

ANNALES DE L'INSTITUT FOURIER

JOEL H. SHAPIRO

Zeros of random functions in Bergman spaces

Annales de l'institut Fourier, tome 29, n° 4 (1979), p. 159-171

http://www.numdam.org/item?id=AIF_1979__29_4_159_0

© Annales de l'institut Fourier, 1979, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ZEROS OF RANDOM FUNCTIONS IN BERGMAN SPACES

by Joel H. SHAPIRO (*)

1. Introduction.

Let μ be a finite, positive, rotation invariant Borel measure on the open unit disc Δ of the complex plane, and suppose that μ gives positive mass to each annulus $r < |z| < 1$. For $0 < p < \infty$ the *weighted Bergman space* A_μ^p is the collection of functions f holomorphic in Δ with

$$\|f\|_p^p = \int |f|^p d\mu < \infty.$$

Let $A_\mu^{p+} = \bigcup_{q>p} A_\mu^q$, so $A_\mu^{p+} \subset A_\mu^p$. For f holomorphic in Δ , let $Z(f)$ denote the *zero set* of f , with each zero counted according to its multiplicity. If f belongs to some class \mathfrak{F} of holomorphic functions we frequently refer to $Z(f)$ as an \mathfrak{F} -*zero set*.

Recently we showed [7] that for each such μ and p there exists f in A_μ^p such that :

- (a) $Z(f)$ is contained in no A_μ^{p+} zero set, and
- (b) $Z(f+1) \cup Z(f-1)$ lies in no $A_\mu^{(p/2)+}$ zero set, hence in no A_μ^p zero set.

These results continued the work of Charles Horowitz [2] and Walter Rudin [4]. Horowitz considered the special measures

$$d\mu(z) = (1 - |z|)^\alpha dx dy \quad (\alpha > -1),$$

(*) Research partially supported by the National science Foundation.

and used infinite products to construct the desired functions ; while Rudin got similar results for Hardy spaces on the unit ball and polydisc in \mathbf{C}^n by means of an ingenious « multiplier argument ». The proof in [7] used Rudin's idea, with the desired function f constructed as a gap series.

The point of this paper is that Rudin's method (which we will describe in the next section) also works very naturally in the context of *random* power series. We show that a *Gaussian power series which almost surely lies in $A_\mu^p \setminus A_\mu^{p+}$ must almost surely have properties (a) and (b) listed above.*

More precisely, let $(\zeta_n)_0^\infty$ be a sequence of independent complex Gaussian random variables with mean zero and variance one [3 ; Ch. 9, sec. 3, p. 118]. Suppose $(a_n)_0^\infty$ is a sequence of complex numbers with

$$(1.1) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq 1,$$

and consider the random power series

$$(1.2) \quad f(z) = \sum_{n=0}^{\infty} \zeta_n a_n z^n.$$

Since almost surely $|\zeta_n| = O(\sqrt{\log n})$ [3 ; Ch. XI, sec. 4, p. 121, Prop. 3], condition (1.1) insures that with probability one the series (1.2) converges uniformly on compact subsets of Δ to a holomorphic function. The quantity which controls the random behavior of f is its variance $\sigma_f^2(z)$, defined for $z \in \Delta$ by

$$(1.3) \quad \sigma_f^2(z) = \mathcal{E}\{|f(z)|^2\} = \sum_{n=0}^{\infty} |a_n|^2 |z|^{2n}.$$

The main result of this paper is the following.

THEOREM 1. — *Suppose f is defined by formulas (1.1) and (1.2). Then*

(a) *the following are equivalent :*

(i) $\sigma_f \in L^p(\mu)$ but $\notin L^{p+}(\mu)$.

(ii) *With probability one :* $f \in A_\mu^p$ but $\notin A_\mu^{p+}$.

(iii) *With probability one :* $f \in A_\mu^p$ but $Z(f)$ is not contained in any A_μ^{p+} zero set.

(b) *If any (hence all) of the above conditions hold, then with probability one : $Z(f+1)$ and $Z(f-1)$ are A_μ^p zero sets, but their union is not even an $A_\mu^{(p/2)^+}$ zero set.*

The most important of these results are (b), and the implication (i) \rightarrow (iii) of (a) : these imply the corresponding results in [7]. For their proof we require only the most basic facts about Gaussian random variables. The other non-trivial implication in (a) is (ii) \rightarrow (i), which follows from a beautiful result of X. Fernique concerning moments of vector valued Gaussian random variables. These matters, Rudin's multiplier argument, and some other preliminaries are reviewed in section 2. Theorem 1 is proved in the third section, and the paper closes with some remarks and open problems.

