ANNALES DE L'INSTITUT FOURIER

S. K. PICHORIDES

On the L^1 norm of exponential sums

Annales de l'institut Fourier, tome 30, nº 2 (1980), p. 79-89

<http://www.numdam.org/item?id=AIF_1980__30_2_79_0>

Ann. Inst. Fourier, Grenoble
30, 2 (1980), 79-89.

ON THE L^1 NORM OF EXPONENTIAL SUMS

by S. K. PICHORIDES

1. Introduction.

The problem of finding a lower bound for the L^1 norm of trigonometric polynomials of the form

$$F(x) = a_1 \exp(in_1x) + \cdots + a_N \exp(in_Nx)$$

where $0 < n_1 < \ldots < n_N$ are integers and $|a_j| \geqslant 1, j = 1, 2, \ldots, N$, depending only on N has a relatively long history (see [1] for details). The existence of such a bound of the order of $\log N$ (in the case of coefficients equal to 1) is known as Littlewood's conjecture. The best result in this direction up to now is a bound of the order of $(\log N)^{1/2}$ (see [1] and [2]). The purpose of this paper is to improve the above estimate by establishing the following

THEOREM 1. — *There is an absolute positive constant* C *such that*

$$(1.1) \qquad (2\pi)^{-1} \int_0^{2\pi} |F| \geqslant C (\log N)/(\log \log N)^2 .$$

The method of the proof is closely related to that of [2]. However the basic lemma 3 of [2] is replaced now by lemma 5 which is considerably stronger and its proof constitutes the main part of the present paper.

Loosely speaking we can summarize the main idea of the proof as follows : After a suitable translation of the sequence $\{n_1, \ldots n_N\}$ we consider the parts of F corresponding to those frequencies which are multiples of distinct powers of 2. We prove that the L^1 norm of F exceeds an average of the L^1 norms of some of these parts by a « sufficiently large » quantity except in the case of a « big » number of such parts. In the first case the proof is completed by induction while in the second we use standard methods of the theory of lacunary trigonometric series.

A few words concerning our notation. The letter C will denote an absolute positive constant not necessarily always the same. The same letter C with a subscript will denote an absolute positive constant which remains the same in all its occurences. All integrals will be understood with respect to the normalized Lebesgue measure $(2\pi)^{-1}\,dx$ and the absence of limits of integration will mean integration over $[0,2\pi]$. For any 2π-periodic measurable function g, $\|g\|$ will denote its L^1 norm $\int_0^{2\pi}|g|$. Finally for any subset E of $[0,2\pi]$ its normalized measure will be denoted by $|\mathrm{E}|$ and for any finite sequence Γ of integers $|\Gamma|$ will denote the number of its elements.

We shall give a detailed proof only in the case of coefficients equal to 1 (exponential sums). Trivial changes are needed for the proof of the general case (see remark 1 of section 5). In section 2 we give some auxiliary lemmas and in section 3 the main lemma 5. Section 4 is devoted to the completion of the proof of theorem 1 and in the last one we offer some comments.

2. Auxiliary lemmas.

LEMMA 1. — *Let* E *be a measurable subset of* $[0,2\pi]$ *such that* $0 < |\mathrm{E}| < 1$ *and let* $\mathrm{G}(x) = 1 + a_1 \exp(ix) + \cdots + a_k \exp(ikx)$. *Then*

$$\left\{(|\mathrm{E}|^{-1})\int_{\mathrm{E}}|\mathrm{G}|\right\}^{|\mathrm{E}|}\left\{(|\mathrm{E}'|^{-1})\int_{\mathrm{E}'}|\mathrm{G}|\right\}^{|\mathrm{E}'|} \geqslant 1 \tag{2.1}$$

where E′ *is the complement of* E *in* $[0,2\pi]$.

