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BJÖRN E. J. DAHLBERG

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APPROXIMATION OF HARMONIC FUNCTIONS

by Björn E. J. DAHLBERG

1. Introduction.

In this note we shall study the following approximation problem : Let u be harmonic in a domain D that has a regular boundary. When is it possible to find functions f_j of bounded variation in D (that is functions whose gradients are bounded in D) such that $\sup_D |f - f_j| \rightarrow 0$ as $j \rightarrow +\infty$? The main result of this paper is that this approximation is always possible if u is the Poisson integral of a function $f \in L^p(\sigma)$, $p \geq 2$, where σ denotes the surface measure of ∂D and is not always possible if $f \in L^p(\sigma)$, $p < 2$.

This type of approximation appears implicitly in the main step of the proof of the Corona theorem, see Carleson [1, 2], for the case when u is a bounded and holomorphic function. For the case when u is the Poisson integral of a function of bounded mean oscillation BMO this type of approximation has been carried out by Varopoulos [9] and Garnett [5]. In these cases it is required that the approximands f_j have gradients that are Carleson measures.

THEOREM 1. — *Suppose u is harmonic in a bounded Lipschitz domain $D \subset \mathbf{R}^n$, $n \geq 2$. Then for every $\varepsilon > 0$ there is a function φ such that $|u - \varphi| < \varepsilon$ in D and for all $P \in \partial D$ we have that*

$$\int_{\beta(r)} |\nabla \varphi| dQ \leq C[\varepsilon^{-1} \int_{\beta(Cr)} |\nabla u|^2 \text{dist}\{Q, \partial D\} dQ + \varepsilon r^{n-1}].$$

Here $\beta(r) = \{Q \in D : |Q - P| < r\}$ and $\nabla \varphi$ denotes the gradient of φ . The constant C only depends on D .

We remark that this result means that φ is of bounded variation if $\int_D |\nabla u|^2 \text{dist}\{Q, \partial D\} dQ < \infty$. It's known that this happens if and only if u

is the Poisson integral of a function $f \in L^2(\sigma)$, see Stein [8] for the case of domains with smooth boundaries and Dahlberg [3] for the case of Lipschitz domains.

We recall that a measure μ is called a Carleson measure if $|\mu|(\beta(P,r)) \leq Cr^{n-1}$ for all $P \in \partial D$. It's known that a harmonic function u is the Poisson integral of a function of bounded mean oscillation if and only if $|\nabla u|^2 \text{ dist}\{Q,\partial\}$ is a Carleson measure, see Fefferman-Stein [4] for the case of smooth domains and this has recently been shown to hold for Lipschitz domains by E. Fabes and U. Neri (unpublished). Therefore $|\nabla\varphi| dQ$ is a Carleson measure if and only if u is the Poisson integral of a BMO-function, see Varopoulos [9].

THEOREM 2. — *Let U denote the unit disk in \mathbf{R}^2 . If $p < 2$ then there is an $f \in L^p(\sigma, \partial U)$ such that if $u = Pf$ then*

$$\sup_U |u - \varphi| = \infty$$

for all φ that are of bounded variation in U .

In addition to this example it's known that there are bounded holomorphic functions that are not of bounded variation, see Rudin [7].

2. The method of approximation.

We start by recalling that a bounded domain $D \subset \mathbf{R}^n$ is called a Lipschitz domain if ∂D can be covered by finitely many open right circular cylinders whose bases have a positive distance from ∂D and corresponding to each cylinder L there is a coordinate system (x,y) with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, with the y -axis parallel to the axis of L and a function $\varphi : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ satisfying a Lipschitz condition (i.e. $|\varphi(x) - \varphi(z)| \leq M|x - z|$) such that

$$L \cap D = \{(x,y) : y > \varphi(x)\} \cap L$$

and

$$D \cap L = \{(x,y) : y = \varphi(x)\} \cap L.$$

We recall that a Lipschitz domain D is starshaped with star center P^* and with standard inner cone Γ if $P^* \in \Gamma(P) \subset D$ for all $P \in \partial D$, where $\Gamma(P)$ denotes the cone with vertex P having its axis along the line through P and P^* and being congruent to Γ . (With a cone we mean an open, non empty,

