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TISCHLER FIBRATIONS OF OPEN, FOLIATED SETS

by J. CANTWELL (*) and L. CONLON (**)

Introduction.

Let M be a smooth, closed n-manifold, $\mathcal{F}$ a foliation of M of codimension one. Unless otherwise specified, we will assume only that $\mathcal{F}$ has C^∞ leaves integral to a C^0 hyperplane field ($\mathcal{F}$ is said to be of class C^{0+}). We will further require that M be orientable and that $\mathcal{F}$ be transversely orientable.

If each leaf of $\mathcal{F}$ is everywhere dense without holonomy, then [10., Theorem 4] implies the existence of a transverse, holonomy invariant, positive measure, finite on compact sets. As in the proof of [10., Theorem 6], it follows that M admits a possibly new C^∞ structure in which the C^∞ structures of the leaves of $\mathcal{F}$ are unchanged and in which $\mathcal{F}$ is defined by a closed, nonsingular 1-form ω. By a theorem of D. Tischler [11], the manifold M, in this new structure, fibers smoothly over S^1 and such fibrations can be found arbitrarily C^∞-close to $\mathcal{F}$. Also, as seems to be well known to experts, these approximating fibrations can be chosen so that the leaves of $\mathcal{F}$ are regular coverings of the fibers in a very natural way, the covering group being a subgroup of co-rank 1 in the group $P(\omega) = \mathrm{Im}(\omega : \pi_1(M) \to \mathbf{R})$ of periods of ω.

More generally, suppose that $U \subset M$ is an open, connected, $\mathcal{F}$-saturated subset, each leaf of $\mathcal{F}|U$ being dense in U with trivial holonomy. Such sets are prominent among the fundamental building blocks of C^2 foliations [1], [13]. For instance, such a set U is the

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necessary ambience for any leaf at finite level with an « exotic » nonexponential growth type [1, (3.6) and (3.7)]. Let $\hat{U}$ be the completion of U in the sense of G. Hector [8] and P. Dippolito [5]. This is a manifold with finitely many boundary components [5, Proposition 2] and, generally, it is not compact. The foliation $\mathscr{F}$ induces a C^{0+} foliation $\hat{\mathscr{F}}$ of $\hat{U}$ having each component of $\partial\hat{U}$ as a leaf. The above method of finding a new C^∞ structure generalizes to $\hat{U}$, making $\hat{\mathscr{F}}$ a C^∞ foliation, C^∞-trivial at $\partial\hat{U}$, such that $\hat{\mathscr{F}}|U(=\mathscr{F}|U)$ is defined by a closed, nonsingular 1-form ω on U.

Here we investigate the possibility of smoothly approximating $\hat{\mathscr{F}}$ over precompact regions by a C^∞ foliation $\mathscr{F}^*$ (called a *Tischler foliation*) of $\hat{U}$, C^∞-trivial at $\partial\hat{U}$, such that $\mathscr{F}^*|U$ fibers U over S^1. When that is possible, we further investigate the possibility of choosing these fiberings of U so that the leaves of $\mathscr{F}|U$ are regular coverings of the fibers in a suitably natural way. These questions are of interest, of course, only for $\dim(M) > 2$.

If $\dim(M) = 3$, we find that Tischler foliations always exist (2.1), but we give smooth counterexamples in all dimensions greater than three (4.5). A condition guaranteeing the existence of Tischler foliations in arbitrary dimensions is that the period group $P(\omega)$ be free abelian (2.2). In particular, this gives Tischler foliations if (1) $\hat{U}$ is compact, or (2) each leaf of $\mathscr{F}|U$ has two dense ends, or (3) $\mathscr{F}$ is transversely analytic (cf. (3.10), (3.11), and Remark (2) following (3.11)). This condition on $P(\omega)$ also implies the result about regular coverings (3.8), but even on 3-manifolds, where Tischler foliations always exist, the regular covering property often fails when $P(\omega)$ is not free abelian (3.9).

1. Technical preliminaries.

Fix $M, \mathscr{F}$, and $U \subset M$ as in the introduction. Fix a transverse, smooth, 1-dimensional foliation $\mathscr{L}$. As in [1, (1.6)], obtain the transverse, invariant measure μ for $\mathscr{F}|U$ and the associated C^0 flow $\Phi : \mathbf{R} \times M \to M$, nonsingular precisely on U, having as flow lines in U the leaves of $\mathscr{L}|U$, and preserving the foliation $\mathscr{F}$. Let $P(\mu) \subset \mathbf{R}$ be the additive subgroup of periods of μ [1, (1.7)]. That is, $t \in P(\mu)$ if and only if Φ_t carries some (hence every) leaf of $\mathscr{F}|U$ onto itself.

The following is proven by reasoning, familiar-to-specialists, entirely similar to that in [10, Theorem 6].

(1.1) LEMMA. — *There is a possibly new differentiable structure on* $\hat{U}$ *under which*

(1) $\hat{\mathscr{F}}$ *is of class* C^∞ *and is* C^∞*-trivial at* $\partial\hat{U}$;

(2) *The differentiable structure on each leaf of* $\hat{\mathscr{F}}$ *remains unchanged;*

(3) $\mathscr{F}|U$ *is defined by a closed, nonsingular form* $\omega \in A^1(U)$, *and* $P(\mu) = P(\omega)$.

Indeed, a new C^∞ structure is chosen in $\hat{U}$ so that the local leaves of $\hat{\mathscr{L}}$ (the 1-dimensional foliation of $\hat{U}$ induced by $\mathscr{L}$) are the level sets of the first $n-1$ local coordinates, and the flow parameter of Φ provides the n^{th} coordinate. Of course, at the boundary this n^{th} coordinate takes values $\pm\infty$, where we use a smooth structure on $[-\infty, \infty]$ relative to which the group of translations acts smoothly and is C^∞-flat at $\pm\infty$. The coordinate transformations are of the form $x_i = x_i(\bar{x}_1, \dots, \bar{x}_{n-1})$, $1 \leqslant i \leqslant n-1$, $x_n = \bar{x}_n + c$, c constant, so (1) and (2) follow. The form ω will be well defined on U by the local formulas $\omega = dx_n$. The equality of $P(\mu)$ and $P(\omega)$ is elementary.

We are going to express ω in terms of a carefully chosen basis of $H^1(\hat{U};\mathbf{R})$.

Decomposition of ω. — Recall Dippolito's decomposition [5, Theorem 1] of $\hat{U}$ into a compact, connected manifold K with corners, called the *nucleus*, and noncompact « arms » $\hat{U}_j \cong B_j \times [-1,1]$, $1 \leqslant j \leqslant r$, where B_j is a complete, non-compact, connected, $(n-1)$-*dimensional submanifold of a component of* $\partial\hat{U}$, ∂B_j is compact and connected, and each $\{x\} \times [-1,1]$ is a leaf of $\hat{\mathscr{L}}$. By attaching to K successively larger chunks of the arms, we construct a sequence of nuclei

$$K = K_0 \subset K_1 \subset \cdots \subset K_i \subset \cdots$$

such that $\hat{U} = \bigcup_{i \geqslant 0} K_i$ and each $K_i \subset \operatorname{int}(K_{i+1})$ (interior relative to $\hat{U}$). Remark that the number of arms attached to K_i may become unbounded as $i \to \infty$.

The inclusions $K_i \hookrightarrow \hat{U}$ induce homomorphisms λ_i :

$$H_1(K_i;\mathbf{R}) \longrightarrow H_1(\hat{U};\mathbf{R}),$$

and we set $A_i = \operatorname{Im}(\lambda_i)$, a subspace of $H_1(\hat{U};\mathbf{R})$ of finite dimension $n(i)$. Remark that $H_1(\hat{U};\mathbf{R}) = \bigcup_{i \geqslant 0} A_i$.

Choose *integral* cycles $\sigma_1, \ldots, \sigma_{n(0)}$ in U which represent a basis of A_0, integral cycles $\sigma_{n(0)+1}, \ldots, \sigma_{n(1)}$, $n(1) \geqslant n(0)$, in U which represent a possibly trivial extension of this basis to a basis of A_1, etc. This gives rise to a possibly infinite basis $[\sigma_1], [\sigma_2], \ldots, [\sigma_k], \ldots$ of $H_1(\hat{U};\mathbf{R})$.

