for $v = u_{x_n}$.

Collecting together these bounds we obtain

$$\|Du\|_{W^{2,p}(\Omega)} \leqslant C(\|u\|^{N}_{C^{2,\beta}(\Omega)} + 1)(\|Df\|_{L^p(\Omega)} + \|D_xF\|_{L^p(\Omega)} + \|u\|_{W^{2,p}(\Omega)}).$$

Applying a standard interpolation inequality completes the proof. □

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