convex and possibly truncated cone). If u is harmonic in D and $u(P^*) = 0$ we have the following result from Dahlberg [4] : Let γ be a cone with the same vertex P_0 as Γ and assume that $\bar{\gamma} - \{P_0\} \subset \Gamma$. Let $\gamma(P)$ be constructed as $\Gamma(P)$ and put

$$M(P) = \sup \{ |u(Q)| : Q \in \gamma(P) \}.$$

Then

$$(2.1) \quad C^{-1} \int_{\partial D} M^2 d\sigma \leq \int_D |\nabla u|^2 \text{dist} \{Q, \partial D\} dQ \leq C \int_{\partial D} M^2 d\sigma,$$

where C only depends on γ and Γ .

We shall first suppose that u is a function in the cube

$$U = \{ (x,y) : 0 < x_i < 1, i=1, 2, \dots, n-1, 0 < y < 1 \}.$$

We let Ω_m denote the collection of all dyadic cubes of side 2^{-m} in $\{x \in \mathbb{R}^{n-1} : 0 < x_i < 1\}$. If $Q \in \Omega_m$ we put $T(Q) = \{(x,y) : x \in Q, 2^{-m-1} \leq y < 2^{-m}\}$. The collection of all $T(Q)$, when Q runs over $\bigcup_{m \geq 0} \Omega_m$

is denoted by Λ . If $T_1, T_2 \in \Lambda$ and $T_i = T(Q_i)$ we say that $T_1 < T_2$ if $Q_1 \subset Q_2$ and the side of Q_2 is twice the side of Q_1 . We shall fix the number $a > 0$ and put $\Gamma = \{(x,y) : |x| < ay\}$. For $P \in \mathbb{R}^n$ we set $\Gamma_p = P + \Gamma = \{P+Q : Q \in \Gamma\}$. For $T \in \Lambda$ we put

$$L(T) = \left\{ V \in \Lambda : V \cap \left[\bigcup_{p \in T} \Gamma_p \right] \neq \emptyset \right\}.$$

We observe that if $T_1 < T_2$ and $T_1 \in L(T)$ then $T_2 \in L(T)$ also.

We shall next describe the method for approximating u . We say that a $T \in \Lambda$ is red if $\text{diam}(T) \sup_T |Ou| \geq k\varepsilon$. Otherwise it's called blue. (Here k is

a small number to be chosen later.) The main step now is to put together the blue intervals into domains of Lipschitz character, where the oscillation of u is $\leq \varepsilon$.

Let $S = \left\{ (x,y) : 0 < x_i < 1, \frac{1}{2} < y < 1 \right\}$ and suppose that S is blue. We

shall now define $K(S) \subset \Lambda$ inductively as follows : First $S \in K(S)$ and a $T \in \Lambda$ is added to $K(S)$ provided there is a $T' \in K(S)$ such that $T < T'$, all elements of $L(T)$ are blue and $|u(P_S) - u(P_T)| \leq m\varepsilon$, where P_T is the center of T . Put $H(S) = \bigcup_{T \in K(S)} L(T)$ and let $D(S)$ denote the interior of the closure of

$\bigcup_{T \in H(S)} T$. Suppose now that $T \in \Lambda$, $T \subset U - D(S)$, and $\partial T \cap d(S) \neq \emptyset$,