Choose closed forms $\omega_1, \omega_2, \ldots$ in $A^1(\hat{U})$ such that $\omega_i(\sigma_j) = \delta_{ij}$. If σ_j does not represent an element of A_i, then $\omega_j | K_i = dh$ for some smooth $h : K_i \to \mathbf{R}$. One smoothly extends h to $\hat{h} : \hat{U} \to \mathbf{R}$ by standard techniques and replaces ω_j by $\omega_j - d\hat{h}$ so as to guarantee that $\omega_j | K_i \equiv 0$. Thus, each point of $\hat{U}$ has a neighborhood on which only finitely many of the forms ω_j are not identically zero.

A further wrinkle is needed in the choice of these forms. Let W be a neighborhood of $\partial\hat{U}$ in $\hat{U}$ such that (see figure 1) :

(*a*) $\hat{U} - K_0 \subset W$;

(*b*) the components of $W \cap K_0$ are disjoint collar neighborhoods of the respective components of $\partial\hat{U} \cap K_0$, fibered by $\hat{\mathscr{L}} | (W \cap K_0)$.

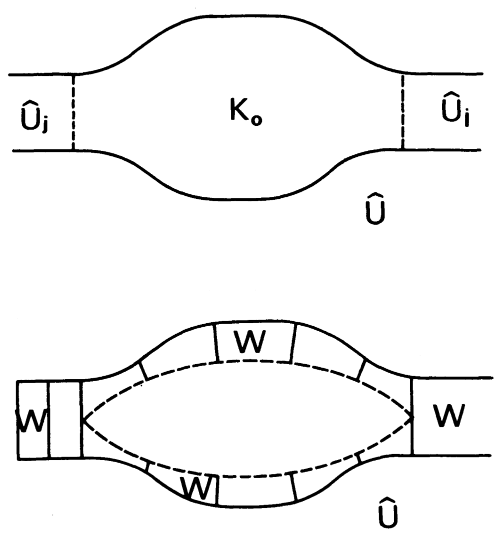

Fig. 1.

Thus, in each component of $W \cap K_0$, we have a canonical choice of projection p into $\partial\hat{U}$ along the leaves of $\hat{\mathscr{L}}$. In each component of $\hat{U} - K_0$, we have two such choices of p.

Fix ω_j. We will find a closed form $\eta \in A^1(\partial\hat{U})$ and a smooth function $h : W \to \mathbf{R}$ such that $\omega_j | W = p^*(\eta) + dh$ unambigously. Damping h

off to zero near the boundary of W in U and extending by 0 defines a smooth function $\hat{h} : \hat{U} \to \mathbf{R}$ such that $\omega_j - d\hat{h}$ vanishes on the tangents to $\hat{\mathscr{L}}$ both near $\partial\hat{U}$ and outside of (say) K_1. We replace ω_j with $\omega_j - d\hat{h}$. We have to take precautions to insure that the local finiteness of $\{\omega_j\}_{j\geqslant 1}$ is not destroyed. Here are more details.

(1) For each component L_k of $\partial\hat{U}$, choose $\eta_k \in A^1(L_k)$ that pulls back via p to the appropriate part of W as a form cohomologous to ω_j.

(2) If L_k and L_q are two components of $\partial\hat{U}$ such that some arm $\hat{U}_i \trianglelefteq B_i \times [-1,1]$ has $B_i \times \{-1\} \subset L_k$, $B_i \times \{1\} \subset L_q$, the forms η_k and η_q restrict to cohomologous forms on B_i, so similar adjustments as above allow us to assume that these restrictions are equal. This guarantees the non-ambiguity of $p^*(\eta)$.

(3) If $\omega_j | K_i \equiv 0$, we can choose η to vanish on $K_i \cap \partial\hat{U}$ and h to vanish on $W \cap K_i$. This guarantees the local finiteness.

Let $c_j = \omega(\sigma_j)$ and consider the sum $\hat{\omega} = \sum_j c_j\omega_j$. This sum is locally finite and each ω_j is closed, so $\hat{\omega}$ is a closed 1-form on $\hat{U}$. Also, $\hat{\omega}$ vanishes on the tangents to the leaves of $\hat{\mathscr{L}}$ both near $\partial\hat{U}$ and in $\hat{U} - K_1$.

Since $H^1(U;\mathbf{R})$ is the dual vector space to $H_1(U;\mathbf{R})$ and $U \hookrightarrow \hat{U}$ is a homotopy equivalence, the following lemmas are easy consequences of our constructions.

(1.2) LEMMA. — *There is a smooth function* $g : U \to \mathbf{R}$ *such that* $\omega = \hat{\omega}|U + dg$. *Near* $\partial\hat{U}$ and in $\hat{U} - K_1$, *the restrictions of* ω *to the leaves of* $\mathscr{L}|U$ agree with those of dg. *In particular,* dg *is nonsingular in those regions and it is unbounded near* $\partial\hat{U}$.

(1.3) LEMMA. — *Let* $W_0 \subset \hat{U}$ *be an open, relatively compact set. Fix* $i \geqslant 1$ *such that* $W_0 \subset K_i$. *If numbers* $\tilde{c}_j \in \mathbf{R}$ *are chosen,* $j \geqslant 1$, *so that* $\tilde{c}_1, \ldots, \tilde{c}_{n(i)}$ *are sufficiently near* $c_1, \ldots, c_{n(i)}$ *respectively, then* $\tilde{\omega} = \sum_j \tilde{c}_j(\omega_j|U) + dg$ *is a closed, nonsingular* 1*-form on* U, *defining a foliation* $\tilde{\mathscr{F}}$ *transverse to* $\mathscr{L}|U$, *and* $\tilde{\omega}|(W_0 \cap U)$ *is as* C^∞*-close to* $\omega|(W_0 \cap U)$ *as desired.*

Practically as immediate is the following.

(1.4) LEMMA. — *The foliation* $\tilde{\mathscr{F}}$ *of* (1.3) *can be extended to a* C^∞ foliation $\mathscr{F}^*$ *of* $\hat{U}$, C^∞*-trivial at* $\partial\hat{U}$, *by letting each component of* $\partial\hat{U}$ *be a leaf.*

Indeed, the local flows on U produced by $\tilde{\omega}$ and having flow lines along $\mathscr{L}|U$ agree with Φ outside a compact subset of U, hence they can be assembled into a smooth global flow $\tilde{\Phi}$ on U that preserves $\tilde{\mathscr{F}}$. Since $\tilde{\Phi}$ and Φ agree near $\partial\hat{U}$, any coordinate system $x_1, \ldots, x_n$ in a neighborhood of $\partial\hat{U}$, having as $\hat{\mathscr{F}}$-plaques the level sets of x_n, $0 \leqslant x_n \leqslant \infty$ (or $-\infty \leqslant x_n \leqslant 0$), is readily C^∞-transformed to a coordinate system

$$\tilde{x}_1 = x_1, \ldots, \tilde{x}_{n-1} = x_{n-1}, \qquad \tilde{x}_n = x_n + \tau(x_1, \ldots, x_{n-1})$$

having as $\mathscr{F}^*$-plaques the level sets of $\tilde{x}_n$. On overlaps, the coordinate transformations are of the form

$$\tilde{x}_i = \tilde{x}_i(\tilde{y}_1, \ldots, \tilde{y}_{n-1}), \qquad 1 \leqslant i \leqslant n-1, \qquad \tilde{x}_n = \tilde{y}_n + c.$$

Of course, as usual, we stipulate that the level sets of the first $n-1$ coordinates be plaques of $\mathscr{L}$.

Remarks. — (1) the foliation $\mathscr{F}^*$ extends over M to a C^0 foliation, again denoted $\mathscr{F}^*$, such that $\mathscr{F}^*|(M-U) = \mathscr{F}|(M-U)$. One can then show that, in a certain reasonable sense, $\mathscr{F}^*$ is uniformly close to $\mathscr{F}$.

(2) The group $P(\tilde{\omega})$ of periods is equal to the set of numbers $t \in \mathbf{R}$ such that $\tilde{\Phi}_t$ carries each leaf of $\tilde{\mathscr{F}}$ onto itself. It is elementary that the foliation $\tilde{\mathscr{F}}$ fibers U over S^1 if and only if $P(\tilde{\omega})$ is infinite cyclic.

2. Existence of Tischler foliations.

We keep all of the same conventions and notations as in Section 1.