I want to thank my colleague Joel Zinn of Michigan State University for several interesting discussions, and especially for pointing out Fernique's theorem to me.

2. Preliminaries.

(a) *Rudin's multiplier argument.* As exploited in both this paper and [7], Rudin's idea is this : if the zero set of $f \in A_\mu^p \setminus A_\mu^{p+}$ is contained in some A_μ^{p+} zero set, then $fh \in A_\mu^{p+}$ for some h holomorphic in Δ . Since h decreases the growth of f , it must have relatively small values where f is large. Assuming (without loss of generality) that $h(0) = 1$, we obtain from the subharmonicity of h :

$$(2.1) \quad 0 = \log |h(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta$$

for $0 \leq r < 1$, which forces h on the circle $|z| = r$ to balance out any small values with appropriate large ones. Therefore if $f \in A_\mu^p$ does not get into A_μ^{p+} because it has large values on substantial portions of certain circles $|z| = r_n (r_n \rightarrow 1-)$, then we should expect that no h holomorphic in Δ can multiply f into A_μ^{p+} . We will show in the next section that any Gaussian series (1.2) which almost surely lies in $A_\mu^p \setminus A_\mu^{p+}$ will almost surely be such an f . This complements the work in [7] where such f 's were constructed as gap series.

(b) *Gaussian random variables.* The reference for all of this material is [3; Ch. XI, sec. 1-4]. From now on $(\zeta_n)_0^\infty$ denotes a sequence of independent complex Gaussian random variables with mean zero and variance one, defined on a probability space $(\Omega, \mathfrak{F}, \text{Pr})$. In particular, (ζ_n) is an orthonor-

mal sequence in $L^2(\Omega, \mathfrak{F}, \text{Pr})$, and for each Borel subset B of the complex plane :

$$\text{Pr}\{\zeta_n \in B\} = \frac{1}{2\pi} \iint_B e^{-(x^2+y^2)/2} dx dy.$$

From this it follows quickly that for $0 \leq \lambda < \infty$,

$$\text{Pr}\{|\zeta_n| > \lambda\} = e^{-\pi\lambda^2}$$

[3; Ch. XI, sec. 4, p. 121, formula (3.1)]. A crucial property of the sequence (ζ_n) is that if (a_n) is a complex sequence with $\|a\|_2^2 = \sum |a_n|^2 < \infty$, and if $Z = \sum a_n \zeta_n$, then the random variable $Z/\|a\|_2$ has the same distribution as ζ_n . In particular :

$$(2.2) \quad \text{Pr}\{|Z| > \lambda \|a\|_2\} = e^{-\pi\lambda^2},$$

and for $0 < p < \infty$:

$$(2.3) \quad \mathcal{E}\{|Z|^p\} = C_p^p (\mathcal{E}\{|Z|^2\})^{p/2} = C_p^p \|a\|_2^p,$$

where C_p is independent of (a_n) , and \mathcal{E} denotes integration with respect to Pr . These are the only properties of (ζ_n) that we require for the main part of the proof of Theorem 1.

We remark in passing that the statement « f has property Q with probability one» (or «almost surely») means that there exists $E \in \mathfrak{F}$ with $\text{Pr}\{E\} = 1$ such that f has property Q for every $\omega \in E$. We do not require $\{\omega \in \Omega : f \text{ has property } Q\}$ to belong to \mathfrak{F} . Similar remarks apply to statements like «with probability $\geq \delta$, f has property Q ».

(c) *Interchanging measure and probability.* Some form of the next result occurs frequently in applications of probability to analysis.