where $d(S) = U \cap \partial D(S)$. Let T_i , $0 \leq i \leq N$, be such that $T = T_0 < T_1 < \dots < T_N = S$ and let j be the smallest integer such that $T_j \in K(S)$. Since $T_{j-1} \notin K(S)$ there are two cases to consider. If $L(T_{j-1})$ contains a red interval R we say that $T \in A(S)$ and if this is not the case we say that $T \in B(S)$. Also, we define $\alpha(S)$ and $\beta(S)$ as $U(\partial T \cap \partial D(S))$ where T runs over $A(S)$ and $B(S)$ respectively. We observe that there is a number $M > 0$ only depending on Γ such that the projection T' of T into \mathbf{R}^{n-1} is contained in R^* , where $R^* \subset \mathbf{R}^{n-1}$ is the cube with the same center as R^1 but with a side that is M times the side of R' . (Here R is the red interval contained in $L(T_{j-1})$.) Also there is a $v \in H(S)$ such that $\text{diam } R \leq \text{diam } V \leq 2 \text{ diam } R$ and $|P_R - P_V| \leq M \text{ diam } R$ (we'll say that R touches $D(S)$). Let $|E|$ denote the $(n-1)$ -dimensional Hausdorff measure of a set $E \subset \mathbf{R}^n$. The Lipschitz character of $D(S)$ implies that $|\alpha(S)| \leq C \left| \bigcup_{T \in A(S)} T' \right|$, which together with the above observations show that

$$(2.2) \quad |\alpha(S)| \leq C \Sigma |\partial R|,$$

where the sum is taken over all red intervals that touch $D(S)$. Let $b > a$ be sufficiently large and put $\gamma = \{(x,y) : |x| < -by\}$. If $\Omega = \bigcup_{P \in d(S)} \gamma_P$, then $D_1 = D(S) - \bar{\Omega}$ is again a Lipschitz domain. It's easily seen that if $a > 0$ has been chosen sufficiently small then b can be chosen so that D_1 is a starshaped Lipschitz domain with starcenter P_S and a standard inner cone P' that only depends on a and b . We have also that

$$\left| \bigcup_{T \in B(S)} \partial T \cap d(S) \cap \partial D_1 \right| \geq c |\beta(S)|$$

where $c > 0$ only depends on a and b .

For $P \in \partial D_1$ we put $M_S(P) = \sup |u(Q) - u(P_S)|$, where Q runs over all points on the line segment between P and P_S . Suppose now that $T \in B(S)$ and $T = T_0 < T_1 < \dots < T_N = S$. If j is the smallest index for which $T_j \in K(S)$ it follows that $L(T_{j-1})$ does not contain any red cube. If P_{j-1} denotes the center of T_{j-1} it follows that $|u(P_{j-1}) - u(P_S)| \geq m\varepsilon$.

If $j = 1$ it follows that $|u(P) - u(P_S)| \geq (m-k)\varepsilon$ for all $P \in T = T_0$ and hence $M_S(P) \geq (m-k)\varepsilon$ for all $P \in \partial T \cap \partial D_1$. Suppose now that $j > 1$ and $P \in \partial T \cap d(S) \cap \partial D_1$. Let Q denote the point on the line segment between P and P_S that has the same y -coordinate as P_{j-1} . Since the line segment between P_{j-1} and Q is contained in $D(S)$ it follows that

$$|u(P_{j-1}) - u(Q)| \leq k\varepsilon |P_{j-1} - Q| (\text{diam } T_{j-1})^{-1} < m\varepsilon/2$$

if k has been chosen sufficiently small. Hence we have in all cases that

$$(2.3) \quad |\beta(S)| \leq C|\{P \in \partial D_1 : M(P) > m\epsilon/2\}|.$$

If there is an interval in $\Lambda - H(S)$ that's not red let S_1 denote one with maximal diameter. After making a change of scale we construct $H(S_1)$ as above and in this way we get a decomposition $\Lambda = \Lambda_R \cup \left[\bigcup_j H(S_j) \right]$ into pairwise disjoint sets, where Λ_R denotes the collection of all red intervals in Λ . We claim that if u is harmonic and $L_j = |\partial D(S_j)|$ then

$$(2.4) \quad \Sigma L_j \leq C \left[1 + \epsilon^{-2} \iint_0 y |\nabla u|^2 dx dy \right]$$

where C is independent of u and ϵ , $\tilde{U} = \{(x,y) : -1 < x_i < 2, 0 < y < 2\}$. Following Garnett [5] we first observe that if $R \in \Lambda$ is red then

$$(2.5) \quad |\partial R| \leq C\epsilon^{-2} \iint_{R^*} y |\nabla u|^2 dx dy,$$

where $R^* = \bigcup_{P \in R} B(P, \delta/2)$, $\delta = \text{dist} \{R, R^{n-1}\}$

and $B(P,r) = \{Q : |P-Q| < r\}$.

