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A THEOREM ON WEAK TYPE ESTIMATES FOR RIESZ TRANSFORMS AND MARTINGALE TRANSFORMS

by Nicolas Th. VAROPOULOS

1. Riesz transforms.

Let $\mu \in \mathrm{M}(\mathbf{R}^n)$ be a bounded Radon measure on $\mathbf{R}^n$ and let $d\mu = fdx + d\nu$, $f \in \mathrm{L}^1(\mathbf{R}^n)$ and ν singular, be its Lebesgue decomposition. Let us further denote by $u(x, y) = u_0(x, y)$ $x \in \mathbf{R}^n$, $y > 0$ the Poisson integral of μ on the upper half space, by $u_1(x, y), \ldots, u_n(x, y)$ the Riesz conjugate system of u_0, and by $\mathrm{R}_j\mu(x)$ $(x \in \mathbf{R}^n)$ $j = 1, \ldots, n$ the Riesz transforms of μ. It is well known then that there exists C a constant depending only on the dimension n such that

$$\mathrm{N}(\lambda) = \lambda\, m\left[x\,;\sum_{j=1}^{n} |\mathrm{R}_j\mu(x)|^2 > \lambda^2\right] \leqslant \mathrm{C}\,\|\mu\|$$

where m denotes Lebesgue measure (cf. [1] Ch. 1). In this note I shall prove the following

THEOREM. – *There exists a numerical constant $k > 0$ only depending on the dimension n such that*

$$\varliminf_{\lambda\to\infty} \mathrm{N}(\lambda) \geqslant k\,\|\nu\|\,.$$

When $n = 1$ a stronger version of the above theorem is due to P. Jones (unpublished).

THEOREM (P. Jones). – *When $n = 1$ and μ, ν and $\mathrm{N}(\lambda)$ are as above we have:*

$$\lim_{\lambda\to\infty} \mathrm{N}(\lambda) = \frac{2}{\pi}\,\|\nu\|\,.$$

A weaker version of P. Jones's theorem is due to Cereteli [2] (cf. [2], he assumed that $\lim_{\lambda\to\infty} N(\lambda) = 0)$. Let us denote by

$$u_k^*(\xi) = \sup\{|u_k(x, y)| ; (x, y) \in T_\alpha(\xi)\} \ \xi \in \mathbf{R}^n$$

the non-tangential maximal function of $u_k (k = 0, 1, \dots, n)$ where $T_\alpha(\xi)$ is the standard conical region in $\mathbf{R}_+^{n+1}$ with vertex at $\xi \in \mathbf{R}^n$ vertical axis and opening α. Our theorem above contains then the following theorem of R.F. Gundy [3].

THEOREM (R.F. Gundy). – *Let $\mu \in M(\mathbf{R}^n)$ be as above and let us suppose that $m[u_k^* > \lambda] = o(1/\lambda)$ for $k = 1, 2, \dots, n$. Then ν, the singular part of μ vanishes.*

R.F. Gundy actually stated the above theorem when $n = 1$ but his proof can easily be adapted to any dimension.

The proof of our Theorem will need the following Lemma which already appeared in a weaker form in [3].

LEMMA. – *Let μ, ν and $u = u_0$ be as above and let*

$$u^*(x) = \sup_{y>0} |u(x, y)| .$$

We then have: $\varliminf_{\lambda\to\infty} \lambda\, m[u^* > \lambda] \geqslant c\,\|\nu\|$ where $c > 0$ is a numerical constant depending only on the dimension.

The proof is easy and for completeness I shall outline it:

Proof. – By an easy reduction we can suppose that $\mu = \nu > 0$ and that $\operatorname{supp} \mu$ is contained in the unit cube. We can then define the diadic maximal function of μ by

$$u^{\#}(x) = \sup \left\{ \frac{1}{|I|} \int_I d\mu \, ; \, x \in I, \ I \text{ closed diadic cube} \right\} .$$

By the positivity of μ it then follows that

$$u^*(x) \geqslant A u^{\#}(x) \tag{1.1}$$

where $A > 0$ is numerical depending only on the dimension.