LEMMA A [3; Ch. V, sec. 4, p. 42]. — Suppose $(\Omega, \mathfrak{F}, P)$ and (T, \mathfrak{B}, m) are probability spaces, and $E \in \mathfrak{F} \otimes \mathfrak{B}$ (product sigma-algebra). Define the usual cross-sections;

$$E^\omega = \{t \in T : (\omega, t) \in E\} \quad (\omega \in \Omega)$$

$$E_t = \{\omega \in \Omega : (\omega, t) \in E\} \quad (t \in T),$$

and suppose $0 \leq \theta, \eta \leq 1$. If $P\{E_t\} \geq \eta$ for $[m]$ a.e. t in T , then

$$P\{\omega \in \Omega : m(E^\omega) \geq \theta\eta\} \geq \frac{(1-\theta)\eta}{1-\theta\eta}.$$

Proof. — Let $A = \{\omega \in \Omega : m(E^\omega) \geq \theta\eta\}$. Then by Fubini's theorem :

$$\begin{aligned} \eta &\leq \int_T P\{E_t\} dm(t) \\ &= \int_\Omega m(E^\omega) dP(\omega) \\ &= \int_A + \int_{\Omega \setminus A} m(E^\omega) dP(\omega) \\ &\leq P(A) + \theta\eta[1 - P(A)], \end{aligned}$$

and the result follows upon solving for $P(A)$.

(d) *Fernique's Theorem.* For $f \in A_\mu^p$ let $\|f\| = \|f\|_p$ if $p \geq 1$, and $\|f\|_p^p$ if $0 < p < 1$. Then $\|\cdot\|$ is a norm on A_μ^p if $p \geq 1$, and a « p -norm » if $0 < p < 1$ (that is, $\|af\| = |a|^p\|f\|$ when $0 < p < 1$). It is not difficult to use the subharmonicity of $|f|^p$ to check that for each $z \in \Delta$ the linear functional of « evaluation at z »

$$f \rightarrow f(z) \quad (f \in A_\mu^p)$$

is continuous on A_μ^p ($0 < p < \infty$). From this it follows that A_μ^p , in the metric induced by $\|\cdot\|$, is complete, i.e., it is a Banach space when $p \geq 1$ and a « p -Banach space » when $0 < p < 1$. Even when $0 < p < 1$ there are enough continuous linear functionals to separate points (the point evaluations, for example), and the Borel structure induced on A_μ^p by the « norm » topology coincides with the one induced by the topology of uniform convergence on compact subsets of Δ (since the closed unit ball of A_μ^p is closed in this weaker topology).

From these considerations it follows routinely that if (u_n) is a sequence in A_μ^p for which the Gaussian series $Z = \sum \zeta_n u_n$ converges almost surely in A_μ^p , then, even when $0 < p < 1$, Z is an A_μ^p -valued Gaussian random variable in the following sense : if Z' and Z'' are independent and similar to Z , then so are $(Z' + Z'')/\sqrt{2}$ and $(Z' - Z'')/\sqrt{2}$.

Thus X. Fernique's Theorem [1] (or more precisely when $0 < p < 1$, its proof) applies to Z , and shows that the tail distribution $\Pr\{\|Z\| > \lambda\}$ decays exponentially as $\lambda \rightarrow \infty$. In particular,

$$(2.4) \quad \mathcal{E}\{\|Z\|_p^p\} < \infty,$$

which yields the following characterization of Gaussian Taylor series which a.s. belong to A_μ^p .

LEMMA B. — Suppose f and σ_f are given by (1.1)-(1.3). Then $f \in A_\mu^p$ almost surely if and only if $\sigma_f \in L^p(\mu)$.

Proof. — For any f given by (1.1) and (1.2), we have from (2.3) and Fubini's Theorem :

$$\int \sigma_f^p d\mu = C_p^{-p} \int \mathcal{E}\{|f|^p\} d\mu = C_p^{-p} \mathcal{E}\{\|f\|_p^p\}.$$

Thus $\sigma_f \in L^p(\mu)$ implies $\mathcal{E}\{\|f\|_p^p\} < \infty$, hence $\|f\|_p^p < \infty$ a.s. Conversely, suppose $\|f\|_p^p < \infty$ a.s. We claim f is an A_μ^p -valued Gaussian random variable. Indeed, the fact that the integral means

$$(2.5) \quad M_p^p(f; r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

increase with r [6; Theorem 17.6, p. 363], along with the monotone convergence theorem, show that the Taylor series (1.2) of f is a.s. Abel summable to f in A_μ^p . Thus by [3; Theorem 1, Ch. II, p. 11] (whose proof works even when $0 < p < 1$), the series converges a.s. in A_μ^p to f ; hence f is Gaussian. From (2.4) we see $\mathcal{E}\{\|f\|_p^p\} < \infty$, hence by the calculations at the beginning of this proof, $\sigma_f \in L^p(\mu)$. This completes the proof.