Proof. — We write χ_{E} and $\chi_{\mathrm{E}'}$ for the characteristic functions of E and E′ respectively. On applying Jensen's formula ([4], VII 7.8) we obtain

$$0 \leqslant \int \log|\mathrm{G}| = |\mathrm{E}|\int(\chi_{\mathrm{E}}/|\mathrm{E}|)\log|\mathrm{G}| + |\mathrm{E}'|\int(\chi_{\mathrm{E}'}/|\mathrm{E}'|)\log|\mathrm{G}|.$$

Jensen's convexity inequality ([4], I 10.8) applied to the last two integrals yields

$$0 \leqslant |\mathrm{E}|\log\left\{(|\mathrm{E}|^{-1})\int_{\mathrm{E}}|\mathrm{G}|\right\} + |\mathrm{E}'|\log\left\{(|\mathrm{E}'|^{-1})\int_{\mathrm{E}'}|\mathrm{G}|\right\}$$

from which (2.1) follows.

LEMMA 2. — *Let*

$$G(x) = a_0 + a_1 \exp(ix) + \cdots + a_k \exp(ikx) = g(x) + i\tilde{g}(x)$$

and write $\exp(imx)G(x) = g_m(x) + i\tilde{g}_m(x)$. *Then*

$$\lim \|g_m\| = (2/\pi)\|G\|, \text{ as } m \to \infty. \tag{2.2}$$

Proof. — Let $I_j = (a_j, b_j) = (2\pi j/m, 2\pi(j+1)/m), j = 0, 1, \ldots, m-1$. We take m so big that the variation of g and $\tilde{g}$ in I_j is less than $\varepsilon(>0)$. It follows that $g_m(x) = g(x)\cos mx - \tilde{g}(x)\sin mx$ differs from

$$g(a_j)\cos mx - \tilde{g}(a_j)\sin mx$$

by less than 2ε. The last expression equals $|G(a_j)|\cos(mx+t_j)$ where t_j is such that $\tan t_j = \tilde{g}(a_j)/g(a_j)$. It follows

$$\begin{aligned}\int_{I_j} |g_m| &= |G(a_j)| \int_{I_j} |\cos(mx+t_j)| \, dx + |I_j| O(\varepsilon) \\ &= (2/\pi)|G(a_j)||I_j| + |I_j| O(\varepsilon).\end{aligned}$$

Adding for $j = 0, 1, \cdots, m-1$ and letting $m \to \infty$ and $\varepsilon \to 0$ we obtain (2.2).

LEMMA 3. — *Suppose that* $m_1, m_2, \ldots$ *is a sequence of positive integers such that any integer can be writen in at most one way in the form* $b_1 m_1 + \cdots + b_n m_n$, *for some integer* n, *where* $b_j \in \{-1, 0, 1\}$, $j = 1, 2, \ldots$. *Then for any* g *of the form* $g(x) = a_0 + a_1 \exp(ix) + \cdots + a_n \exp(inx) + \cdots$ *we have*

$$\left\{\sum_{j=1}^{\infty} |a_{m_j}|^2\right\}^{1/2} \leqslant C \int |g| (\log^+ |g|)^{1/2} + C \tag{2.3}$$

where $\log^+ a = \log a$ *if* $a \geqslant 1$ *and* $\log^+ a = 0$ *if* $0 < a \leqslant 1$.

Proof. — In the case of a lacunary sequence m_j, i.e. $m_{j+1}/m_j \geqslant a > 1$ for all j, this result, due to Zygmund, is known. The proof given in [4], XII 7.6 works word for word in the more general case we need here.

LEMMA 4. —

$$\sum_{s=n}^{2n} (1/2^s)\binom{s}{n} = 1. \tag{2.4}$$

Proof. – We write $A_n = \binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \cdots + \binom{2n}{n}$ and observe that (2.4) is equivalent to $A_n = 2^{2n}$. Using the formula

$$\binom{a+1}{b+1} = \binom{a}{b} + \binom{a}{b+1}$$

we obtain easily $A_{n+1} = 2A_n + (1/2)A_{n+1}$ i.e. $A_{n+1} = 4A_n$. The proof is completed by induction on n.