Let now $\lambda > 0$ be fixed but large enough and let us apply the usual Calderon-Zygmund argument (cf. [1] Ch. 1) on the unit cube with respect to μ at the level $\lambda > 0$. We obtain then a disjoint family of closed diadic cubes $I_1, I_2, \dots$ (i.e. disjoint interiors) such that

$$[u^{\#}(x) \geqslant \lambda] = \bigcup_{j=1}^{\infty} \mathrm{I}_j \; ; \; \lambda \leqslant \frac{1}{|\mathrm{I}_j|} \int_{\mathrm{I}_j} d\mu < 2^n\lambda \, , \; j = 1, 2, \ldots$$

It follows in particular that

$$\|\nu\| = \nu\Big(\bigcup_{j=1}^{\infty} \mathrm{I}_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} \mathrm{I}_j\Big) \leqslant 2^n\lambda m\Big(\bigcup_{j=1}^{\infty} \mathrm{I}_j\Big) = 2^n\lambda m[u^{\#}(x) \geqslant \lambda] \, .$$

From this and (1.1) the Lemma follows.

Proof of the Theorem. – Let $\mu, f, \nu, u = u_0, u_1, \ldots$ be as above and let us suppose, as we clearly may, that μ is compactly supported and that $\|\nu\| = 1$. Let us also fix α s.t. $\dfrac{n-1}{n} < \alpha < 1$ (n is the dimension) and define :

$$\mathrm{F} = (|u_0|^2 + \ldots + |u_n|^2)^{\alpha/2}$$

$$\mathrm{F}_0 = (|f|^2 + |\mathrm{R}_1\mu|^2 + \ldots + |\mathrm{R}_n\mu|^2)^{\alpha/2} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n) \, .$$

We clearly have $\lim\limits_{y \to 0} \mathrm{F}(x, y) = \mathrm{F}_0(x)$ for a.a. $x \in \mathbf{R}^n$ and $|\mathrm{F}_0(x)| = 0(|x|^{-n\alpha})$ as $x \longrightarrow \infty$. We also have

$$\mathrm{F} \leqslant \Phi = \mathrm{P.I.F}_0 \tag{1.2}$$

where P.I. denotes the Poisson integral of the function F_0. The above inequality follows by harmonic majorization if we observe that:

(a) F is a subharmonic function in $\mathbf{R}^{n+1}_+$ (cf. [1] Ch. VII)

(b) F satisfies the following boundedness conditions :

$$\sup_{x \in \mathbf{R}^n ; \, y > y_0} |\mathrm{F}(x, y)| < +\infty \quad \forall y_0 > 0$$

$$\sup_{|x| > \mathrm{A}, \, y > 0} |\mathrm{F}(x, y)| < +\infty \quad \text{for some } \mathrm{A} > 0$$

$$\sup_{y > 0} \int_{|x| < \mathrm{A}} |\mathrm{F}(x, y)|^q dx < \infty \quad \forall \mathrm{A} > 0 \quad 1 < q < 1/\alpha \, .$$

We clearly have also:

$$\overline{\lim_{\lambda \to \infty}} \lambda^p m[\mathrm{F}_0 > \lambda] = \overline{\lim_{\lambda \to \infty}} \mathrm{N}(\lambda) \leqslant c\|\nu\| = c \quad (p = 1/\alpha) \tag{1.3}$$

where c is numerical only depending on the dimension.

Let us now denote by MF_0 the Hardy-Littlewood maximal function of F_0 (cf. [1] Ch. I) and for every $\lambda > 0$ let us consider the decomposition:

$$F_0 = F_\lambda + F^\lambda = F_0 \chi_{[F_0 < \lambda/2]} + F_0 \chi_{[F_0 \geqslant \lambda/2]} \cdot$$

We clearly have then $[MF_0 > \lambda] \subset [MF^\lambda > \lambda/2]$ which implies by the weak-L^1 inequality that

$$m[MF_0 > \lambda] \leqslant \frac{c}{\lambda} \|F^\lambda\|_1 \quad \lambda > 0 \tag{1.4}$$

where c only depends on the dimension. To estimate $\|F^\lambda\|_1$ let us denote by $m(\lambda) = m[F_0 > \lambda]$ we then have:

$$\begin{aligned}\|F^\lambda\|_1 &= \lambda/2 m(\lambda/2) + \int_{\lambda/2}^{\infty} m(t)\,dt \\ &= \lambda/2 m(\lambda/2) + \int_{\lambda/2}^{a\lambda/2} m(t)\,dt + \int_{a\lambda/2}^{\infty} m(t)\,dt \\ &\leqslant \frac{a\lambda}{2} m(\lambda/2) + \int_{a\lambda/2}^{\infty} m(t)\,dt\end{aligned}$$

for all $a > 1$ because $m(t)$ is a decreasing function. This by (1.3) implies that for λ large enough we have:

$$\|F^\lambda\|_1 \leqslant \frac{a\lambda}{2} m(\lambda/2) + \frac{2C}{p-1}\left(\frac{a\lambda}{2}\right)^{1-p} \quad (p = 1/\alpha)$$

which together with (1.4) gives that

$$\lambda^p m[MF_0 > \lambda] \leqslant C\left[\frac{a^{1-p}}{p-1} + a\left(\frac{\lambda}{2}\right)^p m\left(\frac{\lambda}{2}\right)\right]$$

for all $a > 1$, where C again only depends on the dimension. The conclusion is that:

$$\varlimsup_{\lambda\to\infty} \lambda^p m[MF_0 > \lambda] \leqslant C\left[\frac{a^{1-p}}{p-1} + a \varlimsup_{\lambda\to\infty} N(\lambda)\right]$$

and if we suppose that $\Lambda = \varlimsup_{\lambda\to\infty} N(\lambda) < 1$ and set $a = \Lambda^{-1/p}$ we obtain that

$$\varlimsup_{\lambda\to\infty} \lambda^p m[MF_0 > \lambda] \leqslant \frac{Cp}{p-1}\left(\varlimsup_{\lambda\to\infty} N(\lambda)\right)^{1-1/p} \tag{1.5}$$

(1.2) now implies that:

$$(u^*(x))^\alpha = (\sup_{y>0} |u(x,y)|)^\alpha \leqslant \sup_{y>0} F(x,y) \leqslant \sup_{y>0} \Phi(x,y) \leqslant MF_0(x).$$

This together with (1.5) gives:

$$\varlimsup_{\lambda\to\infty} \lambda m[[u^* > \lambda] \leqslant \frac{Cp}{p-1} (\varlimsup_{\lambda\to\infty} N(\lambda))^{1-1/p}$$

which together with the Lemma finally implies that:

$$\varliminf_{\lambda \to \infty} N(\lambda) \geqslant A > 0$$

where A is a numerical constant only depending on the dimension. This proves our theorem.

The above theorem should be compared with the well known theorem of Loomis (cf. [4]) that asserts that if $\mu = \sum_{j=1}^{m} c_j \delta_{x_j} \in M(\mathbf{R})$, $x_1, \ldots, x_m \in \mathbf{R}$, $c_1, \ldots, c_m > 0$ then the Hilbert transform $\widetilde{\mu}(x)$ of μ satisfies

$$\lambda[|\widetilde{\mu}(x)| > \lambda] = \frac{2}{\pi} \|\mu\| \qquad \forall \lambda > 0 .$$

It is from this fact, by approximating arbitrary positive singular measures by discrete measures as above, that P. Jones was able to prove his more precise theorem.

The above method if followed through will yield the following version of Loomis's theorem for higher dimensions:

THEOREM. – *Let* ν *be a positive singular bounded measure of* $\mathbf{R}^n$ *we then have:*

$$k \|\nu\| \leqslant \lambda m \left[\sum_{j=1}^{n} |R_j \nu(x)| > \lambda \right] \leqslant c \|\nu\| ; \quad \forall \lambda > 0$$

where c, $k > 0$ *only depend on the dimension.*

Indeed by an easy reduction argument that involves dilatation of the space, throwing away a negligible (i.e. mass less than ϵ) piece and multiplying by a constant we see that it is enough to show that there exists some positive constant $a > 0$ depending only on the dimension such that has every positive singular measure ν of mass 1 supported by the unit cube in $\mathbf{R}^n$ we have:

$$\lambda m \left[\sum_{j=1}^{n} |R_j \nu(x)| > \lambda \right] \geqslant a .$$

Now, if we follow the previous argument with care, we see that for such a measure there exist two positive constants a_0, b_0 that only depend on the dimension such that:

$$\lambda m \left[\sum_{j=1}^{n} |R_j \nu(x)| > \lambda \right] \geqslant a_0 > 0, \qquad \forall \lambda \geqslant b_0 > 0 .$$

Examining the behavior of $R_j\nu(x)$ as $x \longrightarrow \infty$ we also see that there exist two other positive constants a_1, b_1 such that:

$$\lambda m\left[\sum_{j=1}^{n} |R_j\,\nu(x)| > \lambda\right] \geqslant a_1 > 0 \qquad \forall 0 < \lambda < b_1 .$$

The interval $b_1 \leqslant \lambda \leqslant b_0$ if not empty can be dealt with trivially. This completes the proof.