(e) *Two technical lemmas.* We close this section with two lemmas needed to deduce Theorem 1 from the essential probabilistic arguments, which will be isolated in Proposition 2 of the next section.

LEMMA C. — Given μ as usual, and $0 < p < \infty$, there exists a finite positive rotation invariant Borel measure ν whose closed support is $\{|z| \leq 1\}$, and such that $A_\mu^p = A_\nu^p$.

Proof. — For f holomorphic in Δ the integral mean $M_p^p(f; r)$ defined by (2.5) increases with r , so if $f \in A_\mu^p$ then

$$(2.6) \quad 2\pi M_p^p(f; r) \leq \|f\|_p^p \mu(r)^{-1}$$

where

$$\mu(r) = \mu\{z : r \leq |z| < 1\}.$$

Our standing hypotheses on the measure μ insure that $\mu(r) > 0$ for each $0 < r < 1$, and $\mu(r) \downarrow 0$ as $r \uparrow 1$. In particular μ is a bounded, strictly positive, measurable function on $[0,1)$, hence the measure

$$d\nu(z) = d\mu(z) + \mu(|z|) dx dy/\pi$$

has closed support equal to $\{|z| \leq 1\}$, and dominates μ . Since (2.6) insures that

$$\int_{\Delta} |f|^p dv \leq 2\|f\|_p^p$$

we see that $A_{\mu}^p = A_{\nu}^p$, as desired.

LEMMA D. — Suppose γ is a finite positive Borel measure on the interval $[0,1]$ which is either (i) purely atomic, or (ii) continuous with closed support equal to $[0,1]$. Then for $0 < \alpha < 1$:

$$(2.7) \quad \int_0^1 \gamma[r,1]^{-\alpha} d\gamma(r) < \infty.$$

Proof. — (i) Suppose γ is purely atomic, say with mass γ_n at r_n ($n = 1, 2, \dots$), and no mass anywhere else. Let $\rho_n = \sum_{k \geq n} \gamma_k$. Then the integral in (2.7) is just the series $\sum \rho_n^{-\alpha} \gamma_n$, whose convergence for $0 < \alpha < 1$ is a standard exercise in advanced calculus (see [5; Ch. 3, pp. 79-80, problem 12(b)] for the special case $\alpha = 1/2$).

(ii) If γ is continuous with closed support $= [0,1]$, then the function

$$\gamma(r) = \gamma([r,1]) \quad (0 \leq r < 1)$$

is continuous and strictly decreasing on $[0,1]$. The integral in (2.7) can then be interpreted as the Riemann-Stieltjes integral

$$- \int_0^1 v(r)^{-\alpha} dv(r)$$

which, after making the change of variable $x = v^{-1}(r)$ (composition inverse), and paying due respect to the singularity at $r = 1$, becomes [5; Theorem 6.19, p. 132]

$$\int_0^1 x^{-\alpha} dx < \infty.$$

This completes the proof.

3. Proof of the Main Theorem.

We isolate the essential part of Theorem 1 in the following proposition, which we state in somewhat more generality than actually required. The

following notations help the exposition. As in the proof of Lemma C, let

$$\mu(r) = \mu\{z \in \Delta : r \leq |z| < 1\}.$$

For b holomorphic in Δ and $0 \leq r < 1$, let

$$M_\infty(b; r) = \max\{|b(z)| : |z| = r\}$$

and write

$$b_r(e^{i\theta}) = b(re^{i\theta}).$$

From now on, f always represents a Gaussian power series as given by (1.1) and (1.2), with σ_f given by (1.3). We also assume that the measure μ has total mass 1, so $0 < \mu(r) \leq 1$.