3. The main lemma.

Let $n_1 < n_2 < \dots < n_N$ be N integers and write

$$\Gamma = \{n_1,n_2,\dots,n_N\},\ f(x) = \sum_{m\in\Gamma} \cos mx .$$

Let $k_0 < k_1 < \dots < k_{2n}$ be $2n+1$ non negative distinct integers such that for each $r = 0, 1, \dots, 2n$ there are elements of Γ which are odd multiples of 2^{k_r}. We write

$$f_r(x) = \sum_{m\in\Gamma_r} \cos mx, \quad \Gamma_r = \{m \in \Gamma : m = \text{odd multiple of } 2^{k_r}\}.$$

We define now the function q to be 1 if more than n of the f_r's, $r = 0, 1, \dots, 2n$, are positive and -1 otherwise. Thus $q(x) = 2\chi(x) - 1$ where $\chi(x)$ is the characteristic function of the set where the number of positive f_r's exceeds the number of negative ones.

LEMMA 5. – *There is a set* $E \subset [0,2\pi]$ *and positive constants* $b_n, b_{n+1}, \dots, b_{2n}$ *such that* $|E| = 1 - 1/2^n$, $b_n + b_{n+1} + \cdots + b_{2n} = 1$ *and*

$$\int fq \geqslant \sum_{s=n}^{2n} b_s\|f_s\| + (1/2^{n-1})\int_E |f_{n-1}|. \tag{3.1}$$

Since the proof of this lemma is relatively long, it may preferably be read after paragraph 4 where its role to the proof of theorem 1 is explained.

Proof. – We write $g_r(x) = \operatorname{sgn} f_r(x)$ and observe that

(3.2) *The spectrum of* g_r *and of any product of the form* $g_r g_s \dots g_t$, $r < s < \dots < t$, *contains only odd multiples of* 2^{k_r}.

This property is an immediate consequence of the trivial fact that an integrable real function f has a spectrum consisting of odd multiples of 2^k if and only if f is $(2\pi/2^k)$-periodic and $f\left(\frac{\pi}{2^k}+x\right) = -f(x)$.

Let $\varepsilon = \{\varepsilon_0,\ldots,\varepsilon_{2n}\}$ be a sequence of $+1$'s and -1's and let Φ be the set of such sequences which contain more $+1$ than -1. The function $\chi(x)$ mentioned above is then given by the formula

$$\chi(x) = \sum_{\varepsilon\in\Phi}\prod_{k=0}^{2n}\left(\frac{1+\varepsilon_k g_k}{2}\right).$$

Since $\int f = 0$ we have

$$\int fq = 2\int f\chi = 2\int (f_0+\cdots+f_{2n})\chi$$

where the last equality is a consequence of (3.2). Thus

$$\int fq = 2\sum_{s=0}^{2n}\int f_s\chi.$$

In order to prove (3.1) we shall show that all integrals $A_s = \int f_s\chi$ are positive and that $2(A_n+A_{n+1}+\cdots A_{2n})$ and $2A_{n-1}$ exceed the first and second term respectively of the right hand side of (3.1). To this end we examine A_s for a fixed s. A_s is the sum of the integrals

$$\int f_s\prod_{k=0}^{2n}\left(\frac{1+\varepsilon_k g_k}{2}\right)$$

for all choices of ε in Φ. We observe now that

$$\int f_s\prod_{k=0}^{2n}\left(\frac{1+\varepsilon_k g_k}{2}\right) = \int\prod_{k<s}\left(\frac{1+\varepsilon_k g_k}{2}\right)\cdot f_s\left(\frac{1+\varepsilon_s g_s}{2}\right)\cdot\prod_{k>s}\frac{1+\varepsilon_k g_k}{2}\Bigg).$$