To see (2.5), we first observe that there is a number c_n only depending on n such that there is $P \in \bar{R}$ with $|\nabla u(P)| \geq c_n k \epsilon \delta^{-1}$. Since $|\nabla u|^2$ is subharmonic it follows that

$$\iint_{R^*} |\nabla u|^2 y dx dy \geq \frac{1}{2} \delta \iint_{B(P, \delta/2)} |\nabla u|^2 dx dy \geq c\epsilon^2 |\partial R|,$$

which gives (2.5). We also observe that from Cauchy's inequality follows that

$\left(\iint_R |\nabla u| dx dy \right)^2 \leq C |\partial R| \iint_R |\nabla u|^2 y dx dy$ which together with (2.5) gives

$$(2.6) \quad \iint_R |\nabla u| dx dy \leq C\epsilon^{-1} \iint_{R^*} |\nabla u|^2 y dx dy.$$

Let $\theta > 0$ be a small fixed number and let I denote those $j : s$ for which $|\partial D(S_j) \cap R^{n-1}| \geq \theta L_j$. Since the domains $D(S_j)$ are pairwise disjoint it follows that

$$(2.7) \quad \sum_I L_j \leq \theta^{-1}.$$

Let II denote those $j : s$ for which $|\alpha(S_j)| \geq \theta L_j$. Since the domains $\{R^*\}_{R \in \Lambda_R}$ have uniformly bounded overlap and there is a fixed number N such that no red interval $R \in \Lambda_R$ touches more than N of the domains $D(S_j)$ it follows from (2.2) and (2.5) that

$$(2.8) \quad \sum_{II} L_j \leq \theta^{-1} \Sigma |\alpha(S_j)| \leq C \varepsilon^{-2} \iint_{\Omega} y |\nabla u|^2 dx dy.$$

Finally let III be those $j : s$ for which $|\beta(S_j)| \geq \theta L_j$. From (2.1) and (2.3) follows that

$$|\beta(S_j)| \leq C \varepsilon^{-2} \int_{D_j} \text{dist} \{Q, \partial D_j\} |\nabla u|^2 dQ \leq C \varepsilon^{-2} \iint_{D_j} y |\nabla u|^2 dx dy$$

so we have that

$$(2.9) \quad \sum_{III} L_j \leq C \varepsilon^{-2} \iint_{\Omega} y |\nabla u|^2 dx dy.$$

If the constant θ has been chosen small enough then each $D(S_j)$ belongs to one of the categories I, II or III. Hence (2.4) follows from (2.7-9).

We now define $\varphi = uh + \Sigma u(P_j)h_j$, where h is the characteristic function of $\bigcup_{R \in \Lambda_R} \bar{R}$, h_j is the characteristic function of $D(S_j)$ and P_j is the center of S_j . Clearly $|u - \varphi| \leq \varepsilon$. It remains to estimate $|\nabla \varphi|$. To this end let λ_j be the surface measure of $\partial D(S_j)$ and if $\{R_j\}_{j=1}^{\infty} = \Lambda_R$ we let σ_j denote the surface measure of ∂R_j . With this notation we have that $|\nabla \varphi| \leq C[|\nabla u|h + \varepsilon \Sigma(\sigma_j + \lambda_j)]$, where the ε in front of the sum appears because the jump of φ at a common boundary point of domains of the form $D(S_j)$ or R_k is less than ε .