PROPOSITION 2. — *Suppose that*

$$(3.1) \quad \limsup_{r \rightarrow 1^-} \frac{\sigma_f(r)\mu(r)^{1/p}}{-\log \mu(r)} > 0.$$

Then the following holds with probability one : for each positive integer N , every b holomorphic in Δ with

$$\limsup_{r \rightarrow 1^-} M_\infty(b; r)\mu(r)^{N/p} < 1,$$

and every h holomorphic in Δ , we have

$$(f^N + b)h \notin A_\mu^{p/N}.$$

Remark. — For part (a) of Theorem 1 we need only the case $N = 1$, $b \equiv 0$, while for part (b) we require $N = 2$, $b \equiv -1$. However, these special cases are no easier to prove than the general proposition, which gives some further information regarding remark (i), section 5 of [7].

Proof. — Let T denote the unit circle $\{|z| = 1\}$, m normalized Lebesgue measure on T , and μ_1 the unique finite positive Borel measure on $[0,1)$ such that

$$\int_\Delta g \, d\mu = \int_{[0,1)} \left\{ \int_T g(rt) \, dm(t) \right\} d\mu_1(r)$$

for each $g \in C_0([0,1))$.

Fix $k > 0$. We are going to show that the desired result holds with probability at least $k/(k+1)$; hence with probability one, since k is

arbitrary. In this regard the reader should note that although the set $\{\omega : f^N \notin A_\mu^{p/N}\}$ is a tail event, the set we are interested in :

$$\{\omega : (f^N + b)h \notin A_\mu^{p/N} \text{ for all } h, b \text{ as in the Proposition}\}$$

is *not* (in fact it is not even clear that it is an event), so the zero-one law does not apply.

According to the hypothesis (3.1) there is a positive number δ and a positive sequence $r_n \rightarrow 1 -$ such that

$$\sigma_f(r_n)\mu(r_n)^{1/p} \geq -\delta \log \mu(r_n).$$

Let $\lambda_n^{-1} = \sigma_f(r_n)\mu(r_n)^{1/p}$, so

$$(3.2) \quad 0 < \lambda_n \leq \frac{1}{-\delta \log \mu(r_n)} \rightarrow 0.$$

For each positive integer n , (2.2) insures that for all $t \in T$ we have

$$(3.3) \quad \Pr\{|f(r_n t)| > \lambda_n \sigma_f(r_n)\} = e^{-\pi \lambda_n} = \eta_n,$$

where $\eta_n \rightarrow 1$ because $\lambda_n \rightarrow 0$. Let

$$E(n) = \{(\omega, t) \in \Omega \times T : |f(r_n t)| > \lambda_n \sigma_f(r_n)\}.$$

Then using the notation of Lemma A, equation (3.3) asserts that $\Pr\{E_t(n)\} = \eta_n$ for every t in T ; hence Lemma A, with $\theta = \eta_n^k$, shows that with probability at least

$$\beta_n = \frac{\eta_n(1 - \eta_n^k)}{1 - \eta_n^{k+1}}$$

we have $m\{E^\omega(n)\} \geq \eta_n^{k+1}$ ($n = 1, 2, \dots$). Since $\beta_n \rightarrow k/(k+1)$ as $n \rightarrow \infty$, it follows that with probability $\geq k/(k+1)$:

$$(3.4) \quad m\{E^\omega(n)\} \geq \eta_n^{k+1} \text{ for infinitely many } n.$$

Let $F = \{\omega \in \Omega : (3.4) \text{ holds}\}$; then $\Pr\{F\} \geq k/(k+1)$. We are going to show that for each $\omega \in F$;

$$(f^N + b)h \notin A_\mu^{p/N}$$

whenever h, b, N are as in the hypothesis of the proposition. This will complete the proof.

To this end, fix $\omega \in F$ and b, N , and h . Suppose, as we may, that $h(0) = 1$, and choose $0 < \varepsilon < 1$ so that

$$\limsup_{r \rightarrow 1^-} M_\infty(b; r)\mu(r)^{N/p} < \varepsilon.$$

Letting $R_n = \{r_n \leq |z| < 1\}$ we have :

$$\begin{aligned} \int_{R_n} |(f^N + b)h|^{p/N} d\mu &= \int_{[r_n, 1)} \left\{ \int_T |(f_r^N + b_r)h_r|^{p/N} dm \right\} d\mu_1(r) \\ &\geq \int_{[r_n, 1)} \exp \left\{ (p/N) \int_T \log |(f_r^N + b_r)h_r| dm \right\} d\mu_1(r) \end{aligned}$$

by the arithmetic-geometric mean inequality. Let $I(r)$ denote the integral inside the braces. Then using (2.1) and the fact that $\int \log |g_r| dm$ increases with r for any holomorphic function g on Δ [6; Theorems 17.3 and 17.5, pp. 362-363] we obtain for $r_n \leq r < 1$:

$$\begin{aligned} I(r) &\geq \int_T \log |f_r^N + b_r| dm \\ &\geq \int_T \log |f_{r_n}^N + b_{r_n}| dm \\ &\geq \int_{E^{\omega(n)}} \log \|f_{r_n}^N - |b_{r_n}|\| dm. \end{aligned}$$