First, assume that $\dim(M) = 3$. Fix an open, relatively compact subset $W_0 \subset \hat{U}$ and fix $i > 0$ such that $W_0 \subset K_i$. Consider the decomposition of $\hat{U}$ into the nucleus K_i and arms $\hat{U}_j \cong B_j \times [-1,1]$, $1 \leqslant j \leqslant r$. Thus, each $\partial B_j \cong S^1$, so $K_i \cap \hat{U}_j \cong S^1 \times [-1,1]$. Also, the homomorphism $H_*(K_i \cap \hat{U}_j;\mathbf{Z}) \to H_*(\hat{U}_j;\mathbf{Z})$ identifies with $H_*(\partial B_j;\mathbf{Z}) \to H_*(B_j;\mathbf{Z})$ and this is one-one.

In this situation, the Mayer-Vietoris sequence yields a short exact sequence

$$0 \to \mathbf{Z}^r \to H_1(K_i;\mathbf{Z}) \oplus H_1(\hat{U}_1;\mathbf{Z}) \oplus \cdots \oplus H_1(\hat{U}_r;\mathbf{Z}) \to H_1(\hat{U};\mathbf{Z}) \to 0.$$

Here, $\mathbf{Z}^r = \mathrm{H}_1\left(\bigcup_{j=1}^{r} (\mathrm{K}_i \cap \hat{\mathrm{U}}_j);\mathbf{Z}\right)$ is generated by the cycles $\partial \mathrm{B}_j$ and each $\mathrm{H}_1(\hat{\mathrm{U}}_j;\mathbf{Z}) = \mathrm{H}_1(\mathrm{B}_j;\mathbf{Z})$ is free abelian on a basis that contains the cycle $\partial\mathrm{B}_j$. It follows that $\mathrm{H}_1(\hat{\mathrm{U}};\mathbf{Z}) = \mathrm{A} \oplus \mathrm{B}$ where A is the (finitely generated) image of $\mathrm{H}_1(\mathrm{K}_i;\mathbf{Z})$ induced by the inclusion $\mathrm{K}_i \hookrightarrow \hat{\mathrm{U}}$ and B is free abelian. In the choice of integral cycles $\sigma_1, \sigma_2, \ldots,$ as in Section 1, we can arrange that $\{\sigma_1,\ldots,\sigma_{n(i)}\}$ gives a basis of A/(torsion) and that $\{\sigma_j\}_{j>n(i)}$ gives a basis of B. Thus, the forms ω_j, $j \leqslant n(i)$, annihilate B.

Choose the numbers $\tilde{c}_1, \ldots, \tilde{c}_{n(i)}$ to be rational and as close to $c_1, \ldots, c_{n(i)}$, respectively, as desired. For $j > n(i)$, set $\tilde{c}_j = 0$.

Since $\mathrm{P}(\tilde{\omega}) \subset \mathbf{R}$ is generated by $\tilde{\omega}(\sigma_j) = \tilde{c}_j$, $j \geqslant 1$, the above choices force $\mathrm{P}(\tilde{\omega})$ to be infinite cyclic. By the final remark in Section 1, we obtain the following.

(2.1) THEOREM. — *If* $\dim(\mathrm{M}) = 3$ *and if* $\mathrm{W}_0 \subset \hat{\mathrm{U}}$ *is open and relatively compact, then there exist Tischler foliations* $\mathscr{F}^*$ *of* $\hat{\mathrm{U}}$ *that are arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on* W_0.

By similar, but slightly more delicate choices of the cycles σ_j and the rational numbers $\tilde{c}_j$, we will prove the following.

(2.2) THEOREM. — *If* $\dim(\mathrm{M}) \geqslant 3$, *if* $\mathrm{W}_0 \subset \hat{\mathrm{U}}$ *is open and relatively compact, and if* $\mathrm{P}(\omega)$ *is free abelian, then there exist Tischler foliations* $\mathscr{F}^*$ *of* $\hat{\mathrm{U}}$ *that are arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on* W_0. *Furthermore,* $\mathscr{F}^*$ *can be chosen so that*

$$\mathrm{Ker}\,(\omega : \pi_1(\mathrm{U})\to\mathbf{R}) \subset \mathrm{Ker}\,(\tilde{\omega} : \pi_1(\mathrm{U})\to\mathbf{R}).$$

The final assertion in (2.2) will guarantee that the leaves of $\mathscr{F}|\mathrm{U}$ are regular coverings of the fibers of $\tilde{\mathscr{F}} = \mathscr{F}^*|\mathrm{U}$ in a natural way (3.8). The corresponding assertion is absent from (2.1) due to a wealth of counter-examples (3.9).

Proof of (2.2). — Since $\mathrm{P}(\omega)$ is free abelian, the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\omega) \longrightarrow \mathrm{H}_1(\mathrm{U};\mathbf{Z}) \xrightarrow{\ \omega\ } \mathrm{P}(\omega) \longrightarrow 0$$

can be split. Since $\mathrm{H}_1(\hat{\mathrm{U}};\mathbf{Z}) = \mathrm{H}_1(\mathrm{U};\mathbf{Z})$ canonically, we obtain

$$\mathrm{H}_1(\hat{\mathrm{U}};\mathbf{Z}) = \mathrm{Ker}(\omega) \oplus \mathrm{P}$$
$$\mathrm{H}_1(\hat{\mathrm{U}};\mathbf{R}) = (\mathrm{Ker}(\omega)\otimes\mathbf{R}) \oplus (\mathrm{P}\otimes\mathbf{R})$$

such that ω carries P one-one onto $\mathrm{P}(\omega)$.

Set $\omega_{(j)} = \omega|(K_j \cap U)$. The inclusions $K_j \subset K_{j+1} \subset \hat{U}$ induce commutative diagrams

$$\begin{tikzcd} \mathrm{Ker}(\omega_{(j)}) \arrow[r] \arrow[dr, "\gamma_j"'] & \mathrm{Ker}(\omega_{(j+1)}) \arrow[d, "\gamma_{j+1}"] \\ & \mathrm{Ker}(\omega) \end{tikzcd}$$

and $\mathrm{Ker}(\omega) = \bigcup_{j \geqslant 0} \mathrm{Im}(\gamma_j)$. Set $m(j) = \dim(\mathrm{Im}(\gamma_j) \otimes \mathbf{R})$ and choose integral cycles $\rho_1, \ldots, \rho_{m(0)}$ in $K_0 \cap U$ and $\rho_{m(j)+1}, \ldots, \rho_{m(j+1)}$ in $K_{j+1} \cap U$, $j \geqslant 0$, such that the classes $[\rho_1], [\rho_2], \ldots, [\rho_k], \ldots$ define a possibly infinite basis of $\mathrm{Ker}(\omega) \otimes \mathbf{R}$. We can choose the cycles $\sigma_1, \ldots, \sigma_{n(0)}$ (respectively, $\sigma_{n(j)+1}, \ldots, \sigma_{n(j+1)}$) of Section 1 so that $\rho_1, \ldots, \rho_{m(0)}$ (respectively, $\rho_{m(j)+1}, \ldots, \rho_{m(j+1)}$) are among them. Let $\tau_1, \ldots, \tau_{n(0)-m(0)}$ (respectively, $\tau_{n(j)-m(j)+1}, \ldots, \tau_{n(j+1)-m(j+1)}$) be the remaining σ_k's. Finally, let $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ be a possibly infinite basis of the free abelian summand P. One then has a possibly infinite integer matrix $(M_{jk})_{j,k \geqslant 1}$, each row of which has only finitely many nonzero entries, such that, in $H_1(\hat{U}; \mathbf{Z})$,

$$[\tau_j] = \sum_{k \geqslant 1} M_{jk} \alpha_k \bmod \mathrm{Ker}(\omega), \; j \geqslant 1.$$

The rows of this matrix are linearly independent over $\mathbf{R}$.