Let $Q \subset R^{n-1}$ be a cube and put

$$S(Q) = \{(x,y) : x \in Q, 0 < y < \text{side of } Q\}.$$

We shall now estimate $\iint_{S(Q)} |\nabla \varphi| dx dy$. Let M be a large positive number and let $V \subset R^{n-1}$ be the largest dyadic cube that contains Q for which $|V| \leq 6^n M |Q|$. If M is large enough, then it follows from (2.5) and (2.6) that

$$\iint_{S(Q)} |\nabla u|h dx dy + \varepsilon \Sigma \sigma_j(S(Q)) \leq C \varepsilon^{-1} \iint_{S(V)} |\nabla u|^2 dx dy.$$

From (2.4) and possibly a change of scale we see that

$$\Sigma \lambda_j(S(Q)) \leq C \left[\varepsilon^{-2} \iint_{S(V)} |\nabla u|^2 y dx dy + |Q| \right],$$

where the prime denotes summation over those $j : s$ for which $S_j \subset S(V_1)$, where V_1 is the largest dyadic cube that contains Q for which $|V_1| \leq M|Q|$. If $\lambda_j(S(Q)) > 0$ and if S_j is not contained in $S(V_1)$ then $D(S_j)$ contains (x_Q, Ly_Q) where (x_Q, y_Q) is the center of $S(Q)$ and the constant L only depends on M and the choice of the cone Γ for the construction of $D(S_j)$. Since the domains $D(S_k)$ are pairwise disjoint there is at most one j with this property and from the Lipschitz character of $D(S_j)$ it follows that $\lambda_j(S(Q)) \leq C|Q|$ which concludes the proof of theorem 1 for the case of smooth domains.

The case when u is harmonic in a Lipschitz domain is easily reduced to the case when u is defined in

$$U' = \{(x,y) : 0 < x_i < 1, f(x) < y < f(x) + 1\},$$

where f is a Lipschitz function. Letting $T(x,y) = (x, y - f(x))$ we see that T maps U' onto

$$U = \{(x,y) : 0 < x_i < 1, 0 < y < 1\}.$$

Let $u_1 = u \circ T^{-1}$ and construct ϕ_1 in U as above, this time approximating u_1 . Letting $\phi = \phi_1 \circ T$, it's easily seen that the methods for estimating $\nabla\phi$ work in this case too, which yields theorem 1.

3. An example.

In this section we shall identify \mathbf{R}^2 with the complex plane \mathbf{C} and we'll denote points in \mathbf{C} by $z = x + iy, x, y \in \mathbf{R}$. We'll put $J = \{x : -1 < x < 1\}$ and $Q = \{z : |x| < 2, 0 < y < 4\}$. If $f \in L^p(\mathbf{R})$ we let Pf denote the Poisson integral of f . We shall establish the following result.

THEOREM 3. — *For all $p < 2$ there is an $f \in L^p(\mathbf{R})$ with support in J such that $\sup_Q |Pf - \phi| = \infty$ for all ϕ such that $\iint_Q |\nabla\phi| dx dy < \infty$.*

We shall deduce theorem 3 from the following lemma, the proof of which is given later.

LEMMA 1. — *For $\theta \in (0,1)$ there is a function $g_\theta \in L^2(\mathbf{R})$ with support in J such that if $0 < \varepsilon < 1$ and $|u - \phi| \leq \varepsilon$ in Q , then*

$$\iint_Q |\nabla\phi| dx dy \geq c\varepsilon^{-\theta}, \text{ where } c > 0 \text{ is independent of } \varepsilon.$$

Proof of theorem 3. — We shall first define a sequence of intervals $I_j \subset \mathbf{R}$ by putting $I_1 = [0,1]$ and requiring that I_{j+1} is to the right of I_j , $|I_j| = 2^{-j}$ and $\text{dist} \{I_j, I_{j+1}\} = j^{-2}$. Let c_j denote the center of I_j and put

$$(3.1) \quad g_j(x) = g_\theta(2^{j+1}(x - c_j)),$$

where g_θ is as in lemma 1. It's easily seen that

$$|\nabla P g_j(z)| \leq C 2^{-j} |z - c_j|^{-2}$$

whenever $|z - c_j| > 2^{-j}$. If $Q_j = \{z : |x - c_j| < 2^{-j}, 0 < y < 2^{1-j}\}$ we therefore have

$$(3.2) \quad \sup \{|\nabla P g_k(z)| : z \in Q_j\} \leq C 2^{-k} k^2 (k \neq j).$$