Because of (3.2) the contribution of the first factor in the last integral will be only a factor $1/2^s$ and hence

$$\text{(3.3)}\quad \int f_s\prod_{k=0}^{2n}\left(\frac{1+\varepsilon_k g_k}{2}\right) = (1/2^s)\int f_s\left(\frac{1+\varepsilon_s g_s}{2}\right)\prod_{k>s}\left(\frac{1+\varepsilon_k g_k}{2}\right)$$
$$= (1/2^{s+1})\int \varepsilon_s|f_s|\prod_{k>s}\left(\frac{1+\varepsilon_k g_k}{2}\right)$$

where we used again (3.2) in the last equality.

Consider now the contribution to A_s of those sequences ε which have a definite pattern for $k > s$ and suppose that this pattern contains m, $0 \leqslant m \leqslant \min(n, 2n-s)$ elements equal to -1.

We divide Φ, the class of admissible ε's, into $n+1$ classes Φ_0, $\Phi_1, \ldots, \Phi_n$ according to the number of elements equal to -1 that they contain. The elements ε with the given pattern for $k > s$ are distributed among the Φ_j's in the following manner : (i) There are no such elements for $j < m$, (ii) There is one such element in Φ_m with $\varepsilon_s = 1$ (iii) There are $\binom{s}{j-m-1}$ with $\varepsilon_s = -1$ and $\binom{s}{j-m}$ with $\varepsilon_s = 1$ in Φ_j, $j > m$.

It follows that the contribution to A_s of the ε's with the given pattern for $k > s$ is

$$\text{(3.4)} \quad \left\{(1/2^{s+1}) \int |f_s| \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right)\right\} \left\{1 + \left\{\binom{s}{1} - \binom{s}{0}\right\} + \cdots + \left\{\binom{s}{n-m} - \binom{s}{n-m-1}\right\}\right\}$$

$$= (1/2^{s+1}) \binom{s}{n-m} \int |f_s| \prod_{k>s} \left(\frac{1+\varepsilon_k g_k}{2}\right)$$

where $\binom{s}{a} = 0$ if $a > s$. A_s will be obtained by summing these expressions over all ε's with a pattern having m elements equal to -1 for $k > s$ and then summing over all m with $\max(0, n-s) \leqslant m \leqslant \min(n, 2n-s)$.

Thus it is evident that all A_s's are > 0. We examine now the cases $s = n-1$ and $s \geqslant n$.

<u>$s = n - 1$.</u> Since $\binom{s}{n-m} = 1$, A_{n-1} will exceed

$$(1/2^n) \int |f_{n-1}| \sum \prod_{k>n-1} \left(\frac{1+\varepsilon_k g_k}{2}\right)$$

where the summation exceeds over all choices of $+1$ and -1 for the ε_n, $\varepsilon_{n+1}, \ldots, \varepsilon_{2n}$ except the choices $\varepsilon_n = \varepsilon_{n+1} = \cdots = \varepsilon_{2n} = -1$ and

$$\varepsilon_n = \varepsilon_{n+1} = \cdots = \varepsilon_{2n} = 1.$$

We conclude that the sum appearing inside the integral sign equals 1 for all x except for the set of points where $f_n, f_{n+1}, \cdots, f_{2n}$ are all $\geqslant 0$ or all $\leqslant 0$. The last set has measure

$$\int \prod_{k>n-1} \left(\frac{1-g_k}{2}\right) + \int \prod_{k>n-1} \left(\frac{1+g_k}{2}\right) = 1/2^n$$

and so its complement E has measure $1 - (1/2^n)$. This shows that $2A_{n-1}$ exceeds the last term in the right hand side of (3.1).