2. Martingale transforms.

In this section I shall be brief. The reader should look at Gundy's paper [3] and also at [5] where analogous problems are treated and also at [6] where Martingale transforms are examined in more details.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_n, \mathbf{P})$ be a probability space with a filtration $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots \subset \mathfrak{F}$ and let us assume that for every $n \geqslant 1$ we can find $r_1^{(n)}, r_2^{(n)}, \ldots, r_p^{(n)}$ (for some fixed p) real functions bounded by c some fixed constant that are measurable w.r.t. $\mathfrak{F}_n$ and relatively orthonormal (i.e. s.t. $\mathbf{E}(r_j^{(n)} r_k^{(n)} // \mathfrak{F}_{n-1}) = \delta_{jk}$) with relative mean zero (i.e. $\mathbf{E}(r_j^{(n)} // \mathfrak{F}_{n-1}) = 0$) and also that they span the martingales over $(\mathfrak{F}_n)_{n \geqslant 1}$ in the sense that every martingale X on the above space (w.r.t. the filtration) can be written as:

$$X_n = d_1 + d_2 + \ldots + d_n \; ; \; d_n = X_n - X_{n-1} \tag{2.1}$$

with:

$$d_n = a_{n-1}^{(1)} r_1^{(n)} + \ldots + a_{n-1}^{(p)} r_p^{(n)}$$

where the a_{n-1}^{j} $(j = 1, \ldots, p)$ are $\mathfrak{F}_{n-1}$ measurable and are then uniquely determined.

The above situation arises in many natural martingales e.g. diadic martingales have this property and the $r^{(n)}$ are just the Radamacher sequence.

Let now $M = (m_{ij})_{i,j=1}^{p}$ be a complex matrix with constant coefficients; given a martingale as in (2.1) we shall then define its transform by M which will be a new martingale $MX = Y$ that is defined by:

$$(MX)_n = Y_n = \delta_1 + \delta_2 + \ldots + \delta_n$$

with:

$$\delta_n = b^{(1)}_{n-1} r^{(n)}_1 + \ldots + b^{(p)}_{n-1} r^n_p$$

where

$$b^{(i)}_{n-1} = \sum_{j=1}^{p} m_{ij} a^{(j)}_{n-1} \qquad i = 1, 2, \ldots, p$$

cf. [6] for more details. We have then

THEOREM. – *Let* $M_1, M_2, \ldots, M_k$ *be matrices that do not have a common real eigenvector. An arbitrary* L^1*-bounded martingale as in (2.1) is then uniformly integrable (i.e. satisfies* $X_n = \mathbf{E}(X // \mathfrak{F}_n)$ $n = 1, 2, \ldots$ *for some* $X \in L^1(\Omega)$*) if and only if its* k *transforms* $M_j X = Y^{(j)}$ $j = 1, 2, \ldots, k$ *satisfy* $\mathbf{P}[|\lim_{n\to\infty} Y^{(j)}_n| > \lambda] = o\left(\frac{1}{\lambda}\right)$ *as* $\lambda \longrightarrow \infty$.

The proof of the above theorem is entirely analogous to the one we gave for the Riesz transforms. The key point is of course that we can use the Chao-Taibleson-Janson subharmonicity Lemma (cf. [6]). We also have

THEOREM. – *Let* $M_1, M_2, \ldots, M_k$ *be* k *matrices whose ideal generates the identity i.e. such that there exist matrices* $P_1, P_2, \ldots, P_k$ *such that* $P_1 M_1 + P_2 M_2 + \ldots + P_k M_k = I$. *Then an* L^1*-bounded martingale as in (2.1) is uniformly integrable if the* k *"maximal transforms"* $M_j^* X = \sup_n |(M_j X)_n|$ $j = 1, 2, \ldots, k$ *satisfy:*

$$\mathbf{P}[M_j^* X > \lambda] = o\left(\frac{1}{\lambda}\right).$$

This is the analogue of R.F. Gundy's theorem and the proof is analogous.

Proof (Outline). – The fact that $\mathbf{P}[M_j^* X > \lambda] = o(\lambda^{-1})$ and the Good-λ inequalities imply (just as in the theorems of Gundy in [3]) that: $\mathbf{P}[\sup_n |(P_j M_j X)_n| > \lambda] = o(\lambda^{-1})$. This and our hypothesis clearly implies that: $\mathbf{P}[\sup |X_n| > \lambda] = o(\lambda^{-1})$. This gives the uniform integrability of X by [3] [5]. (In fact the proof of the Lemma in § 1 if properly interpreted proves just that.)

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