Let $b_j > 0$ be defined by $b_j^{p2^{-j}} = j^{-2}$ and put $f = \sum b_j g_j$. Clearly $f \in L^p(\mathbf{R})$ and the support of f is bounded. From (3.2) follows $u = b_j P f_j + R_j$, where $u = P f$ and

$$(3.3) \quad \sup \{|\nabla R_j(z)| : z \in Q_j\} \leq C \sum_k b_k k^2 2^{-k} = M < \infty.$$

Suppose now that $|u - \varphi| \leq L < \infty$ in $\bigcup_{j \geq 1} Q_j$. We shall next show that this

implies that $\sum_j \iint_{Q_j} |\nabla \varphi| dx dy = \infty$ whenever $\theta > p - 1$.

If z_j denotes the center of Q_j it follows from (3.3) that

$$\sup \{|m_j - R_j(z)| : z \in Q_j\} \leq M \text{diam}(Q_j) \rightarrow 0 \text{ as } j \rightarrow \infty,$$

where $m_j = R_j(z_j)$. Therefore there is a j_0 such that if $j \geq j_0$ then $|P f_j - \varphi_j| \leq 2L b_j^{-1}$ in Q_j , where $\varphi_j = (\varphi - m_j) b_j^{-1}$. From lemma 1 follows now that

$$\sum_j \iint_{Q_j} |\nabla \varphi| dx dy \geq C \sum_{j \geq j_0} 2^{-j} b_j^{1+\theta} = \infty \text{ if } \theta > p - 1$$

which yields the theorem.

We remark that by using a suitable conformal mapping it's easily seen that theorem 2 follows from theorem 3.

We'll need the following lemma for the proof of lemma 1.

LEMMA 2. — *Suppose u is harmonic in $B = B(z_0, 5r) \subset \mathbf{C}$. If $|u - \varphi| \leq \varepsilon$ in B and if $\sup \{|u(z_1) - u(z_2)| : z_1, z_2 \in B(z_0, r)\}$ then $\iint_B |\nabla \varphi| dx dy \geq c \varepsilon r$, where $c > 0$ is a universal constant.*

Proof. — Pick $z_1, z_2 \in B(z_0, r)$ such that $|u(z_1) - u(z_2)| \geq 7\varepsilon$. Since the function $z \rightarrow |u(z) - u(z_2)|^2$ is subharmonic it follows that

$$\int_{B(z_1, r)} |u(z) - u(z_2)|^2 dx dy \geq 7^2 \pi \varepsilon^2 r^2.$$

Since $B(z_1, r) \subset \tilde{B} = B(z_2, 3r)$ we therefore have that

$$\int_{\tilde{B}} |\varphi - \tilde{\varphi}|^2 dx dy \geq \pi \varepsilon^2 r^2,$$

where $\tilde{\varphi} = \frac{\int_B \varphi dx dy}{\int_B dx dy}$. The Poincaré-Sobolev inequality (see Meyers and Ziemer [6] for general versions) says that there is a constant C such that for all balls

$$\tilde{B} \left(\int_{\tilde{B}} |\varphi - \tilde{\varphi}|^2 dx dy \right)^{1/2} \leq C \int_{\tilde{B}} |\nabla \varphi| dx dy,$$

which yields lemma 2.