$s \geqslant n.$ In this case all choices of ε_k, $k \geqslant n$, are permissible and all the factors $\binom{s}{n-m}$ are not less than $\binom{s}{n}$. Arguing as in the previous case we obtain

$$2A_s \geqslant 2(1/2^{s+1})\binom{s}{n}\int|f_s| = (1/2^s)\binom{s}{n}\|f_s\|.$$

Writing $b_s = (1/2^s)\binom{s}{n}$, $s = n, n+1, \ldots, 2n$ and using lemma 4 we see that the condition $b_n + b_{n+1} + \cdots + b_{2n} = 1$ is satisfied. This completes the proof of lemma 5.

4. Proof of theorem 1.

We recall our notation

$$\Gamma = \{n_1, n_2, \ldots, n_N\}, \quad F(x) = f(x) + i\tilde{f}(x) = \sum_{j=1}^{N} \exp(in_j x).$$

We shall prove theorem 1 by induction on N, so we assume that

$$\|H\| \geqslant C_0 (\log M)/(\log\log M)^2$$

for all exponential sums H with $M < N$ non zero terms, where C_0 will be determined in the course of the proof. We also recall that in this paragraph we shall give the proof for exponential sums only. The case of more general polynomials is essentially the same (see section 5). To avoid trivialities we assume, as we may, that N is large.

We shall first replace Γ by one of its translates. This obviously does not affect F and allows a profitable use of lemma 5.

We write the numbers $n_1, \ldots, n_N$ in the dyadic system and substract their common tail (if any). This amounts to a translation which leaves all the n_j's positive. Let 2^{k_0} be the highest power of 2 which divides all the elements of the so translated sequence. There are certainly odd and even multiples of 2^{k_0}. If the number of odd ones exceeds the number of the even ones then we add 2^{k_0} to all the elements of Γ. We write Γ_0 for the set of odd multiples of 2^{k_0}. Now we consider the set of even multiples of 2^{k_0} and substract again their common tail from all the elements of Γ. Let 2^{k_1} $(k_1 > k_0)$ be the highest power of 2 which divides all the even multiples of 2^{k_0}. If there are more odd than even multiples of 2^{k_1} (among the even multiples of 2^{k_0}) we add 2^{k_1} to all elements of $'\Gamma$. It is obvious that the operations in the second step do not affect $|\Gamma_0|$ and 2^{k_0}. We write Γ_1 for the set of odd multiples of 2^{k_1} and we continue in the same way until the sequence Γ is exhausted. We obtain thus a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ of disjoint subsets of Γ such that Γ_j contains only odd multiples of $2^{k_j}, j = 0, 1, \ldots, r$, $|\Gamma_j| \leqslant |\Gamma_{j+1}| + \cdots + |\Gamma_r|, j < r$ and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_r$.

Finally we translate the sequence Γ by a large multiple of 2^{k_r} so that the L^1 norm of any part of F will differ from $\pi/2$ times the L^1 norm of its real part by less than $1/N$. This is possible because of lemma 2.

Using now lemma 5 we have

$$\begin{aligned}(4.1)\quad ||F|| &= (\pi/2)||f|| + 0(N^{-1}) \geqslant (\pi/2)\int fq + 0(N^{-1}) \\ &\geqslant (\pi/2)\sum_{s=n}^{2n} b_s||f_s|| + (\pi/2)(1/2^{n-1})\int_E |f_{n-1}| + 0(N^{-1}) \\ &= \sum_{s=n}^{2n} b_s||F_s|| + (C/2^{n-1})\int_E |f_{n-1}| + 0(N^{-1})\end{aligned}$$

where $F_s = \sum_{m\in\Gamma_s} \exp(imx)$ and E depends on the choice of f_s's, $s = 0, 1, \ldots, 2n$. We shall now consider two cases :

Case a. « There are $2n + 1$, with n an integer satisfying the inequalities $\log\log\log N < n \log 2 < 1 + \log\log\log N$, sets Γ_j such that

$$|\Gamma_j| \geqslant N/(\log N)^4 ».$$

In this case the induction hypothesis and the fact that

$$b_n + b_{n+1} + \cdots + b_{2n} = 1 \text{ imply}$$

$$\begin{aligned}(4.2)\quad \sum_{s=n}^{2n} b_s||F_s|| &\geqslant C_0\left(\frac{\log N - 4\log\log N}{(\log\log N)^2}\right)(1 + 0(1/\log N)) \\ &\geqslant C_0\left(\frac{\log N}{(\log\log N)^2}\right) - \frac{5C_0}{\log\log N}\end{aligned}$$

provided that N is sufficiently large.