Since $\{\sigma_1, \sigma_2, \ldots\} = \{\rho_1, \rho_2, \ldots\} \cup \{\tau_1, \tau_2, \ldots\}$, we can define $p(j)$, $j \geqslant 1$, so that $\sigma_{p(j)} = \tau_j$. If $\sigma_p = \rho_k$, *then* $c_p = 0$ and we set $\tilde{c}_p = 0$. Fix K_i such that $W_0 \subset K_i$ and choose $\tilde{c}_{p(j)}$, $1 \leqslant j < n(i) - m(i)$, rational and as close as desired to $c_{p(j)}$. There exists $r \geqslant n(i) - m(i)$ such that

$$[\tau_j] = \sum_{k=1}^{r} M_{jk} \alpha_k \bmod \mathrm{Ker}(\omega), \qquad 1 \leqslant j \leqslant n(i) - m(i),$$

and there are (not necessarily unique) rational numbers d_k, $1 \leqslant k \leqslant r$, such that

$$\tilde{c}_{p(j)} = \sum_{k=1}^{r} M_{jk}\, d_k, \quad 1 \leqslant j \leqslant n(i) - m(i).$$

If $k > r$, set $d_k = 0$ and define rational numbers

$$\tilde{c}_{p(j)} = \sum_{k\geqslant 1} M_{jk}\, d_k, \quad j \geqslant 1.$$

This defines $\tilde{c}_p$ for all $p \geqslant 1$ and the corresponding 1-form

$$\tilde{\omega} = \sum_{p\geqslant 1} \tilde{c}_p(\omega_p | U) + dg$$

as in (1.3). Then

$$\hat{\omega} : H_1(\hat{U};\mathbf{Z}) \to \mathbf{R}$$

annihilates every $[\rho_j]$, hence $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\tilde{\omega})$. Furthermore, $\tilde{\omega}[\tau_j] = \tilde{c}_{p(j)}$. There is a unique cohomology class $[\gamma] \in H^1(\hat{U};\mathbf{R})$ that vanishes on $\mathrm{Ker}(\omega)$ and assigns to each α_k the rational number d_k. By the above, $[\gamma]$ assigns to each $[\tau_j]$ the number $\tilde{c}_{p(j)}$, so $[\gamma] = [\tilde{\omega}]$. Thus, $P(\tilde{\omega}) = P(\gamma)$ and this is generated by the finite set $\{d_1,\ldots,d_r\}$ of rational numbers, so $P(\tilde{\omega})$ is infinite cyclic.

Finally, since $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\tilde{\omega})$ at the level of homology, the corresponding inclusion holds at the level of homotopy. $\square$

3. The regular covering property.

Let L be a leaf of $\mathscr{F}|U$ and let F be a fiber of $\tilde{\mathscr{F}} = \mathscr{F}^*|U$. Fix a reference point $x_0 \in L$ and choose $t_0 \in \mathbf{R}$ such that $\Phi_{t_0}(x_0) \in F$. Consider

Condition (*). There exists a smooth function $\tau : L \to \mathbf{R}$ such that $\tau(x_0) = t_0$ and $\Phi_{\tau(x)}(x) \in F$, $\forall x \in L$.

If Condition (*) is satisfied, we will define $p : L \to F$ by $p(x) = \Phi_{\tau(x)}(x)$ and prove that this is a regular covering space with covering group $G \subset P(\mu) = P(\omega)$ such that $P(\mu) \cong G \oplus \mathbf{Z}$. Since one easily produces countably generated, additive subgroups $P \subset \mathbf{R}$ that do not admit $\mathbf{Z}$ as a direct summand, and since $P(\mu)$ can be any such subgroup [1, (5.5)], we cannot expect Condition (*) always to be satisfied.

(3.1) LEMMA. — *Condition* (*) *holds if and only if*

$$\mathrm{Ker}(\omega : \pi_1(\hat{U}) \to \mathbf{R}) \subset \mathrm{Ker}(\tilde{\omega} : \pi_1(\hat{U}) \to \mathbf{R}).$$

Furthermore, τ *is uniquely determined by* x_0 *and* t_0.

(3.2) Corollary. — *Condition* (*) *holds for one choice of initial conditions* L, F, x_0, t_0 *if and only if it holds for all such choices.*

By the final assertion in (2.2) we also have

(3.3) Corollary. — *If* $P(\omega)$ *is free abelian, then Tischler foliations can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on any preassigned precompact region, such that Condition* (*) *holds.*

Proof of (3.1). — Fix a leaf L of $\mathscr{F}$ and a basepoint $x_0 \in L$. Let σ be a piecewise smooth loop in U based at x_0. In standard fashion, using the transverse flow Φ_t, we deform σ to a loop at x_0 of the form $\sigma_1 + \sigma_2$, where σ_1 is a path in L and σ_2 lies along the flow line through x_0. Thus, $\omega(\sigma) = \int_{\sigma_2} \omega$ and this is zero if and only if σ_2 reduces to the single point x_0. Thus, the image of $i_* : \pi_1(L,x_0) \to \pi_1(\hat{U},x_0)$, where i is the inclusion, is exactly $\mathrm{Ker}(\omega)$. The condition that $\mathrm{Ker}(\omega) \subset \mathrm{Ker}(\tilde{\omega})$ becomes the condition that $\tilde{\omega}(\sigma) = 0$ for every piecewise smooth loop σ lying on L.

If Condition (*) holds, define $p_t : L \to \hat{U}$ by $p_t(x) = \Phi_{t\tau(x)}(x)$, $0 \leqslant t \leqslant 1$. This homotopy can be used to deform any 1-cycle σ on L to a 1-cycle $\tilde{\sigma}$ on F, all within U. Thus, $\tilde{\omega}(\sigma) = \tilde{\omega}(\tilde{\sigma}) = 0$.

Conversely, suppose $\tilde{\omega}(\sigma) = 0$ for each piecewise smooth loop σ on L. Fix t_0 so that $\Phi_{t_0}(x_0) \in F$. Given $x \in L$, choose a piecewise smooth path $\gamma : [0,1] \to L$, $\gamma(0) = x_0$ and $\gamma(1) = x$. We want to project γ smoothly along the leaves of $\mathscr{L}|U$ to $\tilde{\gamma} : [0,1] \to F$, $\tilde{\gamma}(0) = \Phi_{t_0}(x_0)$. More precisely, we want to define a piecewise smooth function $\tau_\gamma : [0,1] \to \mathbf{R}$, $\tau_\gamma(0) = t_0$, such that

$$\Phi_{\tau_\gamma(t)}(\gamma(t)) = \tilde{\gamma}(t) \in F, \qquad 0 \leqslant t \leqslant 1.$$

The mere fact that $\mathscr{L}|U$ is transverse to $\hat{\mathscr{F}} = \mathscr{F}^*|U$ does not guarantee that this is possible, but the additional fact that $\hat{\mathscr{F}}$ fibers U over S^1 makes it a straightforward exercise to prove the existence and uniqueness of τ_γ. If $\rho : [0,1] \to L$ also satisfies $\rho(0) = x_0$ and $\rho(1) = x$, then we claim that $\tau_\rho(1) = \tau_\gamma(1)$. Indeed, let $\lambda : [0,1] \to U$ be the curve (along a leaf of $\mathscr{L}|U$)

$$\lambda(t) = \Phi_{t\tau_\gamma(1)+(1-t)\tau_\rho(1)}(x).$$

Either this curve is constant (i.e., $\tau_\gamma(1) = \tau_\rho(1)$) or it is nonsingular and

$\int_\lambda \tilde{\omega} \neq 0$. The cycle $\tilde{\sigma} = \tilde{\rho} + \lambda + \tilde{\gamma}^{-1}$ is homologous in U to the cycle $\sigma = \rho + \gamma^{-1}$. Since σ is a cycle on L,

$$0 = \tilde{\omega}(\sigma) = \tilde{\omega}(\tilde{\sigma}) = \int_\lambda \tilde{\omega},$$

so λ is constant. Consequently, we can define $\tau(x) = \tau_\gamma(1)$ unambiguously, τ is smooth, and $\Phi_{\tau(x)}(x) = \tilde{\gamma}(1) \in \mathrm{F}$. Also, τ is unique since each τ_γ is unique. □

Assuming that Condition (*) holds, we fix the choices of L, F, and τ and we define $p : \mathrm{L} \to \mathrm{F}$ as above. Our candidate for the covering group $\mathrm{G} \subset \mathrm{P}(\mu)$ is as follows.

DEFINITION. – $\mathrm{G} = \{\tau(x_1) - \tau(x_2) | p(x_1) = p(x_2)\}$.

(3.4) LEMMA. – G *is a subgroup of* $\mathrm{P}(\mu)$ *and* $\mathrm{P}(\Phi_t(z)) = p(z)$, $\forall t \in \mathrm{G}$, $\forall z \in \mathrm{L}$.