Since $\omega \in F$, this yields for infinitely many n :

$$\begin{aligned} I(r) &\geq \int_{E^{\omega(n)}} \log \|[\lambda_n \sigma_f(r_n)]^N - M_\infty(b; r_n)\| dm \\ &\geq m\{E^{\omega(n)}\} \log [\mu(r_n)^{-N/p} - \varepsilon \mu(r_n)^{-N/p}] \\ &\geq \eta_n^{k+1} \log [(1 - \varepsilon)\mu(r_n)^{-N/p}] \quad (\text{by (3.4)}) \end{aligned}$$

whenever $r_n \leq r < 1$. Thus, infinitely often :

$$\begin{aligned} \int_{R_n} |(f^N + b)h|^{p/N} d\mu &\geq \int_{[r_n, 1)} \exp \{(p/N)I(r)\} d\mu_1(r) \\ &\geq (1 - \varepsilon)^{p/N} \mu(r_n)^{1 - \eta_n^{k+1}}. \end{aligned}$$

Recalling the definition of η_n :

$$1 - \eta_n^{k+1} = 1 - e^{-(k+1)\pi\lambda_n} \leq (k+1)\pi\lambda_n,$$

so

$$\begin{aligned} \mu(r_n)^{1 - \eta_n^{k+1}} &\geq [\mu(r_n)^{\lambda_n}]^{(k+1)\pi} \\ &\geq \{\mu(r_n)^{1/\log \mu(r_n)}\}^{-\delta(k+1)\pi} \quad (\text{by 3.2}) \\ &= e^{-\delta(k+1)\pi}. \end{aligned}$$

Thus for each $\omega \in F$:

$$\limsup_{n \rightarrow \infty} \int_{R_n} |(f^N + b)h|^{p/N} d\mu \geq (1 - \varepsilon)^{p/N} e^{-\delta(k+1)\pi} > 0,$$

hence $(f^N + b)h \notin A_\mu^{p/N}$. This completes the proof.

Deduction of Theorem 1. — In part (a) the equivalence (i) \rightarrow (ii) is immediate from Lemma B (section 2), and the implication (iii) \rightarrow (ii) is trivial. So it remains to show that (i) implies both (iii) and (b). In view of Lemma B it is enough to show that if $\sigma_f \notin L^{p^+}(\mu)$, then with probability one : $Z(f)$ is not contained in any $A_\mu^{p^+}$ zero set, and $Z(f+1) \cup Z(f-1)$ is not contained in any $A_\mu^{(p/2)^+}$ zero set.

So suppose $\sigma_f \notin L^{p^+}(\mu)$. We will show in a moment that this implies

$$(3.5) \quad \limsup_{r \rightarrow 1^-} \sigma_f(r)\mu(r)^{1/q} = \infty$$

for each $q > p$, which yields :

$$\limsup_{r \rightarrow 1^-} \frac{\sigma_f(r)\mu(r)^{1/q}}{-\log \mu(r)} = \infty$$

for each $q > p$. Thus Proposition 2 (with q replacing p) guarantees that for each $q > p$ it is almost sure that

$$(f^N + b)h \notin A_\mu^{q/N}$$

for every h holomorphic in Δ , b constant, and $N = 1, 2, \dots$. Since a countable intersection of sets of probability one again has probability one, it follows upon quoting the above result for a sequence $q_n \downarrow p$ that almost surely : $(f^N + b)h \notin A_\mu^{(p/N)^+}$ for all b, h, N as above.

Taking $N = 1, b \equiv 0$ we see from the discussion of section 2 (a) that a.s. $Z(f)$ is contained in no $A_\mu^{p^+}$ zero set, which proves (iii). Taking $N = 2$ and $b \equiv -1$ we see that a.s.

$$Z(f^2 - 1) = Z(f+1) \cup Z(f-1)$$

lies in no $A_\mu^{(p/2)^+}$ zero set, which proves (b).