We shall show that the remaining term $(1/2^{n-1})\int_E |f_{n-1}|$ of (4.1) exceeds $5C_0/\log\log N$ if C_0 is suitably chosen.

We start with a few simple remarks :

(i) Except for a factor $1/2$, f_{n-1} is an exponential sum and hence lemma 1 applies to $2f_{n-1}$.

(ii) The norm $\|f_{n-1}\|$ is bounded by the norm of the sum of all terms in F corresponding to frequencies which are multiples (even and odd) of $2^{k_{n-1}}$. The norm of this sum is in turn bounded by $\|F\|$.

(iii) We may obviously assume that $\|F\| \leqslant \log N$, since otherwise there is nothing to prove.

(iv) The factors $(1/|E|)^{|E|}$ and $(1/|E'|)^{|E'|}$ appearing in (2.1) are bounded by an absolute constant.

Using these remarks and applying lemma 1 we obtain

$$1 \leqslant C\left(\int_E |f_{n-1}|\right)^{|E|} \|f_{n-1}\|^{|E'|} \leqslant C\left(\int_E |f_{n-1}|\right)^{|E|}$$

where in the last inequality we used (iii) and the estimate

$$|E'| = 1/2^n \leqslant 1/\log\log N .$$

It follows

$$\int_E |f_{n-1}| \geqslant C. \tag{4.3}$$

(4.1), (4.2), (4.3) and the inequality $1/2^{n-1} > 1/\log\log N$ yield

$$\|F\| \geqslant C_0\left(\frac{\log N}{(\log\log N)^2}\right) - \frac{5\,C_0}{\log\log N} + 0(N^{-1}) + \frac{C}{2^{n-1}}$$
$$\geqslant C_0\left(\frac{\log N}{(\log\log N)^2}\right)$$

provided that N is sufficiently large. This completes the proof of case *a*.

Case b. « At most $4\log\log\log N$ Γ_j*'s* are such that

$$|\Gamma_j| \geqslant N/(\log N)^4 \text{ ».}$$

Let $\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_k}$, $j_1 < j_2 < \ldots < j_k$, $k < 4\log\log\log N$ be the Γ_j*'s* with more than $N/(\log N)^4$ elements. We shall show that in this case *there are more than* $(\log N)^3$ *classes* Γ_j. Indeed we may assume that j_1, $j_2 - j_1, \ldots, j_k - j_{k-1}$ are less than $(\log N)^3$. We write

$$\Gamma'_j = \Gamma_{j+1} \cup \ldots \cup \Gamma_r ,$$

i.e. the set of even multiples of 2^{k_j}. Since $|\Gamma_j'| \geqslant |\Gamma_j|$ we have

$$|\Gamma_{j_1}'| \geqslant (1/2)|\Gamma_{j_1}' \cup \Gamma_{j_1}| \geqslant (1/2)\left\{N - \frac{N}{(\log N)^4}(\log N)^3\right| = (1/2)\left\{N - \frac{N}{\log N}\right\}.$$

Continuing in the same way we obtain

$$|\Gamma_{j_h}'| \geqslant (1/2^h)\left\{N - \frac{N}{\log N}(1+2+\cdots+2^{h-1})\right\}, \quad 1 \leqslant h \leqslant k.$$