Proof. – If $p(x_1) = p(x_2)$, then

$$\Phi_{\tau(x_1) - \tau(x_2)}(x_1) = \Phi_{-\tau(x_2)}(p(x_1)) = x_2 .$$

In particular, $\Phi_{\tau(x_1)-\tau(x_2)}(\mathrm{L}) = \mathrm{L}$, proving that $\mathrm{G} \subset \mathrm{P}(\mu)$.

Let $t = \tau(x_1) - \tau(x_2) \in \mathrm{G}$. Define $\bar{\tau} : \mathrm{L} \to \mathbf{R}$ by

$$\bar{\tau}(z) = \tau(\Phi_t(z)) + t .$$

Then $\bar{\tau}(x_1) = \tau(x_2) + t = \tau(x_1)$ and

$$\begin{aligned}\Phi_{\bar{\tau}(z)}(z) &= \Phi_{\tau(\Phi_t(z))}(\Phi_t(z)) \\ &= p(\Phi_t(z)) \in \mathrm{F} .\end{aligned}$$

By the uniqueness assertion in (3.1), $\bar{\tau} \equiv \tau$ and, in particular,

$p(z) = p(\Phi_t(z))$, $\forall t \in \mathrm{G}$, $\forall z \in \mathrm{L}$.

Evidently $0 \in \mathrm{G}$. Also, if $t \in \mathrm{G}$ then $-t \in \mathrm{G}$. Let $p(x_1) = p(x_2)$ and $p(y_1) = p(y_2)$. We must show that

$$(\tau(x_1) - \tau(x_2)) + (\tau(y_1) - \tau(y_2)) \in \mathrm{G} .$$

Let $u = \Phi_{\tau(y_1)-\tau(y_2)}(x_2)$. Then $p(u) = p(x_2)$. As above, for $z \in L$,

$$\tau(\Phi_{\tau(y_1)-\tau(y_2)}(z)) + \tau(y_1) - \tau(y_2) = \tau(z) = \tau(\Phi_{\tau(x_2)-\tau(u)}(z)) + \tau(x_2) - \tau(u).$$

By letting $z = x_2$, we obtain

$$\tau(u) + \tau(y_1) - \tau(y_2) = \tau(u) + \tau(x_2) - \tau(u),$$

hence

$$\tau(y_1) - \tau(y_2) = \tau(x_2) - \tau(u).$$

Consequently,

$$\tau(x_1) - \tau(x_2) + \tau(y_1) - \tau(y_2) = \tau(x_1) - \tau(u)$$

and this is an element of G. □

(3.5) LEMMA. — *For each* $y \in F$, *the natural action* $G \times L \to L$ *induces a simply transitive action of* G *on* $p^{-1}(y)$.

Proof. — Let $t \in G$ and $x \in L$, and suppose that $\Phi_t(x) = x$. Then, as in the proof of (3.4),

$$\tau(x) = \tau(\Phi_t(x)) + t = \tau(x) + t,$$

so $t = 0$. That is, G acts on L without fixed points. If $y_1, y_2 \in p^{-1}(y)$, then $\tau(y_1) - \tau(y_2) \in G$ and $\Phi_{\tau(y_1)-\tau(y_2)}(y_1) = y_2$. □

(3.6) PROPOSITION. — *The map* $p : L \to F$ *is a regular covering and* $G \subset P(\mu)$ *is the group of covering transformations.*

Proof. — A finite biregular cover of M relative to $(\mathscr{F}, \mathscr{L})$ (cf. [2, Section 1], [5]) defines a (generally infinite) biregular cover $\{W_\alpha\}_{\alpha \in A}$ of $\hat{U}$ relative to $(\hat{\mathscr{F}}, \hat{\mathscr{L}})$. Fix a biregular cover $\{V_\beta\}_{\beta \in B}$ of $\hat{U}$ for $(\mathscr{F}^*, \hat{\mathscr{L}})$ such that each $\overline{V_\beta}$ lies in some W_α. Given $y \in F$, $x \in p^{-1}(y) \subset L$, and a plaque P_y^* around y coming from a suitable V_β, there is a neighborhood P_x of x in L carried diffeomorphically by p onto P_y^*. Indeed, by a small deformation of $\overline{V_\beta}$ within a surrounding W_α, holding y fixed, we produce a compact biregular neighborhood for $(\hat{\mathscr{F}}, \hat{\mathscr{L}})$ meeting exactly the same local flow lines as $\overline{V_\beta}$. If P is an $\hat{\mathscr{F}}$-plaque of this biregular neighborhood, there is some $t \in \mathbf{R}$ such that $\Phi_t(P)$ has interior P_x as desired.

Let P_x and P_y^* be as above. Let $t \in G$ be such that

$$\Phi_t(P_x) \cap P_x \neq \varnothing .$$

Let $z_1, z_2 \in P_x$ such that $z_1 = \Phi_t(z_2)$. Then

$$p(z_1) = p(\Phi_t(z_2)) = p(z_2), \qquad \text{so} \qquad z_1 = z_2 .$$

By (3.5), $t = 0$. It follows that P_y^* is evenly covered by

$$p^{-1}(P_y^*) = \bigcup_{t \in G} \Phi_t(P_x). \qquad \square$$

(3.7) PROPOSITION. — *If* $G \subset P(\mu) = P(\omega)$ *is the group of covering transformations as above, then* $P(\omega) = G \oplus \mathbf{Z}$.

Proof. — Without loss of generality, we assume there is a basepoint $x_0 \in L \cap F$ such that $p(x_0) = x_0$. Indeed, given arbitrary $x_0 \in L$, we can, if necessary, replace (L, x_0) with $(\Phi_{\tau(x_0)}(L), \Phi_{\tau(x_0)}(x_0))$ and τ with $\tau \circ \Phi_{-\tau(x_0)} - \tau(x_0)$. This leaves the subgroup $G \subset P(\omega)$ unchanged.

Both $\mathscr{F}|U$ and $\hat{\mathscr{F}}$ are transversely complete e-foliations of U (cf. [4]). Thus the leaf inclusions induce monomorphisms of fundamental groups and we obtain exact sequences

$$0 \longrightarrow \pi_1(L, x_0) \longrightarrow \pi_1(\hat{U}, x_0) \xrightarrow{\ \omega\ } P(\omega) \longrightarrow 0$$

$$0 \longrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(\hat{U}, x_0) \xrightarrow{\ \tilde{\omega}\ } P(\tilde{\omega}) \longrightarrow 0 .$$

By the first of these, we identify $P(\omega)$ with $\pi_1(\hat{U}, x_0)/\pi_1(L, x_0)$. Since $p : L \to F$ is a regular covering and $p(x_0) = x_0$, we obtain a commutative diagram of inclusions

$$\begin{tikzcd} \pi_1(L, x_0) \arrow[r, hook] \arrow[d, hook, "p_*"'] & \pi_1(\hat{U}, x_0) \\ \pi_1(F, x_0) \arrow[ur, hook] & \end{tikzcd}$$

and

$$G = \pi_1(F, x_0)/\pi_1(L, x_0) \subset \pi_1(\hat{U}, x_0)/\pi_1(L, x_0) = P(\omega).$$

By (3.1), $\tilde{\omega}$ vanishes on $\pi_1(L,x_0)$, so the second of the above sequences yields an exact sequence

$$0 \longrightarrow G \longrightarrow P(\omega) \xrightarrow{\tilde{\omega}} P(\tilde{\omega}) \longrightarrow 0.$$

But $P(\tilde{\omega}) \cong \mathbf{Z}$ and this sequence splits. □

Combining (3.3), (3.6), and (3.7), we obtain

(3.8) THEOREM. – *If* $P(\omega)$ *is free abelian, then Tischler foliations* $\mathscr{F}^*$ *can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on any preassigned precompact region, such that there is a natural regular covering map* $p : L \to F$, L *a leaf of* $\mathscr{F}|U$ *and* F *a fiber of* $\hat{\mathscr{F}} = \mathscr{F}^*|U$, *with covering group* G *a direct summand:* $P(\omega) \cong G \oplus \mathbf{Z}$.