We shall next prove lemma 1. Let $\alpha > 0$ be defined by $(1 - 2\alpha) = \theta(1 + 2\alpha)$ and put $a_k = k^{-1/2 - \alpha}$ for $k = 1, 2, \dots$. Let $\delta > 0$ be a given number. We claim that there is a sequence of positive integers $n_k + \infty$ such that if $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ and if

$$S_k = \{z : n_k^{-1} \leq 1 - |z| \leq 4n_k^{-1}\}$$

then $f'(z) = a_k n_k z^{n_k - 1} + R_k(z)$, where

$$\sup \{|R_k(z)| : z \in S_k\} \leq \delta a_k n_k.$$

To see this choose $n_1 = 100$ and if n_1, \dots, n_{k-1} have been chosen then

$$\left| \sum_{j < k} a_j n_j z^{n_j - 1} \right| \leq k n_{k-1} < \delta / 2 a_k n_k$$

if n_k has been chosen large enough. If we also require that $n_{j+1} \geq n_j + 2$ and

$$(1 - n_j^{-1})^{\frac{1}{2} n_j} + 1^{-1} n_{j+1} \leq \min(1, a_j \delta / 2)$$

we have for $z \in S_k$ that

$$\left| \sum_{j>k} a_j n_j z^{n_j-1} \right| \leq \sum_{j>k} a_j n_j (1 - n_k^{-1})^{n_j-1} \leq \sum_{j>k} (1 - n_k^{-1})^{\frac{1}{2}n_j} \leq \sum_{s=1}^{\infty} (1 - n_k^{-1})^{\frac{1}{2}n_k+s} \leq \frac{1}{2} \delta n_k a_k$$

and adding these estimates yields the claim.

Hence if δ has been chosen sufficiently small then whenever $B \subset S_k$ is disk of radius $(10n_k)^{-1}$ we have that

$$(3.4) \quad \sup \{ |f(z_1) - f(z_2)| : z_1, z_2 \in B \} > ca_k$$

where $c > 0$ is independent of k .

Let $u = P(fh)$, where h is the characteristic function of

$$\{z : |z|=1, \operatorname{Re} z > 0\} = L.$$

Since $u - f$ has boundary values zero on L it follows that $u - f$ has a harmonic extension to all of $\{z : \operatorname{Re} z > 0\}$. We therefore have that if B is a disk of radius $(10n_k)^{-1}$ such that

$$B \subset S_k \cap \{z : |\arg z| \leq \pi/3\} = S_k^*$$

then it follows from (3.4) that

$$(3.5) \quad \sup \{ |u(z_1) - u(z_2)| : z_1, z_2 \in B \} \geq da_k$$

for $k \geq k_0$, where $d > 0$ is independent of k .

Suppose now that $\varepsilon > 0$ is a small number and that

$$|u - \varphi| \leq \varepsilon \text{ in } \Omega = \{z : |z| < 1, \operatorname{Re} z > -1/2\}.$$

There is a number λ_0 such that we can find more than λn_k disks $B(j,k)$ of radius $(10n_k)^{-1}$ such that $10B(j,k) \subset S_k^*$ whenever $1 \leq j \leq \lambda n_k$ and the disks $B(j,k)$ are pairwise disjoint. It's easily seen from (3.5) that there is an $m > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then

$$\sup \{ |u(z_1) - u(z_2)| : z_1, z_2 \in B(j,k) \} > 10\varepsilon$$

whenever $1 \leq j \leq \lambda n_k$, $k_0 < k < L(\varepsilon)$, where $L(\varepsilon) = m\varepsilon^{-\beta}$, $\beta = 2(1 + 2\alpha)^{-1}$. From lemma 2 follows now that

$$\iint_{\Omega} |\nabla \varphi| dx dy \geq \sum_{k=k_0}^{L(\varepsilon)} \sum_{j=1}^{\lambda n_k} \iint_{10B(j,k)} |\nabla \varphi| dx dy \geq c' L(\varepsilon) \varepsilon = c\varepsilon^{-\theta}.$$

Finally, mapping the unit disk conformally onto the upper half plane yields lemma 1.

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Björn E. J. DAHLBERG .

Chalmers University of Technology
and University of Göteborg
Department of Mathematics
Sven Multins gata 6
S-402 20 Göteborg.