It remains only to prove that (3.5) holds for each $q > p$. Suppose not. Then for some $q > p$:

$$(3.6) \quad \sigma_f(r) = O(\mu(r)^{-1/q}) \quad (r \rightarrow 1^-).$$

Fix $p < s < q$. We will show that $\sigma_f \in L^s(\mu)$, contrary to the hypothesis on σ_f . By Lemma C we may assume that the measure μ has $\{|z| \leq 1\}$ as its closed support, hence the closed support of μ_1 is the interval $[0,1]$. Thus $\mu_1 = \gamma_1 + \gamma_2$, where γ_1 is purely atomic and γ_2 is continuous with closed support $[0,1]$. By (3.6) we have

$$\sigma_f(r) = O(\gamma_i[r,1]^{-1/q}) \quad (r \rightarrow 1 -)$$

for $i = 1,2$; hence by Lemma D,

$$\int_0^1 \sigma_f(r)^s d\gamma_i(r) < \infty \quad (i = 1,2),$$

hence $\sigma_f \in L^s(\mu)$: a contradiction. This completes the proof of Theorem 1.

4. Concluding Remarks.

Lemma B suggests that Proposition 2 should be capable of improvement.

CONJECTURE. — *If f is not a.s. in A_μ^p (hence by the zero-one law, a.s. not in A_μ^p), then a.s. $(f^N + b)h \notin A_\mu^{p/N}$ for all b, h, N as in the statement of Proposition 2.*

The arithmetic-geometric mean inequality seems to give away too much to get this result: In the case $N = 1, b \equiv 0$, Fernique's inequality might be a possibility. It is not difficult to check that if $fh \in A_\mu^p$ a.s. for some fixed holomorphic h in Δ , then fh is an A_μ^p -valued Gaussian random variable. Then Fernique's inequality, the rotational symmetry of $\sigma_f(z)$, and the monotonicity of $M_p^p(h,r)$ yield:

$$\begin{aligned} \infty &> \mathcal{E}\{\|fh\|_p^p\} \\ &= \int \mathcal{E}\{|fh|^p\} d\mu \\ &= \int \mathcal{E}\{|f|^p\} |h|^p d\mu \\ &= C_p^p \int \sigma_f^p |h|^p d\mu \\ &= C_p^p \int_0^1 \sigma_f(r)^p M_p^p(h;r) d\mu_1(r) \\ &\geq C_p^p \int \sigma_f^p d\mu, \end{aligned}$$

hence $f \in A_{\mu}^p$ a.s. But this merely shows that :

$$f \notin A_{\mu}^p \text{ a.s.} \Rightarrow \forall h \text{ holomorphic in } \Delta; fh \notin A_{\mu}^p \text{ a.s.}$$

whereas the desired result is :

$$f \notin A_{\mu}^p \text{ a.s.} \Rightarrow \text{a.s. : } fh \notin A_{\mu}^p \forall h \text{ holomorphic in } \Delta.$$

BIBLIOGRAPHY

- [1] X. FERNIQUE, Intégrabilité des vecteurs gaussiens, *C.R. Acad. Sci.*, Paris, 270 (1970), 1698-1699.
- [2] Ch. HOROWITZ, Zeros of functions in Bergman spaces, *Duke Math. J.*, 41 (1974), 693-710.
- [3] J.P. KAHANE, Some Random Series of Functions, D.C. Heath and Co., Lexington, MA, 1968.
- [4] W. RUDIN, Zeros of holomorphic functions in balls, *Indag. Math.*, 38 (1976), 57-65.
- [5] W. RUDIN, Principles of Real Analysis, 3rd ed., McGraw-Hill, New York, 1976.
- [6] W. RUDIN, Real and Complex Analysis, McGraw-Hill, New York, 1974.
- [7] J. H. SHAPIRO, Zeros of functions in weighted Bergman spaces, *Michigan Math. J.*, 24 (1977), 243-256.

Manuscrit reçu le 5 juin 1979.

Joel H. SHAPIRO,
Department of Mathematics
Michigan State University
East Lansing, MI 48824 (USA).