If $P \subset \mathbf{R}$ is a countably generated, additive subgroup, an element $a \in P$, $a \neq 0$, will be called *infinitely divisible* if, for suitable, arbitrarily large integers m, one can find $b_m \in P$ such that $mb_m = a$. The group P contains an infinitely divisible element if and only if P is not free abelian (cf. [7], Theorem 19.1, page 93).

(3.9) PROPOSITION. – *If* $\dim(M) = 3$ *and* $P \subset \mathbf{R}$ *is a countably generated, additive subgroup that is not free abelian, then* M *admits a transversely orientable* C^∞*-foliation* $\mathscr{F}$ *with* $U \subset M$ *as usual such that* $P(\omega) = P$ *and such that no choice of Tischler foliation* $\mathscr{F}^*$ *satisfies Condition* (*).

Proof. – Exactly as in [1, (5.5)], construct $\mathscr{F}$ such that $\mathscr{F}|U$ has dense leaves without holonomy and such that $P(\omega) = P$. In choosing the representation $\omega = \Sigma c_j(\omega_j|U) + dg$ of Section 1, it is easy to arrange that c_1 be an infinitely divisible element of $P(\omega)$. In fact, we can arrange that $c_1 = mc_j$, for suitable arbitrarily large integers m and suitable $j > 1$. Furthermore, since $c_1 \neq 0$, we can choose the integral cycle σ_1 (such that $\omega_j(\sigma_1) = \delta_{j1}, j \geqslant 1$) to be a closed transversal to $\mathscr{F}|U$. By performing the standard modification of $\mathscr{F}$ along σ_1, introducing a Reeb component with σ_1 as core transversal, we change U so that $\partial\hat{U}$ has one new component, a torus. The new foliation $\mathscr{F}|U$ has the same properties, including the same period group $P(\omega)$, as before. Perturb σ_1 so that it lies in U near the toral boundary component and is transverse to $\mathscr{F}|U$. Let σ_0 also lie in U near the toral boundary, a perturbed meridian circle relative to the Reeb component and lying on a leaf of $\mathscr{F}|U$. Thus,

$\omega(\sigma_0) = 0$. The new system of basic cycles is either unchanged or it is obtained by adjoining σ_0 to $\{\sigma_1,\sigma_2,\dots\}$, in which case $c_0 = 0$.

Suppose there is a choice of $\mathscr{F}^*$ so that Condition (*) holds. By (3.1), $\tilde{\omega}(\sigma_0) = 0$. Since $\hat{\mathscr{F}}$ fibers U over S^1, $\mathscr{F}^*$ cannot be a product foliation near the new toral component of $\partial\hat{U}$. Thus, $\tilde{\omega}$ is not exact near this torus and it follows that $\tilde{c}_1 = \tilde{\omega}(\sigma_1) \neq 0$. For suitable, arbitrarily large integers m and $j > 1$, we have $\omega(\sigma_1 - m\sigma_j) = 0$, hence $\tilde{\omega}(\sigma_1 - m\sigma_j) = 0$ by (3.1). That is, in $P(\tilde{\omega})$ there are elements $\tilde{c}_1 = \tilde{\omega}(\sigma_1) \neq 0$ and $\tilde{c}_j = \tilde{\omega}(\sigma_j)$ such that $m\tilde{c}_j = \tilde{c}_1$. This contradicts the fact that $P(\tilde{\omega})$ is infinite cyclic. □

Returning to the positive result (3.8), we describe a fairly general situation in which that result applies.

DEFINITION. — *Let* $U \subset M$ *be as usual. If the nucleus* $K \subset \hat{U}$ *can be chosen so that, in each arm* $\hat{U}_j \cong B_j \times [-1,1]$, $\hat{\mathscr{F}}$ *restricts to the product foliation by leaves* $B_j \times \{t\}$, *then* $\hat{\mathscr{F}}$ *is said to be almost trivial.*

(3.10) PROPOSITION. — *The foliation* $\hat{\mathscr{F}}$ *is almost trivial in each of the following cases:*

(*a*) $\hat{U}$ *is compact;*

(*b*) $\mathscr{F}$ *is of class at least* C^2 *and each leaf of* $\mathscr{F}|U$ *has two dense ends;*

(*c*) $\mathscr{F}$ *is transversely analytic.*

Indeed, case (*a*) is vacuously true and, under the additional hypothesis that $\bar{U} - U$ is a union of proper leaves, case (*b*) was proven in [1, (6.9)] and, under the same hypothesis, case (*c*) was pointed out in that same reference. The additional hypothesis can be avoided by using a result of G. Duminy [6] on the structure of semi-proper, exceptional leaves.

(3.11) THEOREM. — *If* $\hat{\mathscr{F}}$ *is almost trivial, then Tischler foliations* $\mathscr{F}^*$ *can be chosen, arbitrarily* C^∞*-close to* $\hat{\mathscr{F}}$ *on any preassigned, precompact region, such that there is a natural regular covering* $p: L \to F$ *with* covering group $G \cong \mathbf{Z}^k$, *some integer* $k \geqslant 1$.

Proof. — If σ is an integral 1-cycle contained in an arm $\hat{U}_j$, then $\omega(\sigma) = 0$. Thus, $P(\omega)$ is the finitely generated image of $\omega: H_1(K;\mathbf{Z}) \to \mathbf{R}$ and (3.8) applies. □

Remarks. — (1) In case (*a*) of (3.10), if $\partial\hat{U} = \varnothing$ (i.e., $U = M$), then a famous result of H. Hopf [9], together with (3.11), implies that each leaf of

$\mathscr{F}(=\mathscr{F}|U)$ has the same number of ends as does the covering group $G \cong \mathbf{Z}^k$. This number is two if $k = 1$, and it is one if $k > 1$. The fact that the number of ends is either one or two is also a consequence of [3, Proposition 1], in which it is shown that, generally (whether or not Tischler foliations exist), each leaf of $\mathscr{F}|U$ has either one dense end or two such ends. The proof is similar to Hopf's proof, so one might expect to show that, at least when $G \cong \mathbf{Z}^k$, the number of dense ends is the same as the number of ends of G. This often fails, however, even when $\hat{U}$ is compact. For instance, let $\hat{U} \cong S^1 \times S^1 \times [-1,1]$, the leaves of $\mathscr{F}|U$ being dense planes. These leaves have one dense end, the Tischler fibers are cylinders $S^1 \times \mathbf{R}$, and the covering $p : \mathbf{R}^2 \to S^1 \times \mathbf{R}$ has covering group $G \cong \mathbf{Z}$.

(2) In case (*b*) of (3.10), if we assume only that $\mathscr{F}$ is of class C^{0+}, we can apply the argument in [1, Section 6] to show that $P(\mu) \cong \mathbf{Z} \times \mathbf{Z}$. Thus, (3.8) applies to the case of two dense ends without the smoothness hypothesis. In this case, $G \cong \mathbf{Z}$.

(3) It is natural to ask whether the covering map $p : L \to F$, when it exists, respects the growth types of L and F, at least when $G \cong \mathbf{Z}^k$. That is, if g_L, $g_F : \mathbf{Z}^+ \to \mathbf{R}^+$ are growth functions for L and F respectively, and if $G \cong \mathbf{Z}^k$, do $g_L(m)$ and $m^k g_F(m)$ have the same growth type ? If $\hat{\mathscr{F}}$ is almost trivial, the answer is « yes », as is easily deduced from [1, (2.8) and (6.10)]. In general, however, the answer is « no », as the constructive proof of [1, (5.5)] clearly implies.

4. An example.

Without some condition on $P(\omega)$, Tischler foliations do not generally exist. Here we show how to construct an appropiate example in which $\dim(M)$ can be an arbitrary integer greater than three. By (2.1), such examples are impossible when $\dim(M) = 3$. In our example, $P(\omega)$ will be the dyadic rationals $\mathbf{Z}[1/2]$. The method of construction may be of some independent interest.