Taking $h = k$ in the last inequality we see that $|\Gamma_{j_k}'|$ exceeds

$$CN/(\log \log N)^4.$$

Since $|\Gamma_j| \leqslant N/(\log N)^4$ if $j \geqslant j_k$, the number of Γ_j's with $j > j_k$ exceeds $\dfrac{C(\log N)^4}{(\log \log N)^4} \geqslant (\log N)^3$. On choosing now one frequency from each such class we obtain a subsequence of Γ which satisfies the hypotheses of lemma 3. Applying this lemma and the obvious inequality $\log |F| \leqslant \log N$ we have

$$(\log N)^{1/2}||F|| \geqslant C(\log N)^{3/2}$$

provided than N is large enough. If C_0 is chosen less than the constant C in the above inequality then (1.1) follows.

Since the cases a and b are obviously complementary the proof of theorem 1 is complete.

5. Remarks.

(i) Minor changes are needed for the proof of the general case of theorem 1 (coefficients not necessarily equal to 1). Only the application of lemma 3 in case b of section 4 deserves to be mentioned. In this case the assumption $||F|| \leqslant \log N$ implies $||F||_\infty \leqslant N \log N$ and hence

$$(\log^+ |F|)^{1/2} \leqslant 2(\log N)^{1/2}.$$

The extra factor 2 has obviously no effect to the proof.

(ii) Examining the proof of lemma 5 we see that more terms of the form $a_{rs}\displaystyle\int_{E_r} |f_s|$ with $a_{rs} > 0$ and $0 < |E_r| < 1$ could be added to the right hand side of (3.1). This can conceivably lead to an improvement of the factor $(\log \log N)^{-2}$ in (1.1). However, as far as we have to rely on lemmas 1 and 3 (or even lemma 4 of [2] which has not been used here), there is no hope that we can obtain the conjectured best bound $C \log N$. It is because of this reason that we did not base our proof on the refinements of lemma 5 mentioned above, but we prefered to keep the exposition as simple as possible at the cost of a probably better final estimate (see also remark b of [2]).

(iii) A common feature of the present proof, as well those used previously in connection with Littlewood's conjecture, is the following : Directly or indirectly we try to find a function q such that $\|q\|_\infty \leqslant 1$ and $\int fq$ is as large as possible. We assume that

$$f(x) = \cos(n_1 x) + \cos(n_2 x) + \cdots + \cos(n_N x).$$

The best choice of q is obviously $sgnf$, in which case $\|f\|$ is the sum of the $\hat{q}(n_j)$'s. However to obtain information on $\widehat{sgnf}(n_j)$'s appears to be a difficult task. We are thus led, very losely speaking, to « approximate » $sgnf$ by another bounded function which is easier to handle. The choice in the present paper was the characteristic function of the set where there are more positive than negative parts of a suitably chosen decomposition of f.

The above choice suggests another possible candidate for q ; namely the characteristic function of the set where there are more positive than negative terms $\cos n_j x$. Again it appears to be a difficult task to obtain information on the n_j, $1 \leqslant j \leqslant N$, Fourier coefficients of this q.

BIBLIOGRAPHY

[1] J. F. Fourier, On a theorem of Paley and the Littlewood conjecture, To appear in *Arkiv för Matematik*.

[2] S. K. Pichorides, On a conjecture of Littlewood concerning exponential sums (I), *Bull. Greek Math. Soc.*, Vol. 18 (1977), 8-16.

[3] S. K. Pichorides, On a conjecture of Littlewood concerning exponential suns (II), *Bull. Greek Math. Soc.*, Vol. 19 (1978), 274-277.

[4] A. Zygmund, Trigonometric Series. Vol. I, II. Cambridge University Press, 1968.

October 1979.

Manuscrit reçu le 12 novembre 1979.

S. K. Pichorides,
Perm. address
N. R. C. Demokritos
Aghia Paraskevi
Athens, Greece.
Pres. address
Univ. Paris Sud-bât. 425,
91405 Orsay, France.