(A) *Generalized Reeb components.* — Let L be an open, connected manifold of dimension $n-1$, $n \geqslant 3$. Suppose that there is a decomposition

$$L = A \cup B_1 \cup B_2 \cup \cdots \cup B_k \cup \cdots$$

such that

(1) A is a compact, connected, $(n-1)$-dimensional manifold with ∂A connected;

(2) $B_i \cong B_{i+1}$, $i \geqslant 1$, and B_i is a compact, connected, $(n-1)$-dimensional manifold such that ∂B_i has two components, $\partial_+ B_i$ and $\partial_- B_i$;

(3) $A \cap B_1 = \partial A = \partial_- B_1$ and $A \cap B_i = \emptyset$, $i > 1$;

(4) $B_i \cap B_{i+1} = \partial_+ B_i = \partial_- B_{i+1}$, $i \geqslant 1$, and $B_i \cap B_{i+k} = \emptyset$, $i \geqslant 1$, $k \geqslant 2$;

(5) there is a diffeomorphism γ of L onto itself such that $\gamma(A \cup B_1) = A$ and $\gamma(B_{i+1}) = B_i$, $i \geqslant 1$.

Example. – Let $L = \mathbf{R}^2$, let $A = \{v \in \mathbf{R}^2 : \|v\| \leqslant 2\}$, and let

$$B_i = \{v \in \mathbf{R}^2 : 2^i \leqslant \|v\| \leqslant 2^{i+1}\}, \quad i \geqslant 1.$$

Finally, let $\gamma(v) = v/2$.

Under these circumstances, we have a proper nest of compact sets

$$A \supset \gamma(A) \supset \gamma^2(A) \supset \cdots \supset \gamma^k(A) \supset \cdots.$$

The intersection of these sets is a compact, nonempty, γ-invariant set K and γ is a contraction of L to K. In the above example, $K = \{0\}$. In all cases, γ generates a properly discontinuous action of $\mathbf{Z}$ on $L - K$ and $(L-K)/\mathbf{Z}$ is a closed, connected, $(n-1)$-dimensional manifold T. Indeed, T is obtained from B_i by identifying $\partial_+ B_i$ to $\partial_- B_i$ via γ.

Let $I = [0,1]$ and let $h : I \to I$ be a diffeomorphism (into) such that $h(0) = 0$ and $h(t) < t$, $0 < t \leqslant 1$. Thus, h is a contraction to 0. We also assume that h is C^∞-tangent to the identity at $t = 0$.

Let $\varphi : L \times I \to L \times I$ be the diffeomorphism (into) defined by

$$\varphi(x,t) = (\gamma(x), h(t)).$$

Then φ contracts $L \times I$ to $K \times \{0\}$. Let $X = (L \times I) - (K \times \{0\})$. Then X is an n-manifold with boundary and $\varphi : X \to X$ has no fixed points. Indeed, $\{\varphi^k\}_{k \geqslant 0} = \mathbf{Z}^+$ is a properly discontinuous semigroup of diffeomorphisms of X *into* itself. The boundary component $\partial_0 X = (L \times \{0\}) - (K \times \{0\})$ is invariant under this semigroup. The quotient $Y = X/\mathbf{Z}^+$ is an n-manifold with one boundary component,

$$\partial Y = \partial_0 X/\mathbf{Z}^+ \cong (L-K)/\mathbf{Z} = T.$$

The quotient map $X \to Y$ carries $A \times [h(1),1] \cup B_1 \times [0,1]$ onto Y, hence Y is compact. Finally, the foliation of X by leaves $L \times \{t\}$, $0 < t \leqslant 1$, together with the leaf $\partial_0 X$, is invariant under this semigroup and passes to a C^∞ foliation of Y with $\partial Y \cong T$ as one leaf and all other leaves diffeomorphic to L. The noncompact leaves wind in on ∂Y in a very regular way. Indeed, these leaves each have one end and that end is periodic of period ∂Y, in the sense of [2, (6.1)].

Since h is assumed to be C^∞-tangent to the identity at $t = 0$, it follows that the above foliation is C^∞-trivial at ∂Y. Thus, the double of Y yields a closed, C^∞-foliated n-manifold M having exactly one compact leaf, all other leaves being diffeomorphic to L.

Example. – Applying our construction to $L = \mathbf{R}^2$, $\gamma(v) = v/2$, we obtain the Reeb-foliated solid torus with double the standard Reeb foliation of $S^1 \times S^2$.

We call Y, together with the above foliation, a generalized Reeb component. The doubling construction shows that generalized Reeb components do appear as components in C^∞ foliations of suitable closed n-manifolds M.

(B) *A special example.* – Here, we require that $n \geqslant 4$. Let D denote the closed unit disk in $\mathbf{R}^{n-2}$ and let $R = S^1 \times D = \{(\theta,x)\}$, where θ is well defined mod 2π. Choose a smooth map $i : S^1 \times D \to D$ such that, for each θ, $i_\theta : D \to D$ is an imbedding into $\operatorname{int}(D)$ and $i_\theta(D) \cap i_{\theta+\pi}(D) = \varnothing$. It is here that the condition $n \geqslant 4$ is needed (Borsuk-Ulam). Finally, define

$$\psi : R \longrightarrow R$$
$$\psi(\theta,x) = (2\theta, i_\theta(x)).$$

Thus, ψ imbeds R into $\operatorname{int}(R)$ as indicated in figure 2.

Let s denote the successor function, $s(i) = i + 1$, and consider the sequence of imbeddings

$$R \times \{0\} \xrightarrow{\psi \times s} R \times \{1\} \xrightarrow{\psi \times s} \cdots \longrightarrow R \times \{i\} \longrightarrow \cdots.$$

Let L be the $(n-1)$-manifold obtained by passing to the direct limit of this sequence and consider the natural imbeddings $R \times \{i\} \to L$. Let A be the imbedded $R \times \{0\}$ and define B_i inductively by letting

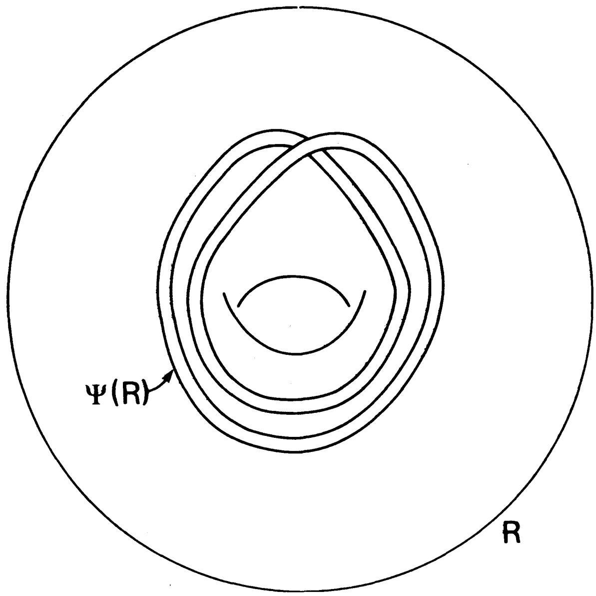

Fig. 2.

$A \cup B_1 \cup \cdots \cup B_i$ be the imbedded $R \times \{i\}$. Finally, define the diffeomorphism $\gamma : L \to L$ via the commutative diagram

$$\begin{tikzcd}
R \times \{0\} \arrow[r, "\psi \times s"] \arrow[d, "\psi \times id"'] & R \times \{1\} \arrow[r, "\psi \times s"] \arrow[dl, "id \times s^{-1}"] & R \times \{2\} \arrow[r] \arrow[dl, "id \times s^{-1}"] & \cdots \\
R \times \{0\} \arrow[r, "\psi \times s"] & R \times \{1\} \arrow[r] & \cdots &
\end{tikzcd}$$

It is elementary to check the hypotheses (1) through (5) of (A).

For use in (C), remark that the sequence of fundamental groups

$$\pi_1(R \times \{0\}) \longrightarrow \pi_1(R \times \{1\} \longrightarrow \cdots$$

is exactly

$$\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \cdots$$

hence $\pi_1(L) = H_1(L) = \mathbf{Z}[1/2]$.

(C) *The promised example.* – In the generalized Reeb component of (B), we modify the foliation so that the compact leaf ∂Y remains a leaf, as does the diffeomorphic image in Y of $L \times \{1\}$, but the remainder of the foliation consists of dense leaves without holonomy. Then U will be the diffeomorphic image of $L \times (h(1),1)$ under the quotient map $X \to Y$. Since $\pi_1(U) = \pi_1(L) = \mathbf{Z}[1/2]$, there exists no fibration of U by connected manifolds over S^1. Doubling Y will complete our example.

Let $d\theta \in A^1(R)$ be the closed, nonsingular form pieced together out of the exterior derivatives of the branches of θ. Evidently, $\psi^*(d\theta) = 2\,d\theta$, so we obtain a closed, nonsingular form $\pi \in A^1(L)$ that « restricts » to $2^{-i}\,d\theta$ on $R \times \{i\}$, $i \geqslant 0$. The following is a direct computation.

(4.1) LEMMA. — *The form* $\eta \in A^1(L)$ *satisfies* $\gamma^*(\eta) = 2\eta$ *and* $P(\eta) = \mathbf{Z}[1/2]$.

Define the contraction $h : I \to I$ so that it imbeds in a flow. More precisely, let $f : I \to \mathbf{R}$ be a smooth map, C^∞-tangent to 0 at $t = 0$, such that $f(t) < 0$, $0 < t \leqslant 1$, let $h_u(t)$ be the local flow on I generated by the vector field $f(t)\dfrac{d}{dt}$ (always defined on all of I for $u \geqslant 0$), and set $h = h_1$. The following is standard.

(4.2) LEMMA. — $h^*(dt/f) = dt/f$ *on* $(0,1]$.

Let $J = [h(1), 1]$ and let $g_0 : J \to \mathbf{R}$ be C^∞ and C^∞-tangent to 0 at the endpoints, $g_0 | \operatorname{int}(J)$ strictly positive. Let $g_k : h^k(J) \to \mathbf{R}$ be given by

$$g_k(h^k(t)) = 2^{-k} g_0(t), \qquad k \in \mathbf{Z}^+, \qquad t \in J.$$

Finally, define $g : I \to \mathbf{R}$ by

$$g | h^k(J) = g_k$$
$$g(0) = 0.$$

(4.3) LEMMA. — *The function* g *is continuous,* $g|(0,1]$ *is* C^∞ *and* C^∞-*tangent to* 0 *at* $h^k(1)$, $k \geqslant 0$, *and* $h^*(g) = g/2$. *For an appropriate choice of the vector field* $f(t)\dfrac{d}{dt}$, *the function* g *is also* C^∞ *at* $t = 0$ *and* C^∞-*tangent to* 0 *there.*

Proof. — Every assertion is trivial except those concerning the behavior of g at $t = 0$. For each real number $u \geqslant 0$, define $g_u : h_u(J) \to \mathbf{R}$ by $g_u(h_u(t)) = 2^{-u} g_0(t)$. When $u = k \in \mathbf{Z}^+$, this definition agrees with that of

g_k. We want to assure that, for each integer $n \geqslant 1$,

$$\lim_{u \to \infty} g_u^{(n)}(h_u(t)) = 0,$$

uniformly for $t \in \mathrm{J}$.

Inductively, on $\mathrm{J} \times [0,\infty)$ define

$$\begin{aligned} Q_1(t,u) &= g_0'(t)f(t) \\ Q_{n+1}(t,u) &= Q_n'(t,u)f(t) - nf'(h_u(t))Q_n(t,u) \end{aligned}$$

where Q_n' denotes the derivative with respect to t. Since

$$h_u^*(dt/f) = dt/f, \qquad \forall u \geqslant 0,$$

we have

$$h_u'(t) = f(h_u(t))/f(t), \quad t \in \mathrm{J}.$$

With the aid of this formula, one verifies

$$(*) \qquad g_u^{(n)}(h_u(t)) = Q_n(t,u)/2^u(f(h_u(t)))^n$$

by induction on $n \geqslant 1$.

If $Q_n^{(k)}(t,u)$ denotes the k^{th} derivative of Q_n with respect to t, then an elementary induction on n shows that $Q_n^{(k)}(t,u)$ is uniformly bounded on $\mathrm{J} \times [0,\infty)$ for each fixed integer $k \geqslant 0$. In particular, $Q_n(t,u)$ is so bounded. Thus, by (*), we must choose f so that $|2^u(f(h_u(t)))^n|$ becomes arbitrarily large, uniformly for $t \in \mathrm{J}$, as $u \to \infty$, for each integer $n \geqslant 1$. This is easily arranged. For example,

$$f(t) = \begin{cases} -\,t^2 e^{-1/t}, & 0 < t \leqslant 1, \\ 0, & t = 0 \end{cases}$$

generates the flow

$$h_u(t) = \begin{cases} (\log(u + e^{1/t}))^{-1}, & 0 < t \leqslant 1 \\ 0, & t = 0 \end{cases}$$

hence

$$|2^u(f(h_u(t)))^n| = 2^u(u + e^{1/t})^{-n}(\log(u + e^{1/t}))^{-2n}. \qquad \square$$

On $\mathrm{L} \times \mathrm{I}$, consider the smooth, nonsingular 1-form $\alpha = fg\eta + dt$.

We also denote by α the restriction of this form to X. Let $U = L \times (h(1),1)$.

(4.4) LEMMA. — *The form* α *is completely integrable and the associated foliation* $\mathscr{H}$ *of* $L \times I$ *is transverse to the intervals* $\{x\} \times I$. *The foliation* $\mathscr{H}|X$ *has the following properties :*

(*a*) $\partial_0 X$ *and* $L \times \{h^k(1)\}$ *are leaves,* $k \in \mathbf{Z}^+$, *and* $\mathscr{H}|X$ is C^∞*-trivial at these leaves;*

(*b*) $\varphi^*(\mathscr{H}|X) = \mathscr{H}|X$;

(*c*) $\mathscr{H}|U$ *is defined by a closed, transversely complete, nonsingular* 1*-form* ω *such that* $P(\omega) = \mathbf{Z}[1/2]$.

Proof. — Since η is closed, $d\alpha = \alpha \wedge (fg)'\eta$, so α is completely integrable. Also, $\alpha(\partial/\partial t) \equiv 1$, so $\mathscr{H}$ is transverse to the interval fibers. Since g is C^∞-tangent to 0 at $t = 0$ and at $t = h^k(1)$, $k \in \mathbf{Z}^+$, (*a*) follows. On $X - \partial_0 X$, $\mathscr{H}$ is also defined by $\alpha/f = g\eta + dt/f$. By (4.1), (4.2), and (4.3), $\varphi^*(\alpha/f) = \alpha/f$. Since $\varphi(\partial_0 X) = \partial_0 X$, (*b*) follows. Finally, $\mathscr{H}|U$ is defined by the closed form $\omega = \eta + dt/fg$. To say that ω is transversely complete means that there is a complete vector field v on U such that $\omega(v) \equiv 1$ (equivalently, $\mathscr{H}|U$ is a transversely complete *e*-foliation in the sense of [4]). The vector field $v = fg\, \partial/\partial t$ satisfies this. For any piecewise smooth 1-cycle σ in U, $\int_\sigma \eta = \int_\sigma \omega$. Thus, $P(\omega) = P(\eta)$ and (*c*) follows from (4.1). □

By part (*b*) of (4.4), $\mathscr{H}|X$ passes to a C^∞ foliation $\mathscr{F}$ of Y. The quotient map imbeds U as an open, $\mathscr{F}$-saturated subset of Y and $\mathscr{F}|U = \mathscr{H}|U$. By parts (*a*) and (*c*) of (4.4), $\hat{\mathscr{F}}$ has all of the properties that we have been assuming in this paper. Also, α has contact of infinite order with dt along $\partial_0 X$, so $\mathscr{F}$ is C^∞-trivial at ∂Y and we can pass to the double M of Y, with the doubled foliation also being denoted by $\mathscr{F}$. As earlier remarked, U does not fiber over S^1 with connected fibers, so we have proven the following.

(4.5) THEOREM. — *For each integer* $n \geqslant 4$, *there exists a closed, orientable n-manifold* M *with a transversely orientable,* C^∞ *foliation* $\mathscr{F}$ *of codimension one and an open, connected,* $\mathscr{F}$*-saturated set* U *of locally dense leaves without holonomy, such that* $\hat{U}$ *admits no associated Tischler foliation.*

Remarks. — (1) One can show that the leaves of $\mathscr{F}|U$ are diffeomorphic to $\mathbf{R}^{n-1}$ and have exponential growth.

(2) Although the product foliation of $\hat{U} \cong L \times [h(1), 1]$ does fiber U over $(h(1), 1) \cong \mathbf{R}$, a simple foliated surgery along a closed transversal to $\mathscr{F}|U$ will alter the example so that the new manifold $\hat{U}$ admits no foliation, tangent to $\partial\hat{U}$, that fibers U over a 1-manifold.

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